Continuous Quantum Walks

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Outline

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Cosmology

Quote

"Hydrogen is a colorless, odorless gas which given sufficient time, turns into people." (Henry Hiebert)

Don't Worry About The Physics

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Quote

"a vector is not the same thing as the list of its components. The vector has a . . . meaning."

Outline

A Transition Operator

Let A be the adjacency matrix of the graph X.

Definition

The transition operator $U(t)$ is defined by

$$U(t) := \exp(itA)$$

It determines a continuous quantum walk.

Perfect State Transfer

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If there is such a time, we say that we have perfect state transfer on X from a to b.

Pretty Good State Transfer

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An Example: K_2

If $X = K_2$, then

$$U(t) = \begin{pmatrix} \cos(t) & i\sin(t) \\ i\sin(t) & \cos(t) \end{pmatrix}.$$

and

$$U(\pi/2) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Composite Systems

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- The Cartesian product of X and Y has adjacency matrix $A_X \otimes I + I \otimes A_Y$. Since $A_X \otimes I$ and $I \otimes A_Y$ commute,

$$U_{X \square Y}(t) = U_X(t) \otimes U_Y(t).$$

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- Since we have pst at time $\pi/2$, we have pst at $\pi/2$ on the d-cube for all d.

Outline

1. Perfect State Transfer
   - Physics 101
   - State Transfer
   - Basics of State Transfer
2. Pretty Good State Transfer
   - Basics
   - PGST on Paths

Properties of $U(t)$

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2. $\overline{U(t)} = U(-t)$.
3. $U(t)$ is unitary.
4. If $|U(t)_{a,b}| = 1$, then $|U(t)_{b,a}| = 1$ and so ab-pst implies ba-pst.

Automorphisms

We identify the automorphism group $\mathrm{Aut}(X)$ with the set of permutation matrices that commute with A. If $a \in V(X)$, then $\mathrm{Aut}(X)_a$ denote the subgroup of all automorphisms of X that fix a. Note that $P \in \mathrm{Aut}(X)_a$ if and only if $Pe_a = e_a$.

Lemma

If we have perfect state transfer from a to b in X, then $\mathrm{Aut}(X)_a = \mathrm{Aut}(X)_b$.

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Lemma

If we have perfect state transfer from a to b in X, then $\mathrm{Aut}(X)_a = \mathrm{Aut}(X)_b$.

It follows that if $n > 2$, we do not have pst on K_n.

Cospectral Vertices

Lemma

If we have perfect state transfer from a to b in X, then for all $m \geq 0$,

$$e_a^T A^m e_a = e_b^T A^m e_b.$$

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Corollary

If we have perfect state transfer from a to b in X, then $X \setminus a$ and $X \setminus b$ are cospectral.

Spectral Decomposition

If A is a symmetric matrix with distinct eigenvalues $\theta_1, \ldots, \theta_m$, there are symmetric idempotent matrices $E_1, \ldots, E_m$ such that, for any function f that is defined on the eigenvalues of A, we have

$$f(A) = \sum_r f(\theta_r) E_r$$

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Consequences:

1. $I = \sum_r E_r$.
2. $E_r E_s = 0$ if $r \neq s$.
3. $U(t) = \sum_r e^{i\theta_r t} E_r$.

Eigenvalue Support

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- If X is vertex transitive, the eigenvalue support of a vertex consists of all eigenvalues of X.
- The eigenvalue support of u is the set of poles of the rational function

$$\frac{\phi(X \setminus u, t)}{\phi(X, t)}.$$

The Ratio Condition

Lemma

If we have pst on X at u and θ_r, θ_s, θ_k, θ_ℓ belong to the eigenvalue support of u and $\theta_k \neq \theta_\ell$, then

$$\frac{\theta_r - \theta_s}{\theta_k - \theta_\ell} \in \mathbb{Q}.$$

No PST on P_4

The eigenvalues $\theta_1, \dots, \theta_4$ of P_4 are respectively

$$\frac{1+\sqrt{5}}{2},\ \frac{-1+\sqrt{5}}{2},\ \frac{1-\sqrt{5}}{2},\ \frac{-1-\sqrt{5}}{2}.$$

and

$$\frac{\theta_1-\theta_4}{\theta_1-\theta_2} = 1+\sqrt{5} \notin \mathbb{Q}.$$

The eigenvalue support of 1 is the spectrum of P_4, and so we conclude that we do not have 12-pst on P_4. With more work we can show that pst does not occur in P_n when $n \geq 4$.

