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Space-time problems in plasma physics mathematical understanding of "vorticity"

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Space-time problems in plasma physics —mathematical understanding of "vorticity"

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1 Introduction

The aim of this lecture is to elucidate a basic mathematical structure that provides a plasma with microscopic complexity and, at the same time, some ordered structures on macroscopic scale. To describe a theory of the co-existence of microscopic complexity (disorder) and macroscopic order, we must start by asking what *scale hierarchy* is. Here, we formulate scale hierarchy invoking the notion of *phase-space foliation* —macroscopic hierarchy is a *leaf* immersed in the total microscopic phase-space; macroscopic description of mechanics is *non-canonical*, while the microscopic world is *canonical* (as we lean in basic physics courses).

The problem is the "geometry" that dictates the mechanics. In comparison with some complicated Hamiltonians invoked to describe strongly coupled systems such as various condensed matters (a typical example is the Ginzburg-Landau energy of superconductivity), "normal" (weakly coupled) a fluid or plasma assumes a simple Hamiltonian (basically a quadratic function on phase space, which parallels the *norm* of the metric). Therefore, the conventional Hamiltonian does not describe a "program" that enables a plasma to produce nontrivial structures or dynamics on macroscopic scales, i.e. the mechanism creating a non-trivial structure (which is typically *vortical*) is not due to some structure written in a Hamiltonian. The creation mechanism must be, then, found in the *geometry* of space time that hosts the matter, rather than the energy, the matter itself.

Let us start by a simple example. A charged particle's Hamiltonian H is the sum of the kinetic energy $|\boldsymbol{P} - q\boldsymbol{A}|^2/2m$ ($\boldsymbol{P}$ is the canonical momentum,

$\boldsymbol{A}$ is the vector potential, m is the mass, and q is the charge) and the potential energy $q\phi$. The Boltzmann distribution is $n_0 e^{-\beta H} = n_0 e^{-(V^2/2m+q\phi)}$, where $\boldsymbol{V} = \boldsymbol{P} - q\boldsymbol{A}$. Hence, the magnetic field does not influence the equilibrium distribution (this is simply because magnetic force does not change energy). If we assume charge neutrality and put $\phi = 0$, we obtain a spatially homogeneous density distribution. Needless to say, this simple model does not apply to describe rich structures (inhomogeneous distributions of various physical quantities) of plasmas created by magnetic fields. We thus proffer a theory that attributes creation mechanism to "space". The notion of particle (an element sitting in space) is, then, deformed to some "quasi-particle" that better describes macroscopic hierarchy of a plasma. In Sec. 2, we will construct a mathematical framework which enables us to formulate an appropriate space = macroscopic hierarchy on which a plasma creates a significant structure.

2 Scale hierarchy and self-organization

2.1 Co-existence of order and disorder

Superficially, the process of self-organization of a structure may appear to be an antithesis of the maximum entropy *ansatz*. And yet various nonlinear systems display what may be viewed as the simultaneous existence of order and disorder. This co-existence will begin to make sense if the self-organization processes and the entropy principle were to manifest on different scales; disorder can still develop at a microscopic scale while an ordered structure emerges on some appropriate macroscopic scale. Writing a theory of self-organization, then, will be an exercise in delineating and understanding the characteristic *scale hierarchy* of the physical system.

Here we introduce a mathematical framework to describe a theory of self-organization on hierarchical phase space. To delineate the emergence of a clear and distinct scale hierarchy, we adopt an approach by Hamiltonian mechanics; we will investigate a magnetospheric plasma as a typical example of systems in which ordered structures can emerge while maximizing entropy.

Magnetospheric plasmas (the naturally occurring ones such as the planetary magnetospheres, as well as their laboratory simulations [24, 16, 17, 27]) are self-organized around the dipole magnetic fields in which charged particles cause a variety of interesting phenomena: the *inward diffusion* (or up-hill diffusion) is of particular interest. This process is driven by some spontaneous fluctuations (symmetry breaking) that violate the constancy of angular momentum. In a strong enough magnetic field, the canonical angular momentum P_θ is dominated by the magnetic part $q\psi$: the charge multiplied by the flux function (in the r-θ-z cylindrical coordinates, $\psi = rA_\theta$, where A_θ is the θ component of the vector potential). The conservation of $P_\theta \approx q\psi$, therefore, restricts the particle motion to the magnetic surface (level-set of ψ). It is only via randomly-phased fluctuations that the particles can diffuse across magnetic surfaces. Although the "diffusion" is normally a process that diminishes gradients, numerical experiments do exhibit preferential inward shifts through random motions of test particles [3, 21]. Detailed specification of the fluctuations or the microscopic motion of particles is not the subject of present effort. We plan to construct, instead, a clear-cut description of equilibria that maximize entropy simultaneously with bearing steep density gradients [27].

Such an equilibrium will be formulated as a grand-canonical distribution on a leaf of "foliated phase space" that represents a macroscopic hierarchy. Heterogeneity is created by the distortion of the metric (invariant measure) dictating *equipartition* on the leaf. In a strongly inhomogeneous magnetic field (typically a dipole magnetic field), the phase-space metric of *magnetized particles* is distorted; thus the projection of the equipartition distribution onto the flat space of the laboratory frame yields peaked profile because of the connecting inhomogeneous Jacobian weight.

2.2 Hamiltonian of charged particle

The Hamiltonian of a charged particle is a sum of the kinetic energy and the potential energy:

$$H = \frac{m}{2}v^2 + q\phi, \tag{1}$$

where $\boldsymbol{v} := (\boldsymbol{P} - q\boldsymbol{A})/m$ is the velocity, $\boldsymbol{P}$ is the canonical momentum, $(\phi, \boldsymbol{A})$ is the electromagnetic 4-potential, m (q) is the particle mass (charge). In the present work, we may treat electrons and ions equally (in later discussion, we will neglect ϕ assuming charge neutrality, but generalization to a non-neutral plasma will be interesting [24, 5]). Denoting by $\boldsymbol{v}_{\|}$ and $\boldsymbol{v}_\perp$ the parallel and perpendicular (with respect to the local magnetic field) components of the velocity, we may write

$$H = \frac{m}{2}v_\perp^2 + \frac{m}{2}v_{\|}^2 + q\phi. \tag{2}$$

The velocities are related to the mechanical momentum as $\boldsymbol{p} := m\boldsymbol{v}$, $\boldsymbol{p}_{\|} := m\boldsymbol{v}_{\|}$, and $\boldsymbol{p}_\perp := m\boldsymbol{v}_\perp$.

In a strong magnetic field, $\boldsymbol{v}_\perp$ can be decomposed into a small-scale cyclotron motion $\boldsymbol{v}_c$ and a macroscopic guiding-center drift motion $\boldsymbol{v}_d$. The periodic cyclotron motion $\boldsymbol{v}_c$ can be *quantized* to write $(m/2)v_c^2 = \mu\omega_c(\boldsymbol{x})$ in terms of the magnetic moment μ and the cyclotron frequency $\omega_c(\boldsymbol{x})$; the adiabatic invariant μ and the gyration phase $\vartheta_c := \omega_c t$ constitute an action-angle pair. In the standard interpretation, in analogy with the Landau levels in quantum theory, ω_c is the *energy level* and μ is the *number* of quasi-particles (quantized periodic motions) at the corresponding energy level.

