Packing and Covering

in Uniform Hypergraphs

Penny Haxell

University of Waterloo
Let $\mathcal{H}$ be a hypergraph. A **packing** or **matching** of $\mathcal{H}$ is a set of pairwise disjoint edges of $\mathcal{H}$.

The parameter $\nu(\mathcal{H})$ is defined to be the maximum size of a packing in $\mathcal{H}$. 
Covering

A cover of the hypergraph \( \mathcal{H} \) is a set of vertices \( C \) of \( \mathcal{H} \) such that every edge of \( \mathcal{H} \) contains a vertex of \( C \).

The parameter \( \tau(\mathcal{H}) \) is defined to be the minimum size of a cover of \( \mathcal{H} \).
Comparing \( \nu(\mathcal{H}) \) and \( \tau(\mathcal{H}) \)

For every hypergraph \( \mathcal{H} \) we have \( \nu(\mathcal{H}) \leq \tau(\mathcal{H}) \).

For every \( r \)-uniform hypergraph \( \mathcal{H} \) we have \( \tau(\mathcal{H}) \leq r\nu(\mathcal{H}) \).
The upper bound $\tau(\mathcal{H}) \leq r\nu(\mathcal{H})$ is attained for certain hypergraphs, for example for the complete $r$-uniform hypergraph $\mathcal{K}^r_{rt+r-1}$ with $rt+r-1$ vertices, in which $\nu = t$ and $\tau = rt$. 
**Ryser’s Conjecture**

**Conjecture:** Let $\mathcal{H}$ be an $r$-partite $r$-uniform hypergraph. Then

$$\tau(\mathcal{H}) \leq (r - 1)\nu(\mathcal{H}).$$

This conjecture dates from the early 1970’s.
A stronger conjecture

Conjecture (Lovász): Let $\mathcal{H}$ be an $r$-partite $r$-uniform hypergraph. Then there exists a set $S$ of at most $r - 1$ vertices such that the hypergraph $\mathcal{H}'$ formed by removing $S$ satisfies

$$\nu(\mathcal{H}') \leq \nu(\mathcal{H}) - 1.$$
Results on Ryser’s Conjecture

• $r = 2$: This is König’s Theorem for bipartite graphs.

• $r = 3$: Known (proved by Aharoni, 2001)

• $r = 4$ and $r = 5$: Known for small values of $\nu(\mathcal{H})$, namely for $\nu(\mathcal{H}) \leq 2$ when $r = 4$ and for $\nu(\mathcal{H}) = 1$ when $r = 5$. (Tuza)

• whenever $r - 1$ is a prime power: If true, the upper bound is best possible.
Here \( \nu(\mathcal{H}) = 1 \) and \( \tau(\mathcal{H}) = r - 1 \).
On Ryser’s Conjecture

**Theorem (PH, Scott 2012)** There exists $\epsilon > 0$ such that for every 4-partite 4-uniform hypergraph $\mathcal{H}$ we have

$$\tau(\mathcal{H}) \leq (4 - \epsilon)\nu(\mathcal{H}).$$

**Theorem (PH, Scott 2012)** There exists $\epsilon > 0$ such that for every 5-partite 5-uniform hypergraph $\mathcal{H}$ we have

$$\tau(\mathcal{H}) \leq (5 - \epsilon)\nu(\mathcal{H}).$$
On Ryser’s Conjecture for $r = 3$

**Theorem (Aharoni 2001):** Let $\mathcal{H}$ be a 3-partite 3-uniform hypergraph. Then

$$\tau(\mathcal{H}) \leq 2\nu(\mathcal{H}).$$

**Proof:** Uses topological connectivity of matching complexes of bipartite graphs.

**Q:** What is $\mathcal{H}$ like if it is a 3-partite 3-uniform hypergraph with $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$?
Extremal hypergraphs for Ryser’s Conjecture
Home base hypergraphs
Extremal hypergraphs for Ryser’s Conjecture

Theorem (PH, Narins, Szabó): Let $\mathcal{H}$ be a 3-partite 3-uniform hypergraph with $\tau(\mathcal{H}) = 2\nu(\mathcal{H})$. Then $\mathcal{H}$ is a home base hypergraph.
Some proof ingredients

The nontrivial bounds for Ryser’s conjecture for $r = 4$ and $r = 5$ use Tuza’s result that the conjecture is true for intersecting hypergraphs, together with several classical results.

