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Hall-MHD and applications

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Hall – MHD

and applications

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General Motivation

- Many laboratory, astrophysical and space plasmas can properly be described within the theoretical framework of Magnetohydrodynamics (MHD).

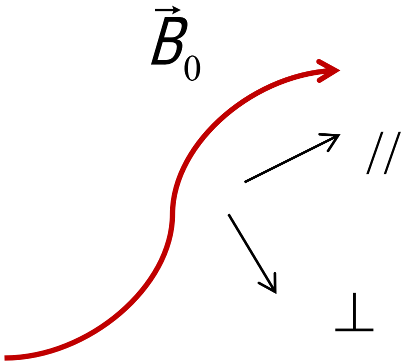
- MHD is a fluidistic approach to describe the large scale dynamics of plasmas.

- The standard approach is also known as one-fluid MHD.

- We are going to start from a somewhat more general approach known as two-fluid MHD, which acknowledges the presence of ions and electrons and considers kinetic effects such as Hall, electron pressure and electron inertia.

- Physical processes that can be addressed with MHD include:

  - Magnetic reconnection
  - Magnetic confinement
  - Magnetic dynamo
  - MHD turbulence



- We will also address the case of plasmas embedded in strong external magnetic fields, which allow for an approximation known as reduced MHD, both for one-fluid MHD (RMHD) and two-fluid MHD (RHMHD).

Fluid equations for multi-species plasmas

- For each species s we have (Goldston & Rutherford 1995):
  - Mass conservation $$\frac{\partial n_s}{\partial t} + \vec{\nabla} \cdot (n_s \vec{U}_s) = 0$$
  - Equation of motion $$m_s n_s \frac{d\vec{U}_s}{dt} = q_s n_s (\vec{E} + \frac{1}{c}\vec{U}_s \times \vec{B}) - \vec{\nabla} p_s + \vec{\nabla} \bullet \vec{\sigma}_s + \sum_{s'} \vec{R}_{ss'}$$
  - Momentum exchange rate $$\vec{R}_{ss'} = -m_s n_s \upsilon_{ss'} (\vec{U}_s - \vec{U}_{s'})$$
- These moving charges act as sources for electric and magnetic fields:
  - Charge density $$\rho_c = \sum_s q_s n_s \approx 0$$
  - Electric current density $$\vec{J} = \frac{c}{4\pi} \vec{\nabla} \times \vec{B} = \sum_s q_s n_s \vec{U}_s$$

Two-fluid MHD equations

- For a fully ionized plasma with ions of mass m_i and massless electrons (since $m_e << m_i$):
  - Mass conservation: $$0 = \frac{\partial n}{\partial t} + \vec{\nabla}\cdot(n\vec{U}) \quad , \qquad n_e \cong n_i \cong n$$
  - Ions: $$m_i n \frac{d\vec{U}}{dt} = en(\vec{E} + \frac{1}{c}\vec{U}\times\vec{B}) - \vec{\nabla} p_i + \vec{\nabla}\bullet\overleftrightarrow{\sigma} + \vec{R}$$
  - Electrons: $$0 = -en(\vec{E} + \frac{1}{c}\vec{U}_e\times\vec{B}) - \vec{\nabla} p_e - \vec{R}$$
  - Friction force: $$\vec{R} = -m_i n \nu_{ie}(\vec{U} - \vec{U}_e)$$
  - Ampere's law: $$\vec{J} = \frac{c}{4\pi}\vec{\nabla}\times\vec{B} = en(\vec{U} - \vec{U}_e) \quad \Rightarrow \quad \vec{R} = -\frac{m\upsilon_{ie}}{e}\vec{J}$$
  - Polytropic laws: $$p_i \propto n^{\gamma} \quad , \qquad p_e \propto n^{\gamma}$$
  - Newtonian viscosity: $$\sigma_{ij} = \mu\,(\partial_i U_j + \partial_j U_i)$$

Hall-MHD equations

- The dimensionless version, for a length scale L_0, density n_0 and Alfven speed $v_A = B_0 / \sqrt{4\pi m_i n_0}$

$$\frac{d\vec{U}}{dt} = \frac{1}{\varepsilon}(\vec{E} + \vec{U} \times \vec{B}) - \frac{\beta}{n}\vec{\nabla} p_i - \frac{\eta}{\varepsilon n}\vec{J} + \nu\nabla^2\vec{U} \qquad \nu = \frac{\mu}{m_i n v_A L_0}$$

$$0 = -\frac{1}{\varepsilon}(\vec{E} + \vec{U}_e \times \vec{B}) - \frac{\beta}{n}\vec{\nabla} p_e + \frac{\eta}{\varepsilon n}\vec{J}$$

where $\vec{J} = \vec{\nabla} \times \vec{B} = \frac{n}{\varepsilon}(\vec{U} - \vec{U}_e)$

- We define the Hall parameter $\varepsilon = \frac{c}{\omega_{pi} L_0}$

as well as the plasma *beta* $\beta = \frac{p_0}{m_i n_0 v_A^2}$ and the electric resistivity $\eta = \frac{c^2 \nu_{ie}}{\omega_{pi}^2 L_0 v_A}$

- Adding these two equations yields:

$$\boxed{n\frac{d\vec{U}}{dt} = (\vec{\nabla} \times \vec{B}) \times \vec{B} - \beta\vec{\nabla}(p_i + p_e) + \nu\nabla^2\vec{U}}$$

