Packing and Covering

in Uniform Hypergraphs

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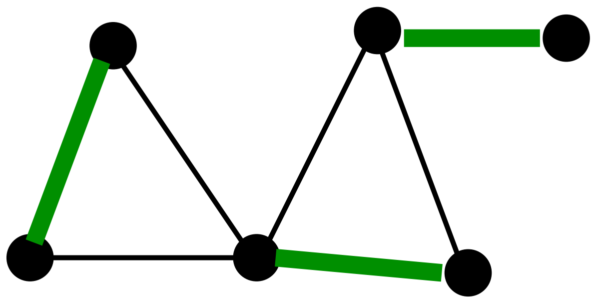
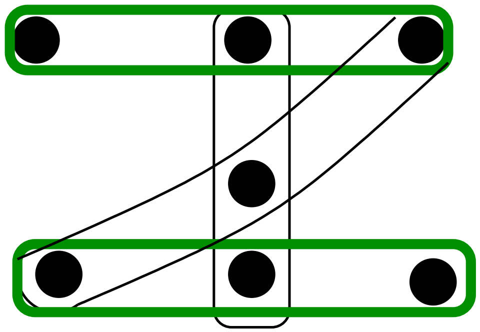
Packing

Let $\mathcal{H}$ be a hypergraph. A packing or matching of $\mathcal{H}$ is a set of pairwise disjoint edges of $\mathcal{H}$.

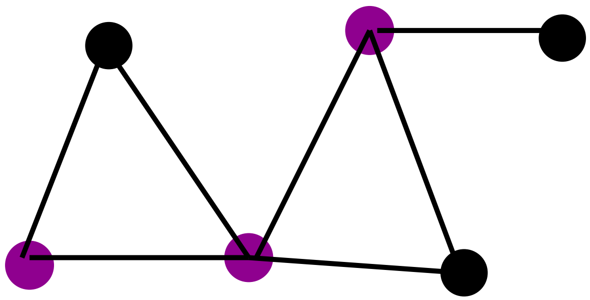


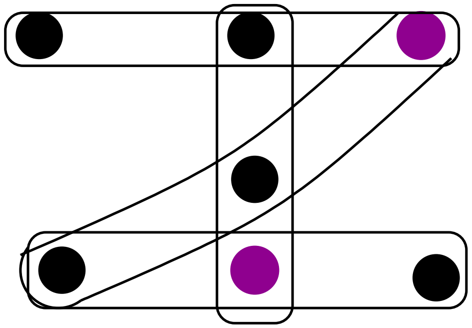


The parameter $\nu(\mathcal{H})$ is defined to be the maximum size of a packing in $\mathcal{H}$.

Covering

A cover of the hypergraph $\mathcal{H}$ is a set of vertices C of $\mathcal{H}$ such that every edge of $\mathcal{H}$ contains a vertex of C.

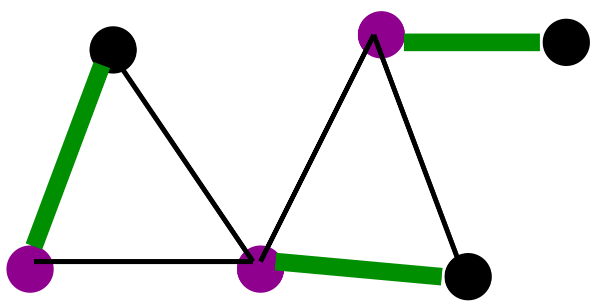


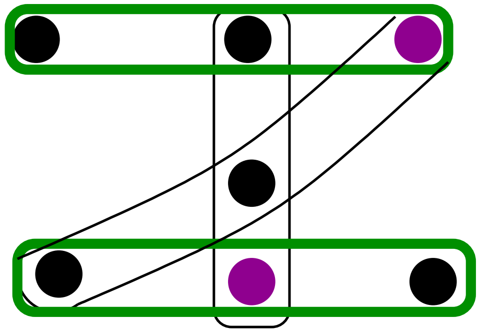


The parameter $\tau(\mathcal{H})$ is defined to be the minimum size of a cover of $\mathcal{H}$.

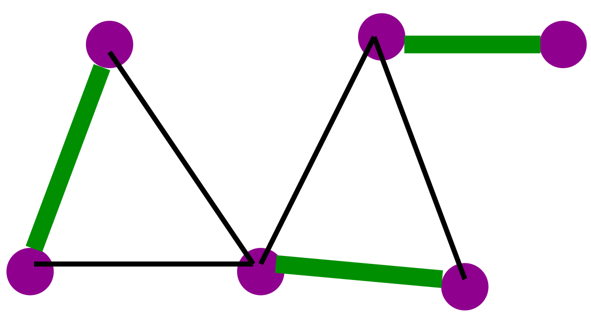
Comparing $\nu(\mathcal{H})$ and $\tau(\mathcal{H})$

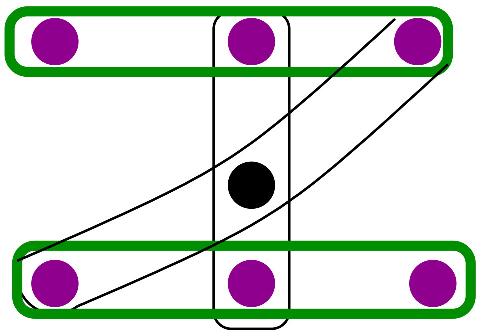
For every hypergraph $\mathcal{H}$ we have $\nu(\mathcal{H}) \leq \tau(\mathcal{H})$.



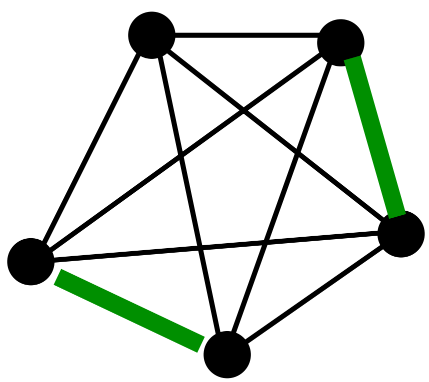


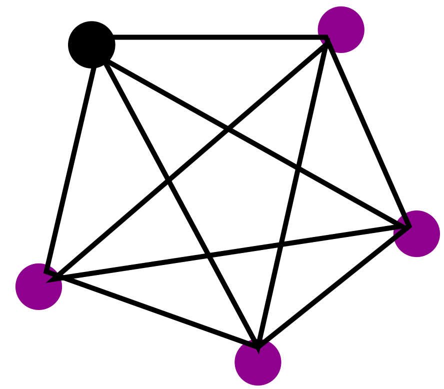
For every r-uniform hypergraph $\mathcal{H}$ we have $\tau(\mathcal{H}) \leq r\nu(\mathcal{H})$.





The upper bound $\tau(\mathcal{H}) \le r\nu(\mathcal{H})$ is attained for certain hypergraphs, for example for the complete r-uniform hypergraph $\mathcal{K}^r_{rt+r-1}$ with $rt+r-1$ vertices, in which $\nu = t$ and $\tau = rt$.

