Discrete symmetries and lepton mixing

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D.H, A. Yu. Smirnov; 1204.0445
D.H, A. Yu. Smirnov; in prep.

It all begins with large mixing angles in the leptonic sector

$$|U_{l\nu}|^2 = \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \left(\begin{array}{ccc} \sim 0.7 & \sim 0.3 & \sim 0 \\ \sim 0.1 & \sim 0.5 & \sim 0.4 \\ \sim 0.2 & \sim 0.2 & \sim 0.6 \end{array} \right) \end{array}$$

$$|U_{TBM}|^2 = \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \left(\begin{array}{ccc} 2/3 & 1/3 & 0 \\ 1/6 & 1/3 & 1/2 \\ 1/6 & 1/3 & 1/2 \end{array} \right) \end{array}$$

TriBimaximal Mixing

Harrison, Perkins, Scott, 2002

Even before and for different reasons,
Bimaximal mixing had been proposed

Bimaximal Mixing

Vissani, 1997

$$|U_{BM}|^2 = \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \left(\begin{array}{ccc} 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{array} \right) \end{array}$$

Do TBM or BM have something to do with the actual neutrino mass matrix??

1. Is is possible to reproduce those mixing patterns from a fundamental Lagrangian??

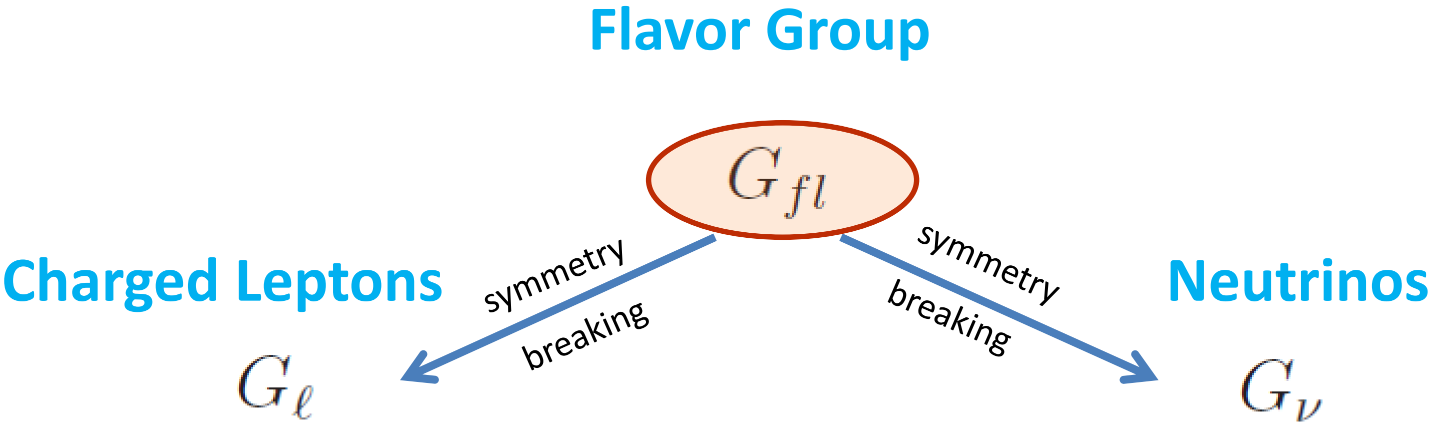
Yes! Using discrete symmetries

2. Is is possible to produce a realistic theory?

This is rather difficult.

Can we make model-independent statements about the use of discrete symmetries in flavor??

General framework



Flavor Group

G_{fl}

symmetry breaking

symmetry breaking

Charged Leptons

G_ℓ

Neutrinos

G_ν

Bottom-up approach: Identify G_ℓ and G_ν with accidental symmetries of the mass terms. Use them to define the flavor group G_{fl}

old game!

Identifying the accidental symmetries

$$\mathscr{L} = \frac{g}{\sqrt{2}}\bar{\ell}_L U_{PMNS}\gamma^\mu \nu_L W_\mu^+ + \bar{E}_R m_\ell \ell_L + \frac{1}{2}\bar{\nu^c}_L m_\nu \nu_L + \ldots + \text{h.c.}$$

Focus on the mass terms

Charged Leptons

$\bar{E}_R m_\ell \ell_L$ is invariant under $U(1)^3$ **accidental**

$$E_R \to T E_R\,, \quad \ell_L \to T \ell_L \qquad T = \text{diag}\{e^{i\alpha}, e^{i\beta}, e^{i\gamma}\}$$

Identifying the flavor symmetry

$$\mathscr{L} = \frac{g}{\sqrt{2}}\bar{\ell}_L U_{PMNS}\gamma^\mu \nu_L W_\mu^+ + \bar{E}_R m_\ell \ell_L + \frac{1}{2}\bar{\nu^c}_L m_\nu \nu_L + \ldots + \text{h.c.}$$

Focus on the mass terms

Neutrinos

$\frac{1}{2}\bar{\nu^c}_L m_\nu \nu_L$ invariant under $Z_2 \otimes Z_2$ accidental

$$S_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \quad S_3 = S_1 S_2 = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

Enter mixing matrix

$$\mathscr{L} = \frac{g}{\sqrt{2}}\bar{\ell}_L U_{PMNS}\gamma^\mu \nu_L W^+_\mu + \bar{E}_R m_\ell \ell_L + \frac{1}{2}\bar{\nu^c}_L m_\nu \nu_L + \ldots + \mathrm{h.c.}$$

Change of basis

$$\mathscr{L} = \frac{g}{\sqrt{2}}\bar{\ell}_L \gamma^\mu \nu_L W^+_\mu + \bar{E}_R M_\ell \ell_L + \frac{1}{2}\bar{\nu^c}_L M_\nu \nu_L + \cdots + \mathrm{h.c.}$$

$$M_\nu = U^* m_\nu U^\dagger$$

$$M_\ell = m_\ell V$$

$$U_{PMNS} = VU$$

Take $U \equiv U_{PMNS} \quad V \equiv 1$

Invariance of M_ν under $Z_2 \otimes Z_2$ accidental ⇒ $S^\dagger_{iU} M_\nu S_{iU} = M_\nu$ with $S_{iU} = U S_i U^\dagger$

