S_4 and CP

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Outline

- Existing approaches with flavor symmetry G_f and CP
- S_4 and CP as showcase
  - Z_2 and the diagonal subgroup of Z_2 and CP for neutrinos
  - Generalization: Z_2 and CP for neutrinos
- Conclusions

Approaches with G_f and CP in Literature

We would like to investigate in more detail an idea dubbed "$\mu\tau$ reflection symmetry". ***(Harrison/Scott ('02,'04), Grimus/Lavoura ('03))***

$$Q_{23} m_\nu Q_{23} = m_\nu^\star$$

imposes as conditions on the mixing angles:

$$\sin\theta_{23} = \cos\theta_{23} \quad \text{and} \quad \sin\theta_{13}\cos\delta_{CP} = 0$$

A mass matrix m_ν for neutrinos with similar properties has been found by ***Babu/Ma/Valle ('02)***.

Harrison/Scott ('02,'04) have discussed also "triχmaximal mixing"; for some model building see ***Harrison (ICHEP12)***.

Approaches with G_f and CP in Literature

There have been several further approaches - among them are

- ***Ferreira et al. ('12)*** get with $\Delta(27)$ maximal CP violation and θ_{23}
- Combination of S_4 and CP ***(Mohapatra/Nishi ('12))***
- T' and CP violation ***(Chen/Mahanthappa ('09), Meroni et al. ('12))***
- S_2 and CP as acc. symmetries in potential ***(Babu/Kubo ('04,'11))***
- "Geometrical CP violation" ***(Branco et al. ('83))***
  (recently discussed by ***de Medeiros Varzielas et al.***)

S_4 Basis

In the following we choose a basis in which the generators are real and furthermore two of them are diagonal.

$$\rho(S) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad , \quad \rho(T) = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ -1 & \sqrt{2} & -1 \end{pmatrix} \quad , \quad \rho(U) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The generators S, T and U fulfill the relations

$$S^2 = E \ , \quad T^3 = E \ , \quad U^2 = E \ ,$$
$$(ST)^3 = E \ , \quad (SU)^2 = E \ , \quad (TU)^2 = E \ , \quad (STU)^4 = E \ .$$

Standard Way to get TB Mixing from $\boldsymbol{S_4}$

- Require invariance of charged lepton sector under Z_3 generated by T
- Require invariance of neutrino sector under $Z_2 \times Z_2$ generated by S and U

Lepton mixing is given by the matrix V which diagonalizes $\rho(T)$:
$V\rho(T)V^\dagger = \mathrm{diag}\left(1, \omega^2, \omega\right)$ with $\omega = e^{2\pi i/3}$

$$V = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -i/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & i/\sqrt{2} \end{pmatrix}$$

Definition of CP Transformation

Form of CP transformation ***(Grimus et al. ('87), Haber/Surujon ('12))***

$$\phi \xrightarrow{\text{CP}} X\phi^\star$$

with X being unitary and symmetric, i.e.

$$XX^\star = \mathbb{1}$$

We have to fulfill consistency conditions, if we wish to study theories with a flavor symmetry and CP. We exemplify this in the case of S_4.

Consistency Conditions

Consider the following sequence of transformations
ϕ transforms as rep. ρ of S_4

$$\phi \xrightarrow{\text{CP}} X\phi^\star \xrightarrow{S_4} \rho(g)X\phi^\star \xrightarrow{\text{CP}} X\left(\rho(g)X\phi^\star\right)^\star = \left(X^\star\rho(g)X\right)^\star \phi$$

Thus we have to fulfill the requirement

$$\left(X^{-1}\rho(g)X\right)^\star = \rho(g') \quad \text{with} \quad g, g' \in S_4 \ , \quad \text{but in general} \quad g \neq g'$$

It is sufficient to fulfill this for the generators of the group S_4.

Consistency Conditions II

We are interested in a particular case in which one of the Z_2 which we like to preserve in the neutrino sector "combines" with the transformation CP:

$$(Z_2)_I \times [(Z_2)_{II} \times \text{CP}]_{diag}$$

with $(Z_2)_I$ generated by S and $(Z_2)_{II}$ generated by U.

Again, we have to fulfill some consistency conditions:

$$\phi \xrightarrow{\text{CP}} X\phi^\star \xrightarrow{S} \rho(S)X\phi^\star \quad \text{and} \quad \phi \xrightarrow{S} \rho(S)\phi \xrightarrow{\text{CP}} X\left(\rho(S)\phi\right)^\star$$

thus we need to have: $\rho(S)X - X\rho(S)^\star = 0$ and similar for U.

For our choice of $\rho(S)$ and $\rho(U)$: $[\rho(S), X] = 0$ and $[\rho(U), X] = 0$.

Possible Solutions for $\boldsymbol{X}$

We find the following four different solutions for X - up to overall phase:

$$X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \rho(E) \ , \quad X_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \rho(S) \ ,$$

$$X_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \rho(U) \ , \quad X_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \rho(SU) \ .$$

We find that all X belong to S_4 itself and that they are elements of the Klein group $(Z_2)_I \times (Z_2)_{II}$.