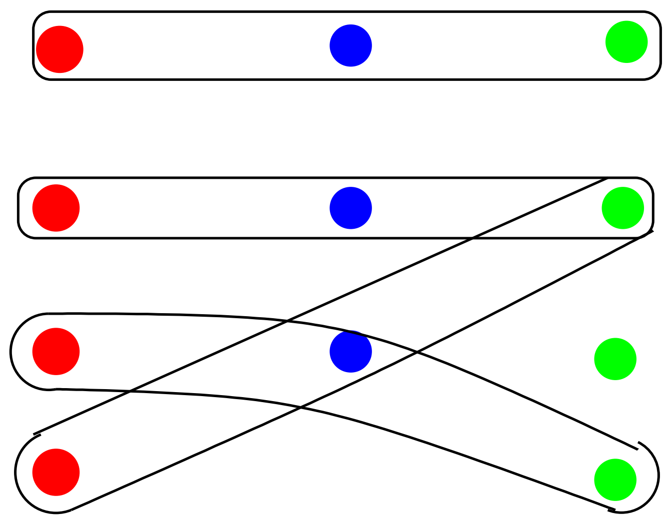




Ryser's Conjecture

Conjecture: Let $\mathcal{H}$ be an r-partite r-uniform hypergraph. Then

$$\tau(\mathcal{H}) \leq (r-1)\nu(\mathcal{H}).$$



This conjecture dates from the early 1970's.

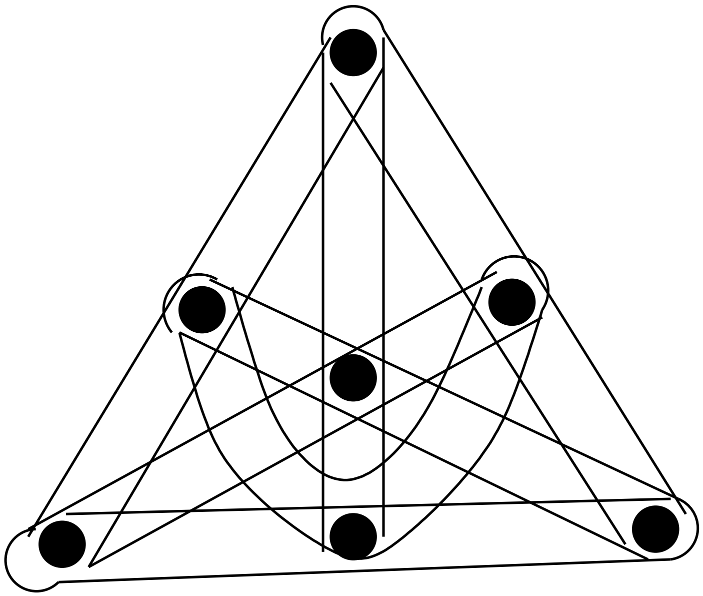
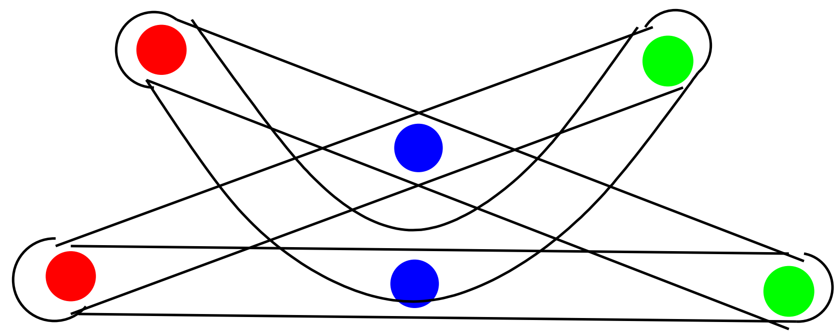
A stronger conjecture

Conjecture (Lovász): Let $\mathcal{H}$ be an r-partite r-uniform hypergraph. Then there exists a set S of at most $r-1$ vertices such that the hypergraph $\mathcal{H}'$ formed by removing S satisfies

$$\nu(\mathcal{H}') \leq \nu(\mathcal{H}) - 1.$$

Results on Ryser's Conjecture

- $r = 2$: This is König's Theorem for bipartite graphs.
- $r = 3$: Known (proved by Aharoni, 2001)
- $r = 4$ and $r = 5$: Known for small values of $\nu(\mathcal{H})$, namely for $\nu(\mathcal{H}) \leq 2$ when $r = 4$ and for $\nu(\mathcal{H}) = 1$ when $r = 5$. (Tuza)
- whenever $r - 1$ is a prime power: If true, the upper bound is best possible.

Here $\nu(\mathcal{H}) = 1$ and $\tau(\mathcal{H}) = r - 1$.

On Ryser's Conjecture

Theorem (PH, Scott 2012) There exists $\epsilon > 0$ such that for every 4-partite 4-uniform hypergraph $\mathcal{H}$ we have

$$\tau(\mathcal{H}) \leq (4 - \epsilon)\nu(\mathcal{H}).$$

Theorem (PH, Scott 2012) There exists $\epsilon > 0$ such that for every 5-partite 5-uniform hypergraph $\mathcal{H}$ we have

$$\tau(\mathcal{H}) \leq (5 - \epsilon)\nu(\mathcal{H}).$$

On Ryser's Conjecture for $r = 3$

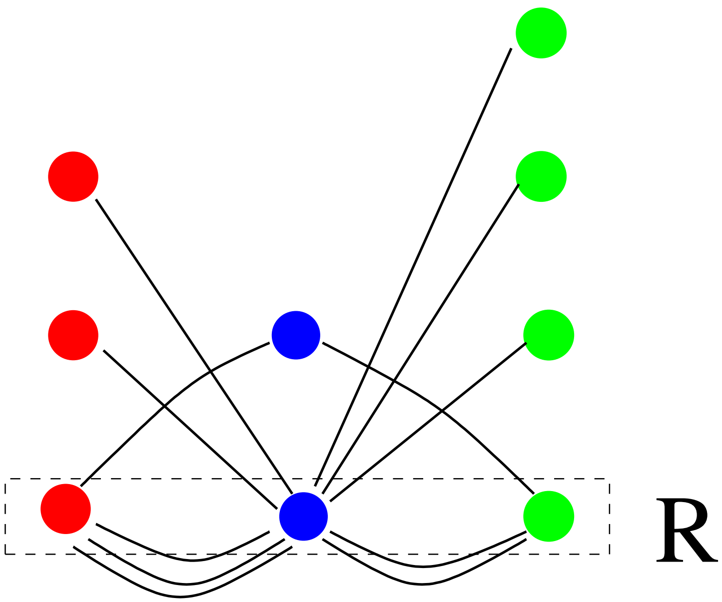
Theorem (Aharoni 2001): Let $\mathcal{H}$ be a 3-partite 3-uniform hypergraph. Then

