Nowhere-zero vectors in the row space or null space of certain incidence matrices

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Part I

A *zero-sum flow* for a matrix N is a nowhere-zero vector in the null space of N.

These are equivalent:

- N admits a zero-sum flow.
- The deletion of any column of N does not decrease the rank.

For a real matrix N, let P be the matrix of the orthogonal projection from $\mathbb{R}^m$ onto the row space of N, and $Q = I - P$ the matrix of the orthogonal projection onto the null space of N. (When N has full row rank, $P = N^\top (NN^\top)^{-1} N$.)

These are equivalent:

- N admits a zero-sum flow.
- The deletion of any column of N does not decrease the rank.
- The diagonal entries of Q are nonzero.

Part of a proof: Check that the deletion of the first column of N reduces the rank if and only if $(1, 0, 0, \ldots, 0) \in \text{row}(N)$. This is the case if and only if $(1, 0, 0, \ldots, 0)Q = (0, 0, \ldots, 0)$. $\square$

Theorem 1. (W, 1982) Let $N = W_{tk}$ be the incidence matrix of t-subsets versus k-subsets of an n-set, and let $\overline{W}_{tk}$ be the "disjointness matrix". If $t \le k \le n - t$, then

$$P = \sum_{i=0}^{t} (-1)^i \frac{\binom{k-i}{t-i}}{b_t^i} \; \overline{W}_{ik}^{\top} W_{ik}.$$

Theorem *q*. Let $N = N_{tk}$ be the incidence matrix of t-dimensional subspaces versus k-dimensional subspaces of an n-dimensional space over $\mathbb{F}_q$, and let $\overline{N}_{tk}$ be the "skewness matrix". If $t \le k \le n - t$, then

$$P = \sum_{i=0}^{t} (-1)^i q^{\binom{i}{2}} \frac{\begin{bmatrix} k-i \\ t-i \end{bmatrix}_q}{b_t^i} \; \overline{N}_{ik}^{\top} N_{ik}.$$

Here b_t^i denotes the number of k-subsets that contain t of the points but none of i other points in Theorem 1, and has the q-analogous meaning in Theorem q.

We can see that the diagonal terms of Q are positive when $t < k < n - t$. Thus the "linear matrices" (or "projective matrices") N_{tk} admit zero-sum flows whenever they have more columns than rows. This answers a question mentioned in the talk of S. Shahriari.

In general, given a set of S columns, the drop in rank when the columns S are deleted from N is the nullity of the principal submatrix of Q with rows and columns indexed by S.

If N is the inidence matrix of a block design, then

$$r(r-\lambda)Q = r(r-\lambda)I - rN^\top N + \lambda kJ$$

The diagonal entries are $(r-k)(r-\lambda)$, and this is positive if $r > k$, i.e. if $b > v$. Cf. Akbari, Khosrovshahi, Mofidi "Zero-sum flows in designs".

If we delete two columns of N, corresponding to blocks A and B, the rank will decrease if and only if

$$\begin{pmatrix} (r-k)(r-\lambda) & k\lambda - r\mu \\ k\lambda - r\mu & (r-k)(r-\lambda) \end{pmatrix}$$

is singular, where $\mu = |A \cap B|$. That is, if and only if equality hold in Connor's Inequalities

$$|k\lambda - r\mu| \leq (r-k)(r-\lambda).$$

Part II

A zero-sum Ramsey-type problem.

For this talk, we motivate our results on diagonal forms of certain incidence matrices by a zero-sum Ramsey-type problem of Alon and Caro (1993).

What is the least integer $R(H;m)$ so that if $n \geq R(H;m)$ and the edges of the complete t-uniform hypergraph $K_n^{(t)}$ on n vertices are colored with integers from $\{0, 1, \ldots, m-1\}$, then there exists a subhypergraph H' isomorphic to H so that the sum of the colors on the edges of H' is 0 modulo m?

The classical Ramsey's Theorem implies that such an integer exists when the number of edges of H is $\equiv 0 \pmod{m}$.

Caro proved (1996) that when $\binom{k}{t}$ is even,

$$R(K_k^{(t)}; 2) \leq k + t.$$

When $t = 2$, $R(K_k; 2) = k + 2$. (See the blackboard.)