A sunflower $S$ with centre $C$ in a hypergraph is a set of edges satisfying $S \cap S' = C$ for all $S \neq S'$ in $S$. 
The Sunflower Theorem

**Theorem (Erdős and Rado):** Every hypergraph of rank $r$ with more than $(t - 1)^r r!$ edges contains a sunflower with $t$ petals.

**Fact:** If an intersecting hypergraph $H$ of rank $r$ contains a sunflower $S$ with $r + 1$ petals with centre $C$, then $H \setminus S \cup \{C\}$ is also intersecting.
The Bollobás Theorem

Theorem (Bollobás): Let \( \{A_1, \ldots, A_t\} \) and \( \{B_1, \ldots, B_t\} \) be such that

- \( A_i \cap B_i = \emptyset \) for each \( i \),
- \( A_i \cap B_j \neq \emptyset \) for each \( i \neq j \),
- \( |A_i| \leq k \) and \( |B_i| \leq l \) for each \( i \).

Then \( t \leq \binom{k+l}{k} \).
Some proof ingredients

The extremal result for Ryser’s conjecture for \( r = 3 \) initially follows Aharoni’s proof of the conjecture for \( r = 3 \), which uses Hall’s Theorem for hypergraphs together with König’s Theorem.

Hall’s Theorem: The bipartite graph \( G \) has a complete matching if and only if: For every subset \( S \subseteq A \), the neighbourhood \( \Gamma(S) \) is big enough.

Here big enough means \( |\Gamma(S)| \geq |S| \).
bipartite 3-uniform hypergraphs
def: A complete packing:
def: The neighbourhood of the subset $S$ of $A$ is the graph with vertex set $X$ and edge set $\{\{x, y\} : \{z, x, y\} \in H \text{ for some } z \in S\}$.
neighbourhood of $S$
What should **big enough** mean?
Hall’s Theorem for 3-uniform hypergraphs

Theorem (Aharoni, PH, 2000): The bipartite 3-uniform hypergraph $H$ has a complete packing if: For every subset $S \subseteq A$, the neighbourhood $\Gamma(S)$ has a matching of size at least $2(|S| - 1) + 1$. 
Aharoni’s proof of Ryser for $r = 3$

Let $H$ be a 3-partite 3-uniform hypergraph. Let $\tau = \tau(H)$. Then by König’s Theorem, for every subset $S$ of $A$, the neighbourhood graph $\Gamma(S)$ has a matching of size at least $|S| - (|A| - \tau)$.

Then by a defect version of Hall’s Theorem for hypergraphs, we find that $H$ has a packing of size $\lceil \tau/2 \rceil$. 
Proof of Hall’s Theorem for hypergraphs

The proof has two main steps.

**Step 1:** The bipartite 3-uniform hypergraph $H$ has a complete packing if: For every subset $S \subseteq A$, the topological connectivity of the matching complex of the neighbourhood graph $\Gamma(S)$ is at least $|S| - 2$.

**Step 2:** If the graph $G$ has a matching of size at least $2(|S| - 1) + 1$ then the topological connectivity of the matching complex of $G$ is at least $|S| - 2$.

The matching complex of $G$ is the abstract simplicial complex with vertex set $E(G)$, whose simplices are the matchings in $G$. 
Topological connectivity

One way to describe topological connectivity of an abstract simplicial complex $\Sigma$, as it is used here:

We say $\Sigma$ is $k$-connected if for each $-1 \leq d \leq k$ and each triangulation $T$ of the boundary of a $(d + 1)$-simplex, and each function $f$ that labels each point of $T$ with a point of $\Sigma$ such that the set of labels on each simplex of $T$ forms a simplex of $\Sigma$, the triangulation $T$ can be extended to a triangulation $T'$ of the whole $(d + 1)$-simplex, and $f$ can be extended to a full labelling $f'$ of $T'$ with the same property.

Hall’s Theorem for hypergraphs uses this together with Sperner’s Lemma.

The topological connectivity of the matching complex of $G$ is not a monotone parameter.
Extremal hypergraphs for Ryser’s Conjecture

Two main parts are needed in understanding the extremal hypergraphs for Ryser’s Conjecture for $r = 3$.