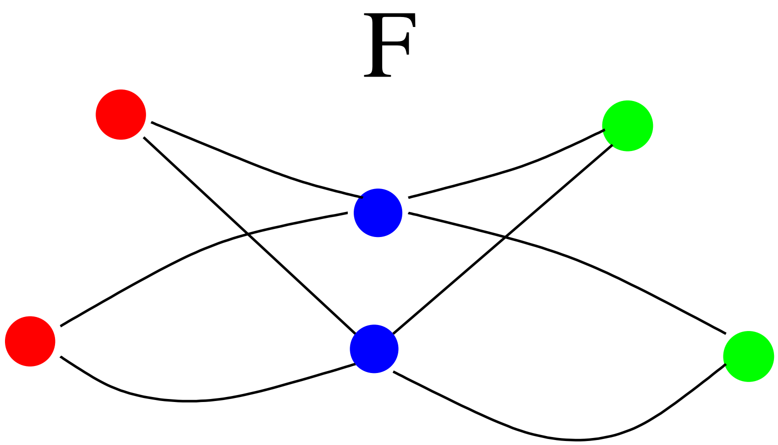
$$\tau(\mathcal{H}) \leq 2\nu(\mathcal{H}).$$

Proof: Uses topological connectivity of matching complexes of bipartite graphs.

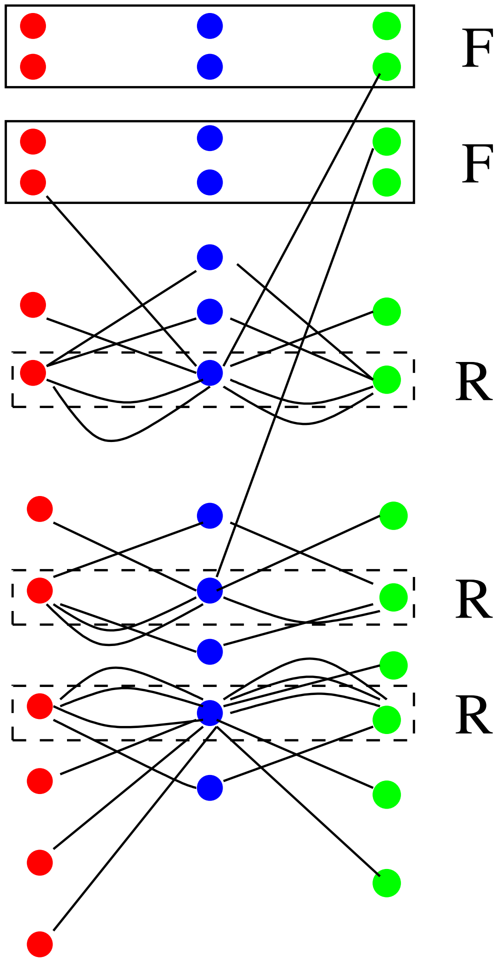
Q: What is $\mathcal{H}$ like if it is a 3-partite 3-uniform hypergraph with $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$?

Extremal hypergraphs for Ryser's Conjecture





Home base hypergraphs



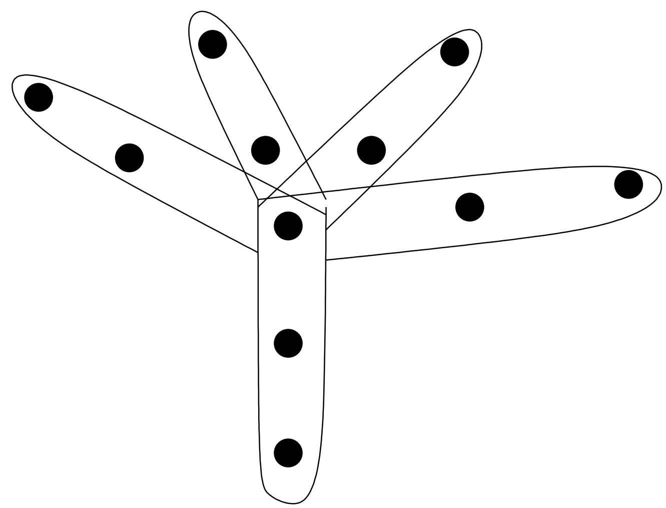
Extremal hypergraphs for Ryser's Conjecture

Theorem (PH, Narins, Szabó): Let $\mathcal{H}$ be a 3-partite 3-uniform hypergraph with $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$. Then $\mathcal{H}$ is a home base hypergraph.

Some proof ingredients

The nontrivial bounds for Ryser's conjecture for $r = 4$ and $r = 5$ use Tuza's result that the conjecture is true for intersecting hypergraphs, together with several classical results.

A sunflower $\mathcal{S}$ with centre C in a hypergraph is a set of edges satisfying $S \cap S' = C$ for all $S \neq S'$ in $\mathcal{S}$.



The Sunflower Theorem

Theorem (Erdös and Rado): Every hypergraph of rank r with more than $(t-1)^r r!$ edges contains a sunflower with t petals.

Fact: If an intersecting hypergraph $\mathcal{H}$ of rank r contains a sunflower $\mathcal{S}$ with $r+1$ petals with centre C, then $\mathcal{H} \setminus \mathcal{S} \cup \{C\}$ is also intersecting.

The Bollobás Theorem

Theorem (Bollobás): Let $\{A_1, \ldots, A_t\}$ and $\{B_1, \ldots, B_t\}$ be such that

- $A_i \cap B_i = \emptyset$ for each i,
- $A_i \cap B_j \neq \emptyset$ for each $i \neq j$,
- $|A_i| \leq k$ and $|B_i| \leq l$ for each i.

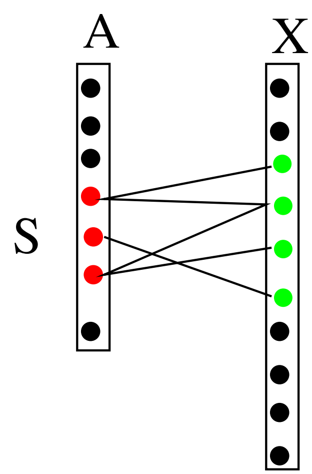
Then $t \leq \binom{k+l}{k}$.

Some proof ingredients

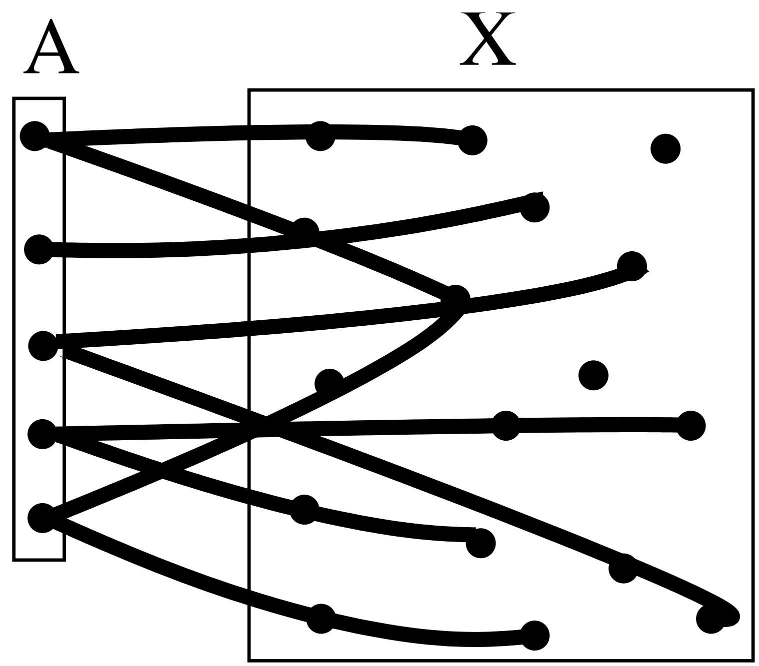
The extremal result for Ryser's conjecture for $r = 3$ initially follows Aharoni's proof of the conjecture for $r = 3$, which uses Hall's Theorem for hypergraphs together with König's Theorem.