Hall-MHD equations

- On the other hand, using

$$\left.\begin{aligned} \vec{E} &= -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi \\ \vec{B} &= \vec{\nabla} \times \vec{A} \end{aligned}\right\} \rightarrow \boxed{\frac{\partial \vec{A}}{\partial t} = (\vec{U} - \frac{\varepsilon}{n}\vec{\nabla} \times \vec{B}) \times \vec{B} - \vec{\nabla}\phi + \frac{\varepsilon\beta}{n}\vec{\nabla} p_e - \frac{\eta}{n}\vec{\nabla} \times \vec{B}}$$

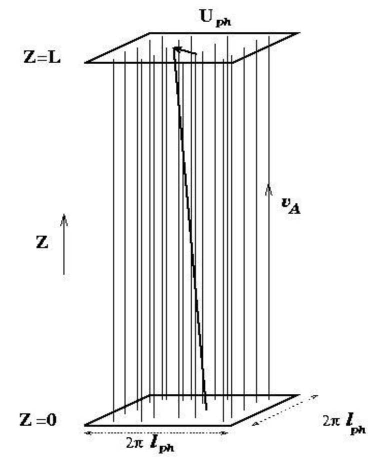
Hall MHD in a strong field

- Let us assume a strong magnetic field along $\hat{z}$ so that

$$\vec{B} = \hat{z} + \delta\vec{B} \quad , \quad |\delta\vec{B}| \approx \alpha << 1$$

where α represents the typical tilt of field lines with respect to $\hat{z}$. We assume

$$\nabla_\perp \approx 1 \quad , \quad \partial_z \approx \alpha << 1$$



- The magnetic and velocity fields can be expanded in terms of potentials of order α:

$$\vec{B} = \hat{z} + \vec{\nabla} \times (a\hat{z} + g\hat{x}) = [a_y, -a_x, 1+b] \quad , \quad b = -g_y$$

$$\vec{U} = \vec{\nabla}\psi + \vec{\nabla} \times (\varphi\hat{z} + f\hat{x}) = [\varphi_y + \psi_x, -\varphi_x + \psi_y, u + \psi_z] \quad , \quad u = -f_y$$

- We want to eliminate the fast scale dynamics, characterized by $\tau_{A\perp} \approx L_\perp / v_A$, i.e. $\partial_t \approx 1$

- We obtain the following conditions

$$\boxed{\begin{aligned} \nabla_\perp^2 \psi &= 0 \\ \vec{\nabla}_\perp [b + \beta(p_i + p_e)] &= 0 \\ \vec{\nabla}_\perp [\phi + \varphi - \varepsilon(b + \beta p_e)] &= 0 \end{aligned}}$$

Hall MHD in a strong field

- The relatively slower dynamics, characterized by $\tau_{A//} \approx L_{//} / V_A$, i.e. $\partial_t \approx \alpha$

  is given by the following equations (Gomez, Dmitruk & Mahajan 2008):

$$\boxed{\begin{array}{llll}
\partial_t a = \partial_z(\varphi - \varepsilon b) & + [\varphi - \varepsilon b, a] & & + \eta \nabla_\perp^2 a \\
\partial_t \omega = \partial_z j & + [\varphi, \omega] - [a, j] & & + \nu \nabla_\perp^2 \omega \\
\partial_t b = \partial_z(u - \varepsilon j) & + [\varphi, b] + [u - \varepsilon j, a] & & + \eta \nabla_\perp^2 b \\
\partial_t u = \partial_z b & + [\varphi, u] - [a, b] & & + \nu \nabla_\perp^2 u
\end{array}}$$

  where $j = -\nabla_\perp^2 a$ and $\omega = -\nabla_\perp^2 \varphi$

- These are the RHMHD equations. Their ideal invariants (just as for 3D HMHD) are:

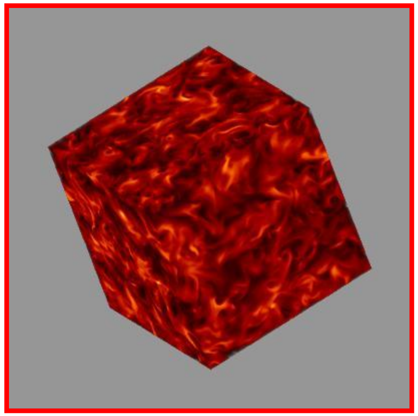
$$E = \frac{1}{2}\int d^3 r (|\vec{U}|^2 + |\vec{B}|^2) = \frac{1}{2}\int d^3 r (|\vec{\nabla}_\perp \varphi|^2 + |\vec{\nabla}_\perp a|^2 + u^2 + b^2) \qquad \text{energy}$$

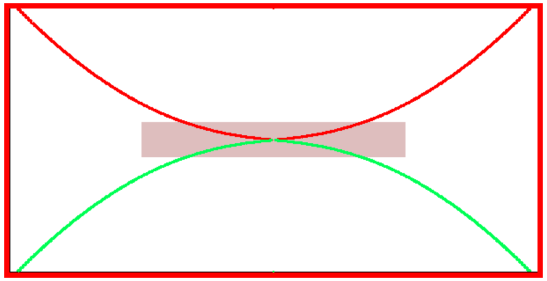
$$H_m = \frac{1}{2}\int d^3 r (\vec{A} \cdot \vec{B}) = \int d^3 r\, a b \qquad \text{magnetic helicity}$$