Theorem 1 (W, 2002) *When $\binom{k}{t}$ is even, $R(K_k^{(t)}; 2)$ is equal to $k + 2^e$ where 2^e is the least power of 2 that appears in the base 2 representation of t but not in the base 2 representation of k.*

Theorem 2 (W, 2002) *For any t-uniform hypergraph H with k vertices and an even number of edges, $R(H; 2) \leq k + t$.*

Note that for a graph G on k vertices, $R(G;2) = k$ means that no matter how the edges of K_k are colored with 0 and 1, there is a copy of G in K_k that has an even number of edges of color 1. It is very common that $R(G;2) = k$.

Theorem 3 (Y. Caro, 1994) *For a simple graph G on k vertices with an even number of edges, $R(G;2) = k$ unless* (i) $G = K_k$, (ii) $G = K_a \cup K_b$ *with* $a + b = k$ *or* (iii) *all vertices of G have odd degree.*

Theorem 4 (W. and Tony Wong, 2012) *For a t-uniform hypergraph H on k vertices with an even number of edges, $R(H;2) = k$ almost always.*

The matrices $N_t(\mathbf{h})$

Fix t and consider integer column vectors $\mathbf{h}$ where the coordinates of $\mathbf{h}$ are indexed by the t-subsets of an n-set X. We may call such a vector $\mathbf{h}$ a t-*vector* based on the set X. As an important instance, $\mathbf{h}$ may be the characteristic (0,1)-vector of a simple t-uniform hypergraph.

Given an integer t-vector $\mathbf{h}$ based on a n-set X, we consider the matrix $N_t(\mathbf{h})$, or simply N_t, whose columns are all images of $\mathbf{h}$ under the symmetric group S_n. An example when $n = 3$ and

$t = 1$ is

$$N_1 = \begin{pmatrix} 3 & 5 & 9 & 3 & 5 & 9 \\ 5 & 9 & 3 & 9 & 3 & 5 \\ 9 & 3 & 5 & 5 & 9 & 3 \end{pmatrix}.$$

Normally, one need only use the *distinct* images of **h** as the columns of N_t, but, for our purposes, it will not matter if N_t has repeated columns. In fact, it is sometimes convenient for the purposes of induction to assume that N_t has $n!$ columns indexed by the set of all permutations of X.

Given a t-uniform hypergraph with vertex set X, let $N_t(H) = N_t(\mathbf{h})$ where **h** is the *characteristic t-vector* of H. Here $N_t(H)$ is a $(0,1)$-matrix.

Example: Let $n = 4$ and let G be the path of length 2 plus an isolated vertex. Then $N_2(G)$ is the 6×12 matrix below.

	213	214	314	123	124	324	132	234	134	...
$\{1,2\}$	1	1	0	1	1	...				
$\{1,3\}$	1	0	1	0	0	...				
$\{1,4\}$	0	1	1	0	0	...				
$\{2,3\}$	0	0	0	1	0	...				
$\{2,4\}$	0	0	0	0	1	...				
$\{3,4\}$	0	0	0	0	0	...				

$W_{23}(6) = N_2(\Delta + \cdots)$ Each column is a 2-vector.

$$
\begin{array}{c}
\{1,2\} \\ \{1,3\} \\ \{2,3\} \\ \{1,4\} \\ \{2,4\} \\ \{3,4\} \\ \{1,5\} \\ \{2,5\} \\ \{3,5\} \\ \{4,5\} \\ \{1,6\} \\ \{2,6\} \\ \{3,6\} \\ \{4,6\} \\ \{5,6\}
\end{array}
\begin{pmatrix}
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{pmatrix}
$$

Let $H^{\uparrow n}$ denote the hypergraph obtained by adjoing isolated vertices to H in order to obtain a total of n vertices.

Given a simple t-uniform hypergraph H on k vertices, the matrix $N_t(H^{\uparrow n})$ has as columns the characteristic vectors of all subhypergraphs of $K_n^{(t)}$ isomorphic to H. A coloring of the edges of $K_n^{(t)}$ is a t-vector $\mathbf{x}$ based on $V(K_n^{(t)})$. The sum of the colors on the edges of a copy H' of H is the H'-coordinate of $\mathbf{x}N_t(H^{\uparrow n})$. Thus $R(H;m)$ is the least integer n so that the module generated by the rows of $N_t(H^{\uparrow n})$ contains no vectors with all coordinates $\not\equiv 0 \pmod{m}$.