Hall's Theorem: The bipartite graph G has a complete matching if and only if: For every subset $S \subseteq A$, the neighbourhood $\Gamma(S)$ is big enough.

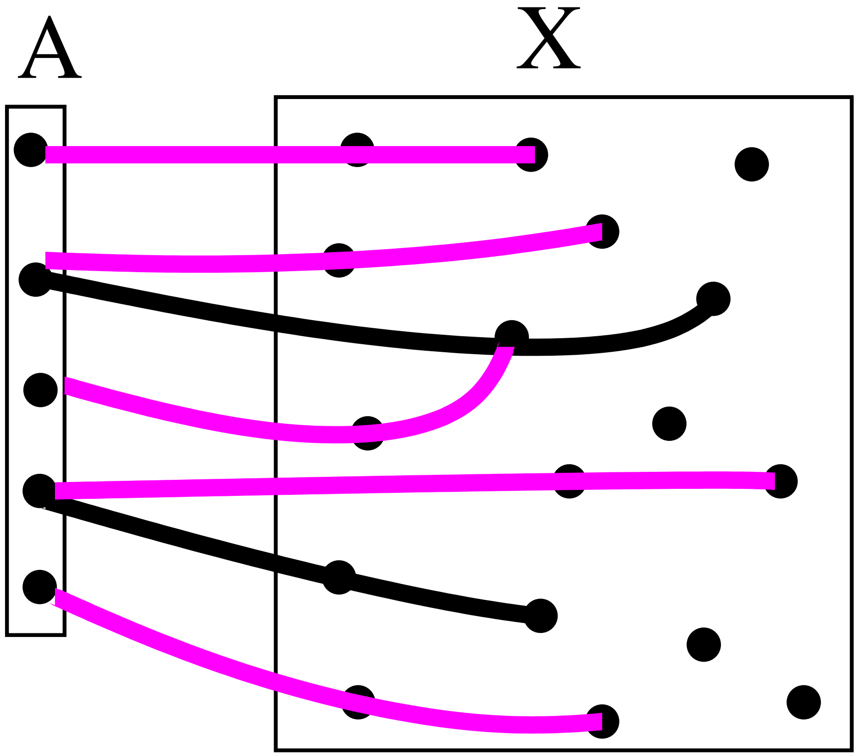
Here big enough means $|\Gamma(S)| \geq |S|$.