For an axisymmetric system with a poloidal (but no toroidal) magnetic field, let (ψ, ζ, θ) be a magnetic coordinate system such that $\boldsymbol{B} = \nabla\psi \times \nabla\theta$ (θ is the toroidal angle). An approximately-vacuum magnetic field may also be written as $\boldsymbol{B} = \nabla\xi = B\nabla\zeta$. For a point dipole of magnetic moment M,

$$\begin{aligned} \psi(r,z) &= Mr^2(r^2+z^2)^{-3/2}, \\ \xi(r,z) &= Mz(r^2+z^2)^{-3/2}. \end{aligned}$$

The magnetic field strength is

$$B = M\sqrt{(r^4 + 5r^2z^2 + 4z^4)/(r^2+z^2)^5}.$$

The macroscopic part of the perpendicular kinetic energy is expressed as $mv_d^2/2 = (P_\theta - q\psi)^2/(2mr^2)$, where P_θ is the angular momentum in the θ direction and r is the radius from the geometric axis. In terms of the canonical-variable set $\boldsymbol{z} = (\vartheta_c, \mu, \zeta, p_{\|}, \theta, P_\theta)$ the Hamiltonian of the guiding center (or, the quasi-particle) becomes

$$H_c = \mu\omega_c + \frac{1}{2m}p_{\|}^2 + \frac{1}{2m}\frac{(P_\theta - q\psi)^2}{r^2} + q\phi. \tag{3}$$

Note that the energy of the cyclotron motion has been quantized in term of the frequency $\omega_c(\boldsymbol{x})$ and the action μ; the gyro-phase ϑ_c has been coarse grained (integrated to yield 2π).

2.3 Boltzmann distribution

The standard Boltzmann distribution function is derived when we assume that $\mathrm{d}^3v\mathrm{d}^3x$ is an invariant measure and the Hamiltonian H is the determinant of the ensemble. Maximizing the entropy $S = -\int f \log f \mathrm{d}^3v\mathrm{d}^3x$ keeping the total energy $E = \int Hf\mathrm{d}^3v\mathrm{d}^3x$ and the total particle number $N = \int f\mathrm{d}^3v\mathrm{d}^3x$ constant, we obtain

$$f(\boldsymbol{x}, \boldsymbol{v}) = Z^{-1}\mathrm{e}^{-\beta H}, \tag{4}$$

where Z is the normalization factor ($\log Z - 1$ is the Lagrange multiplier on N) and β is the inverse temperature (the Lagrange multiplier on E). The corresponding configuration-space density,

$$\rho(\boldsymbol{x}) = \int f \mathrm{d}^3 v \propto \mathrm{e}^{-\beta q \phi}, \tag{5}$$

becomes constant for a charge neutral system ($\phi = 0$).

Needless to say that the Boltzmann distribution or the corresponding configuration-space density, with an appropriate Jacobian multiplication, is independent of the choice of phase-space coordinates. Moreover, the density is invariant no matter whether we quantize the cyclotron morion or not. Let us confirm this fact by a direct calculation. For the Boltzmann distribution of the "guiding-center plasma"

$$\begin{aligned} f(\mu, v_d, v_\parallel; \boldsymbol{x}) &= Z^{-1} \mathrm{e}^{-\beta H_c} \\ &= Z^{-1} \mathrm{e}^{-\beta \left(\mu \omega_c(\boldsymbol{x}) + m v_d^2/2 + m v_\parallel^2/2 + q\phi(\boldsymbol{x}) \right)}, \end{aligned} \tag{6}$$

the density is given by

$$\rho(\boldsymbol{x}) = \int f \mathrm{d}^3 v = \int f \, \frac{2\pi\omega_c}{m} \mathrm{d}\mu \mathrm{d}v_d \mathrm{d}v_\parallel \propto \mathrm{e}^{-\beta q \phi}, \tag{7}$$

exactly reproducing (5).

2.4 Equilibrium on macroscopic hierarchy

Now we formulate the "macroscopic hierarchy" on which the thermal equilibrium creates a structure. The adiabatic invariance of the magnetic moment μ (the *number* of the quantized quasi-particles) imposes a *topological constraint* on the motion of particles; it is this constraint that is the root-cause of a macroscopic hierarchy and of structure formation. Mathematically, the *scale hierarchy* is equivalent to a foliation of the phase space. To explain how the scale hierarchy is formulated, we start by the general (micro-macro total) formulation, and then separate the microscopic action-angle pair μ-ϑ_c; the *macroscopic phase space* is the remaining sub-manifold immersed in the general phase space, which we delineate as a leaf of the foliation in terms of a *Casimir invariant* (if there is a nontrivial function C satisfying $\{G, C\} = 0$ for every G, we say that the Poisson bracket $\{\ ,\ \}$ is *non-canonical*, and call C a *Casimir invariant*; [13]).

The Poisson bracket on the total phase space, spanned by the canonical variables $\boldsymbol{z} = (\vartheta_c, \mu, \zeta, p_{\|}, \theta, P_\theta)$, is

$$\{F, G\} := \langle \partial_{\boldsymbol{z}} F, \mathcal{J}_c \partial_{\boldsymbol{z}} G \rangle,$$

where $\langle \boldsymbol{u}, \boldsymbol{v} \rangle := \int u_j v^j \mathrm{d}^6 z$ is the inner-product and $\mathcal{J}$ is the canonical symplectic matrix (Poisson tensor):

$$\mathcal{J}_c := \begin{pmatrix} J_c & 0 & 0 \\ 0 & J_c & 0 \\ 0 & 0 & J_c \end{pmatrix}, \quad J_c := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{8}$$

The equation of motion for the Hamiltonian H_c is written as $\mathrm{d}z^j/\mathrm{d}t = \{H_c, z^j\}$. Notice that the quantization of the cyclotron motion suppresses change in μ. Liouville's theorem determines the invariant measure $\mathrm{d}^6 z$, by which we obtain the Boltzmann distribution (6).

To extract the macroscopic hierarchy, we "separate" the microscopic variables (ϑ_c, μ) by modifying the symplectic matrix as

$$\mathcal{J}_{nc} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & J_c & 0 \\ 0 & 0 & J_c \end{pmatrix}. \tag{9}$$

The Poisson bracket

$$\{F, G\}_{nc} := \langle \partial_{\boldsymbol{z}} F, \mathcal{J}_{nc} \partial_{\boldsymbol{z}} G \rangle$$

determines the kinematics on the macroscopic hierarchy; the corresponding kinetic equation $\partial_t f + \{H_c, f\}_{nc} = 0$ reproduces the familiar drift-kinetic equation (for example [14]).