Still $S^2_{iU} = 1$

Choosing the flavor subgroups

For the neutrinos

Just choose at least one of the S_{iU}

Choosing the flavor subgroups

For charged leptons, use a **discrete abelian** subgroup of $U(1)^3$ as part of the group of flavor

Impose $T^m = 1$, T unitary

$$T = \begin{pmatrix} e^{2\pi i k_1/m} & & \\ & e^{2\pi i k_2/m} & \\ & & e^{-2\pi i(k_1+k_2)/m} \end{pmatrix}$$

Defining the flavor group

Focus on one-generator subgroups

- Choose one of the S_{iU} and a T
- Define a relation between S_{iU} and T

We had $T^m = 1\,,\quad S_{iU}^2 = 1$

Add $(S_{iU}T)^p = (US_iU^\dagger T)^p = \mathbb{I}$

The relations

$$S_{iU}^n = T^m = (S_{iU}T)^p = \mathbb{I}$$

define the **von Dyck group** $D(n,m,p)$

$D(2,2,p)$ is the dihedral group $\mathbf{D}_p$

$$D(2,2,3) = \mathbf{S}_3$$
$$D(2,3,3) = \mathbf{A}_4$$
$$D(2,3,4) = \mathbf{S}_4$$
$$D(2,3,5) = \mathbf{A}_5$$

Notice that if

$$\frac{1}{n} + \frac{1}{m} + \frac{1}{p} \leq 1$$

The von Dyck group is infinite

Now you know the flavor group and the symmetry breaking pattern, go and construct a model

Constraints on the mixing matrix

$$W_i = S_{iU}T = US_iU^\dagger T\,, \quad W_i^p = 1$$

$$\mathrm{Det}[W_i - \lambda\mathbb{I}] = 0$$ **cubic equation with** $\lambda_i^p = 1$

$$\lambda^3 + a\lambda^2 - a^*\lambda - 1 = 0$$ with $\textcircled{a} = -\mathrm{Tr}[W_i]$

Two equations, one for the real and one for the imaginary part of a

TWO CONSTRAINTS ON THE MIXING MATRIX

$$W_i = S_{iU} T = U S_i U^\dagger T\,, \quad W_i^p = 1$$

So, the constraints on the entries of the mixing matrix depend on

$$a = -\mathrm{Tr}[W_i]$$

$$T = \begin{pmatrix} e^{2\pi i k_1/m} & & \\ & e^{2\pi i k_2/m} & \\ & & e^{-2\pi i (k_1+k_2)/m} \end{pmatrix}$$

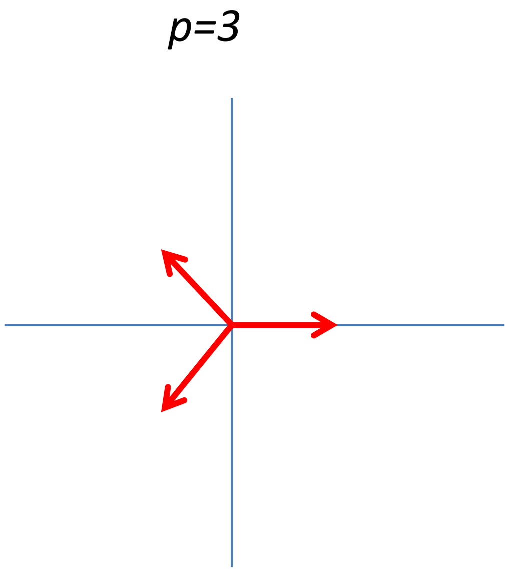
and which S_i is chosen

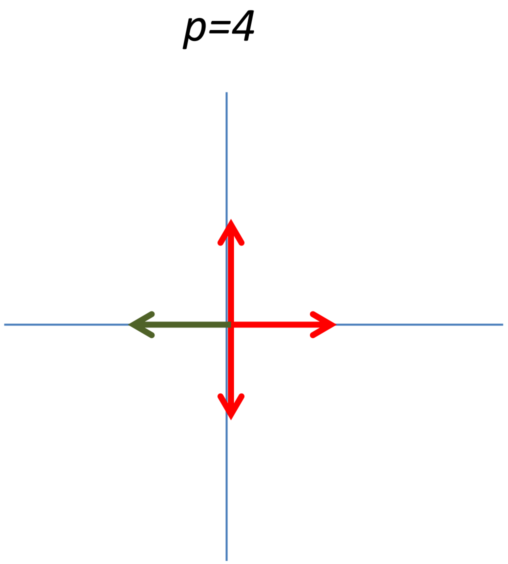
$$S_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \quad S_2 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \quad S_3 = S_1 S_2 = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

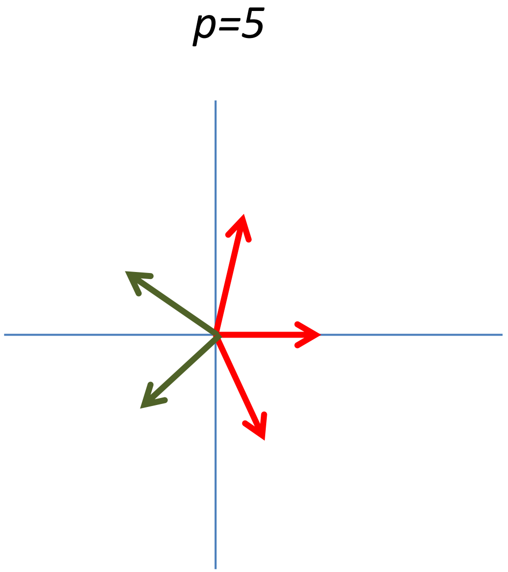
$$W_i = S_{iU} T = U S_i U^\dagger T\,, \quad W_i^p = 1$$

Constraints on the mixing matrix

$$a = -\mathrm{Tr}[W_i]$$







For instance, for $p = 3 \longrightarrow (\lambda - 1)(\lambda - \omega)(\lambda - \omega^2) = \lambda^3 - 1 \longrightarrow a = 0$

or $p = 4 \longrightarrow (\lambda - 1)(\lambda + i)(\lambda - i) = \lambda^3 - \lambda^2 + \lambda - 1 \longrightarrow a = -1$