Not Much PST

Theorem

There are only finitely many connected graphs with maximum valency at most k on which perfect state transfer occurs.

Outline

Pretty Good State Transfer

Definition

We have pretty good state transfer from a to b on X if, for each $\epsilon > 0$, there a time t and scalar γ such that

$$\|U(t)e_a - \gamma e_b\| \leq \epsilon.$$

Constraints

If we have ab-pgst on the graph X, then:

- We have pretty good state transfer from b to a.
- $\mathrm{Aut}(X)_a = \mathrm{Aut}(X)_b$.
- $X \setminus a$ and $X \setminus b$ are cospectral.

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PGST on P_4

Let $E_1, \dots, E_4$ be the idempotents in the spectral decomposition for $A(P_4)$. Then

$$E_1 - E_2 + E_3 - E_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

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If

$$\frac{a}{b} \approx \frac{1+\sqrt{5}}{2}$$

and $a \equiv 3$ and $b \equiv 2$ modulo 4, then

$$U(b\pi/2) \approx -i(E_1 - E_2 + E_3 - E_4).$$

PGST and Signs

If

$$U(t)e_a \approx \gamma e_b$$

then, since $U(t)E_r = e^{i\theta_r t}E_r$,

$$\gamma E_r e_a \approx e^{i\theta_r t}E_r e_b$$

and, since $E_r e_a$ and $E_r e_b$ are real,

$$\gamma^{-1}e^{i\theta_r t} \approx \pm 1.$$

Denote the actual sign of $\gamma^{-1}e^{i\theta_r t}$ by σ_r.

Some Group Theory

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- We have ab-pst if $(\sigma_1\gamma, \dots, \sigma_m\gamma)$ lies in H.
- We have ab-pgst if $(\sigma_1\gamma, \dots, \sigma_m\gamma)$ lies in the closure of H.

PGST on Paths

Lemma

If $n+1$ is a prime, or twice a prime, or a power of two, then we have pgst between the end vertices of the path P_n.

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Proof.

In the stated cases the positive eigenvalues of the path, together with 1, are linearly independent over $\mathbb{Q}$. Now apply Kronecker. $\square$

Phases for P_{3k+2}

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- If $n = 3k+2$ then $\theta_1 = \theta_k + \theta_{k+2}$.
- So if $e^{i\theta_r t} \approx (-1)^{r-1}\gamma$, then

$$\gamma \approx e^{i\theta_1 t} = e^{i\theta_k t} e^{i\theta_{k+2} t} \approx (-1)^{k-1}\gamma\,(-1)^{k+1}\gamma = \gamma^2.$$

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- Therefore $\gamma = 1$.

Phases for P_n

If we have ab-pgst, then

$$U(t) = \sum_r e^{i\theta_r t} E_r \approx \sum_r (-1)^{r-1} \gamma E_r.$$

Hence $\det(U(t)) \approx \gamma^n (-1)^{\lfloor n/2 \rfloor}$.

But $\det(U(t)) = 1$ and this yields three cases:

- $n \equiv 1 \mod 4$ and $\gamma = 1$.
- $n \equiv 3 \mod 4$ and $\gamma = -1$.
- n is even and $\gamma = \pm i$.

The Result

Theorem (Godsil, Kirkland, Severini, Smith)

We have pgst between the end vertices of P_n if and only if $n+1$ is a prime, twice a prime, or a power of two.

Questions

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2. Suppose we use the Laplacian in place of the adjacency matrix: when do we get pretty good state transfer?

The End(s)