Study of Solution X_1

We want to preserve $(Z_2)_I \times [(Z_2)_{II} \times \text{CP}]$ with

- $(Z_2)_I$ generated by S (m_{ij} complex)

$$\rho(S)^T m_\nu \rho(S) = m_\nu \quad \text{fixing the form of} \quad m_\nu = \begin{pmatrix} m_{11} & 0 & m_{13} \\ 0 & m_{22} & 0 \\ m_{13} & 0 & m_{33} \end{pmatrix}$$

- $[(Z_2)_{II} \times \text{CP}]_{diag}$ with U and X_1 (a, b, c, w real)

$$\rho(U)^T m_\nu \rho(U) = m_\nu^\star \quad \text{fixing the form of} \quad m_\nu = \begin{pmatrix} a & 0 & iw \\ 0 & b & 0 \\ iw & 0 & c \end{pmatrix}$$

Study of Solution $\boldsymbol{X_1}$

The neutrino mass matrix m_ν is diagonalized by

$$U_\nu = P R_{13}$$

with

$$P = \text{diag}(1, 1, -i) \quad \text{and} \quad R_{13} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

with θ: $\tan 2\theta = -2w/(a+c)$ - not determined by S_4!

Study of Solution X_1

The PMNS matrix takes the form: $U_{PMNS} = VU_\nu = VPR_{13}$
Results for lepton mixing parameters

$$\sin^2\theta_{13} = \frac{2}{3}\sin^2\theta$$

$$\sin^2\theta_{23} = \frac{1}{2}\left(1 + \frac{\sqrt{3}\sin 2\theta}{2+\cos 2\theta}\right) \approx \frac{1}{2} + \frac{\theta}{\sqrt{3}}$$

$$\sin^2\theta_{12} = \frac{1}{2+\cos 2\theta} \approx \frac{1}{3} + \frac{2\theta^2}{9}$$

$J_{CP} = 0$ and Majorana phases are trivial

Note: We find the same results, if we use instead $X_2 = \rho(S)$.

Study of Solution X_3

We find a physically different result, if we use the solution $X_3 = \rho(U)$:

$$m_\nu = \begin{pmatrix} a & 0 & w \\ 0 & b & 0 \\ w & 0 & c \end{pmatrix} \quad \text{with} \quad a, b, c, w \ \text{real}$$

Thus, a rotation in the (13)-plane is sufficient with the angle θ determined by: $\tan 2\theta = 2w/(c-a)$.

Study of Solution X_3

Results for lepton mixing parameters

$$\sin^2\theta_{13} = \tfrac{2}{3}\,\sin^2\theta$$

$$\sin^2\theta_{23} = \tfrac{1}{2}$$

$$\sin^2\theta_{12} = \tfrac{1}{2+\cos 2\theta} \approx \tfrac{1}{3} + \tfrac{2\theta^2}{9}$$

$$|J_{CP}| = |\sin 2\theta|/(6\sqrt{3})\;, \quad |\delta_{CP}| = \pi/2$$

Majorana phases are trivial

Note: Such results have been found in the literature.

Note2: We get the same result for $X_4 = \rho(SU)$.

Other Choices for $(\boldsymbol{Z_2})_{\boldsymbol{I}}$ and $(\boldsymbol{Z_2})_{\boldsymbol{II}}$

We can equally well choose U or SU for the group $(Z_2)_I$.
Clearly, choice U still leads to $\theta_{13} = 0$ and $\theta_{23} = \pi/4$.
Thus, we discuss instead the case of SU.
Note the following:

- $\rho(SU)$ fulfills the consistency condition
  $\rho(SU)X - X\rho(SU)^\star = 0$
- We can either choose $(Z_2)_{II}$ to be generated by S or U.
  We choose here: U.

Now we can analyze this case as before.
To begin with we choose $X_1 = \rho(E)$.

Other Choices for $(Z_2)_I$ and $(Z_2)_{II}$

- $(Z_2)_I$ generated by SU (m_{ij} complex)

$$\rho(SU)^T m_\nu \rho(SU) = m_\nu \quad \text{fixing the form of} \quad m_\nu = \begin{pmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & m_{23} \\ 0 & m_{23} & m_{33} \end{pmatrix}$$

- $[(Z_2)_{II} \times \text{CP}]_{diag}$ with U and X_1 (a, b, c, w real)

$$\rho(U)^T m_\nu \rho(U) = m_\nu^\star \quad \text{fixing the form of} \quad m_\nu = \begin{pmatrix} a & 0 & 0 \\ 0 & b & iw \\ 0 & iw & c \end{pmatrix}$$

Other Choices for $(Z_2)_I$ and $(Z_2)_{II}$

As you can see, U_ν is then of the form

$$U_\nu = P R_{23}$$

with

$$P = \text{diag}(1, 1, -i) \quad \text{and} \quad R_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

with θ: $\tan 2\theta = -2w/(b+c)$ - not determined by S_4!