The absolute values squared of one column are determined (two constraints plus unitarity)

$$|U_{l\nu}|^2 = \begin{array}{c} \\ \\ \end{array}\left(\begin{array}{ccc} S_1 & S_2 & S_3 \\ |U_{e1}|^2 & |U_{e2}|^2 & |U_{e3}|^2 \\ |U_{\mu 1}|^2 & |U_{\mu 2}|^2 & |U_{\mu 3}|^2 \\ |U_{\tau 1}|^2 & |U_{\tau 3}|^2 & |U_{\tau 3}|^2 \end{array}\right)$$

$$R_i = \mathrm{Re}\{\mathrm{Tr}[W_i + T]\}$$

$$I_i = \mathrm{Im}\{\mathrm{Tr}[W_i + T]\}$$

$$|U_{ei}|^2 = -\frac{R_i \cos\left(\pi \frac{k_1}{m}\right) - 2\cos\left(\pi \frac{k_1+2k_2}{m}\right) - I_i \sin\left(\pi \frac{k_1}{m}\right)}{4\sin\left(\pi \frac{k_1-k_2}{m}\right)\sin\left(\pi \frac{2k_1+k_2}{m}\right)}$$

$$|U_{\mu i}|^2 = \frac{R_i \cos\left(\pi \frac{k_2}{m}\right) - 2\cos\left(\pi \frac{2k_1+k_2}{m}\right) - I_i \sin\left(\pi \frac{k_2}{m}\right)}{4\sin\left(\pi \frac{k_1-k_2}{m}\right)\sin\left(\pi \frac{k_1+2k_2}{m}\right)}$$

$$|U_{\tau i}|^2 = -\frac{R_i \cos\left(\pi \frac{k_1+k_2}{m}\right) - 2\cos\left(\pi \frac{k_1-k_2}{m}\right) + I_i \sin\left(\pi \frac{k_1+k_2}{m}\right)}{4\sin\left(\pi \frac{2k_1+k_2}{m}\right)\sin\left(\pi \frac{k_1+2k_2}{m}\right)}$$

A particular case for *T* (the ‘lazy’ case!)

$$a = -\mathrm{Tr}[W_i]$$

$$T_e = \begin{pmatrix} 1 & & \\ & e^{2\pi ik/m} & \\ & & e^{-2\pi ik/m} \end{pmatrix} \quad T_\mu = \begin{pmatrix} e^{2\pi ik/m} & & \\ & 1 & \\ & & e^{-2\pi ik/m} \end{pmatrix} \quad T_\tau = \begin{pmatrix} e^{2\pi ik/m} & & \\ & e^{-2\pi ik/m} & \\ & & 1 \end{pmatrix}$$

$$|U_{\beta i}|^2 = |U_{\gamma i}|^2$$
$$|U_{\alpha i}|^2 = \eta, \qquad \beta, \gamma \neq \alpha$$

$$\eta \equiv \frac{1-a}{4\sin^2\left(\frac{\pi k}{m}\right)}$$

Remember $$|U_{l\nu}|^2 = \begin{array}{c c} & \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{array}{c} e \\ \mu \\ \tau \end{array} & \begin{pmatrix} \sim 0.7 & \sim 0.3 & \sim 0 \\ \sim 0.1 & \sim 0.5 & \sim 0.4 \\ \sim 0.2 & \sim 0.2 & \sim 0.6 \end{pmatrix} \end{array}$$

Hence in this case, either $i = 2$ or $\alpha = e$

Actually, we have shown that this case is unavoidable if the von Dyck group is finite!

Recapitulating: What I have shown (under some - mostly harmless - assumptions)

After a number of choices have been made

1. The **T-charge** of the charged leptons (k_1 and k_2 value)
2. The **order of T** (m value)
3. The **S-charges** of the neutrinos
4. The **eigenvalues of ST** *(a* value*)*

A two-dimensional surface is cut in the parameter space of the mixing matrix.

Is it possible to fit the measured values of the PMNS matrix??

Either $i = 2$ or $\alpha = e$

Choose $\alpha = e$ in the ‘lazy’ case

$$|U_{\mu i}|^2 = |U_{\tau i}|^2$$

$$|U_{ei}|^2 = \eta$$

$$\eta \equiv \frac{1-a}{4\sin^2\left(\frac{\pi k}{m}\right)}$$

Substituting the standard parameterization for $i = 1$

$$\tan 2\theta_{23} = -\frac{\sin^2\theta_{12} - \cos^2\theta_{12}\sin^2\theta_{13}}{\sin 2\theta_{12}\sin\theta_{13}\cos\delta}$$

$$\cos^2\theta_{12} = \frac{\eta}{\cos^2\theta_{13}}\,.$$

S.F. Ge et al

And for $i = 2$

$$\tan 2\theta_{23} = \frac{\cos^2\theta_{12} - \sin^2\theta_{12}\sin^2\theta_{13}}{\sin 2\theta_{12}\sin\theta_{13}\cos\delta}$$