The nullity of $\mathcal{J}_{nc}$ makes the Poisson bracket $\{\ ,\ \}_{nc}$ *non-canonical* [13]. Evidently, μ is a Casimir invariant (more generally $C = g(\mu)$ with g being any smooth function). The level-set of μ, a leaf of the Casimir foliation, identifies what we may call the *macroscopic hierarchy*. By applying Liouville's theorem to the Poisson bracket $\{\ ,\ \}_{nc}$, the invariant measure on the macroscopic hierarchy is $\mathrm{d}^4 z = \mathrm{d}^6 z/(2\pi \mathrm{d}\mu)$, the the total phase-space measure modulo the microscopic measure. The most probable state (statistical equilibrium) on the macroscopic ensemble must maximize the entropy with respect to this invariant measure. The variational principle is set up following the standard procedure —immersing the macroscopic hierarchy into the general phase space, and incorporating the constraints through the Lagrange multipliers: We maximize entropy $S = -\int f \log f \, \mathrm{d}^6 z$ for a given particle

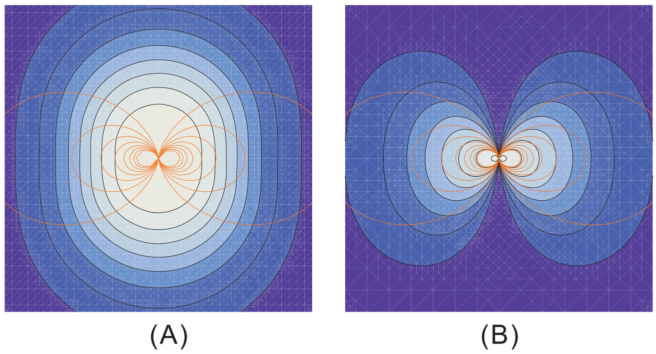

Figure 1: Density distribution (contours) and the magnetic field lines (level-sets of ψ) in the neighborhood of a point dipole. (A) The equilibrium on the leaf of μ-foliation. (B) The equilibrium on the leaf of μ and $J_{\parallel}$-foliation.

number $N = \int f \mathrm{d}^6 z$, a quasi-particle number $M = \int \mu f \mathrm{d}^6 z$, and an energy $E = \int H_c f \mathrm{d}^6 z$, to obtain the distribution function

$$f = f_\alpha := Z^{-1} \mathrm{e}^{-(\beta H_c + \alpha\mu)}, \tag{10}$$

where α, β, and $\log Z - 1$ are, respectively the Lagrange multipliers on M, E, and N. In this "grand-canonical" distribution function, α/β is the chemical potential associated with the quasi-particles.

We can also derive (10) by an *energy-Casimir function*. With a Casimir element μ, we can transform the Hamiltonian as $H_c \mapsto H_\alpha := H_c + \alpha\mu$ (α is an arbitrary constant) without changing the macroscopic dynamics; H_α is called an energy-Casimir function [13]. The Boltzmann distribution with respect to H_α is equivalent to (10).

The factor $\mathrm{e}^{-\alpha\mu}$ in f_α yields a direct ω_c dependence of the configuration-space density:

$$\rho = \int f_\alpha \frac{2\pi\omega_c}{m} \mathrm{d}\mu \mathrm{d}v_d \mathrm{d}v_{\parallel} \propto \frac{\omega_c(\boldsymbol{x})}{\beta\omega_c(\boldsymbol{x}) + \alpha}, \tag{11}$$

which may be compared with the density (7) evaluated for the Boltzmann distribution ($\phi = 0$ assuming charge neutrality). Notice that the Jacobian $(2\pi\omega_c/m)\mathrm{d}\mu$ multiplying the macroscopic measure $\mathrm{d}^4 z$ reflects the distortion of the macroscopic phase space (Casimir leaf) caused by the magnetic field. Figure 1-(A) shows the density distribution and the magnetic field lines.

2.5 Macro-scale action-angle pair

In an axisymmetric system, the quasi-particle motion, periodic in both the parallel and θ directions, may be described in terms of the macroscopic action-angle pairs: $J_{\|}$-$\vartheta_{\|}$ ($:= \sin^{-1}(\zeta/\ell_{\|})$; $\ell_{\|}$ is the bounce orbit length) and P_θ ($\approx q\psi$)-θ (for a hierarchy of adiabatic invariants, see [9] and papers cited there). To find explicit expressions for the parallel action-angle variables, we invoke the Hamiltonian H_c of (3). Neglecting the curvature effect and putting $\phi = 0$, the equation of the parallel motion reads as

$$m\frac{\mathrm{d}^2}{\mathrm{d}t}\zeta = -\mu\nabla_{\|}\omega_c. \tag{12}$$

In the vicinity of $\zeta = 0$, where ω_c has a minimum on each magnetic surface, we may expand

$$\omega_c = \Omega_c(\psi) + \Omega_c''(\psi)\frac{\zeta^2}{2},$$

where $\Omega_c(\psi)$ is the minimum of ω_c and $\Omega_c''(\psi) := \mathrm{d}^2\omega_c/\mathrm{d}\zeta^2|_\psi$. In terms of the length

$$L_{\|}(\psi) := \left(\frac{2\Omega_c(\psi)}{\Omega_c''(\psi)}\right)^{1/2},$$

which scales the variation of ω_c along ζ, (12) is integrated to identify the corresponding action-angle variables: $\zeta = \ell_{\|}\sin\vartheta_{\|}$, $\vartheta_{\|} = \omega_b t$ with the bounce frequency

$$\omega_b = \left(\frac{\Omega_c''(\psi)\mu}{m}\right)^{1/2} = \frac{v_\perp}{L_{\|}(\psi)}.$$

The bounce amplitude $\ell_{\|} = [2E_{\|}/(m\omega_b^2)]^{1/2}$ is evaluated in terms of the parallel energy $E_{\|} := (mv_{\|}^2)/2|_{\zeta=0}$. Assuming $E_{\|} \approx E_\perp := \mu\Omega_c$, we estimate $\ell_{\|} \approx L_{\|}$. The action

$$J_{\|} := \frac{1}{2\pi}\oint mv_{\|}\mathrm{d}\zeta$$

is related to $E_{\|} = J_{\|}\omega_b$, and

$$\mathrm{d}v_{\|} = \left(\frac{\omega_b}{mv_{\|}}\right)\mathrm{d}J_{\|} = \left(\frac{\omega_b}{2mJ_{\|}}\right)^{1/2}\mathrm{d}J_{\|}.$$

The latter, using the relation $\omega_b/(mv_{\|}) = v_\perp/(L_{\|}mv_{\|}) \approx 1/(mL_{\|})$, becomes

$$\mathrm{d}v_{\|} \approx \left(\frac{1}{mL_{\|}}\right)\mathrm{d}J_{\|}.$$

The quantization of the parallel action-angle pair $J_{\|}$-$\vartheta_{\|}$, adds an additional constraint leading to a new equilibrium distribution function:

$$
f_{\alpha,\gamma} = Z^{-1}\mathrm{e}^{-(\beta H_c + \alpha\mu + \gamma J_{\|})}, \tag{13}
$$

and the corresponding density

$$
\begin{aligned}
\rho &= \int f_{\alpha,\gamma} \frac{2\pi\omega_c \mathrm{d}\mu}{m} \frac{\mathrm{d}J_{\|}}{m L_{\|}(\psi)} \mathrm{d}v_d \\
&\propto \frac{\omega_c(\boldsymbol{x})}{m^2} \int_0^\infty \frac{\mathrm{e}^{-(\beta\omega_c+\alpha)\mu}\mathrm{d}\mu}{\beta\sqrt{2\omega_c\mu/m} + \gamma L_{\|}(\psi)}.
\end{aligned} \tag{14}
$$

Through $L_{\|}(\psi)$, the density ρ acquires a dependence on ψ. We may estimate $L_{\|}(\psi) \sim \psi^{-1}$ ($\approx r$ at $z = 0$). Numerical integration of (14) gives a density profile depicted in Fig. 1-(B).

3 Non-canonical mechanics structured by "vortex"

3.1 Symplectic geometry

Vortex is the universal structure that dictates any dynamics of matter encompassing from the canonical particle mechanics (either classical or quantum-mechanical) to the non-canonical macroscopic fluid/plasma mechanics.