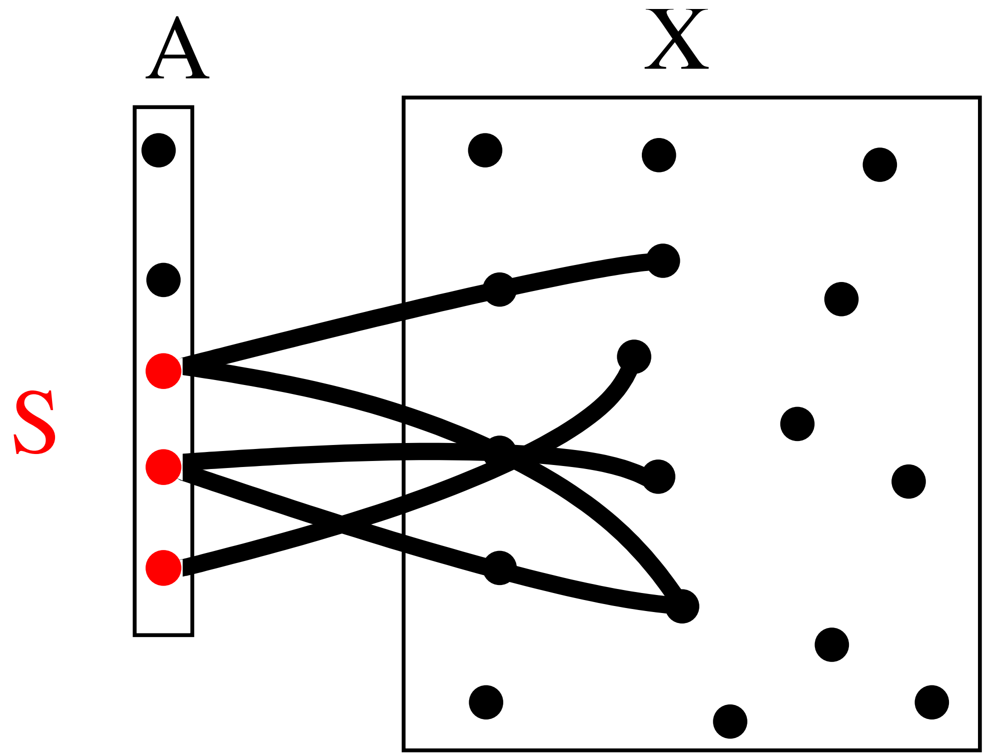
bipartite 3-uniform hypergraphs

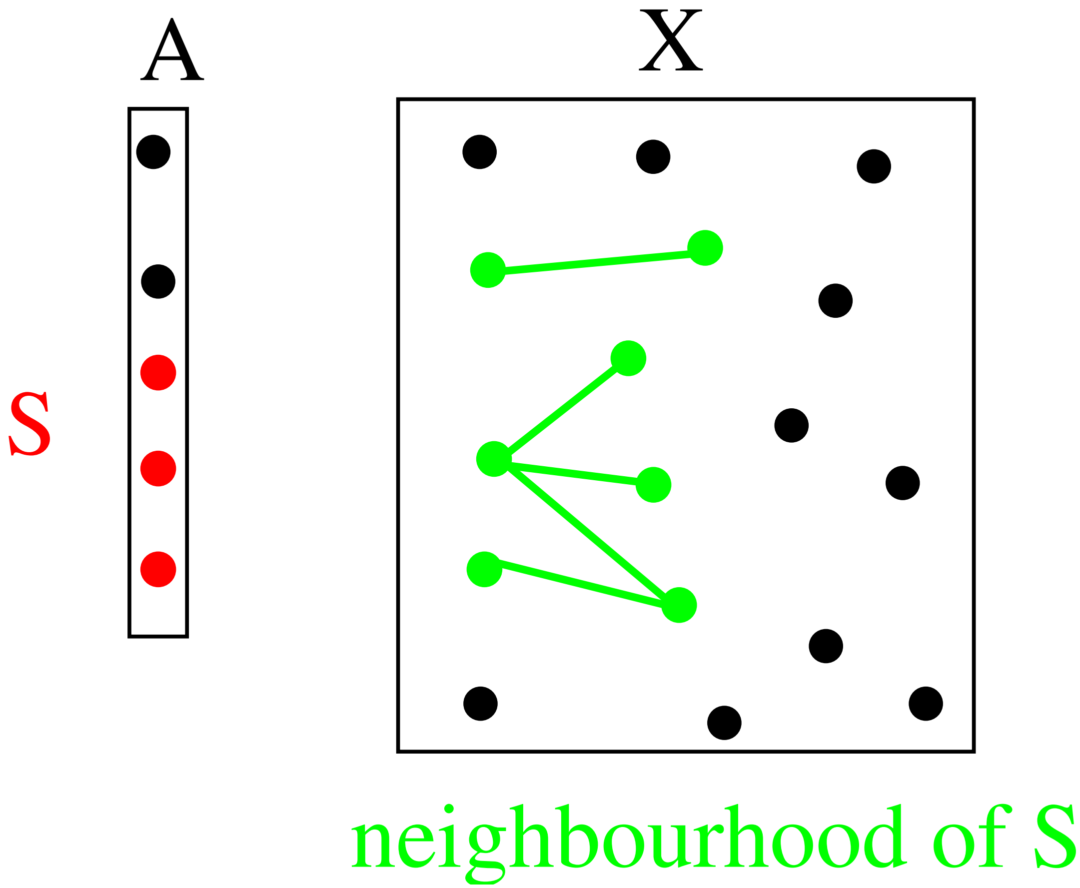


def: A complete packing:

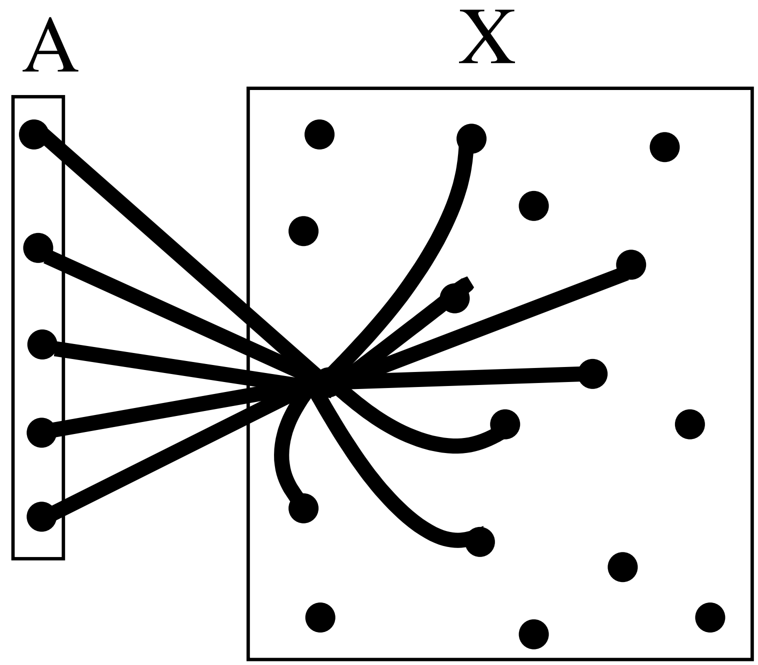


def: The neighbourhood of the subset S of A is the graph with vertex set X and edge set $\{\{x, y\} : \{z, x, y\} \in H$ for some $z \in S\}$.



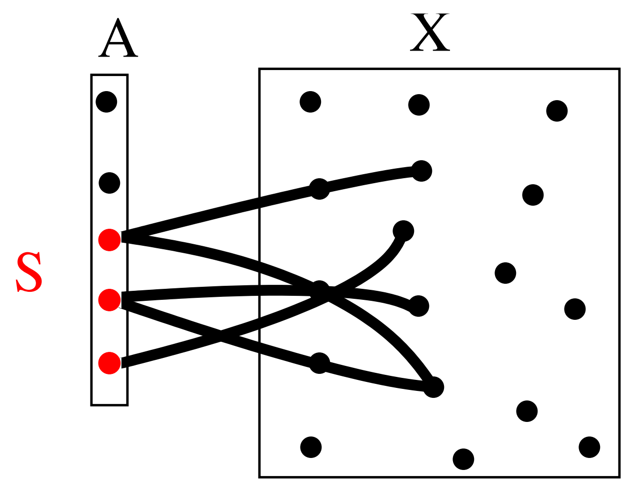

A
X
S
neighbourhood of S

What should big enough mean?

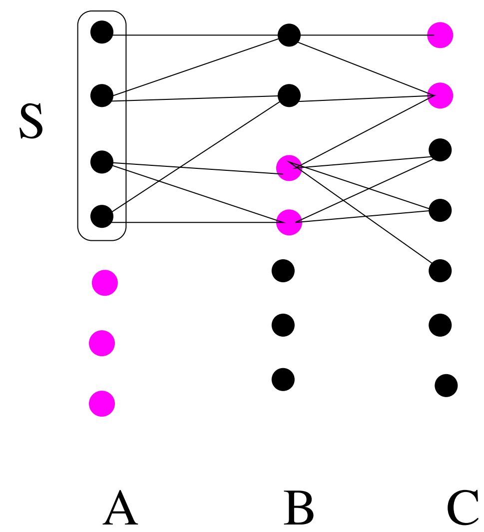


Hall's Theorem for 3-uniform hypergraphs

Theorem (Aharoni, PH, 2000): The bipartite 3-uniform hypergraph H has a complete packing if: For every subset $S \subseteq A$, the neighbourhood $\Gamma(S)$ has a matching of size at least $2(|S|-1)+1$.



Aharoni's proof of Ryser for $r = 3$



Let H be a 3-partite 3-uniform hypergraph. Let $\tau = \tau(H)$. Then by König's Theorem, for every subset S of A, the neighbourhood graph $\Gamma(S)$ has a matching of size at least $|S| - (|A| - \tau)$.

Then by a defect version of Hall's Theorem for hypergraphs, we find that H has a packing of size $\lceil \tau/2 \rceil$.

Proof of Hall's Theorem for hypergraphs

The proof has two main steps.

Step 1: The bipartite 3-uniform hypergraph H has a complete packing if: For every subset $S \subseteq A$, the topological connectivity of the matching complex of the neighbourhood graph $\Gamma(S)$ is at least $|S| - 2$.

Step 2: If the graph G has a matching of size at least $2(|S| - 1) + 1$ then the topological connectivity of the matching complex of G is at least $|S| - 2$.

The matching complex of G is the abstract simplicial complex with vertex set $E(G)$, whose simplices are the matchings in G.