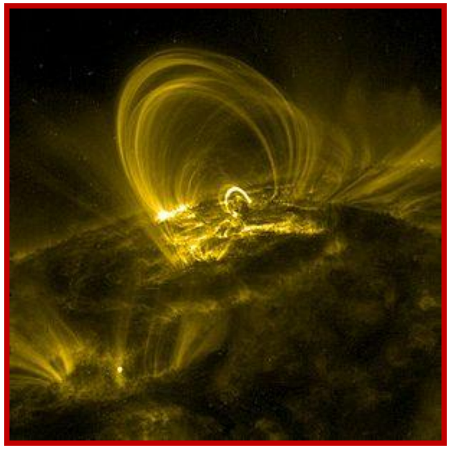
$$H_h = \frac{1}{2}\int d^3 r (\vec{A} + \varepsilon \vec{U}) \cdot (\vec{B} + \varepsilon \vec{\Omega}) = \int d^3 r (ab + \varepsilon(a\omega + ub) + \varepsilon^2 u\omega) \qquad \text{hybrid helicity}$$

Some applications

- We studied a number of astrophysical problems, within the general framework of MHD:
- 3D Hall-MHD turbulent dynamos.
  (Mininni, Gomez & Mahajan 2003, 2005; Gomez, Dmitruk & Mininni 2010)
- 2.5 D Hall-MHD magnetic reconnection in the Earth magnetosphere
  (Morales, Dasso & Gomez 2005, 2006)
- 3D HD helical fluid turbulence
  (Gomez & Mininni 2004)
- RMHD heating of solar coronal loops
  (Dmitruk & Gomez 1997, 1999)
- RHMHD turbulence in the solar wind
  (Martin, Dmitruk & Gomez 2010, 2012)
- Hall magneto-rotational instability in accretion disks
  (Bejarano, Gomez & Brandenburg 2011)







Simulations

- We integrate the RHMHD eqs. numerically, using a spectral scheme in the perpendicular directions and finite differences along the (much smoother) direction z (Gomez, Milano and Dmitruk 2000; also Dmitruk, Gomez & Matthaeus 2003)

- We show results from 512x512x40 runs performed in (CAPS), our linux cluster with 80 cores

- For the horizontal spatial derivatives, we use a pseudo-spectral scheme with 2/3-dealiasing. Spectral codes are well suited for turbulence studies, since they provide exponentially fast convergence. Spatial derivatives along the loop are computed using finite differences.

- Time integration is performed with a second order Runge-Kutta scheme.The time step is chosen to satisfy the CFL condition. This condition is more stringent if Hall is present, since it displays a quadratic dependence with the grid size.

caps

Cluster for Astrophysical Plasma Simulations

Simulations: spatial integration

- We focus on Fourier-Galerkin methods. Let us ilustrate on Burgers equation

$$\boxed{\partial_t u + u\partial_x u = \nu \partial_{xx} u}$$

for u(x,t) on the interval $0 \le x < 2\pi$ assuming periodic boundary conditions and the initial condition $u(x,0) = u_0(x)$

- We expand in a truncated Fourier expansion $\longrightarrow$ $u^N(x,t) = \sum_{k=-N/2}^{N/2} u_k(t)\, e^{ikx}$

- Demanding zero residuals of the solution u(x,t) when projected on the truncated Fourier space

$$\partial_t u_k = -(u\partial_x u)_k - \nu k^2 u_k \quad , \quad (u\partial_x u)_k = \sum_{l+m=k} imu_l u_m$$

- This truncated expansion $u^N(x,t)$ converges exponentially fast to the exact solution as $N \to \infty$

- However, it is computationally very demanding, since it involves $O(N^2)$ operations.

Simulations: spatial integration

- The FFT algorithm yields the discrete set $\{\tilde{u}_k\}$ from the set $\{u(x_j)\}$ after $O(N \log N)$ floating point operations.

$$\{u(x_j)\,,\, x_j = \frac{2\pi}{N} j\,,\; j = 0,\ldots, N-1\} \xleftrightarrow{FFT} \{\tilde{u}_k\,,\, k = -N/2+1,\ldots, N/2\}$$

- The strategy of computing spatial derivatives in Fourier space and nonlinear terms in physical space, is known as pseudo-spectral, i.e.

$$\partial_t u_k = -(u\partial_x u)_k - \nu k^2 u_k \quad , \quad (u\partial_x u)_k = FFT(FFT^{-1}(u_k)\; FFT^{-1}(iku_k))$$

- The relation between discrete Fourier coefficients $\{\tilde{u}_k\}$ and the continuous ones is

$$\tilde{u}_k = u_k + \sum_{m\neq 0} u_{k+Nm}$$

- This sum causes a spurious effect known as aliasing when computing nonlinear terms. Aliasing effects can be suppressed by applying the “two-thirds rule”, i.e.

$$\boxed{\tilde{u}_k = 0 \quad , \quad \forall \quad |k| \geq N/3}$$

Simulations: temporal integration

- We advance the solution through discrete time steps $\longrightarrow$ $t_i = i\Delta t$
- In compact notation, if $$\frac{dU}{dt} = F(U,t)$$

  where F is a nonlinear and spatial differential operator, we use a second order Runge-Kutta scheme.