The *symplectic geometry* is the canonical example: Hamiltonian flow in a phase space (cotangent bundle of some smooth manifold) can be viewed as a "canonical" form of *vortex* implemented to the system by the symplectic (Poisson) matrix: its simplest form

$$
J_c := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{15}
$$

is the generator of the rotation group

$$
A(\theta) := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{16}
$$

Identifying an *exterior derivative* of a 1-form to be a vorticity (generalizing the conventional vorticity $\boldsymbol{\Omega} = \nabla \times \boldsymbol{V}$ of a three-dimensional vector $\boldsymbol{V}$), the symplectic 2-form (the determinant of a canonical Hamiltonian mechanics) is, indeed, the vorticity of a canonical 1-form: Invoking a local coordinate on a smooth n-dimensional manifold M, the canonical 1-form is

$$
\Theta := -\sum_{j=1}^{n} p_j \mathrm{d}q_j \quad (\in T^*M). \tag{17}
$$

The symplectic 2-form is the "vorticity" of Θ:

$$\omega := \mathrm{d}\Theta = \sum_{j=1}^{n} \mathrm{d}q_j \wedge \mathrm{d}p_j. \tag{18}$$

Denoting $z_{2j-1} = q_j$, $z_{2j} = p_j$ $(j = 1, \cdots, n)$, and

$$\mathcal{J}_c := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{19}$$

we may write

$$\omega = \frac{1}{2}\mathcal{J}_c^{k\ell} dz_k \wedge dz_\ell. \tag{20}$$

Given a Hamiltonian $H(\boldsymbol{z})$, the Lagrangian $L = \Theta - H\mathrm{d}t$ determines the equation of motion by the variational principle:

$$\mathcal{J}_c^{k\ell}\frac{dz_\ell}{dt} = -\frac{\partial H}{\partial z_k}. \quad (k = 1, \cdots, 2n) \tag{21}$$

By multiplying $\mathcal{J}_c^{-1} = -\mathcal{J}_c$, we obtain

$$\frac{dz_\ell}{dt} = \mathcal{J}_c^{k\ell}\frac{\partial H}{\partial z_k} \quad (k = 1, \cdots, 2n), \tag{22}$$

which is the canonical Hamilton's equation. Defining a *Poisson bracket* [1] by

$$\{a, b\} := \langle \partial_{\boldsymbol{z}} a, \mathcal{J}_c \partial_{\boldsymbol{z}} a \rangle = (\partial_{z_k} a)\mathcal{J}_c^{k\ell}(\partial_{z_\ell} b), \tag{23}$$

we may evaluate the rate of change of an *observable* $f(\boldsymbol{z})$ by equation

$$\frac{d}{dt} f = \{f, H\}. \tag{24}$$

A remarkable point of the canonical Hamiltonian mechanics is that the vortical motion, generated by the canonical Poisson matrix $\mathcal{J}_c$, is limited between the canonical pairs q_j and p_j. Hamilton-Jacobi equation $\boldsymbol{p} = -\nabla S$ (S is the action) implies $\nabla \times \boldsymbol{p} = 0$ (in general dimension, $\mathrm{d}p = 0$), i.e. the momentum vector is irrotational.

[1] Regarding ∂_{z_i} as the basis of the tangent space, $\mathrm{ad}(a) := J^{ij}(\partial_{z_j} H)\partial_{z_i}$ is a vector field (Hamiltonian flow) on the phase space ($\mathrm{ad}(a)$ is the *adjoint representation* of the Poisson algebra). We may also regard $\mathrm{ad}(a)$ as a differential operator applying to functions on X, which constitute a non-commutative ring $\mathcal{D}(X)$. The Lie bracket $[\mathrm{ad}(a), \mathrm{ad}(b)]$ defines a Lie algebra $\mathcal{A}$; $\mathcal{D}(X)$ is the enveloping algebra of $\mathcal{A}$.

Note 1 (Heisenberg algebra) *The Poisson algebra of Hamiltonian mechanics is quantized by the* correspondence principle*:*

$$\{\mathcal{F}\, mod\, \hbar, \mathcal{G}\, mod\, \hbar\} = \hbar^{-1}[\mathcal{F}, \mathcal{G}]\, mod\, \hbar, , \tag{25}$$

where $\mathcal{F}$ and $\mathcal{G}$ are some q-numbers (operators), $[\mathcal{F}, \mathcal{G}] = \mathcal{F}\mathcal{G} - \mathcal{G}\mathcal{F}$, $\{f, g\}$ is a Poisson bracket, and $mod\, \hbar$ is the principal symbol of the preceding operator. The "vortical motion" on the "space" of quantized variables $\hat{q}_j$ and $\hat{p}_k$ (corresponding to q_j and p_k) is then translated into the commutation rule of such that

$$[\hat{q}_j, \hat{p}_k] = i\hbar^{-1}\delta_{jk}, \quad [\hat{q}_j, \hat{q}_k] = 0, \quad [\hat{p}_j, \hat{p}_k] = 0, \tag{26}$$

which defines a Heisenberg algebra. Here again, the vortical motion (non-commutativity) is limited a to the canonical pairs $\hat{q}_j$ and $\hat{p}_j$.

3.2 Non-canonical Hamiltonian system

We consider a "generalized" Hamiltonian mechanics governed by

$$\frac{d\boldsymbol{z}}{dt} = \mathcal{J}(\boldsymbol{z})\partial_{\boldsymbol{z}} H(\boldsymbol{z}), \tag{27}$$

where $\mathcal{J}(\boldsymbol{z})$ is some *anti-symmetric* operator (may depend on $\boldsymbol{z}$), [2] $H(\boldsymbol{z})$ is a Hamiltonian (a real value regular function on the phase space X), and $\partial_{\boldsymbol{z}} H(\boldsymbol{z})$ is the gradient of $H(\boldsymbol{z})$ in X (we endow X with an inner product to define gradients).

A *non-canonical* Poisson operator $\mathcal{J}(\boldsymbol{z})$ may have a non-trivial kernel:

$$\mathrm{Ker}(\mathcal{J}(\boldsymbol{z})) = \{w \in X;\ \mathcal{J}(\boldsymbol{z})w = 0\}.$$

A function $C(\boldsymbol{z})$ such that

$$\partial_{\boldsymbol{z}} C(\boldsymbol{z}) \in \mathrm{Ker}(\mathcal{J}(\boldsymbol{z}))$$

is called a *Casimir* element, which constitutes the *center* of the Poisson algebra induced by $\mathcal{J}(\boldsymbol{z})$. Since $C(\boldsymbol{z})$ is a constant of motion (independent to H), the phase space is foliated by the level sets of $C(\boldsymbol{z})$. Viewing the phase space as the dual of the Lie algebra, surfaces of constant Casimirs are coadjoint orbits [7].

If the dimension ν of $\mathrm{Ker}(J(\boldsymbol{z}))$ does not change, Casimir elements may be constructed by elements of $\mathrm{Ker}(J(\boldsymbol{z}))$; this is always true if ν is an even number (Lie-Darboux theorem).

[2] We usually demand that the corresponding Poisson bracket satisfies Jacobi's relation.