Topological connectivity

One way to describe topological connectivity of an abstract simplicial complex Σ, as it is used here:

We say Σ is k-connected if for each $-1 \leq d \leq k$ and each triangulation T of the boundary of a $(d+1)$-simplex, and each function f that labels each point of T with a point of Σ such that the set of labels on each simplex of T forms a simplex of Σ, the triangulation T can be extended to a triangulation T' of the whole $(d+1)$-simplex, and f can be extended to a full labelling f' of T' with the same property.

Hall's Theorem for hypergraphs uses this together with Sperner's Lemma.

The topological connectivity of the matching complex of G is not a monotone parameter.

Extremal hypergraphs for Ryser's Conjecture

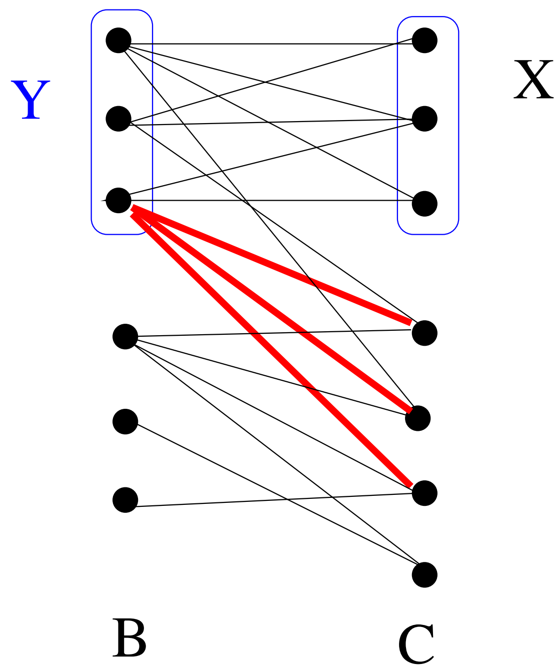
Two main parts are needed in understanding the extremal hypergraphs for Ryser's Conjecture for $r = 3$.

Part A: Show that any bipartite graph G that has a matching of size $2k$ but whose matching complex has the smallest possible topological connectivity (namely $k - 2$) has a very special structure.

Part B: Analyse how the edges of the neighbourhood graph G of A (which has this special structure) extend to A.

Part B (one case)

There exists a subset X of C with $|Y| \leq |X|$, where $Y = \Gamma_G(X)$, such that for each $y \in Y$, if we erase the $(y, C \setminus X)$ edges of G, the topological connectivity of the matching complex goes up.



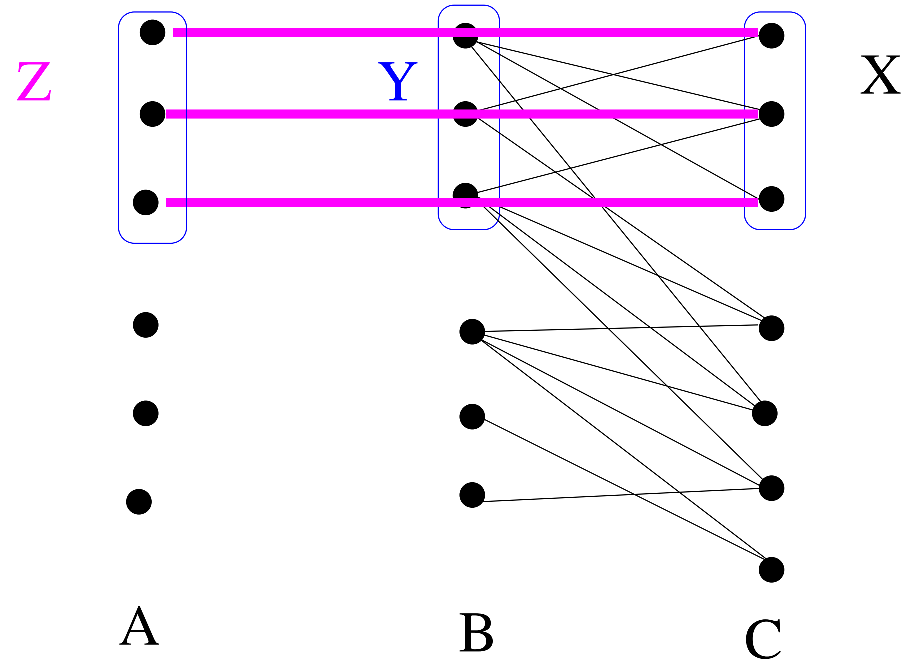
If for each $S \subset A$, the topological connectivity of the matching complex of $\Gamma(S)$ did not go down, then we find H has a packing larger than $\nu(H)$.

So for some S_y, erasing the $(y, C \setminus X)$ edges causes the connectivity to decrease.

Properties of S_y:

- $|S_y| \geq |A| - 1$, which implies $S_y = A \setminus \{a\}$ for some $a \in A$,
- every maximum matching in $\Gamma(S)$ uses an edge of $(y, C \setminus X)$.

What these properties imply



Removing the vertices in Y and Z causes ν to decrease by $|Y|$ and τ to decrease by $2|Y|$. Then we may use induction.

Triangle hypergraphs

Let G be a graph. The triangle hypergraph $\mathcal{H}(G)$ of G is the 3-uniform hypergraph with vertex set $E(G)$. Three edges of G form a hyperedge of $\mathcal{H}(G)$ if and only if they form the edge set of a triangle in G.

Thus a packing in $\mathcal{H}(G)$ is a set of edge-disjoint triangles in G.

A cover in $\mathcal{H}(G)$ is a set S of edges in G such that every triangle contains an edge in S.

Tuza's Conjecture

Conjecture (Tuza 1984): Let $\mathcal{H}$ be a triangle hypergraph. Then

$$\tau(\mathcal{H}) \leq 2\nu(\mathcal{H}).$$

In other words:

Conjecture (Tuza 1984): Suppose the maximum size of a set of pairwise edge-disjoint triangles in a graph G is ν. Then there exists a set of at most 2ν edges in G whose removal makes the graph triangle-free.

Results on Tuza's Conjecture

- known for certain special classes of graphs, including K_5-free chordal graphs (Tuza 1990), odd-wheel-free and four-colourable graphs (Aparna Lakshmanan, Bujtás and Tuza 2011)