In particular, $R(H;2)$ is the least integer n so that the binary code generated by $N_t(H^{\uparrow n})$ does not contain the vector of all ones.

Diagonal form

Given a matrix A, a *diagonal form* for A is a diagonal matrix D of the same dimensions as A so that for some unimodular matrices E and F,

$$EAF = D.$$

The diagonal entries $d_1, d_2, \ldots$ of D may be called (a set of) *diagonal factors* for A. When the d_i's are nonnegative and divide one another successively, i.e. $d_1 \mid d_2 \mid \ldots$, then D is the (integer) Smith (normal) form of A and the diagonal entries are the invariant factors of A. (If j is greater that the number of rows or columns of A, it is convenient to understand $d_j = 0$.)

As a simple example, a diagonal form for $A = \begin{pmatrix} 2 & 1 & -1 \\ 4 & 7 & 3 \end{pmatrix}$ is $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix}$.

Another diagonal form for A is $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 10 & 0 \end{pmatrix}$.

Let A be given. For any unimodular matrix E, let D be the diagonal matrix with d_i equal to the gcd of the elements of the i-th row of EA. Then $EA = DU$ for some integer matrix U. If U is row-unimodular (in which case we call U a *front* for A), then D is a diagonal form for A. As an illustration,

$$\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}2 & 1 & -1\\ 4 & 7 & 3\end{pmatrix} = \begin{pmatrix}6 & 8 & 2\\ 10 & 15 & 5\end{pmatrix} = \begin{pmatrix}2 & 0\\ 0 & 5\end{pmatrix}\begin{pmatrix}3 & 4 & 1\\ 2 & 3 & 1\end{pmatrix}$$

$$= \begin{pmatrix}2 & 0 & 0\\ 0 & 5 & 0\end{pmatrix}\begin{pmatrix}3 & 4 & 1\\ 2 & 3 & 1\\ 0 & 0 & 1\end{pmatrix}.$$

For $t \leq k \leq n$, Let W_{tk} be the inclusion matrix of t-subsets of an n-set versus k-subsets of the n-set. This is $H_t((K_k^{(t)})^{\uparrow n})$, except possibly for repeated columns.

It is possible to find unimodular matrices consisting of $\binom{n}{t}$ rows from the union of the rows of $W_{0t}, W_{1t}, \ldots, W_{tt}$. E.g., when $n = 4$, $t = 2$, one example is

$$E_2(6) \quad = \quad \begin{array}{c} \emptyset \\ \{1\} \\ \{2\} \\ \{3\} \\ \{1{,}3\} \\ \{3{,}4\} \end{array} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Given j, k, v, $j \leq k \leq v - t$, let E_{jk} be obtained from W_{jk} by deleting the rows of W_{jk} corresponding to a $(j-1, j)$-basis. So E_{jk} is of size $\binom{v}{j} - \binom{v}{j-1}$ by $\binom{v}{k}$.

Theorem 5 (W, 1999, 2008) *Given t, k, n, $t \leq k \leq n - t$, the $\binom{n}{t}$ by $\binom{n}{k}$ matrix*

$$\bigsqcup_{j=0}^{t} E_{jk} = \begin{array}{|c|c|} \hline E_{0k} & 1 \text{ row} \\ \hline E_{1k} & n-1 \text{ rows} \\ \hline E_{2k} & \binom{n}{2} - n \text{ rows} \\ \hline & \vdots \\ \hline E_{tk} & \binom{n}{t} - \binom{n}{t-1} \text{ rows} \\ \hline \end{array}$$

is row unimodular.

When $2t \leq n$, the matrix $E_t = \sqcup_{j=0}^{t} E_{jt}$ is unimodular.

