Latin transversals

and the

covering radius of sets of permutations

Ian Wanless / Xiande Zhang

Monash University, Australia

Latin squares

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Each solution is the superposition of two latin squares

$$\begin{array}{cccc} \spadesuit & \heartsuit & \diamondsuit & \clubsuit \\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit \\ \clubsuit & \diamondsuit & \heartsuit & \spadesuit \\ \diamondsuit & \clubsuit & \spadesuit & \heartsuit \end{array} \qquad \begin{array}{cccc} A & K & J & Q \\ Q & J & K & A \\ J & Q & A & K \\ K & A & Q & J \end{array}$$

These squares are *orthogonal mates*.
When we overlay them each ordered pair of symbols occurs once.

Transversals

A *transversal* of a latin square is a set of entries which includes exactly one entry from each row and column and one of each symbol.

♠	**♡**	♢	♣
♡	♠	**♣**	♢
♣	♢	♡	**♠**
♢	♣	♠	♡

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Q	J	**K**	A
J	Q	A	K
K	**A**	Q	J

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Alon *et al.* [1995] lamented that

> *"There have been more conjectures than theorems on latin transversals in the literature."*

Graph theoretic interpretation

A latin square is equivalent to a proper edge colouring of $K_{n,n}$ with n colours.

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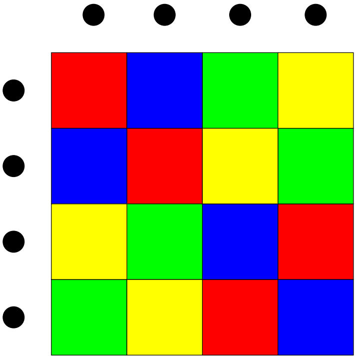

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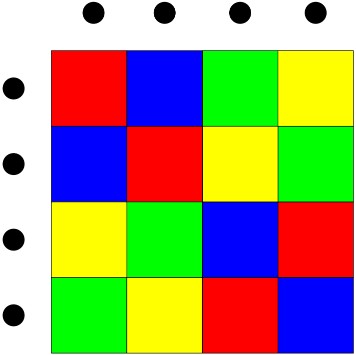

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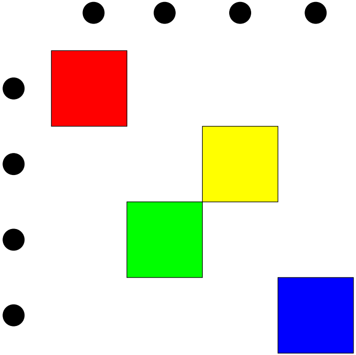

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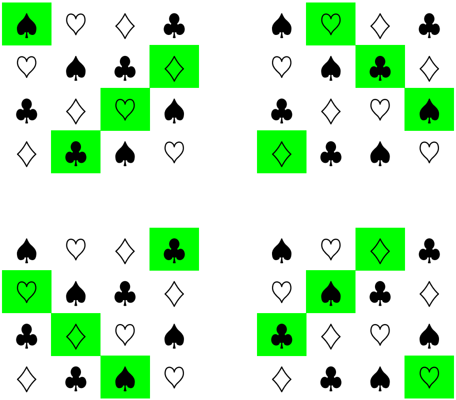
The first theorem

Theorem: A latin square has an orthogonal mate iff it can be decomposed into disjoint transversals.



Some squares have 'em, some squares don't

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Conjecture: [Brualdi] Every latin square has a near transversal.

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Theorem: [Shor '82, Hatami/Shor '08] A latin square of order n has a partial transversal of length at least $n - O(\log^2 n)$.

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Conjecture: A d-dimensional latin hypercube of order n

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Conjecture: A d-dimensional latin hypercube of order n has a transversal unless d and n are even.

Entries not in transversals

#trans-free entries	Order=2	3	4	5	6	7	8	9
0		1	1	1	2	54	267932	19270833530
1						11	13165	18066
2						26	1427	1853
3						12	253	54
4	1					12	508	21
5						6	89	7
6					1	8	65	7
7						3	33	1
8						4	48	1
9							25	
10						1	27	1
11						1	9	
12				1	2	6	9	
13						1	2	
14							2	
16			1		1	1	27	
18							1	
20							1	
28							1	
36					6	1		
64							33	
Total	1	1	2	2	12	147	283657	19270853541

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Lemma: If T is a transversal of L then

$$\sum_{(r,c,s) \in T} \Delta(r, c, s) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

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This immediately shows that $\mathbb{Z}_n$ has no transversal when n is even.