- We first advance half a step $$U^{i+\frac{1}{2}} = U^i + \frac{\Delta t}{2} F(U^i, t_i)$$

  and use $U^{i+\frac{1}{2}}$ to jump the whole step $\longrightarrow$ $$U^{i+1} = U^i + \Delta t \, F(U^{i+\frac{1}{2}}, t_{i+\frac{1}{2}})$$

- This is second order accurate (i.e. $O((\Delta t)^2)$). The size of the step is limited by the CFL condition, i.e.

$$\Delta t \leq \frac{\Delta x}{u_0} \quad \text{for} \quad \partial_t u = u_0 \, \partial_x u$$

RMHD applied to coronal loop heating

- The solar corona is a topologically complex array of loops (TRACE movie 171 A)
- Coronal loops are magnetic flux tubes with their footpoints anchored deep in the convective region.
- They confine a tenuous and hot plasma. Typical densities are n = 10^9 cm^{-3} and temperatures are T = 2-3.10^6 K.

- The magnetic field provides not just the confinement of the plasma, but also the energy to heat it up to coronal temperatures (Parker 1972, 1988; van Ballegooijen 1986; Einaudi et al. 1996).
- One of the key ingredients is the free energy available in the sub-photospheric convective region. Convective motions move the footpoints of fieldlines, thus building up magnetic stresses. See **Mandrini, Demoulin & Klimchuk 2000** for a comprehensive comparison between theoretical models and observations for a large number of active regions.
- However, the typical length scale of these magnetic stresses is way too large for the Ohmic dissipation to do the job, since

$$\boxed{\tau_{diss} \approx \ell^2 / \eta}$$

RMHD Equations

- Reduced MHD is a self-consistent approximation of the full MHD equations whenever:
  (a) one component of the magnetic field is much stronger than the others and,
  (b) spatial variations are smoother along than across (Strauss 1976).

$$\partial_t a = v_A \partial_z \varphi + [\varphi, a] + \eta \nabla_\perp^2 a$$

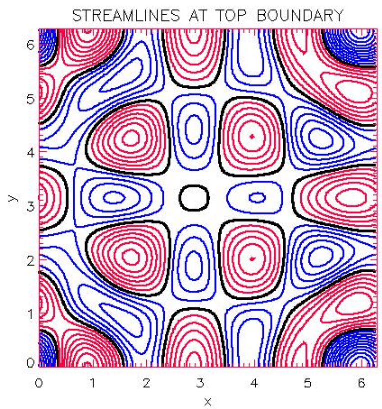
$$\partial_t \omega = v_A \partial_z j + [\varphi, \omega] - [a, j] + \eta \nabla_\perp^2 \omega$$

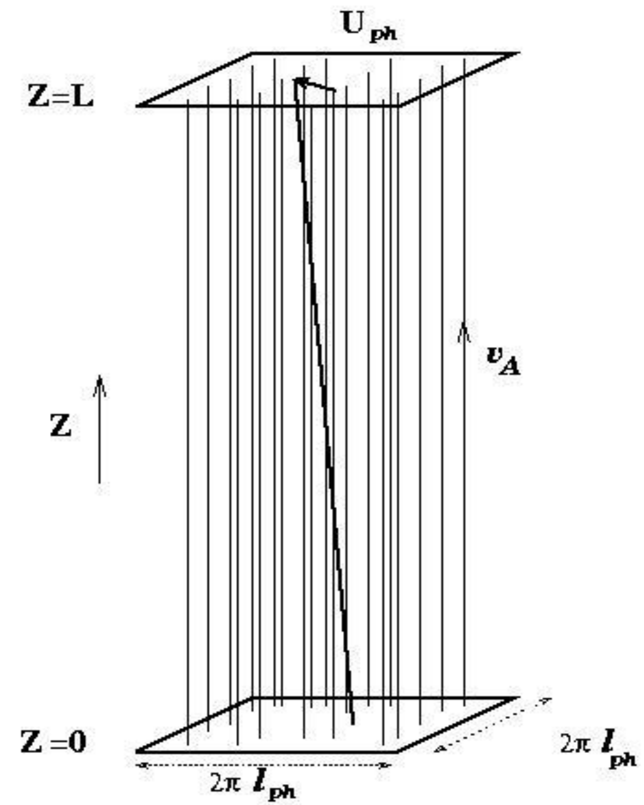
$$\vec{b} = v_A \hat{z} + \vec{\nabla}_\perp \times (a\,\hat{z}) \quad , \quad \vec{u} = \vec{\nabla}_\perp \times (\varphi\,\hat{z})$$

$$\omega = -\nabla_\perp^2 \varphi \qquad , \qquad j = -\nabla_\perp^2 a$$

- These equations describe the evolution of the velocity (**u**) and magnetic field (**b**) inside the box, assuming periodic boundary conditions at the sides.

