Intersection theorems for finite sets

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A Puzzle

Suppose that $a, b, x, y$ are positive real numbers such that

\[
ax \leq 50 \\
ay \leq 100 \\
bx \leq 100 \\
by \leq 100
\]

Prove that

\[
ax + ay + bx + by \leq 300
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The Frankl-Rödl theorem

Let $M$ be a set. A family of sets $\mathcal{A}$ is $M$-intersecting if

$|A \cap B| \in M$ for every $A, B \in \mathcal{A}$
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General Problem of Extremal Set Theory:

Given $\mathcal{A} \subset 2^{[n]}$ and $M \subset \{0, \ldots, n\}$, what is max $|\mathcal{A}|$?

Theorem (Frankl-Rödl (1987), $250$ problem of Erdős)

Suppose that $\mathcal{A} \subset 2^{[n]}$ and $|A \cap B| \neq n/4$ for all $A, B \in \mathcal{A}$, and $n > n_0$. Then $|\mathcal{A}| < (1.99)^n$. 
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Coding theory

- $Q$ is an alphabet
- $q = |Q|$
- $C \subset Q^n$ is a code
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Hamming distance between codewords $C = (c_1, \ldots, c_n)$ and $D = (d_1, \ldots, d_n)$ is

$$d(C, D) := |\{i : c_i \neq d_i\}|$$
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\[ d(C, D) := |\{ i : c_i \neq d_i \}| \]

\[ d(C) = \{ d(C, D) : C, D \in C, C \neq D \} \]
Problem. Find upper and lower bounds for $\max |C|$ given $d(C)$. 

Theorem (Blokhuis, Frankl (1984))
Suppose that $p$ is prime and $d(C)$ is covered by $t$ nonzero residue classes mod $p$. Then 

$$|C| \leq t \sum_{i=0}^{\frac{n-1}{q}} \binom{n}{i}.$$

If $t > (1 + \varepsilon) \frac{n}{q}$, then concentration of the binomial distribution shows that the bound above is $q(1 - o(1))n$, which is rather weak.

Theorem (Frankl-Rödl (1987))
Let $0 < \delta < \frac{1}{2}$ and $\delta n < d < (1 - \delta)n$, and $d$ is even if $q = 2$. If $d \not\in d(C)$, then 

$$|C| < (q - \varepsilon)n,$$

where $\varepsilon = \varepsilon(\delta, q) > 0$. 

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Intersection theorems for finite sets
Borsuk’s Problem - decreasing the diameter

The Diameter of a set $S \subset \mathbb{R}^n$ is $\sup_{x, y \in S} \text{dist}(x, y)$.

Conjecture

Every bounded $S \subset \mathbb{R}^d$ can be partitioned into $d + 1$ sets $S_1, \ldots, S_{d+1}$ of smaller diameter.

If true, then sharp by letting $S$ be the vertices of a regular simplex, for example, $S = \{e_1, \ldots, e_d, v\}$ where $e_i$ is the unit vector with 1 in position $i$, and $v = 1 - \sqrt{n+1} - \frac{1}{n}(1, \ldots, 1)$. 

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Results

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- **Dekster (1995)** for all \( d \) if \( S \) is a body of revolution

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Counterexamples

**Theorem (Kahn-Kalai (1993))**

For large $d$, there exists a bounded $S \subset \mathbb{R}^d$ such that every partition of $S$ into pieces of smaller diameter has at least $(1.2)^{\sqrt{d}}$ parts. In particular, Borsuk’s conjecture fails for $d = 1325$ and each $d > 2014$.

Proof uses Frankl-Wilson (or Frankl-Rödl) theorem.
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Proof uses Frankl-Wilson (or Frankl-Rödl) theorem.

**Conjecture**

There exists $c > 1$ such that for all $d$, there exists a bounded $S \subseteq \mathbb{R}^d$ such that every partition of $S$ into pieces of smaller diameter has at least $c^d$ parts.
More Geometry

How many vectors of the cube in $\mathbb{R}^d$ can be pairwise non-orthogonal?

Conjecture (Larman-Rogers (1972))

Suppose that $d = 4n$. Does every set of $2^{d/2} \pm 1$ vectors in $\mathbb{R}^d$ contain a pair of orthogonal vectors?