Note 2 (singular Casimir elements) *Some* $w \in Ker(\mathcal{J}(\boldsymbol{z}))$ *may not be "integrated" to produce a Casimir element; the existence of a Casimir element implies integrability to foliate the phase space. In general,* $Rank(\mathcal{J}(\boldsymbol{z}))$ *may change as a function of* $\boldsymbol{z}$*. If a* singularity *(where* $Rank(\mathcal{J}(\boldsymbol{z}))$ *changes) exists, we obtain singular Casimir elements. For example, let us consider a one-dimensional system with* $\mathcal{J} = ix$ $(x \in \mathbb{R})$*. The differential equation* $\mathcal{J}\partial_x C(x) = 0$ *produces a hyperfunction solution* $C(x) = Y(x)$ *(Heaviside's step function). More generally, the analysis of the linear partial differential equation* $\mathcal{J}(\boldsymbol{z})\partial_{\boldsymbol{z}} C(\boldsymbol{z}) = 0$*. with singularities leads us to the theory of D-modules [2]; denoting* $\mathcal{P} := \mathcal{J}(\boldsymbol{z})\partial_{\boldsymbol{z}}$*, Casimir elements constitute* $Ker(\mathcal{P}) = Hom_{\mathcal{D}}(Coker(\mathcal{P}), F)$*, where* $\mathcal{D}$ *is the ring of partial differential operators and* F *is the function space on which* $\mathcal{P}$ *operates;* $Coker(\mathcal{P}) = \mathcal{D}/\mathcal{D}\mathcal{P}$ *is the D-module corresponding to the equation* $\mathcal{P}C(\boldsymbol{z}) = 0$*.*

3.3 Generalization to infinite-dimensional phase space

We extend the phase pace to be an infinite-dimensional function space; we consider an evolution equation

$$\partial_t u = \mathcal{J}(u)\partial_u H(u), \tag{28}$$

where u is a state vector (a member of a Hilbert space V; we denote by $\langle a, b\rangle$ the inner product), $H(u)$ is a Hamiltonian (a real functional on V), and ∂_u is the gradient in V). The Poisson operator $\mathcal{J}(u)$ is antisymmetric. In some examples, one can show that formal Jacobi's identity holds for a Poisson bracket $\{F, G\} := \langle \partial_u F, \mathcal{J}(u)\partial_u G\rangle$ on a sheaf of regular functions, while we are not planning to depend much on Lie-Poisson algebra in the present study.

A Casimir functional $C(u)$ is determined by

$$\mathcal{J}(u)\partial_u C(u) = 0, \tag{29}$$

which may be viewed as an infinite-dimensional linear partial differential equation. If $\mathcal{J}$ is independent of u, (29) is a homogeneous equation —a "linearized system" corresponds to this category (see Sec. 3.4). We remark that the "coefficients" of such a homogeneous first-order differential equation are (differential) operators (Note 2,[25]).

3.4 Energy-Casimir functional and linearized system

In a canonical Hamiltonian system ($\mathrm{Ker}(\mathcal{J}) = \{0\}$), an equilibrium point of the dynamics must be a stationary point of the Hamiltonian, i.e. $\partial_u H(u) =$

0. A noncanonical system may have a richer set of equilibrium points that are parameterized by Casimir elements. Given a Casimir element $C(u)$, we can transform the Hamiltonian $H(u)$ as

$$H_\mu(u) = H(u) - \mu C(u) \quad (\mu \in \mathbb{R}) \tag{30}$$

without changing the dynamics (since $\mathcal{J}(u)\partial_u C(u) \equiv 0$). The new Hamiltonian $H_\mu(u)$ is called an energy-Casimir functional, that have been used to construct variational principles for equilibria and stability (the first clear usage of the energy-Casimir method for stability appears to be [8]; see also [13, 1]). For an equilibrium point given by $\partial_u H_\mu(u) = 0$, the parameter μ may be regarded as an eigenvalue. When we determine μ by matching $C(u)$ of the solution with some given number c, the solution (say u_μ) is an equilibrium point on a Casimir leaf $C(u) = c$. We note that a general Hamiltonian system may have even larger class of equilibrium points that may not be parameterized by Casimir elements.

The linearization of the system near an equilibrium point of an energy-Casimir functional has a remarkable simplicity. For a perturbation $u = u_\mu + \tilde{u}$ (we denote by ˜ a perturbed quantity), we linearize the equivalent evolution equation $\partial_t u = \mathcal{J}(u)\partial_u H_\mu(u)$ as (indicating small perturbations by ˜)

$$\partial_t \tilde{u} = \mathcal{J}(u_\mu)\widetilde{\partial_u H_\mu(u)} + \widetilde{\mathcal{J}(u)}\left[\partial_u H_\mu(u)\right]_{u=u_\mu} = \mathcal{J}_\mu \mathcal{H}_\mu \tilde{u}, \tag{31}$$

where $\mathcal{J}_\mu := \mathcal{J}(u_\mu)$ and $\mathcal{H}_\mu$ is the Hessian of $H_\mu(u)$ evaluated at u_μ, i.e. $H_\mu(u_\mu + \tilde{u}) \approx H_\mu(u_\mu) + \langle \tilde{u}, \mathcal{H}_\mu \tilde{u}\rangle/2$. We have used $\left[\partial_u \mathcal{H}_\mu(u)\right]_{u=u_\mu} = 0$. Notice that the Poisson operator $\mathcal{J}_\mu$ is independent of the state vector $\tilde{u}$, hence (31) is a homogeneous linear equation.

3.5 MHD model

We invoke an incompressible MHD model as a simple but sufficiently nontrivial system by which we demonstrate the usefulness of the noncanonical Hamiltonian description of dynamics and the corresponding phase-space foliation. [3]

The state vector is $u := {}^t(\boldsymbol{V}, \boldsymbol{B})$; $\boldsymbol{V}$ is the fluid velocity that is assumed to be incompressible, and $\boldsymbol{B}$ is the magnetic field; they are normalized in the standard Alfvén units. The plasma occupies a smoothly bounded domain $\Omega \subset \mathbb{R}^3$. We impose boundary conditions (denoting by $\boldsymbol{n}$ the unit normal vector onto the boundary $\partial\Omega$)

$$\boldsymbol{n} \cdot \boldsymbol{V} = 0, \quad \boldsymbol{n} \cdot \boldsymbol{B} = 0. \tag{32}$$

[3] The following material draws heavily on Ref. [26].

Let us specify the phase pace of the state vector. We denote by $L^2(\Omega)$ the Hilbert space of Lebesgue-measurable, square-integrable real vector functions on Ω, which is endowed with the standard inner product $\langle \boldsymbol{a}, \boldsymbol{b} \rangle := \int_\Omega \boldsymbol{a} \cdot \boldsymbol{b}\, \mathrm{d}^3x$ and the norm $\|\boldsymbol{a}\| := \langle \boldsymbol{a}, \boldsymbol{a} \rangle^{1/2}$. We will use the same $\langle\ ,\ \rangle$ and $\|\ \|$ regardless of the dimensions of independent and dependent variables. We will also use the standard notation of the Sobolev spaces. Both $\boldsymbol{V}$ and $\boldsymbol{B}$ are members of

$$L^2_\sigma(\Omega) := \{\boldsymbol{u} \in L^2(\Omega);\ \nabla \cdot \boldsymbol{u} = 0,\ \boldsymbol{n} \cdot \boldsymbol{u} = 0\}. \tag{33}$$

Hence our phase space is $V := L^2_\sigma(\Omega) \times L^2_\sigma(\Omega)$.