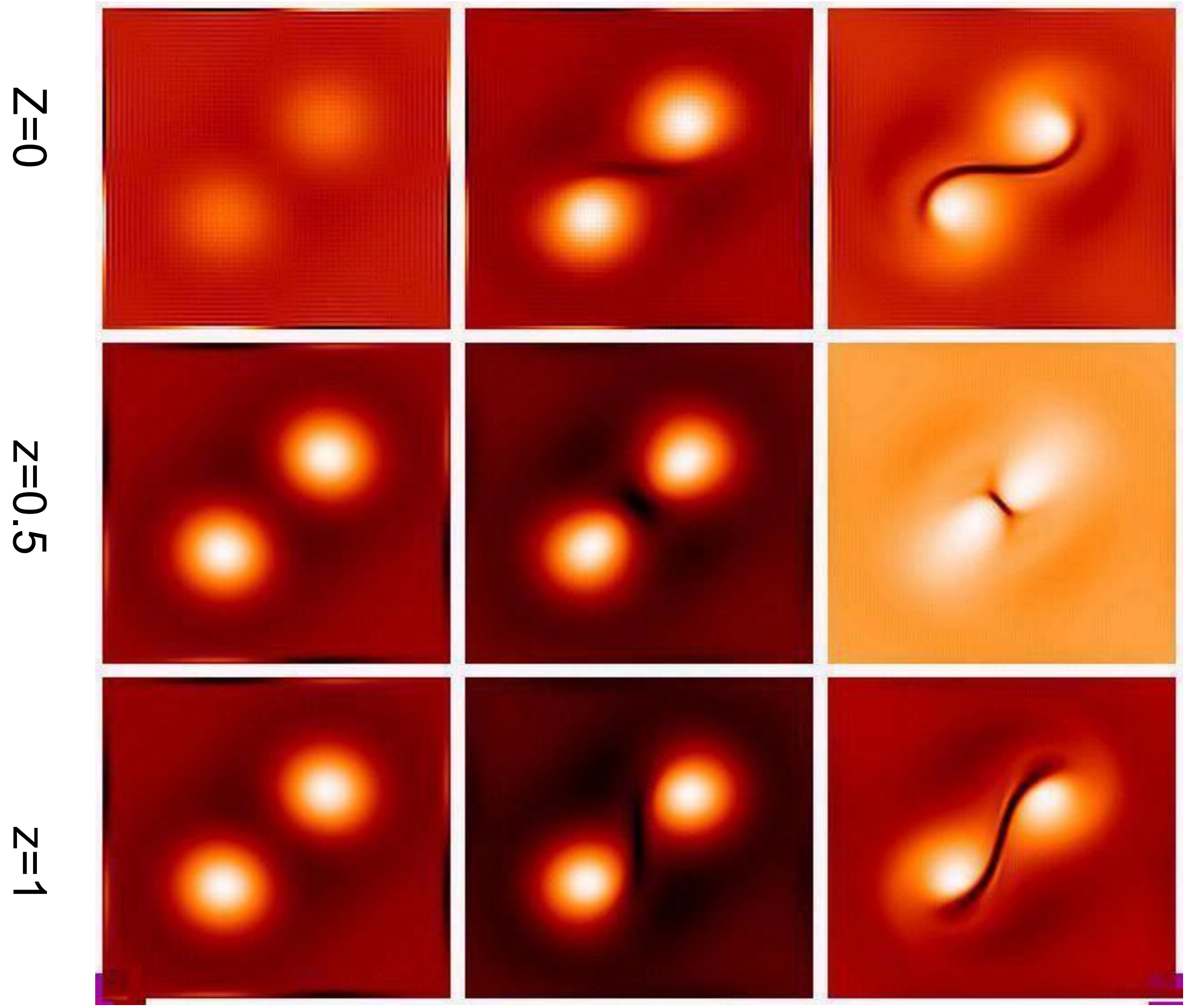
- We enforce stationary velocity fields ($\mathbf{U}_{ph}$) at the top plate.





Current density distribution

Current density



time

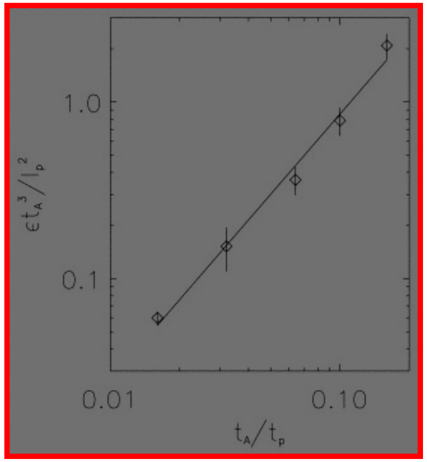
RMHD simulations

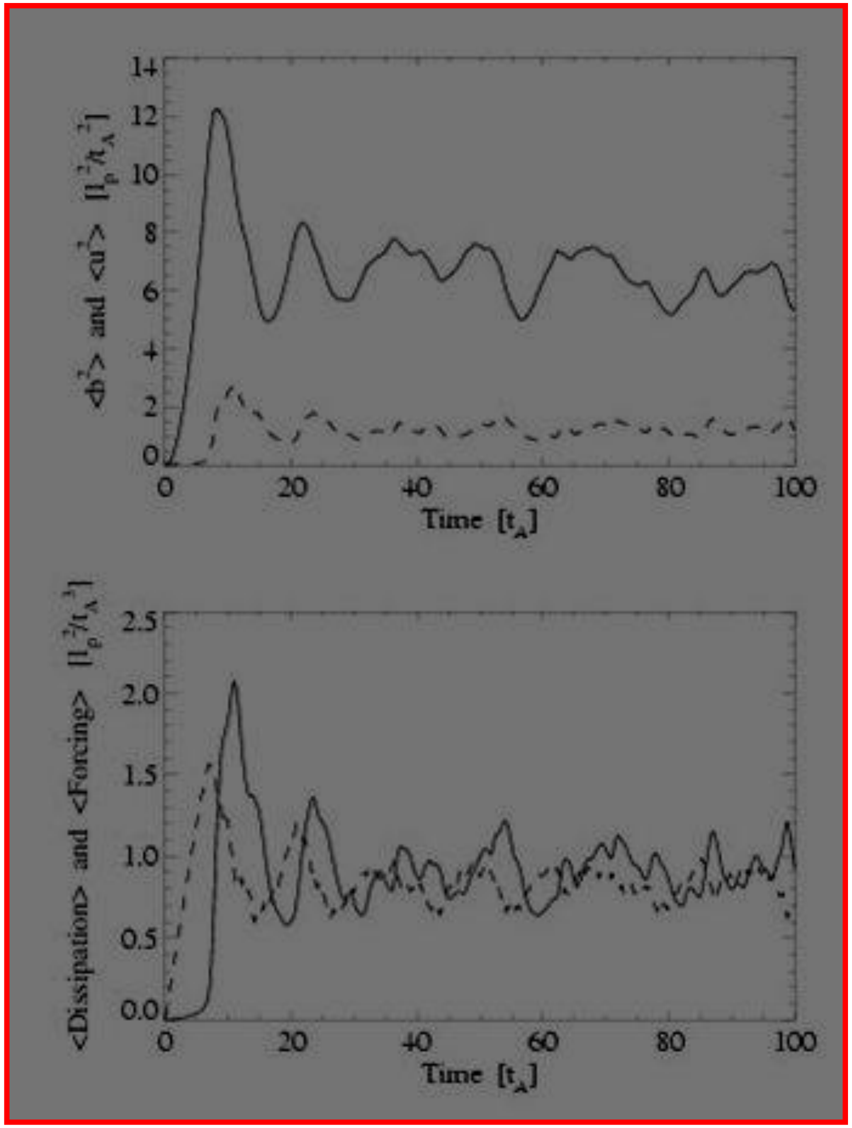
- We perform long time integrations of the RMHD equations. Lengths are in units of the photospheric convective motions (ℓ_{ph}) and times are in units of the Alfven time (t_A) along the loop.