Part A: Show that any bipartite graph $G$ that has a matching of size $2k$ but whose matching complex has the smallest possible topological connectivity (namely $k - 2$) has a very special structure.

Part B: Analyse how the edges of the neighbourhood graph $G$ of $A$ (which has this special structure) extend to $A$. 
Part B (one case)

There exists a subset $X$ of $C$ with $|Y| \leq |X|$, where $Y = \Gamma_G(X)$, such that for each $y \in Y$, if we erase the $(y, C \setminus X)$ edges of $G$, the topological connectivity of the matching complex goes up.
If for each $S \subset A$, the topological connectivity of the matching complex of $\Gamma(S)$ did not go down, then we find $H$ has a packing larger than $\nu(H)$.

So for some $S_y$, erasing the $(y, C \setminus X)$ edges causes the connectivity to decrease.

Properties of $S_y$:

- $|S_y| \geq |A| - 1$, which implies $S_y = A \setminus \{a\}$ for some $a \in A$,
- every maximum matching in $\Gamma(S)$ uses an edge of $(y, C \setminus X)$. 
What these properties imply

Removing the vertices in $Y$ and $Z$ causes $\nu$ to decrease by $|Y|$ and $\tau$ to decrease by $2|Y|$. Then we may use induction.
Triangle hypergraphs

Let $G$ be a graph. The triangle hypergraph $\mathcal{H}(G)$ of $G$ is the 3-uniform hypergraph with vertex set $E(G)$. Three edges of $G$ form a hyperedge of $\mathcal{H}(G)$ if and only if they form the edge set of a triangle in $G$.

Thus a packing in $\mathcal{H}(G)$ is a set of edge-disjoint triangles in $G$.

A cover in $\mathcal{H}(G)$ is a set $S$ of edges in $G$ such that every triangle contains an edge in $S$. 
Tuza’s Conjecture

Conjecture (Tuza 1984): Let $\mathcal{H}$ be a triangle hypergraph. Then

$$\tau(\mathcal{H}) \leq 2\nu(\mathcal{H}).$$

In other words:

Conjecture (Tuza 1984): Suppose the maximum size of a set of pairwise edge-disjoint triangles in a graph $G$ is $\nu$. Then there exists a set of at most $2\nu$ edges in $G$ whose removal makes the graph triangle-free.
Results on Tuza’s Conjecture

• known for certain special classes of graphs, including $K_5$-free chordal graphs (Tuza 1990), odd-wheel-free and four-colourable graphs (Aparna Lakshmanan, Bujtás and Tuza 2011)

• known for planar graphs (Tuza 1990), and more generally graphs without subdivisions of $K_{3,3}$ (Krivelevich 1995)

• weighted versions of the problem have been studied (Chapuy, DeVos, McDonald, Mohar and Scheide 2011)

• for every graph $G$ the triangle hypergraph $\mathcal{H}$ satisfies $\tau(\mathcal{H}) \leq (3 - \frac{3}{19})\nu(\mathcal{H})$.

• If true, Tuza’s Conjecture is best possible.
Tuza’s Conjecture
Fractional versions

Let $\mathcal{H}$ be a hypergraph. A fractional packing of $\mathcal{H}$ is a function $p$ that assigns to each hyperedge $e$ of $\mathcal{H}$ a non-negative real number, such that for each vertex $v$ of $\mathcal{H}$ we have

$$\sum_{e \ni v} p(e) \leq 1.$$ 

Thus a packing (a set $S$ of disjoint hyperedges) corresponds to a fractional packing in which each hyperedge in $S$ gets value 1 and all others get 0.
A fractional cover of $\mathcal{H}$ is a function $c$ that assigns to each vertex of $\mathcal{H}$ a non-negative real number, such that for each hyperedge $e$ of $\mathcal{H}$ we have

$$\sum_{v \in e} c(v) \geq 1.$$ 

Thus a cover of $\mathcal{H}$ (a set $C$ of vertices that meets every hyperedge) corresponds to a fractional cover in which each vertex in $C$ gets value 1 and all other vertices get 0.
The fractional parameter $\nu^*(\mathcal{H})$ is defined to be the maximum of $\sum_{e \in \mathcal{H}} p(e)$ over all fractional packings $p$ of $\mathcal{H}$.