$$\sin^2\theta_{12} = \frac{\eta}{\cos^2\theta_{13}}\,.$$

Choose $\alpha = e$ in the 'lazy' case

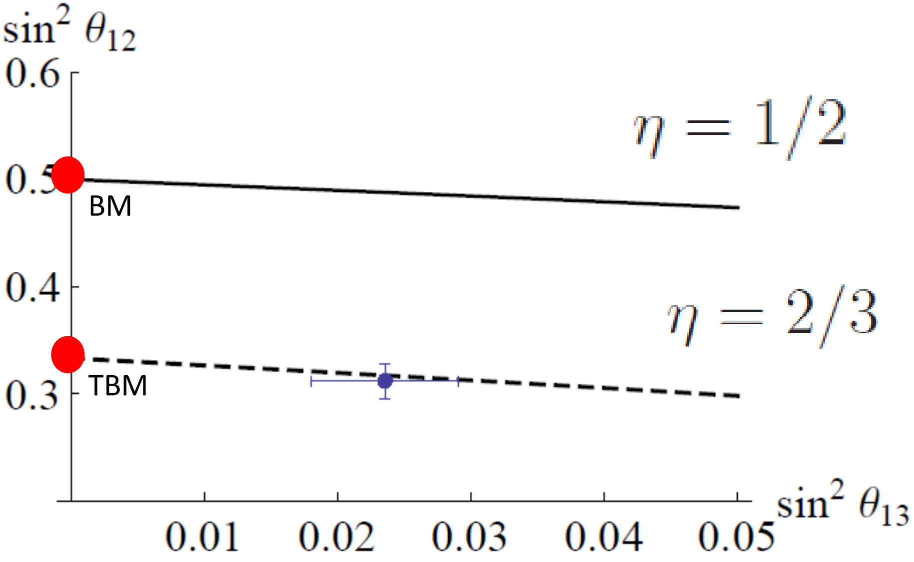
$$S_{iU}^n = T^m = (S_{iU}T)^p = \mathbb{I}$$

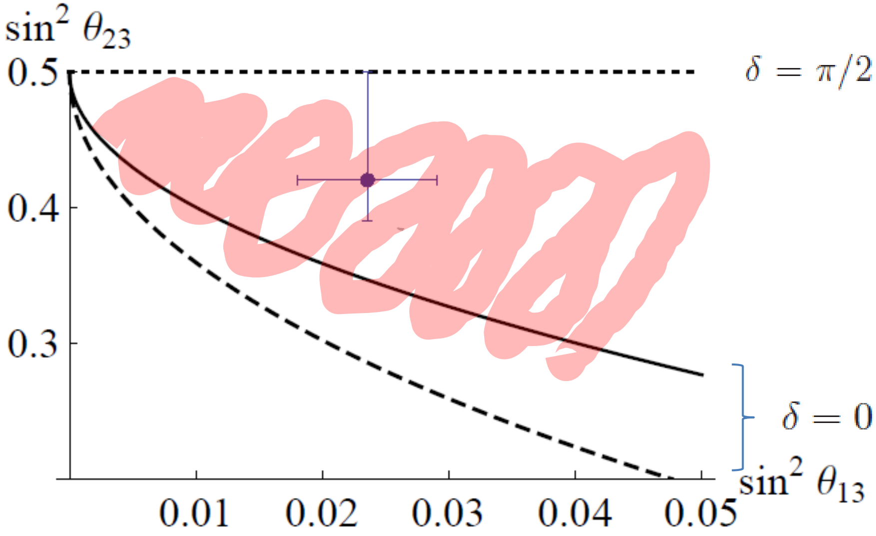
$$\lambda^3 + a\lambda^2 - a^*\lambda - 1 = 0$$

$$\eta \equiv \frac{1-a}{4\sin^2\left(\frac{\pi k}{m}\right)}$$

Taking $i = 1$

- Solid: *m = 4, p = 3. k=1* and from $(\lambda-1)(\lambda-\omega)(\lambda-\omega^2) = \lambda^3 - 1$, *a=0* . Group is **S**$_4$
- Dashed: *m = 3, p = 4. k=1, a=-1.* Group is **S**$_4$





Altarelli, Feruglio, Hagedorn, Merlo,…

Choose $\alpha = e$ in the 'lazy' case

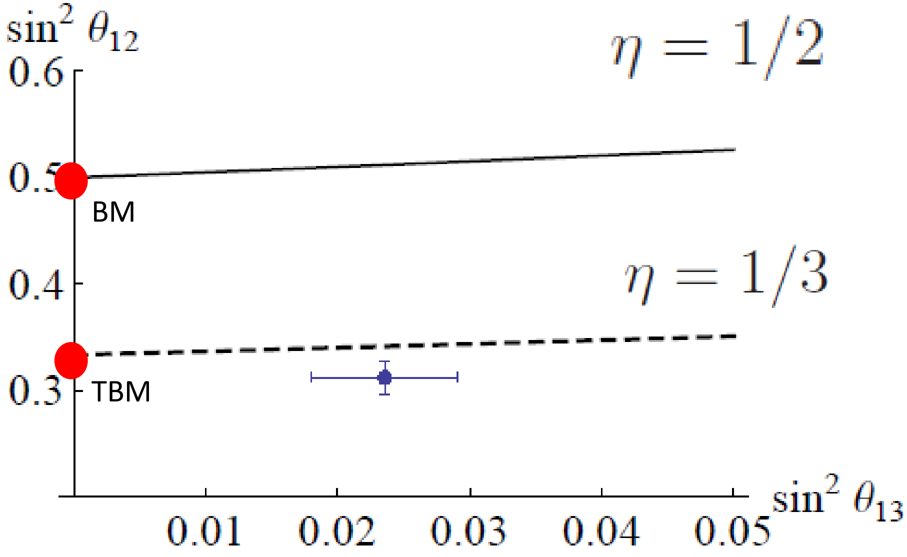
Taking $i = 2$

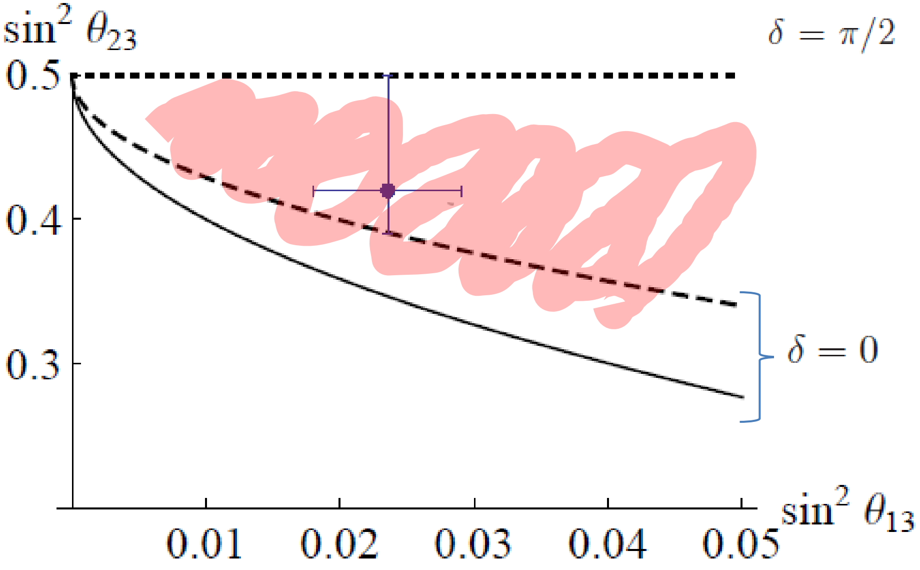
$$S_{iU}^n = T^m = (S_{iU}T)^p = \mathbb{I}$$