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$$\begin{bmatrix} 1 & 3 & 5 & 7 & \cdots & 0 & \\ 0 & & & & & & 1 \\ & 1 & 3 & 5 & \cdots & n-2 & 0 \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{bmatrix}$$

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The highlighted cell is not in any transversal.

Restrictions on transversals

Theorem: [Egan/W.'12] For any odd $m > 1$ there is a LS of order $3m$ in which there is a $(m-1) \times m$ subrectangle that is not hit by any transversal.

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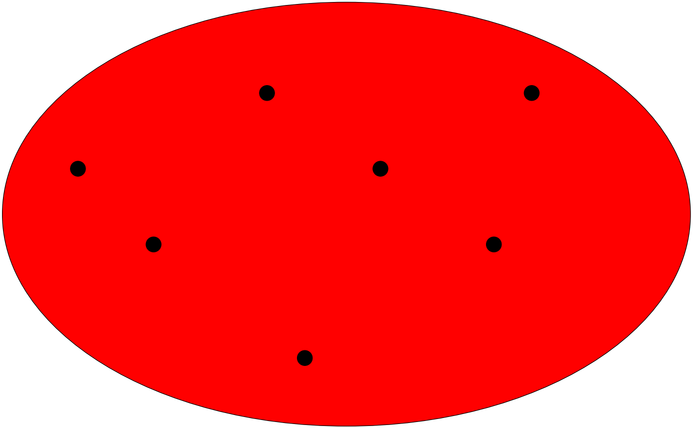
For every order $n > 4$ there is a LS containing a *maximal* set of either 1 or 3 disjoint transversals.

Theorem: [Evans'06] For every n there exists a LS with no set of more than $(n+1)/2$ disjoint transversals.

Covering Radius

For a set of points in a metric space,
the *covering radius* is the smallest r such that
the space is covered by the balls of radius r about the points.



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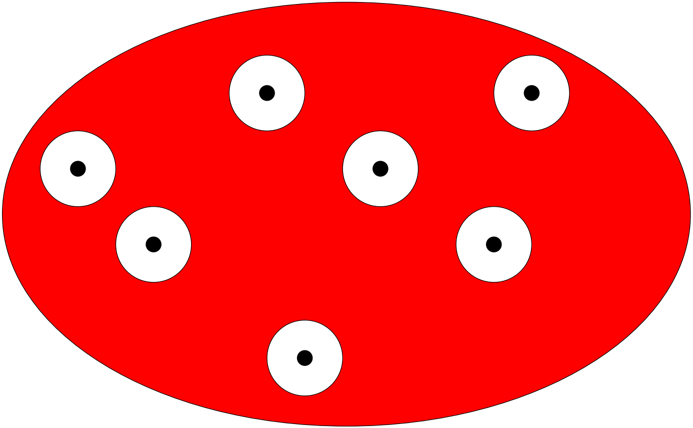

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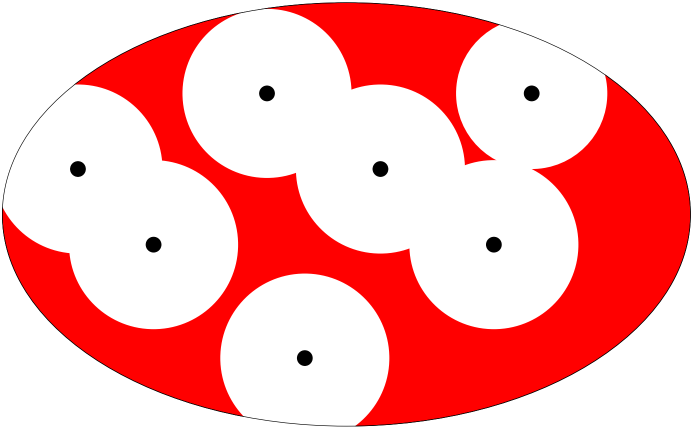