The parameter $\tau^*(\mathcal{H})$ is the minimum of $\sum_{v \in \mathcal{H}} c(v)$ over all fractional covers $c$ of $\mathcal{H}$.

Then we know that $\nu(\mathcal{H}) \leq \nu^*(\mathcal{H})$ and $\tau(\mathcal{H}) \geq \tau^*(\mathcal{H})$.

The Duality Theorem of linear programming tells us that $\tau^*(\mathcal{H}) = \nu^*(\mathcal{H})$. 


Fractional versions

Theorem (Krivelevich 1995): Let $\mathcal{H}$ be a triangle hypergraph. Then

- $\tau^*(\mathcal{H}) \leq 2\nu(\mathcal{H})$.
- $\tau(\mathcal{H}) \leq 2\nu^*(\mathcal{H})$. 
A closer look - the role of $K_4$

(A) Tuza’s Conjecture is true for planar graphs, and best possible because of $K_4$. What can we say about planar graphs for which $\tau(H)$ is close to $2\nu(H)$? Are they close to being disjoint unions of $K_4$’s?

(B) The fractional result $\tau^*(H) \leq 2\nu(H)$ of Krivelevich is best possible because of $K_4$. What can we say about graphs for which $\tau^*(H)$ is close to $2\nu(H)$? Are they close to being disjoint unions of $K_4$’s?
On Question (A)

Theorem (Cui, PH, Ma 2009) Let $\mathcal{H}$ be the triangle hypergraph of a planar graph $G$, and suppose

$$\tau(\mathcal{H}) = 2\nu(\mathcal{H}).$$

Then $G$ is an edge-disjoint union of $K_4$’s and edges, such that every triangle is contained in exactly one of the $K_4$’s.
On Question (A)

Theorem (PH, Kostochka, Thomassé 2011) Let $\mathcal{H}$ be the triangle hypergraph of a $K_4$-free planar graph $G$. Then

$$\tau(\mathcal{H}) \leq \frac{3}{2} \nu(\mathcal{H}).$$

Moreover if equality holds then $G$ is an edge-disjoint union of 5-wheels (plus possibly some edges that are not in triangles).
(B): A stability theorem

Theorem (PH, Kostochka, Thomassé 2011) Let $G$ be a graph such that the triangle hypergraph $\mathcal{H}$ satisfies $\tau^*(\mathcal{H}) \geq 2\nu(\mathcal{H}) - x$. Then $G$ contains $\nu(\mathcal{H}) - \lfloor 10x \rfloor$ edge-disjoint $K_4$-subgraphs plus an additional $\lfloor 10x \rfloor$ edge-disjoint triangles.

Note that just these $K_4$’s and triangles witness the fact that

$$\tau^*(\mathcal{H}) \geq 2\nu(\mathcal{H}) - \lfloor 10x \rfloor.$$ 

The proof also shows that if $G$ is $K_4$-free then

$$\tau^*(\mathcal{H}) \leq 1.8\nu(\mathcal{H}).$$
Stability for Tuza’s conjecture

Could there be a similar stability theorem for Tuza’s Conjecture?

The only known graphs for which equality holds for Tuza’s Conjecture are (disjoint unions of) \( K_4 \) and \( K_5 \). Could it be true that every graph for which \( \tau(\mathcal{H}) \) of the triangle hypergraph is close to \( 2\nu(\mathcal{H}) \) contains many \( K_4 \)’s?
NO.

For each $\epsilon > 0$, there exists a $K_4$-free graph $G_\epsilon$ such that the triangle hypergraph $\mathcal{H}_\epsilon$ satisfies $\tau(\mathcal{H}_\epsilon) > (2 - \epsilon)\nu(\mathcal{H}_\epsilon)$.

For large $n$, let $J$ be an $n$-vertex triangle-free graph with independence number $\alpha(J) < n^{2/3}$. ($R(3, t)$ is of order $t^2 / \log t$.)
Form a graph $G$ by adding a new vertex $v_0$ and joining it to all vertices in $J$.

Then a packing in the triangle hypergraph $\mathcal{H}$ corresponds to a matching in $J$, so

$$\nu(\mathcal{H}) \leq n/2.$$  

A cover in $\mathcal{H}$ corresponds to the complement of an independent vertex set in $J$. Thus

$$\tau(\mathcal{H}) \geq n - n^{2/3}.$$