Theorem (Frankl-Rödl (1987))

Given $r \geq 2$ and $n = d/4 \geq r$, there exists $\varepsilon = \varepsilon(r) > 0$ such that every set of more than $(2^{r} - \varepsilon)$ vectors in $\mathbb{R}^d$ contains $r$ pairwise orthogonal vectors.
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Given $r \geq 2$ and $n = d/4 \geq r$, there exists $\varepsilon = \varepsilon(r) > 0$ such that every set of more than $(2 - \varepsilon)^d \pm 1$ vectors in $R^d$ contains $r$ pairwise orthogonal vectors.
A weak delta system is a collection of sets \( A_1, \ldots, A_r \) such that

\[ |A_i \cap A_j| = |A_1 \cap A_2| \]

for \( 1 \leq i < j \leq r \).
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**Conjecture (Erdős-Szemerédi (1978))**

For every $\varepsilon > 0$, there is $n_0 = n_0(\varepsilon)$ such that if $n > n_0$ and $\mathcal{A} \subset 2^{[n]}$ with $|\mathcal{A}| > (2 - \varepsilon)^n$, then $\mathcal{A}$ contains a weak delta system of size 3.
Theorem (Frankl–Rödl (1987))

Fix $r \geq 3$. Then there are $\eta = \eta(r)$ and $\varepsilon = \varepsilon(r)$ such that if $t = (1/4 \pm \eta)n$ and $A \subseteq 2^{[n]}$ with $|A| > (2 - \varepsilon)^n$, then there are $A_1, \ldots, A_r \in A$ with

$$|A_i \cap A_j| = t$$

for $1 \leq i < j \leq r$. 

Conjecture (Erdős–Szemerédi (1978))

There exists $\varepsilon > 0$ such that if $n$ is sufficiently large and $A \subseteq 2^{[n]}$ with $|A| > (2 - \varepsilon)^n$, then $A$ contains a delta system (not weak!) of size 3.

Recent work of Alon–Shpilka–Umans gives connections between this conjecture and algorithms for Matrix multiplication.
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Computer Science Applications

- Communication Complexity (Sgall 1999)
- Quantum Computing (Buhrman-Cleve-Wigderson 1998)
The Frankl-Rödl theorem

Theorem (Frankl-Rödl (1987))

Let $0 < \eta < \frac{1}{4}$ and $\eta n < t < \left(\frac{1}{2} - \eta\right)n$. There is $\epsilon_0 = \epsilon_0(\eta)$ such that if $A \subset \mathcal{P}[n]$ and $|A \cap B| \neq t$ for all $A, B \in A$, then $|A| < \left(2 - \epsilon_0\right)n$.

How big is $\epsilon_0$ (problem of Erdős)? Frankl-Rödl show it is about $\left(\frac{t}{n}\right)^{2/3}$.
Theorem (Frankl-Rödl (1987))

Let $0 < \eta < 1/4$ and $\eta n < t < (1/2 - \eta)n$. There is $\varepsilon_0 = \varepsilon_0(\eta)$ such that if $\mathcal{A} \subset 2^n$ and $|A \cap B| \neq t$ for all $A, B \in \mathcal{A}$, then

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How big is $\varepsilon_0$ (problem of Erdős)?

Frankl-Rödl show it is about $(t/n)^2/2$. 

Katona’s Theorem

Suppose we forbid all numbers less than $t + 1$ as intersection sizes. Define $\mathcal{A}(n, t)$ to be

\[
\{ A \subseteq [n] : |A| \geq (n + t + 1)/2 \} \quad \text{if } n + t \text{ is odd}
\]

\[
\{ A \subseteq [n] : |A \cap ([n] - \{1\})| \geq (n + t)/2 \} \quad \text{if } n + t \text{ is even.}
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\end{align*}
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Theorem (Katona)

Let $\mathcal{A} \subset 2^{[n]}$ and suppose that $|A \cap A'| > t$ for every $A, A' \in \mathcal{A}$. Then

$$|\mathcal{A}| \leq |\mathcal{A}(n, t)|.$$ 

Moreover, if $t \geq 1$ and $|\mathcal{A}| = |\mathcal{A}(n, t)|$, then $\mathcal{A} = \mathcal{A}(n, t)$. 
The binary entropy function is

\[ H(x) = -x \log_2 x - (1 - x) \log_2 (1 - x). \]
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Conjecture (M-Rödl)