Theorem 6 (W, 1999) *If a t-uniform hypergraph H has at least t isolated vertices, then E_t is a front for $N_t(H)$.*

This means that one set of diagonal factors for $N_t(H)$ is

$$(g_0)^1,\ (g_1)^{n-1},\ (g_2)^{\binom{n}{2}-n} \ldots,\ (g_t)^{\binom{n}{t}-\binom{n}{t-1}},$$

where g_i is the gcd of all entries of $E_{it}N_t(\mathbf{h})$ (this is the same as the gcd of the entries of $W_{it}\mathbf{h}$), where $\mathbf{h}$ is the characteristic vector of H. The number g_i is the gcd of the 'degrees' of i-subsets of the vertices of H. In particular, g_0 is the number of edges of H, g_1 is the gcd of the degrees of the vertices, and g_t is 1 if H is simple with at least one edge.

Theorem 7 (W, Tony Wong, 2012) *If a t-uniform hypergraph H has the property that it and all of its shadows are primitive or multiples of primitive hypergraphs, then E_t is a front for $N_t(H)$.*

Theorem 8 (W, Tony Wong, 2012) *A random t-uniform hypergraph H almost surely has the property that it and all of its shadows are primitive hypergraphs.*

Primitivity and shadows (as we are using the terms here) will be defined later.

We remark that any t-uniform hypergraph with at least one edge and t isolated vertices has the property that it and all of its shadows are primitive or multiples of primitive hypergraphs. A simple t-uniform hypergraph with $t-1$ isolated vertices that is not the union of a complete t-uniform hypergraph and $t-1$ isolated vertices also has the property.

Solutions of systems of congruences

H. J. S. Smith's original paper on Smith form was concerned with integer solutions of linear equations.

Lemma 9 *Let A be an integer matrix and* **b** *a integer column vector. Assume $EAF = D$ where E and F are unimodular and D diagonal with diagonal entries $d_1, d_2, \ldots$. The system of equations $A\mathbf{x} = \mathbf{b}$ has an integer solution* **x** *if and only if the i-th entry of $E\mathbf{b}$ is divisible by d_i for $j = 1, 2, \ldots, n$.*

Lemma 10 *Let* A *be an integer matrix and* $\mathbf{c}$ *an integer row vector. Suppose* $EAF = D$ *where* E *and* F *are unimodular and* D *is diagonal with diagonal entries* $d_1, d_2, \ldots$. *The system of congruences* $\mathbf{y}A \equiv \mathbf{c} \pmod{m}$ *has an integer solution* $\mathbf{y}$ *if and only if the* j*-th entry of* $\mathbf{c}F$ *is divisible by the* $\gcd\ (d_j, m)$ *for* $j = 1, 2, \ldots, n$.

Proof. The congruences $\mathbf{y}A \equiv \mathbf{c} \pmod{m}$ can be written as $(\mathbf{y}E^{-1})EAF \equiv \mathbf{c}F \pmod{m}$, and there is an integer solution $\mathbf{y}$ of this system if and only if there is an integer solution $\mathbf{z}$ of $\mathbf{z}D \equiv \mathbf{c}F \pmod{m}$. $\square$

We want to know if the vector $(1,1,\ldots,1)$ is congruent modulo 2 to some vector $\mathbf{y}N_t(H)$ with $\mathbf{y}$ an integer vector. If E_t is a front for $N_t(H)$, then $E_tN_t(H) = DF^{-1}$ as below, where g_0 is the number of edges of H.

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ & & & \\ & & & \end{pmatrix} N_t(H) = \begin{pmatrix} g_0 & 0 & \cdots & 0 \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ & & & & & \\ & & & & & \end{pmatrix}.$$

Note that $(1,1,\ldots,1)F = (1,0,0,\ldots)$. By Lemma 10, $(1,1,\ldots,1)$ is in the binary code generated by $N_t(H)$ if and only if the gcd of 2 and d_j divides the j-th coordinate of $(1,0,0,\ldots)$, which is the case if and only if g_0 is odd.

More generally, if g_0 is even and there is a front with $(1, 1, \ldots, 1)$ as a row, then $(1, 1, \ldots, 1)$ is not a codeword.

We can describe diagonal factors for any simple graph. For example:

- N_2 for the Petersen graph (n=10) has diagonal factors $1^{35},\ 3^{1},\ 0^{8},\ 15^{1}$.

- N_2 for the Petersen graph plus an isolated vertex (n=11) has diagonal factors $1^{44},\ 3^{10},\ 15^{1}$.

We can reprove Caro's characterization of simple graphs with $R(G;2) = k$.