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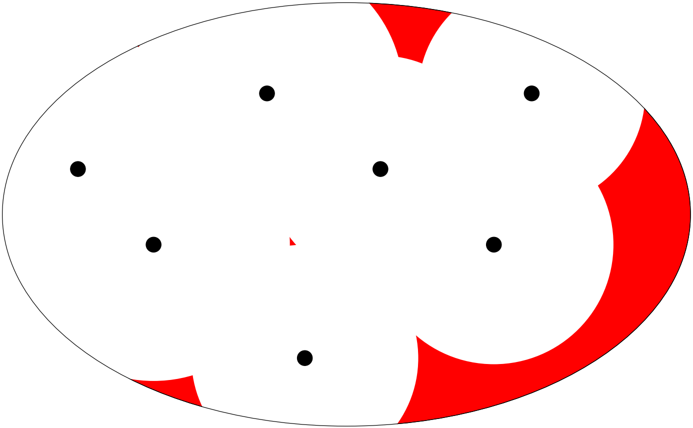

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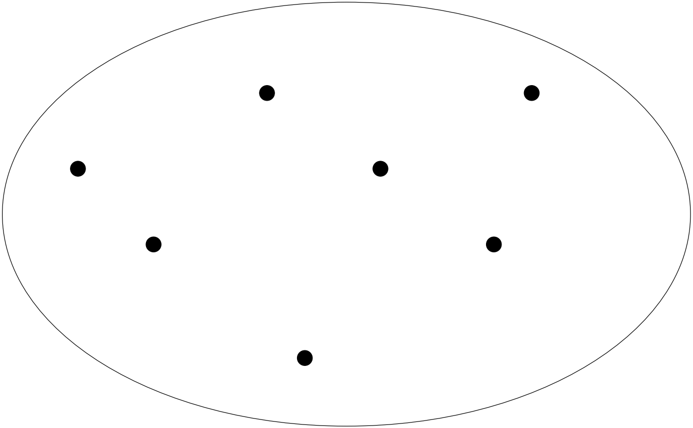

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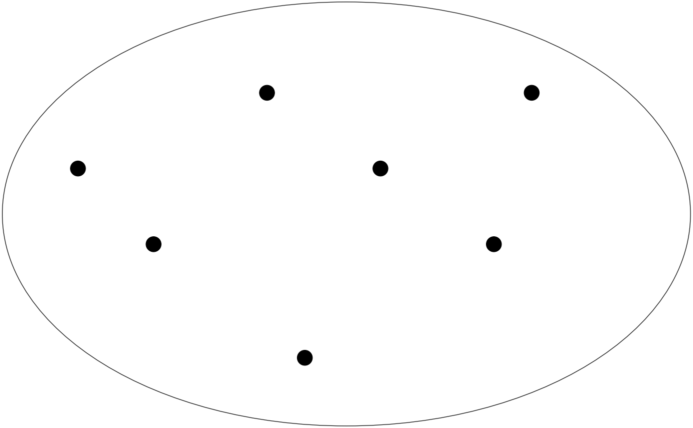

Now apply this to S_n using the Hamming metric.

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Example:

$$P = \left\{ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 3 \\ 2 & 1 & 5 & 3 & 4 \end{array} \right.$$

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Theorem: $f(n,1) = \lfloor n/2 \rfloor + 1$.

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This shows that KS⇒Ryser.

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n	3	4	5	6	7	8	9	10
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$$f(n,2) \leq \begin{cases} n & \text{if } n \text{ is even,} \\ \frac{5}{4}n + O(1) & \text{if } n \equiv 1 \mod 4, \\ \frac{4}{3}n + O(1) & \text{if } n \equiv 3 \mod 4. \end{cases}$$

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For lower bounds, we can't say more than
$f(n,2) > f(n,1) = \frac{1}{2}n + O(1)$.

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- If n is an odd multiple of 3, then there is a LS(n) with no set of more than $n/3 + 2$ disjoint transversals.

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3. Consider the case where the set of permutations is in fact a group. Some results here, but much we don't know...

Our story ends...