Let \( 0 < \eta < 1/2 \), \( \eta n < t < (1/2 - \eta)n \), and \( \mathcal{A} \subset 2^{[n]} \) with \( |A \cap B| \neq t \) for all \( A, B \in \mathcal{A} \). Then

\[ |\mathcal{A}| \leq \left( \frac{n}{(n + t)/2} \right)^{2^{o(n)}} = 2^{H\left(\frac{1}{2} + \frac{t}{2n}\right)n + o(n)}. \]
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If true, the conjecture is sharp as shown by \(\mathcal{A} = \binom{[n]}{>(n+t)/2}\).
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For fixed \( t \) and \( n > n_0(t) \), conjectured by Frankl and proved by Frankl-Füredi.
Theorem (M-Rödl)

Let $0 < \varepsilon < \frac{1}{5}$ be fixed, $n > n_0(\varepsilon)$, $\varepsilon n < t < \frac{n}{5}$ and $A \subset 2^{[n]}$.

Suppose that $|A \cap B| \not\in (t, t + n_0^{25})$ for all $A, B \in A$.

Then $|A| < n \left( \frac{n + t}{n/2} \right)$.

The constant 0.525 is a consequence of the result of Baker-Harman-Pintz that there is a prime in every interval $(s - s_0^{0.525}, s)$ as long as $s$ is sufficiently large.

If we assume the Riemann Hypothesis, then 0.525 could be improved to $\frac{1}{2} + o(1)$ using a result of Cramér.
Forbidding a small interval

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More restricted intersections

Question. Can the upper bound for $M$-intersecting families be improved for more restrictive $M$?

\[ \text{Eventown Theorem} \]

Suppose that $A \subset 2^{[n]}$ such that $|A|$ is even for every $A \in A$, $|A \cap B|$ is even for every $A, B \in A$.

Then $|A| \leq 2 \left\lfloor \frac{n}{2} \right\rfloor + 1$. 

\[ \text{Theorem (Berlekamp (1965), Graver (1975))} \]

Suppose that $A \subset 2^{[n]}$ is $M$-intersecting, where $M = \{0, 2, 4, \ldots\}$.

In other words, $|A \cap B|$ is even for all $A, B \in A$.

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**Eventown Theorem**

Suppose that $\mathcal{A} \subset 2^{[n]}$ such that

- $|A|$ is even for every $A \in \mathcal{A}$
- $|A \cap B|$ is even for every $A, B \in \mathcal{A}$

Then $|\mathcal{A}| \leq 2^{\lceil n/2 \rceil}$.
Proof of Eventown:

- To each $A \in \mathcal{A}$, associate its incidence vector $\nu_A = (\nu_1, \ldots, \nu_n)$ where

$$
\nu_i = \begin{cases} 
1 & i \in A \\
0 & i \notin A 
\end{cases}
$$

Let $S$ be the subspace of $\mathbb{F}^n_2$ spanned by $\{\nu_A\}_{A \in \mathcal{A}}$. $S$ is totally isotropic (meaning $x \cdot y = 0$ for $x, y \in S$) and

$$
\dim(S) \leq \left\lfloor \frac{n}{2} \right\rfloor
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So

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|\mathcal{A}| \leq |S| \leq 2^{\left\lfloor \frac{n}{2} \right\rfloor} = (1.4142^n...)
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Proof of Eventown:

- To each \( A \in \mathcal{A} \), associate its incidence vector \( v_A = (v_1, \ldots, v_n) \) where

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v_i = \begin{cases} 
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- Let \( S \) be the subspace of \( F_2^n \) spanned by \( \{v_A\}_{A \in \mathcal{A}} \).
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The length \( \ell(M) \) of a set \( M \) is the maximum number of consecutive integers contained in \( M \).
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**Definition**

The length \( \ell(M) \) of a set \( M \) is the maximum number of consecutive integers contained in \( M \).