Denoting by $\mathcal{P}_\sigma$ the projector onto the subspace $L^2_\sigma(\Omega)$, the governing equations are

$$\partial_t \boldsymbol{V} = -\mathcal{P}_\sigma(\boldsymbol{V} \cdot \nabla)\boldsymbol{V} + \mathcal{P}_\sigma \left[(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}\right], \tag{34a}$$

$$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{V} \times \boldsymbol{B}). \tag{34b}$$

The Hamiltonian and the Poisson operator are given by

$$H(u) := \frac{1}{2}\left(\|\boldsymbol{V}\|^2 + \|\boldsymbol{B}\|^2\right). \tag{35}$$

$$\mathcal{J}(u) := \begin{pmatrix} -\mathcal{P}_\sigma(\nabla \times \boldsymbol{V})\times & \mathcal{P}_\sigma(\nabla \times \circ) \times \boldsymbol{B} \\ \nabla \times [\circ \times \boldsymbol{B}] & 0 \end{pmatrix}, \tag{36}$$

where $\circ$ implies insertion of the function to the right of the operator. It is readily seen that inserting (35) and (36) into (28) yields the MHD equations (34a) and (34b).

The operator $\mathcal{J}(u)$ has two independent Casimir elements

$$C_1(u) := \frac{1}{2}\int_\Omega \boldsymbol{A} \cdot \boldsymbol{B}\ \mathrm{d}^3x, \quad C_2(u) := \int_\Omega \boldsymbol{V} \cdot \boldsymbol{B}\ \mathrm{d}^3x, \tag{37}$$

which, respectively, represent the magnetic helicity and the cross helicity. They impose topological constraints on the field lines [12]. The "Beltrami equilibrium" is an equilibrium point of the energy-Casimir functional $H(u) - \mu_1 C_1(u) - \mu_2 C_2(u)$. Here we consider a subclass of equilibrium points assuming $\mu_2 = 0$. Then, the determining equation is (denoting $\mu_1 = \mu$)

$$\nabla \times \boldsymbol{B} - \mu \boldsymbol{B} = 0, \tag{38}$$

which reads as an eigenvalue problem of the curl operator. The solution (to be denoted by $\boldsymbol{B}_\mu$) is nothing but the *Taylor relaxed state* [19, 20], which has the connotation of being a "minimum-energy state" on a Casimir leaf.

3.6 Flux condition: decomposition of the harmonic field

While the Beltrami equation (38) together with the homogeneous boundary conditions (32) are seemingly homogeneous equations, there is a "hidden inhomogeneity" when Ω is multiply connected [then, the boundary conditions (32) are insufficient to determine a unique solution]. To delineate the "topological inhomogeneity" of the Beltrami equation, we first make Ω into a simply connected domain Ω_S by inserting cuts Σ_ℓ across each handle of Ω: $\Omega_S := \Omega \setminus (\cup_{\ell=1}^{\nu} \Sigma_\ell)$ (ν is the *genus* of Ω). The *fluxes* of $\boldsymbol{B}$ are given by (denoting by $\mathrm{d}\boldsymbol{\sigma}$ is the surface element on Σ_ℓ)

$$\Phi_\ell(\boldsymbol{B}) := \int_{\Sigma_\ell} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{\sigma}, \tag{39}$$

which are constants of motion. To separate these fixed degrees of freedom, we invoke the Hodge–Kodaira decomposition $L^2_\sigma(\Omega) = L^2_\Sigma(\Omega) \oplus L^2_{\mathrm{H}}(\Omega)$, where

$$L^2_\Sigma(\Omega) := \{\boldsymbol{u} \in L^2(\Omega);\ \nabla \cdot \boldsymbol{u} = 0,\ \boldsymbol{n} \cdot \boldsymbol{u} = 0,\ \Phi_\ell(\boldsymbol{u}) = 0\ (\forall \ell)\}. \tag{40a}$$

$$L^2_{\mathrm{H}}(\Omega) := \{\boldsymbol{u} \in L^2(\Omega);\ \nabla \times \boldsymbol{u} = 0,\ \nabla \cdot \boldsymbol{u} = 0,\ \boldsymbol{n} \cdot \boldsymbol{u} = 0\}. \tag{40b}$$

The dimension of $L^2_{\mathrm{H}}(\Omega)$, the space of *harmonic fields* (or *cohomologies*), is equal to the genus ν of Ω. Now we decompose the total $\boldsymbol{B} \in L^2_\sigma(\Omega)$ into the fixed harmonic "vacuum" field $\boldsymbol{B}_{\mathrm{H}} \in L^2_{\mathrm{H}}(\Omega)$ (which carries the given fluxes $\Phi_1, \cdots, \Phi_\nu$) and a residual component $\boldsymbol{B}_\Sigma$ driven by currents within the plasma volume Ω,

$$\boldsymbol{B} = \boldsymbol{B}_\Sigma + \boldsymbol{B}_{\mathrm{H}}, \quad [\boldsymbol{B}_\Sigma := \mathcal{P}_\Sigma \boldsymbol{B} \in L^2_\Sigma(\Omega),\ \boldsymbol{B}_{\mathrm{H}} \in L^2_{\mathrm{H}}(\Omega)], \tag{41}$$

where $\mathcal{P}_\Sigma$ denotes the orthogonal projector from $L^2(\Omega)$ onto $L^2_\Sigma(\Omega)$.

Now the Beltrami equation (38) reads as an inhomogeneous equation (denoting $\nabla\times$ by curl):

$$(\mathrm{curl} - \mu)\boldsymbol{B}_\Sigma = \mu \boldsymbol{B}_{\mathrm{H}}. \tag{42}$$

When $\boldsymbol{B}_{\mathrm{H}}$ and μ are given, we solve (42) for $\boldsymbol{B}_\Sigma$ (Lemma 1) to obtain the Beltrami magnetic field $\boldsymbol{B}_\mu = \boldsymbol{B}_\Sigma + \boldsymbol{B}_{\mathrm{H}}$. The harmonic field $\boldsymbol{B}_{\mathrm{H}}$ is uniquely determined by the fluxes $\Phi_1, \cdots, \Phi_\nu$. We must also give the parameter μ by some physical condition; here we determine μ by matching the helicity of $\boldsymbol{B}_\mu$ to a prescribed value c_1 of C_1. But the relation between μ and C_1 is somewhat involved and may not be unique; this is the root of the bifurcation problem. In the next subsection, we will see how bifurcations occur in the helicity-matching process.

3.7 Helicity matching

The helicity-matching problem may have solutions under two different situations:

1. If the inhomogeneous equation (42) determines a unique $\boldsymbol{B}_\Sigma$ for a given $\boldsymbol{B}_\mathrm{H}$ ($\neq 0$) and some μ, then (37) evaluates the helicity as a function of μ, which we denote by $\mathcal{C}_\mathrm{A}(\mu)$. For a given value c_1 of the helicity C_1, we must choose an appropriate μ to satisfy $c_1 = \mathcal{C}_\mathrm{A}(\mu)$. The category of these solutions will be called branch-(A).

2. The *homogeneous part* of (42) may have a nontrivial solution (or solutions) $\boldsymbol{\omega}$ for some special $\mu = \lambda$, i.e., $(\mathrm{curl} - \mu)\boldsymbol{\omega} = 0$; this means that λ and $\boldsymbol{\omega}$ are, respectively, an eigenvalue and the corresponding eigenfunction(s) of the self-adjoint curl operator. (The exact definition of the curl operator and its eigenvalues will be described in the next subsection; here we note that the eigenvalues are discrete numbers [22]). If the inhomogeneous equation (42) still has a particular solution $\boldsymbol{G}$, then the general solution of (42) is given by

$$\boldsymbol{B}_\Sigma = \alpha\boldsymbol{\omega} + \boldsymbol{G}, \tag{43}$$

   where α is an arbitrary real number. Substituting this $\boldsymbol{B}_\Sigma$, we evaluate the helicity (37) as a function of α (here, μ is fixed at an eigenvalue λ), which we denote by $\mathcal{C}_\mathrm{B}(\alpha)$. The helicity matching $c_1 = \mathcal{C}_\mathrm{B}(\alpha)$ selects an appropriate amplitude α. The category of these solutions will be called branch-(B).