- Spatial resolution is 256x256x48 and the integration time is 4000 t_A. We use a spectral scheme in the xy-plane and finite differences along z.

- The time averaged dissipation rate is found to scale like (Dmitruk & Gómez 1999)

$$\varepsilon \approx \frac{\rho \ell^2_{ph}}{t_A^3} \left(\frac{t_A}{t_{ph}}\right)^{3/2}$$





- It is essentially independent of the Reynolds number, as expected for stationary turbulence.

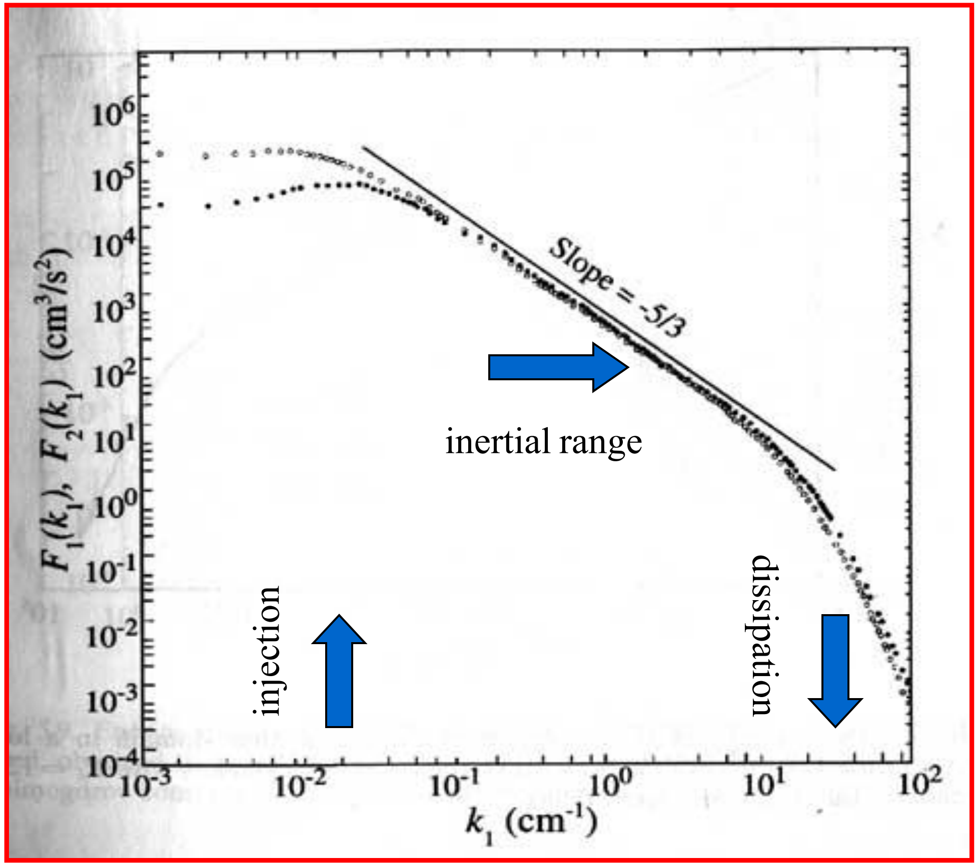
Stationary turbulence

- Energy cascade
  - energy flux toward high k
  - vortex breakdown

- Scale invariance
  - energy flux in k: $\varepsilon_k \approx \dfrac{u_k^2}{\tau_k}$
  - energy power spectrum: $E_k \approx \dfrac{u_k^2}{k}$

$$\tau_k \approx \frac{1}{k u_k} \;,\; \varepsilon_k \approx \frac{u_k^2}{\tau_k} = const.$$