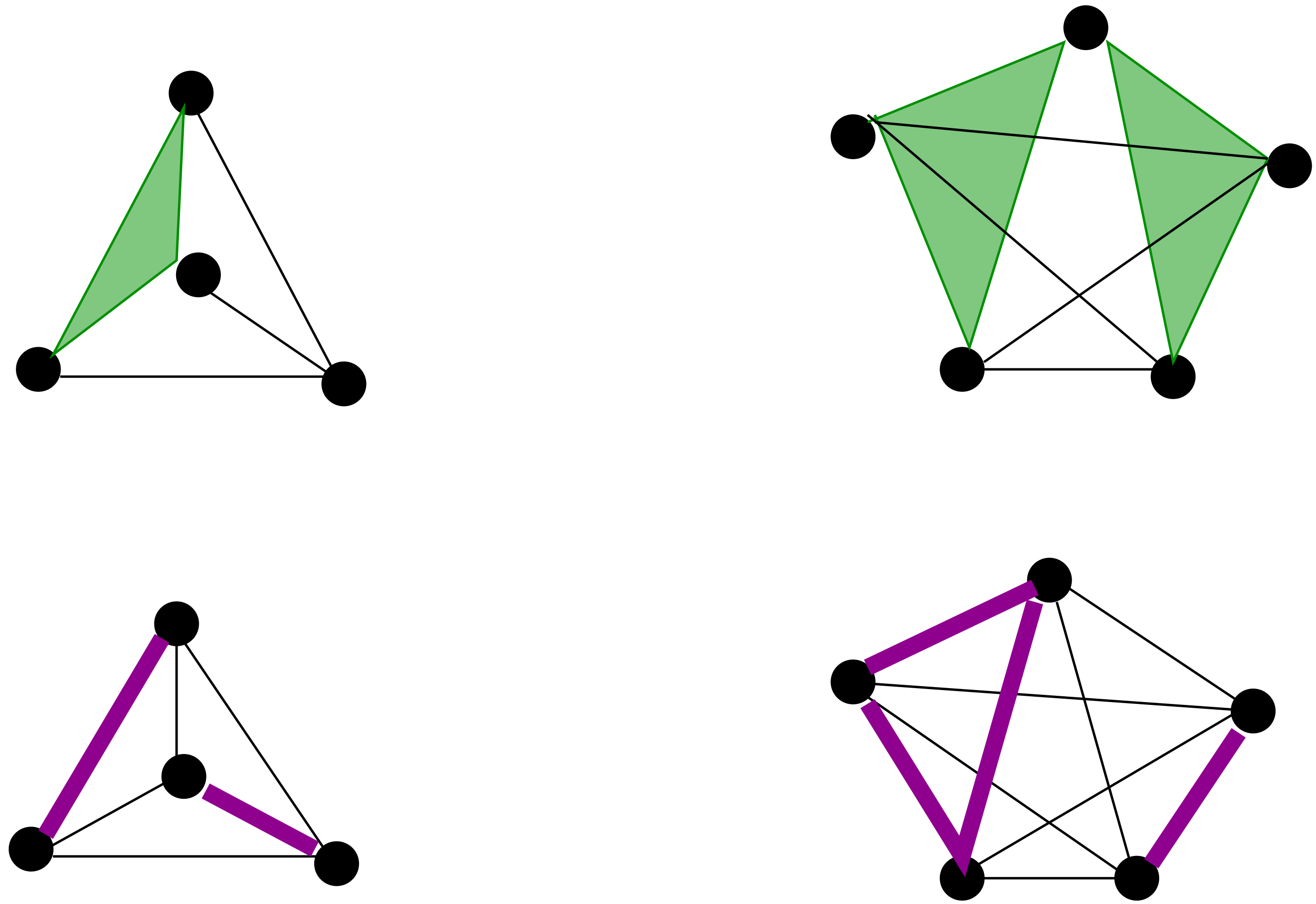
- known for planar graphs (Tuza 1990), and more generally graphs without subdivisions of $K_{3,3}$ (Krivelevich 1995)

- weighted versions of the problem have been studied (Chapuy, DeVos, McDonald, Mohar and Scheide 2011)

- for every graph G the triangle hypergraph $\mathcal{H}$ satisfies $\tau(\mathcal{H}) \leq (3 - \frac{3}{19})\nu(\mathcal{H})$.

- If true, Tuza's Conjecture is best possible.

Tuza’s Conjecture



Fractional versions

Let $\mathcal{H}$ be a hypergraph. A fractional packing of $\mathcal{H}$ is a function p that assigns to each hyperedge e of $\mathcal{H}$ a non-negative real number, such that for each vertex v of $\mathcal{H}$ we have

$$\sum_{e \ni v} p(e) \leq 1.$$

Thus a packing (a set $\mathcal{S}$ of disjoint hyperedges) corresponds to a fractional packing in which each hyperedge in $\mathcal{S}$ gets value 1 and all others get 0.

A fractional cover of $\mathcal{H}$ is a function c that assigns to each vertex of $\mathcal{H}$ a non-negative real number, such that for each hyperedge e of $\mathcal{H}$ we have

$$\sum_{v \in e} c(v) \geq 1.$$

Thus a cover of $\mathcal{H}$ (a set C of vertices that meets every hyperedge) corresponds to a fractional cover in which each vertex in C gets value 1 and all other vertices get 0.

The fractional parameter $\nu^*(\mathcal{H})$ is defined to be the maximum of $\sum_{e \in \mathcal{H}} p(e)$ over all fractional packings p of $\mathcal{H}$.

The parameter $\tau^*(\mathcal{H})$ is the minimum of $\sum_{v \in \mathcal{H}} c(v)$ over all fractional covers c of $\mathcal{H}$.

Then we know that $\nu(\mathcal{H}) \leq \nu^*(\mathcal{H})$ and $\tau(\mathcal{H}) \geq \tau^*(\mathcal{H})$.

The Duality Theorem of linear programming tells us that

$$\tau^*(\mathcal{H}) = \nu^*(\mathcal{H}).$$

Fractional versions

Theorem (Krivelevich 1995): Let $\mathcal{H}$ be a triangle hypergraph. Then

- $\tau^*(\mathcal{H}) \leq 2\nu(\mathcal{H})$.

- $\tau(\mathcal{H}) \leq 2\nu^*(\mathcal{H})$.

A closer look - the role of K_4

(A) Tuza's Conjecture is true for planar graphs, and best possible because of K_4. What can we say about planar graphs for which $\tau(\mathcal{H})$ is close to $2\nu(\mathcal{H})$? Are they close to being disjoint unions of K_4's?

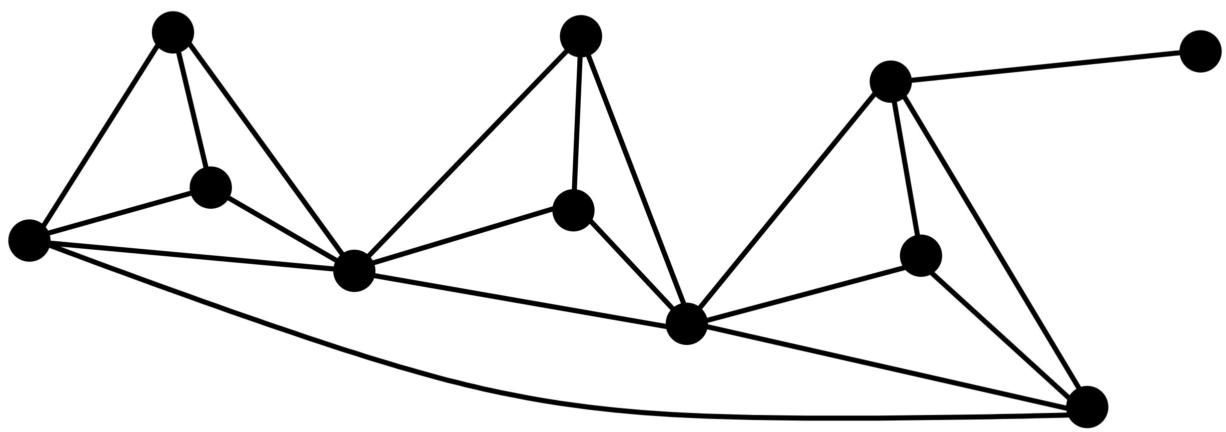
(B) The fractional result $\tau^*(\mathcal{H}) \leq 2\nu(\mathcal{H})$ of Krivelevich is best possible because of K_4. What can we say about graphs for which $\tau^*(\mathcal{H})$ is close to $2\nu(\mathcal{H})$? Are they close to being disjoint unions of K_4's?