$$\lambda^3 + a\lambda^2 - a^*\lambda - 1 = 0$$

$$\eta \equiv \frac{1-a}{4\sin^2\left(\frac{\pi k}{m}\right)}$$

- Dashed: *m = 3, p = 3. k=1* and *a=0* . Group is **A**$_4$
- Solid: *m = 4, p = 3. k=1, a=-1.* Group is **S**$_4$





Ma, Babu, Valle, Altarelli, Feruglio, Merlo,…

Choose $i = 2$ in the 'lazy' case

Hence, either $i = 2$ or $\alpha = e$

$$|U_{e2}|^2 = |U_{\mu(\tau)2}|^2\,, \qquad |U_{\tau(\mu)2}|^2 = \eta$$

$$\sin^2\theta_{12} = \frac{1-\eta}{2\cos^2\theta_{13}}$$

For the case of T_μ

$$\cos\delta = -2\frac{\sin^2\theta_{12}(\cos^2\theta_{23}\sin^2\theta_{13} - \cos^2\theta_{13}) + \cos^2\theta_{12}\sin^2\theta_{23}}{\sin 2\theta_{12}\sin 2\theta_{23}\sin\theta_{13}}$$

For the case of T_τ

$$\cos\delta = 2\frac{\sin^2\theta_{12}(\sin^2\theta_{23}\sin^2\theta_{13} - \cos^2\theta_{13}) + \cos^2\theta_{12}\cos^2\theta_{23}}{\sin 2\theta_{12}\sin 2\theta_{23}\sin\theta_{13}}$$