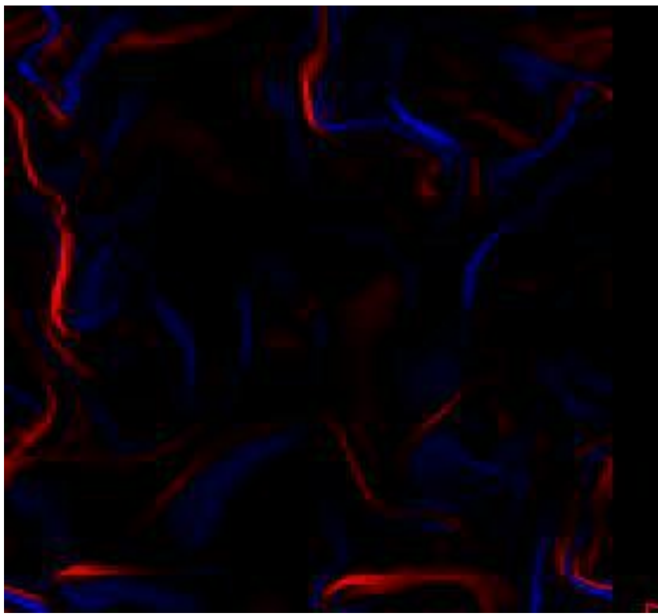
- Therefore $\boxed{E_k \approx \dfrac{u_k^2}{k} = \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}}$

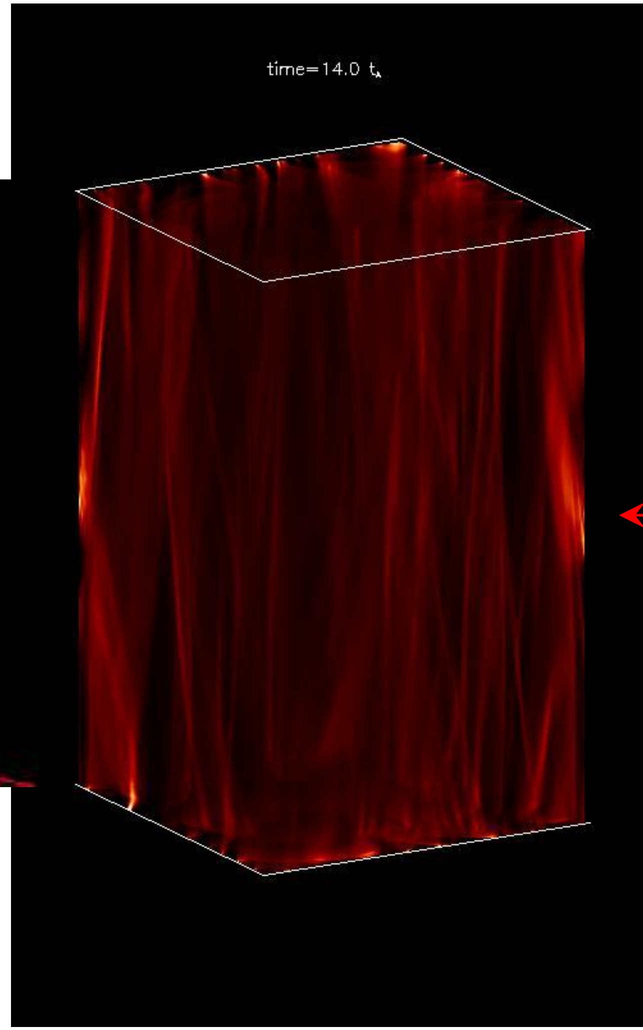


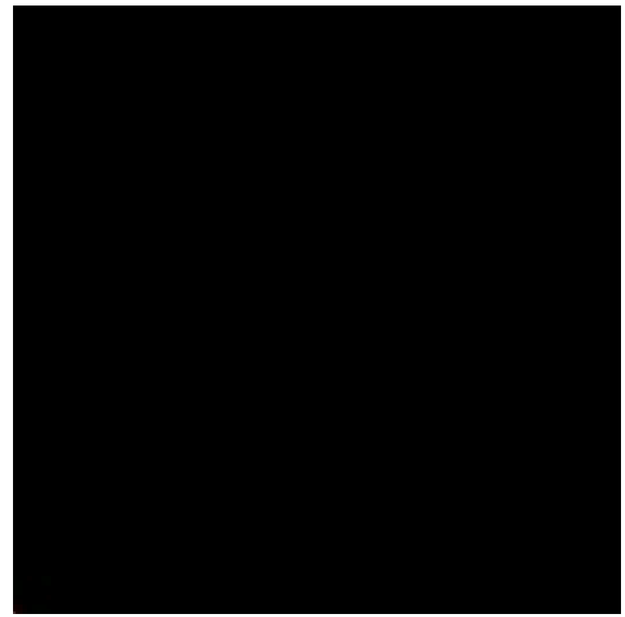
Kolmogorov spectrum (K41)