Primitivity

A t-vector $\mathbf{h}$ will be said to be *primitive* when the gcd of $\langle \mathbf{h}, \mathbf{f} \rangle$ over all integer vectors $\mathbf{f}$ in the null space of $W_{t-1,t}$ is 1.

Spanning sets over the integers for $\text{null}_{\mathbb{Z}}(W_{jt})$ (the module of "null designs" or "trades") have been described circa 1970 by Graham, Li, and Li, and by Graver and Jurkat. For any choice of distinct points $a_1, a_2, \ldots, a_t, b_1, b_2, \ldots, b_t$, consider the t-vector $\mathbf{f}$ where $f(T)$ is the coefficient of the monomial $\prod_{c \in T} c$ in the polynomial

$$(a_1 - b_1)(a_2 - b_2) \cdots (a_t - b_t).$$

We call t-vectors of this form t-pods, and they generate $\text{null}_{\mathbb{Z}}(W_{t-1,t})$.

A 1-vector **h** based on $X = \{1, 2, \ldots, n\}$ is primitive when the gcd of the quantities $\mathbf{h}(i) - \mathbf{h}(j)$ is 1.

A graph (or signed multigraph) G, or 2-vector **g**, is primitive when the gcd of the quantities

$$\mathbf{g}(\{a,b\}) + \mathbf{g}(\{c,d\}) - \Big(\mathbf{g}(\{b,c\}) + \mathbf{g}(\{d,a\})\Big)$$

over all choices of four distinct vertices a, b, c, d is 1.

A 3-uniform hypergraph H or 3-vector **h** is primitive when the gcd of all quantities

$$\mathbf{h}(a_1,a_2,a_3) + \mathbf{h}(a_1,b_2,b_3) + \mathbf{h}(b_1,a_2,b_3) + \mathbf{h}(b_1,b_2,a_3)$$
$$- \left(\mathbf{h}(b_1,b_2,b_3) + \mathbf{h}(b_1,a_2,a_3) + \mathbf{h}(a_1,b_2,a_3) + \mathbf{h}(a_1,a_2,b_3)\right)$$

over all choices of six distinct vertices $a_1, a_2, a_3, b_1, b_2, b_3$ is 1.

Theorem 11 (W, Wong) *A simple graph G is primitive unless G is isomorphic to a complete graph, an edgeless graph, a complete bipartite graph, or the disjoint union of two complete graphs.*

The *shadow* of a t-vector $\mathbf{h}$ is the $(t-1)$-vector $W_{t-1,t}\mathbf{h}$ and the j-th shadow is $W_{t-j,t}\mathbf{h}$.

The shadow of a graph is its vector (1-vector) of its degrees.

Theorem 12 *Let G be a primitive simple graph with m edges and degrees $\delta_1, \delta_2, \ldots, \delta_n$. Let h denote the* gcd *of the degrees δ_i and m; let g denote the* gcd *of all differences $\delta_i - \delta_j$, $i, j = 1, 2, \ldots, n$. Then the invariant factors of $N_2(G)$ are*

$$(1)^{\binom{n}{2}-n}, \quad (h)^1, \quad (g)^{n-2}, \quad (mg/h)^1.$$

• N_2 for the Petersen graph (n=10) has diagonal factors $1^{35},\ 3^1,\ 0^8,\ 15^1$.

• N_2 for the Petersen graph plus an isolated vertex (n=11) has diagonal factors $1^{44},\ 3^{10},\ 15^1$.

Nonprimitive graphs may be considered separately. Here is one case.

Theorem 13 *Let G be the complete bipartite graph $K_{r,n-r}$, where $2 \le r \le n-2$. Define m, g, and h as in the statement of Theorem 12, so in this case*

$$m = r(n-r), \quad g = n-2r, \quad h = \mathrm{gcd}\{r, n-r\}.$$

Then the diagonal entries of one diagonal form for $N_2(G)$ are

$$(1)^{n-2}, \quad (2)^{\binom{n}{2}-2n+2}, \quad (h)^1, \quad (2g)^{n-2}, \quad (mg/h)^1.$$

In the case $r = 2$, the matrix N_2 is square; it is the adjacency matrix of the line graph of the complete graph K_n, and we have reproved a result of Brouwer and Van Eijl.