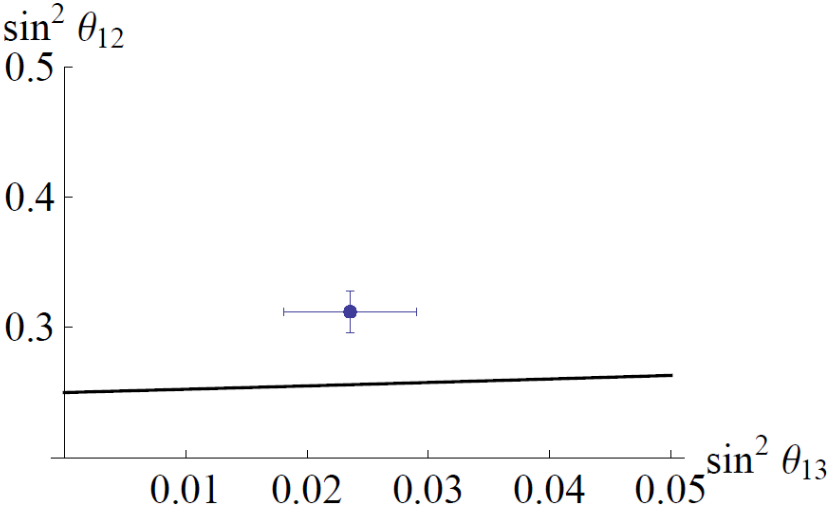
Unexplored

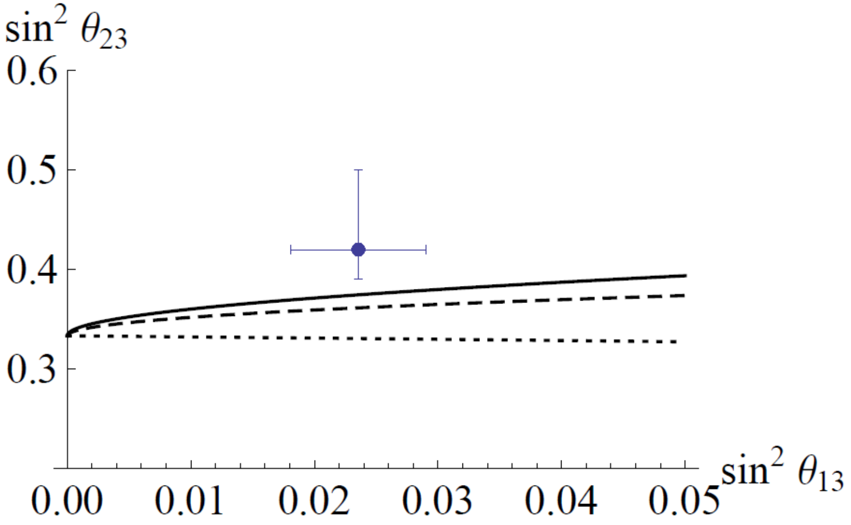
Choose $i = 2$ in the 'lazy' case

$$T_\mu : a = 0\,, \quad p = 3\,, \quad k = 1\,, \quad m = 4$$

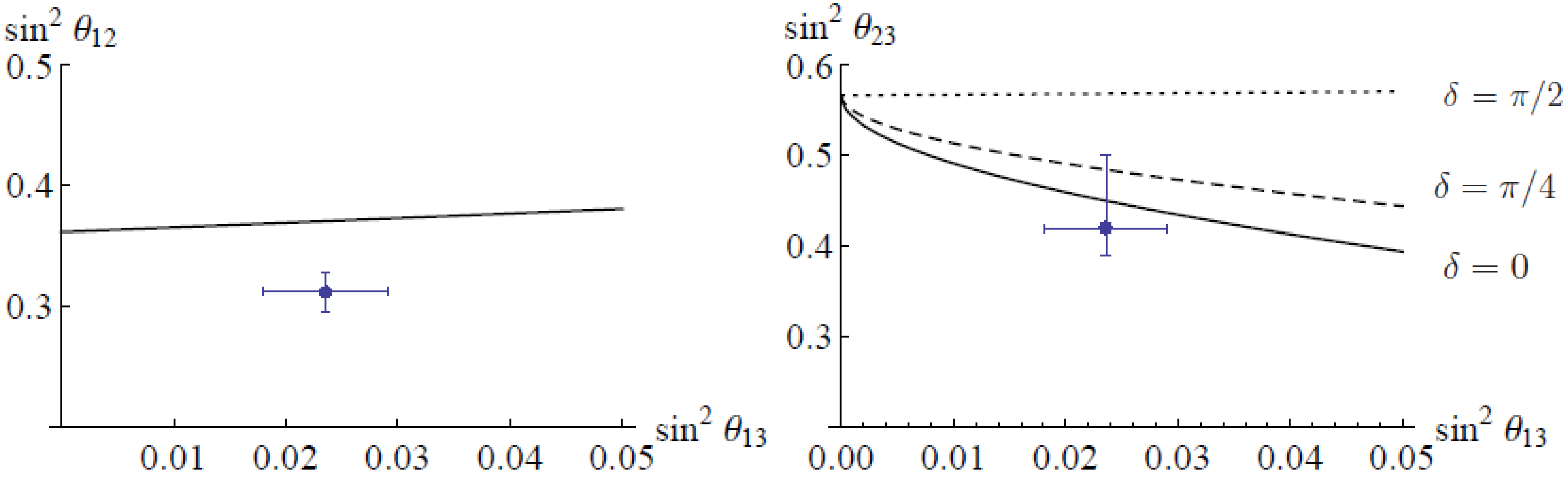
Flipped Bimaximal

$$|U_{l\nu}|^2 = \begin{pmatrix} |U_{e1}|^2 & 1/4 & |U_{e3}|^2 \\ |U_{\mu 1}|^2 & 1/2 & |U_{\mu 3}|^2 \\ |U_{\tau 1}|^2 & 1/4 & |U_{\tau 3}|^2 \end{pmatrix}$$





$T_\mu: \;\; a = 0 \quad k = 2 \quad m = 5$



A few words about TBM

If one imposes that the two Z2 symmetries of the neutrino mass matrix should belong to the flavor group, then 4 relations appear between the entries of the mixing matrix

If they are compatible, they will fix all parameters of the mixing matrix.

TBM is indeed one solution for the case of $\mathbf{S_4}$.

This could be an argument pro TBM.

C. S. Lam

Other Groups

If $\frac{1}{n}+\frac{1}{m}+\frac{1}{p}\leq 1$

the von Dyck group is infinite

$S_{iU}^{n}=T^{m}=(S_{iU}T)^{p}=\mathbb{I}$ → Von Dyck group (may now be infinite)

Add $(S_{iU}T^{\dagger}S_{iU}T)^{q}=\mathbb{I}$ → **Modular group Γ_7, Δ(96)**

Same strategy, for $X=S_{iU}T^{\dagger}S_{iU}T$

$\mathrm{Tr}[X]$ has discrete values

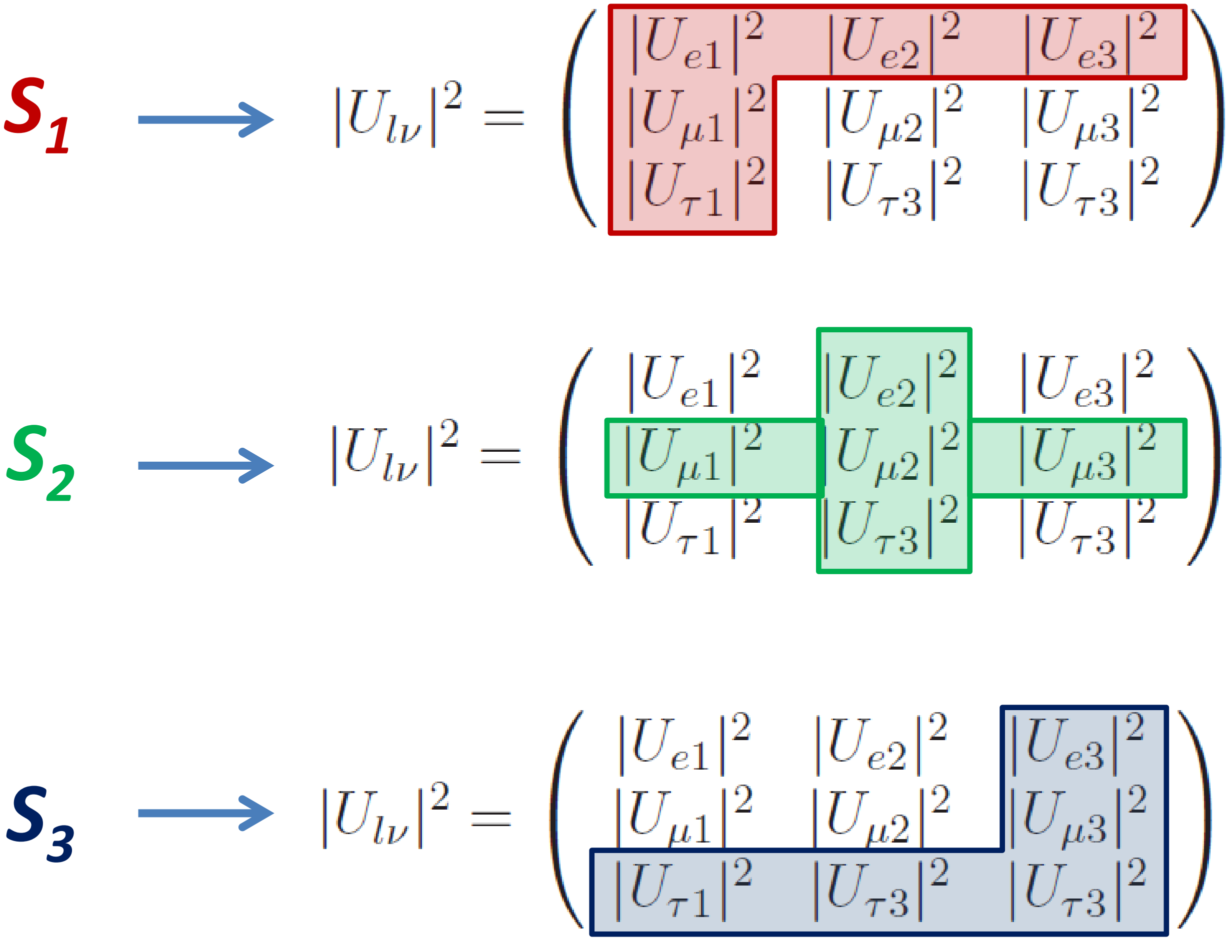
very preliminary!