\[ \ell(M) \leq \ell \quad \text{if and only if} \quad \overline{M} \quad \text{is} \quad (\ell + 1)\text{-syndetic}. \]
Theorem (M-Rödl)

Let $M \subset \{0, 1, \ldots, n\}$ with $\ell(M) = \ell$. Suppose that $\mathcal{A} \subset 2^{[n]}$ is an $M$-intersecting family. Then

$$|\mathcal{A}| < 1.622^n \times 100^\ell.$$
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Bounds for small $\ell(M)$

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- The 1.622 is probably not sharp, just a result of the proof
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Bounds for very small $\ell(M)$

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- This is the first non-linear-algebraic proof of an asymptotic version of the Eventown Theorem; it applies in more general scenarios though doesn’t give bounds as precise as $2^{n/2}$. 

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Dhruv Mubayi  
Intersection theorems for finite sets
Proof Methods

- Prove the result for pairs of families \((A, B)\). This facilitates an induction argument.
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|A \cap B| \in M
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for all \(A \in \mathcal{A}\) and \(B \in \mathcal{B}\)

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Let \(M \subset \{0, 1, \ldots, n\}\) with \(\ell(M) = \ell\). Suppose that \((A, B)\) is an \(M\)-intersecting pair of families in \(2^{[n]}\). Then

\[
|A| \cdot |B| < \min\{2 \cdot 631 \cdot n \times 10^4 \ell, 2^n + 2 \ell \log_2 n\}
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Proof Methods

- Prove the result for pairs of families \((A, B)\). This facilitates an induction argument.
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\[
|A||B| < \min \left\{ 2.631^n \times 10^{4\ell}, \ 2^{n+2\ell \log^2 n} \right\}.
\]
Definition (Sgall)

Say that a function $h : 2^N \rightarrow N \cup \{\infty\}$ is a height function if the following four properties hold:

1. \[ h(L) = 0 \text{ if and only if } L = \emptyset, \]
2. \[ h(L) < \infty \text{ and } L' \subset L \Rightarrow h(L') \leq h(L), \]
3. \[ h(L) < \infty \text{ and } L' \subset L - 1 \Rightarrow h(L') \leq h(L), \]
4. \[ h(L), h(L') \leq s < \infty \Rightarrow \begin{cases} h(L' \cap L) \leq s - 1 \text{ or } h(L' \cap (L - 1)) \leq s - 1. \end{cases} \]
Say that a function $h : 2^N \rightarrow N \cup \{\infty\}$ is a height function if the following four properties hold:

- (A1) $h(L) = 0$ if and only if $L = \emptyset$,
- (A2) if $h(L) < \infty$ and $L' \subset L$, then $h(L') \leq h(L)$,
- (A3) if $h(L) < \infty$ and $L' \subset L - 1$, then $h(L') \leq h(L)$,
- (A4) if $h(L)$, $h(L') \leq s < \infty$, then either $h(L' \cap L) \leq s - 1$ or $h(L' \cap (L - 1)) \leq s - 1$. 

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Sgall’s theorem

**Theorem (Sgall (1999))**

Suppose that $(A, B)$ is an $M$-intersecting pair of families in $2^{[n]}$ and $h(M) \leq s \leq n + 1$. Then

$$|A||B| \leq 2^{n+s-1} \binom{n}{s-1}.$$
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**Theorem (Sgall (1999))**

Suppose that \((A, B)\) is an \(M\)-intersecting pair of families in \(2^{[n]}\) and \(h(M) \leq s \leq n + 1\). Then

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\]

**Theorem (M-Rödl)**

There exists a height function \(h\) such that for every \(M \subset \{0, 1, \ldots, n\}\),

\[
h(M) \leq 1 + 2\ell(M) \log n.
\]

Applying this bound in Sgall’s Theorem yields

\[
|A||B| < 2^{n+2\ell \log^2 n}.
\]
The Height function

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The Height function

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- \( h(L) = 1 + \max\{ A, B \} \)
Lemma (Sgall)

Suppose that $a, b, x, y, p, Q$ are positive real numbers such that

$$ax \leq p \leq Q$$
$$bx \leq Q$$

Then

$$(a + b)(x + y) \leq 2(p + Q).$$
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Sgall’s Lemma and the Puzzle

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