We note that the branch-(B) can appear only if μ is an eigenvalue of the self-adjoint curl operator, and moreover, if the inhomogeneous equation (42) has a particular solution $\boldsymbol{G}$. As is to be shown later, the latter condition does not always apply (depending on the symmetry of the system), i.e., at some eigenvalues, (42) may be solvable only if $\boldsymbol{B}_\mathrm{H} = 0$. On the other hand, it is known that the inhomogeneous equation (42) with a $\boldsymbol{B}_\mathrm{H} \neq 0$ is *uniquely* solvable for every $\mu \in \mathbb{R}$ excepting the eigenvalues of the curl [22], giving the branch-(A) solution; at some eigenvalues, the branch-(B) bifurcates, while for other eigenvalues, the inhomogeneous term $\boldsymbol{B}_\mathrm{H} \neq 0$ prevents a solution.

3.8 Bifurcation of Beltrami equilibrium

To elucidate the mathematical structure around the bifurcation point, we need rigorous analysis of the eigenvalues of the curl operator — for this purpose we first define a *self-adjoint curl operator* $\mathcal{S}$. This section draws heavily on Ref. [22]. We introduce a space

$$H^1_{\Sigma\Sigma}(\Omega) := \{\boldsymbol{u} \in L^2_\Sigma(\Omega) \cap H^1(\Omega);\ \nabla \times \boldsymbol{u} \in L^2_\Sigma(\Omega)\}, \tag{44}$$

which is densely included in $L^2_\Sigma(\Omega)$. The self-adjoint curl operator (which we denote by $\mathcal{S}$) is such that $\mathcal{S}\boldsymbol{u} = \nabla \times \boldsymbol{u}$ for every $\boldsymbol{u}$ in the operator domain $D(\mathcal{S}) = H^1_{\Sigma\Sigma}(\Omega)$. The inverse map $\mathcal{S}^{-1} : L^2_\Sigma(\Omega) \to H^1_{\Sigma\Sigma}(\Omega)$ is a compact operator. We denote by $\sigma_p(\mathcal{S})$ the point spectrum (the set of eigenvalues) of $\mathcal{S}$. Evidently, $0 \notin \sigma_p(\mathcal{S})$. By the compactness of $\mathcal{S}^{-1}$, $\sigma_p(\mathcal{S})$ is a discrete set on $\mathbb{R}$. The eigenvalues of $\mathcal{S}$ are unbounded in both positive and negative directions. The eigenfunctions of $\mathcal{S}$ constitute a complete orthogonal basis of the Hilbert space $L^2_\Sigma(\Omega)$.

To span $L^2_\sigma(\Omega)$, we add the finite-dimensional space $L^2_{\mathrm{H}}(\Omega)$ of "vacuum fields" to the domain of curl, which is, then, regarded as the kernel of the extended curl operator (that is no loner self-adjoint).

Now we solve the inhomogeneous Beltrami equation (42) for a given $\boldsymbol{B}_{\mathrm{H}} \in L^2_{\mathrm{H}}(\Omega)$. We start by reviewing the result of Ref. [22, Sec. 4] on defining a curl operator $\mathcal{T}$ with range and domain extended to include $L^2_{\mathrm{H}}(\Omega)$:

Lemma 1 *Suppose that Ω is a multiply connected smoothly bounded domain, thus $L^2_{\mathrm{H}}(\Omega)$ has a finite dimension.*
(1) For each $\boldsymbol{B}_{\mathrm{H}} \in L^2_{\mathrm{H}}(\Omega)$ there is a vector potential $\boldsymbol{A}_{\mathrm{H}} \in L^2_\Sigma(\Omega)$, i.e., $\boldsymbol{B}_{\mathrm{H}} = \nabla \times \boldsymbol{A}_{\mathrm{H}}$.
(2) Extending the range of curl to include all such $\boldsymbol{B}_{\mathrm{H}}$, and its domain to include the corresponding $\boldsymbol{A}_{\mathrm{H}}$, we extend $\mathcal{S}$ to an operator $\mathcal{T}$, the "non-self-adjoint curl operator," such that $\mathcal{T}\boldsymbol{u} = \nabla \times \boldsymbol{u}$ for every $\boldsymbol{u}$ in the operator domain

$$D(\mathcal{T}) = H^1_{\Sigma\sigma}(\Omega) := \{\boldsymbol{u} \in L^2_\Sigma(\Omega) \cap H^1(\Omega);\ \nabla \times \boldsymbol{u} \in L^2_\sigma(\Omega)\}. \tag{45}$$

(3) For every $\mu \notin \sigma_p(\mathcal{S})$, the inhomogeneous equation

$$(\mathcal{T} - \mu)\boldsymbol{B}_\Sigma = \mu \boldsymbol{B}_{\mathrm{H}} \tag{46}$$

has a unique solution $\boldsymbol{B}_\Sigma = (\mathcal{T} - \mu)^{-1}\mu\boldsymbol{B}_{\mathrm{H}}$, implying that (42) has a unique solution $\boldsymbol{B}_\Sigma \in L^2_\Sigma(\Omega)$.

For $\mu = \lambda_j \in \sigma_p(\mathcal{S})$, we have a nontrivial solution $\boldsymbol{\omega}_j$ of the homogeneous part ($\boldsymbol{B}_{\mathrm{H}} = 0$) of the Beltrami equation (42), i.e., $\boldsymbol{\omega}_j$ is the eigenfunction corresponding to the eigenvalue λ_j.

We are ready to study the existence of a particular solution $\boldsymbol{G}$ of (42) for a given $\boldsymbol{B}_{\mathrm{H}} \neq 0$. As mentioned above, a nontrivial particular solution $\boldsymbol{G}$ becomes the trunk from which the branch-(B) solution bifurcates. We have

Theorem 1 *Let $\lambda_j \in \sigma_p(\mathcal{S})$ and $\mathcal{S}\boldsymbol{\omega}_j = \lambda_j \boldsymbol{\omega}_j$. Iff*

$$\langle \boldsymbol{A}_{\mathrm{H}}, \boldsymbol{\omega}_j \rangle = 0, \tag{47}$$

the inhomogeneous Beltrami equation

$$(\mathcal{T} - \lambda_j)\boldsymbol{G} = \lambda_j \boldsymbol{B}_{\mathrm{H}} \tag{48}$$

has a solution such that $\boldsymbol{G} \in L^2_\Sigma(\Omega)$ and $\langle \boldsymbol{G}, \boldsymbol{\omega}_j \rangle = 0$.