Dissipative structures: current sheets in 2D

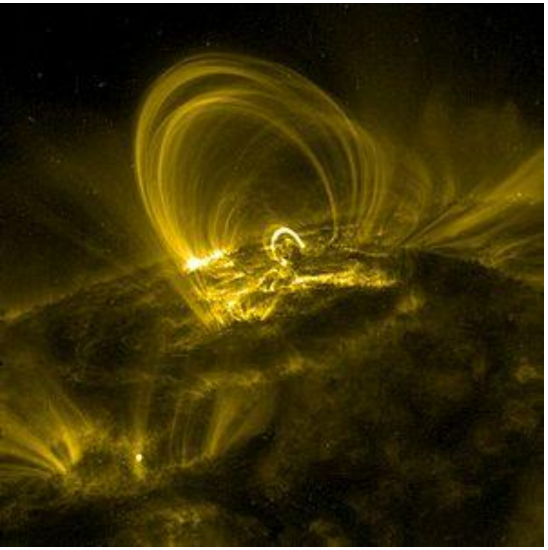
upflows downflows

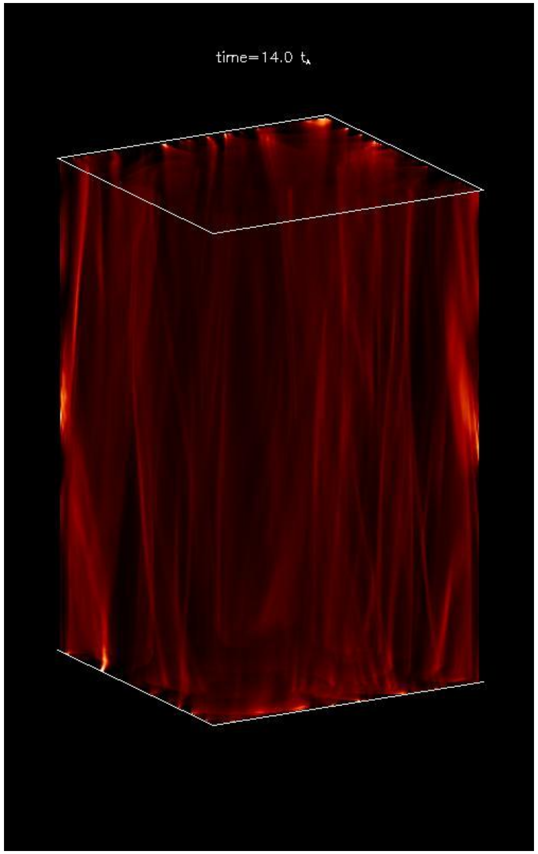


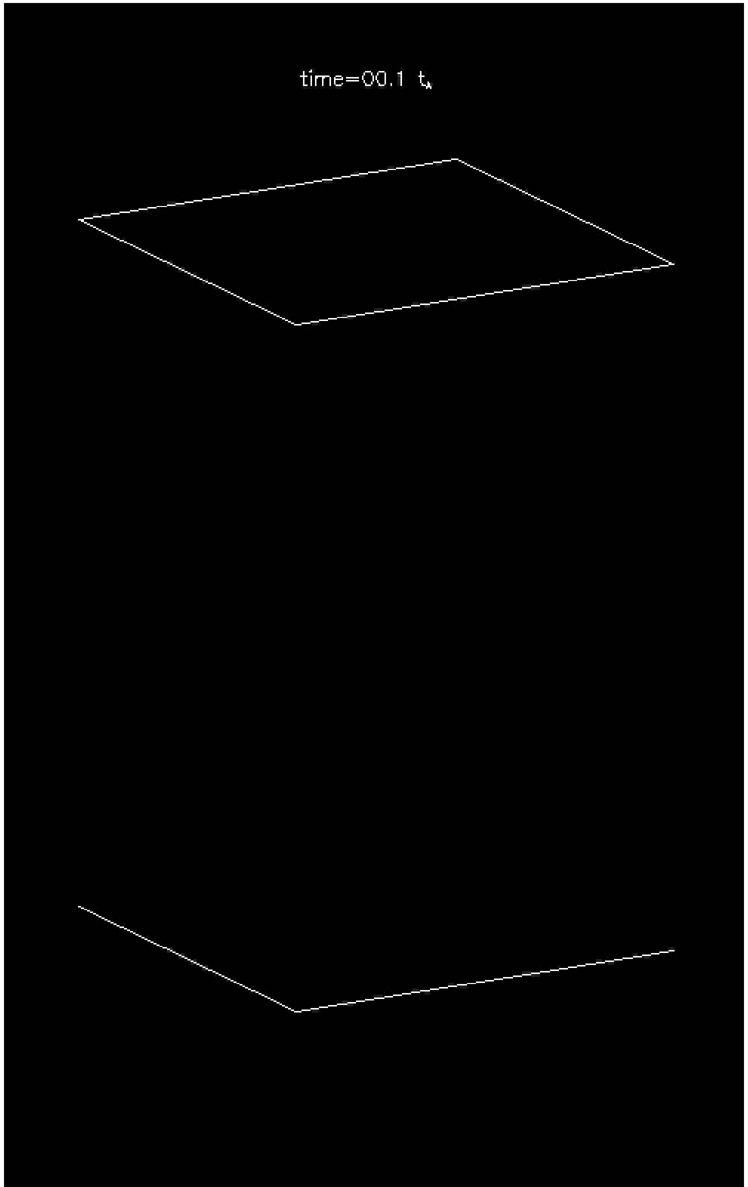




Dissipative structures: current sheets in 3D







Conclusions

- In this first lecture we introduced the Hall-MHD equations, which is an adequate theoretical framework to describe a number of astrophysical and laboratory applications.

- We also presented to so called reduced approximation, which is appropriate for plasmas embedded in relatively strong magnetic fields.

- We briefly showed the numerical techniques used to integrate the Hall-MHD equations (spectral and Runge-Kutta).

- As a first application, we showed RMHD simulations (no Hall effect yet) to study the internal dynamics of magnetic loops in the solar corona.

- In the next lecture, we will include the Hall effect and focus on its influence in magnetic reconnection, dynamo mechanisms or turbulence.