Only one additional constraint in the mixing matrix

De Adelhart, Feruglio, Hagedorn

With the added relation

very preliminar!



$$S_1 \longrightarrow |U_{l\nu}|^2 = \begin{pmatrix} |U_{e1}|^2 & |U_{e2}|^2 & |U_{e3}|^2 \\ |U_{\mu 1}|^2 & |U_{\mu 2}|^2 & |U_{\mu 3}|^2 \\ |U_{\tau 1}|^2 & |U_{\tau 3}|^2 & |U_{\tau 3}|^2 \end{pmatrix}$$

$$S_2 \longrightarrow |U_{l\nu}|^2 = \begin{pmatrix} |U_{e1}|^2 & |U_{e2}|^2 & |U_{e3}|^2 \\ |U_{\mu 1}|^2 & |U_{\mu 2}|^2 & |U_{\mu 3}|^2 \\ |U_{\tau 1}|^2 & |U_{\tau 3}|^2 & |U_{\tau 3}|^2 \end{pmatrix}$$

$$S_3 \longrightarrow |U_{l\nu}|^2 = \begin{pmatrix} |U_{e1}|^2 & |U_{e2}|^2 & |U_{e3}|^2 \\ |U_{\mu 1}|^2 & |U_{\mu 2}|^2 & |U_{\mu 3}|^2 \\ |U_{\tau 1}|^2 & |U_{\tau 3}|^2 & |U_{\tau 3}|^2 \end{pmatrix}$$

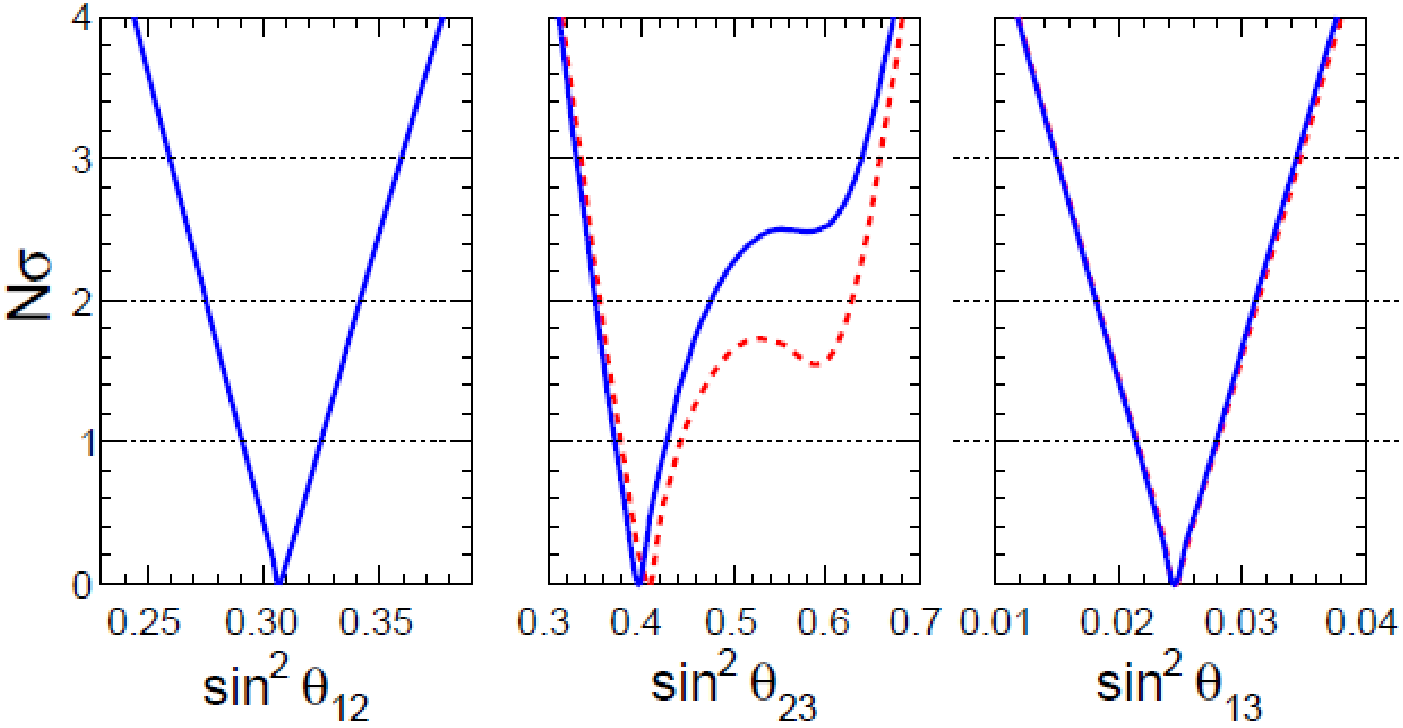
- With these new groups one can put to use the mixing patterns that appear from the infinite von Dyck groups
- The lazy are not wanted anymore (the cases in which no eigenvalue of *T* is 1 are realized)
- The mixing matrix is further constrained

very preliminar!

Conclusions

- Recipe for model building: upgrade the accidental symmetries of the mass terms by making them subgroups of the flavor group.
- The minimal choice of generators (one Z_2 for neutrinos and one Z_N for charged leptons) leads to non-abelian discrete groups of the von Dyck type.
- In this scheme, at least two relations are imposed on the leptonic mixing matrix.
- One case with $\mathbf{S_4}$ shows a very good agreement with the measured values.
- Experimenting with other groups ongoing.