proof. Let V_j be the eigenspace corresponding to λ_j. We define $L^2_{\Sigma\perp}(\Omega) := L^2_\Sigma(\Omega)/V_j$ and $H^1_{\Sigma\Sigma\perp}(\Omega) := H^1_{\Sigma\Sigma}(\Omega)/V_j$, where $H^1(\Omega)$ is the Sobolev space of order 1 (i.e. the Hilbert space of functions in L^2 whose first derivatives are also in L^2). The restriction of $\mathcal{S}$ on $H^1_{\Sigma\Sigma\perp}(\Omega)$ will be denoted by $\mathcal{S}_\perp$. Evidently, $\mathrm{Coker}(\mathcal{S}_\perp - \lambda_j) = V_j$. If the orthogonality condition (47) holds, $\boldsymbol{A}_{\mathrm{H}} \in L^2_{\Sigma\perp}(\Omega)$. We solve (48) applying the method of Proposition 1 of Ref. [22]. Let us write $\boldsymbol{G} = \boldsymbol{W} + \lambda_j \boldsymbol{A}_{\mathrm{H}}$. Inserting this into (48) yields

$$(\mathcal{T} - \lambda_j)\boldsymbol{W} = \lambda_j^2 \boldsymbol{A}_{\mathrm{H}} \quad (\in L^2_{\Sigma\perp}(\Omega)). \tag{49}$$

We can solve (49) by $\boldsymbol{W} = (\mathcal{S}_\perp - \lambda_j)^{-1}\lambda_j^2 \boldsymbol{A}_{\mathrm{H}} \in H^1_{\Sigma\Sigma\perp}(\Omega)$. Thus, the solution of (48) is given by

$$\boldsymbol{G} = (\mathcal{S}_\perp - \lambda_j)^{-1}\lambda_j^2 \boldsymbol{A}_{\mathrm{H}} + \lambda_j \boldsymbol{A}_{\mathrm{H}} \quad (\in L^2_{\Sigma\perp}(\Omega)). \tag{50}$$

In the space $L^2_{\Sigma\perp}(\Omega)$, this solution is unique.

Next, we show that $\langle \boldsymbol{A}_{\mathrm{H}}, \boldsymbol{\omega}_j \rangle \neq 0$ is in contradiction with the solvability of (48). It suffices to assume that V_j is of one dimension. Projecting both sides of (49) onto V_j, we obtain

$$\langle (\mathcal{T} - \lambda_j)\boldsymbol{W}, \boldsymbol{\omega}_j \rangle = \langle (\mathcal{T} - \lambda_j)\boldsymbol{W}_\perp, \boldsymbol{\omega}_j \rangle = \langle \mathcal{T}\boldsymbol{W}_\perp, \boldsymbol{\omega}_j \rangle = \lambda_j^2 \langle \boldsymbol{A}_{\mathrm{H}}, \boldsymbol{\omega}_j \rangle, \tag{51}$$

where $\boldsymbol{W}_\perp$ is the projection of $\boldsymbol{W}$ onto $L^2_{\Sigma\perp}(\Omega)$. For this relation to hold, there must be an element $\boldsymbol{w} \in L^2_{\Sigma\perp}(\Omega)$ such that $\mathcal{T}\boldsymbol{w} = c\boldsymbol{\omega}_j$ $(c \neq 0)$. Substituting $c\boldsymbol{\omega}_j = (c/\lambda_j)\mathcal{T}\boldsymbol{\omega}_j$, we obtain $\mathcal{T}[\boldsymbol{w} - (c/\lambda_j)\boldsymbol{\omega}_j] = 0$. Since $\mathrm{Ker}(\mathcal{T}) = \{0\}$, we deduce $\boldsymbol{w} - (c/\lambda_j)\boldsymbol{\omega}_j = 0$, which contradicts the assumption $\boldsymbol{w} \in L^2_{\Sigma\perp}(\Omega)$. Therefore, if $\langle \boldsymbol{A}_{\mathrm{H}}, \boldsymbol{\omega}_j \rangle \neq 0$, (49) cannot have a solution.

4 Concluding remarks

In this series of lectures, we have encountered some generalized concepts of "state" and "space". Conventionally, a state is represented by a *point* in some phase space X. For example, classical mechanics describes motion of a particle as a curve drawn by a point in phase space. Different recognition of state can be more appropriate to describe macroscopic dynamics and structures of various physical systems. In Sec. 2, we introduced a "quasi-particle" that is a gyro-motion-coarse-grained state moving on a leaf of adiabatic-invariants; this "scale hierarchy" is immersed in a foliated phase space. In Sec. 3, we described the state of a fluid/plasma by Eulerian-variable functions, by which we delineated the space of fluid motion. The phase space (Hilbert space) is foliated by a Casimir = helicity. Back-tracking the argument of Sec. 2, one may interpret a Casimir as an adiabatic invariant (action of integrable cyclic motion). Helicity is, then, an action corresponding to a hidden symmetry in fluid-mechanical representation.

Mathematically, a state can be identified with an *operator* acting on functions defined on X —this concept forms the basis of quantum mechanics (see for example [4]). A *pure state* η_ξ, corresponding to a single point $\xi \in X$, is the operation of δ-function, i.e. an operator $\eta_\xi(f) = f(\xi)$ (for every continuous function $f(x)$) is represented by $\int_X f(x)\delta(x-\xi)\,\mathrm{d}x$. The set of operators, endowed with algebraic structures enabling linear-algebra manipulations, is called a C^* ring. A commutative C^* ring is represented by a ring of (complex valued) continuous functions on X; hence a commutative ring of functions represents some space X. [4] Generalizing a state form a point to some "geometrical object" and considering a more general class of operators (constituting a non-commutative C^* ring), we may describe a far richer phenomena.

Here we describe a simple example in which a "loop" $L(s)$ moving in the Minkowski space-time (s is the proper time) plays the role of "state". Corresponding observable is represented by a 1-form $P = P^0\mathrm{d}x_0 + \cdots P^3\mathrm{d}x_3$, and the corresponding physical quantity is evaluated by

$$C(L(s)) = \oint_{L(s)} P.$$

Physically $C(L(s))$ is the *circulation* of the 1-form (co-vector) P. Denoting by $U = U_\mu\partial^\mu := \mathrm{d}x_\mu/\mathrm{d}s$ the flow velocity (vector field) that transports the loop $L(s)$. We observe

$$\frac{\mathrm{d}}{\mathrm{d}s}C(L(s)) = \oint_{L(s)} L_U P = \oint_{L(s)} i_U \mathrm{d}P.$$

[4] The pure state η_ξ is the quotient ring $C^0(X)/J_\xi$, where J_ξ is the maximal ideal generated by $\|x-\xi\|$.

Denoting $M^{\mu\nu} := \partial^\mu P^\nu - \partial^\nu P^\mu$ (anti-symmetric field tensor), we may write

$$\frac{\mathrm{d}}{\mathrm{d}s} C(L(s)) = \oint_{L(s)} U_\mu M^{\mu\nu}.$$

In an ideal magnetofluid, the canonical momentum $P^\mu = (h/c)U^\mu + qA^\mu$ (h is the enthalpy, q is the charge, A^μ is the electromagnetic 4-potential) obeys $U_\mu P^{\mu\nu} = T\partial^\nu S$, where T is the temperature and S is the entropy [10]. In a barotropic fluid, we may write $T\partial^\nu S = \partial^\nu \Theta$ with a scalar function Θ, thus the circulation $C(L(s))$ conserves, giving an "identity" to the state $L(s)$. This conservation law is a relativistic generalization of Kelvin's circulation law; see [11]. By the relativity, the "loop" is no longer included in a time-slice (reference-time = constant level set) of space-time. Hence, the conventional circulation $c(L(t)) = \oint_{L(t)} P$ ceases to be constant. The change of $c(L(t))$ will, then, be recognized as creation (or annihilation) of non-relativistic (frame dependent) circulation = "vorticity".

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