On Question (A)

Theorem (Cui, PH, Ma 2009) Let $\mathcal{H}$ be the triangle hypergraph of a planar graph G, and suppose

$$\tau(\mathcal{H}) = 2\nu(\mathcal{H}).$$

Then G is an edge-disjoint union of K_4's and edges, such that every triangle is contained in exactly one of the K_4's.

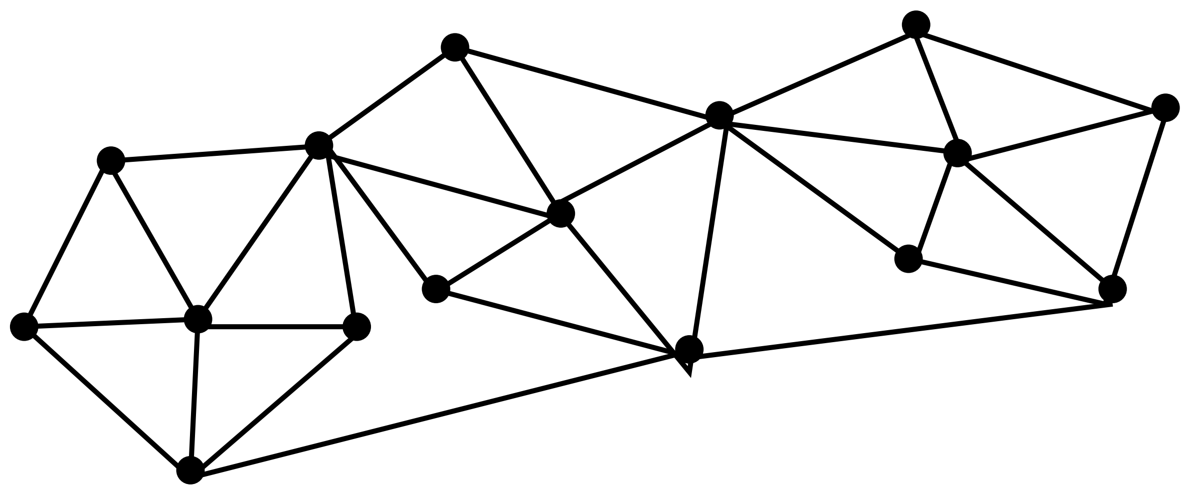


On Question (A)

Theorem (PH, Kostochka, Thomassé 2011) Let $\mathcal{H}$ be the triangle hypergraph of a K_4-free planar graph G. Then

$$\tau(\mathcal{H}) \leq \frac{3}{2}\nu(\mathcal{H}).$$

Moreover if equality holds then G is an edge-disjoint union of 5-wheels (plus possibly some edges that are not in triangles).



(B): A stability theorem

Theorem (PH, Kostochka, Thomassé 2011) Let G be a graph such that the triangle hypergraph $\mathcal{H}$ satisfies $\tau^*(\mathcal{H}) \geq 2\nu(\mathcal{H}) - x$. Then G contains $\nu(\mathcal{H}) - \lfloor 10x \rfloor$ edge-disjoint K_4-subgraphs plus an additional $\lfloor 10x \rfloor$ edge-disjoint triangles.

Note that just these K_4's and triangles witness the fact that

$$\tau^*(\mathcal{H}) \geq 2\nu(\mathcal{H}) - \lfloor 10x \rfloor.$$

The proof also shows that if G is K_4-free then

$$\tau^*(\mathcal{H}) \leq 1.8\nu(\mathcal{H}).$$

Stability for Tuza's conjecture

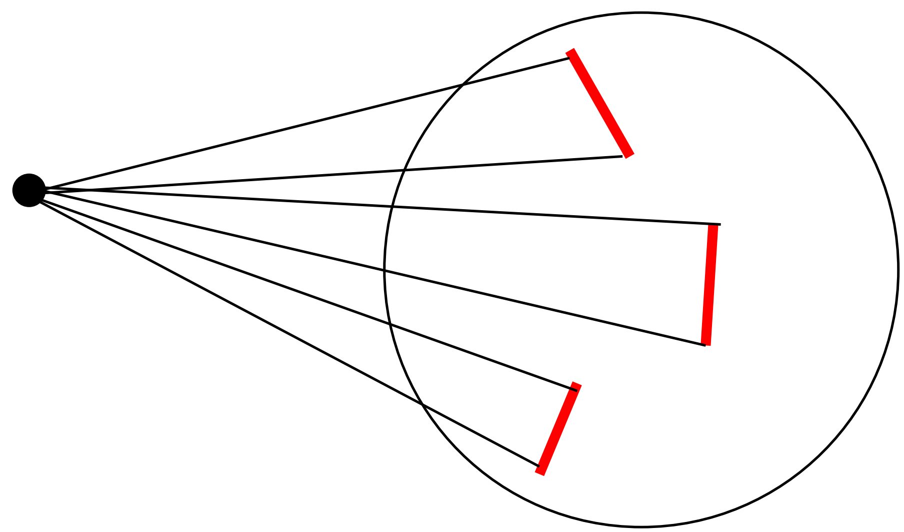
Could there be a similar stability theorem for Tuza's Conjecture?

The only known graphs for which equality holds for Tuza's Conjecture are (disjoint unions of) K_4 and K_5. Could it be true that every graph for which $\tau(\mathcal{H})$ of the triangle hypergraph is close to $2\nu(\mathcal{H})$ contains many K_4's?

NO.

For each $\epsilon > 0$, there exists a K_4-free graph G_ϵ such that the triangle hypergraph $\mathcal{H}_\epsilon$ satisfies $\tau(\mathcal{H}_\epsilon) > (2-\epsilon)\nu(\mathcal{H}_\epsilon)$.

For large n, let J be an n-vertex triangle-free graph with independence number $\alpha(J) < n^{2/3}$. ($R(3,t)$ is of order $t^2/\log t$.)



Form a graph G by adding a new vertex v_0 and joining it to all vertices in J.

Then a packing in the triangle hypergraph $\mathcal{H}$ corresponds to a matching in J, so

$$\nu(\mathcal{H}) \le n/2.$$

A cover in $\mathcal{H}$ corresponds to the complement of an independent vertex set in J. Thus

$$\tau(\mathcal{H}) \ge n - n^{2/3}.$$