TBM now disfavoured



Fogli et al., 1205.5254

Choosing the flavor subgroups

For charged leptons, use a discrete subgroup of $U(1)^3$ as part of the group of flavor

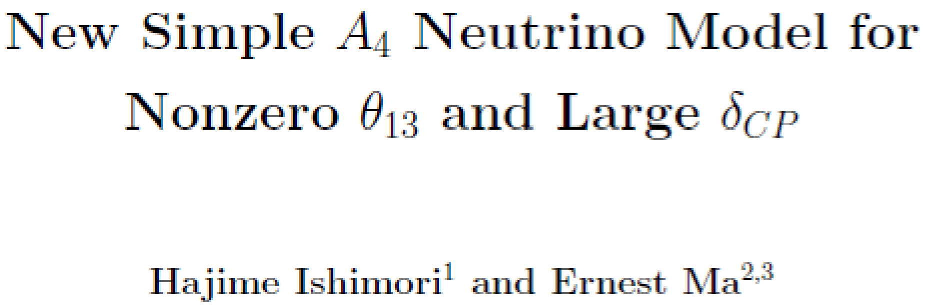
Impose $T^m = 1$

Define T_α

$$T_e = \begin{pmatrix} 1 & & \\ & e^{2\pi ik/m} & \\ & & e^{-2\pi ik/m} \end{pmatrix}, \qquad T_\mu = \begin{pmatrix} e^{2\pi ik/m} & & \\ & 1 & \\ & & e^{-2\pi ik/m} \end{pmatrix},$$

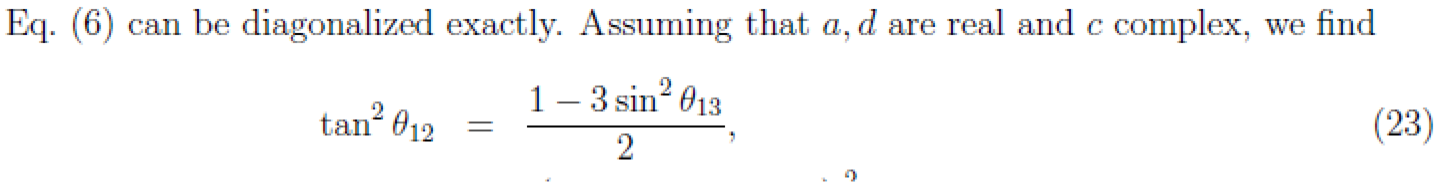
$$T_\tau = \begin{pmatrix} e^{2\pi ik/m} & & \\ & e^{-2\pi ik/m} & \\ & & 1 \end{pmatrix}$$

arXiv:1205.0075v2



New Simple A_4 Neutrino Model for Nonzero θ_{13} and Large δ_{CP}

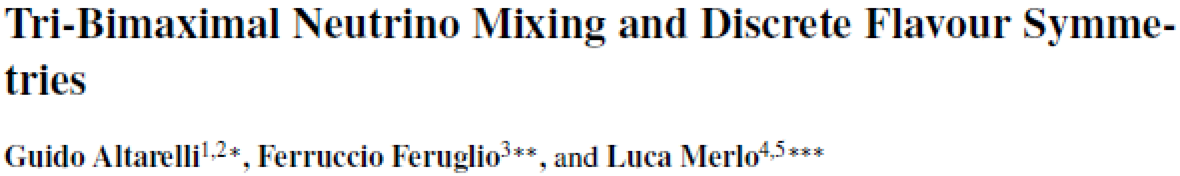
Hajime Ishimori[1] and Ernest Ma[2,3]



Eq. (6) can be diagonalized exactly. Assuming that a, d are real and c complex, we find

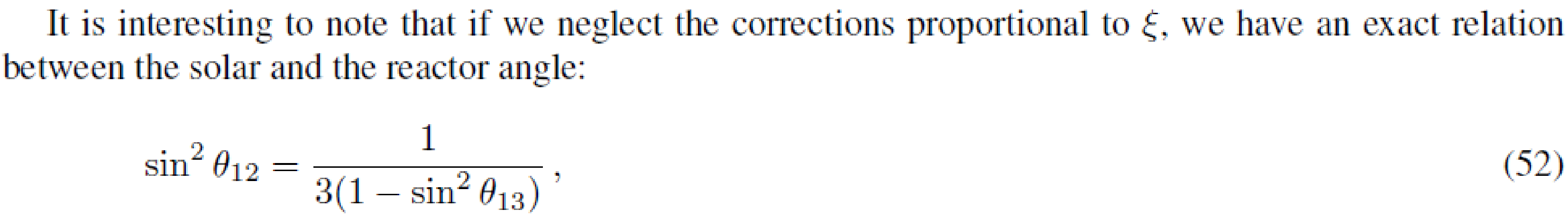
$$\tan^2\theta_{12} = \frac{1-3\sin^2\theta_{13}}{2}, \tag{23}$$

arXiv:1205.5133v1



Tri-Bimaximal Neutrino Mixing and Discrete Flavour Symmetries

Guido Altarelli[1,2]*, Ferruccio Feruglio[3]**, and Luca Merlo[4,5]***



It is interesting to note that if we neglect the corrections proportional to ξ, we have an exact relation between the solar and the reactor angle:

$$\sin^2\theta_{12} = \frac{1}{3(1-\sin^2\theta_{13})}, \tag{52}$$

$$W_{i\alpha} = US_iUT_\alpha\,, \quad W_{i\alpha}^p = 1$$

$$a = -\mathrm{Tr}[W_{i\alpha}]$$

Remember $$|U_{l\nu}|^2 = \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \left(\begin{array}{ccc} \sim 0.7 & \sim 0.3 & \sim 0 \\ \sim 0.1 & \sim 0.3 & \sim 0.5 \\ \sim 0.1 & \sim 0.3 & \sim 0.5 \end{array} \right) \end{array}$$

Hence, either $i = 2$ or $\alpha = e$

Recapitulating: What we assume

First and foremost

- The general framework for building a model with discrete symmetries

Other 'minor' assumptions

1. Neutrinos are Majorana.
2. The flavor symmetry is a subgroup of SU(3).
3. The remaining symmetry in each sector is a one-generator group

Open to discussion!!