The PMNS matrix reads $U_{PMNS} = V P R_{23}$

Other Choices for $(Z_2)_I$ and $(Z_2)_{II}$

Results for lepton mixing parameters

$$\sin^2\theta_{13} = \frac{1}{3}\,\sin^2\theta$$

$$\sin^2\theta_{23} = \frac{1}{2} - \frac{\sqrt{6}\sin 2\theta}{5+\cos 2\theta} \approx \frac{1}{2} - \sqrt{\frac{2}{3}}\theta$$

$$\sin^2\theta_{12} = \frac{\cos^2\theta}{2+\cos^2\theta} \approx \frac{1}{3} - \frac{2\theta^2}{9}$$

$J_{CP} = 0$ and Majorana phases are trivial

We find the same results, if we use instead $X_4 = \rho(SU)$.

Other Choices for $(\boldsymbol{Z_2})_{\boldsymbol{I}}$ and $(\boldsymbol{Z_2})_{\boldsymbol{II}}$

If we use $X_2 = \rho(S)$ or $X_3 = \rho(U)$, we find the general form of m_ν compatible with the symmetry $(Z_2)_I \times [(Z_2)_{II} \times \text{CP}]_{diag}$ to be

$$m_\nu = \begin{pmatrix} a & 0 & 0 \\ 0 & b & w \\ 0 & w & c \end{pmatrix} \quad \text{with} \quad a, b, c, w \ \text{real}$$

Now U_ν is only a rotation in the (23)-plane and thus P also drops from U_{PMNS}.

Other Choices for $(Z_2)_I$ and $(Z_2)_{II}$

Results for lepton mixing parameters

$$\sin^2\theta_{13} = \frac{1}{3}\,\sin^2\theta$$

$$\sin^2\theta_{23} = \frac{1}{2}$$

$$\sin^2\theta_{12} = \frac{\cos^2\theta}{2+\cos^2\theta} \approx \frac{1}{3} - \frac{2\theta^2}{9}$$

$$|J_{CP}| = |\sin 2\theta|/(6\sqrt{6})\;,\quad |\delta_{CP}| = \pi/2$$

and Majorana phases are trivial

Comparison to the case with $(Z_2)_I$ generated by S:

- $\sin^2\theta_{13}$ smaller by factor of 2
- Deviation of $\sin^2\theta_{12}$ from $1/3$ changes sign
- Both allow for θ_{23} and δ_{CP} maximal

Extension of the Analysis: $\boldsymbol{Z_2} \times$ CP

We can also study the case in which the neutrino sector is invariant under $Z_2 \times$ CP.

We note the following

- We find all results mentioned before.
- For Z_2 generated by S we find two additional solutions

$$X_5 = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \rho(TST^2S) \ , \quad X_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \rho(T^2STS)$$

- Phenomenology of these cases is different: "A_4-like".

Extension of the Analysis: $\boldsymbol{Z_2} \times$ CP

Do the analysis as before.

The neutrino mass matrix m_ν invariant under $Z_2 \times$ CP for X_5 or X_6 is

$$m_\nu = \begin{pmatrix} x+iy & 0 & b \\ 0 & a & 0 \\ b & 0 & x-iy \end{pmatrix}$$

with - again - four real parameters a, b, x, y.

Extension of the Analysis: $\boldsymbol{Z_2} \times \mathrm{CP}$

m_ν is diagonalized through

$$U_\nu = \begin{pmatrix} 1/\sqrt{2} & 0 & e^{i\alpha}/\sqrt{2} \\ 0 & 1 & 0 \\ -e^{-i\alpha}/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

with α given by $\tan\alpha = -y/x$.

Extension of the Analysis: $\boldsymbol{Z_2} \times \mathrm{CP}$

Lepton mixing

$$\sin^2\theta_{13} = \frac{1}{3}$$

$$\sin^2\theta_{23} = \frac{1}{2}\left(1 + \frac{\sqrt{3}}{2}\sin\alpha\right) \quad \xrightarrow{\alpha\to 0} \quad \frac{1}{2}$$

$$\sin^2\theta_{12} = \frac{1}{2}$$

$$|J_{CP}| = |\cos\alpha|/(6\sqrt{3}) \quad \xrightarrow{\alpha\to 0} \quad \frac{1}{6\sqrt{3}}$$

Limit $\alpha \to 0$ leads to results of "magic matrix", known from A_4.

Some Example with θ Large

Without having a concrete model we cannot argue that θ is small and thus also cases with θ sizable should be considered. One possible example is:

$$U_{PMNS} = Q_{12} V P R_{13} Q_{13}$$

with Q_{1i} being the permutation of first and i^{th} generation. Lepton mixing is then

$$\sin^2\theta_{13} = \left(\frac{\cos\theta}{\sqrt{6}} - \frac{\sin\theta}{\sqrt{2}}\right)^2 \stackrel{\theta=0.34}{\approx} 0.022 \quad \text{limit } \theta = 0: \quad \frac{1}{6}$$

$$\sin^2\theta_{23} = \frac{4\cos^2\theta}{4 + \cos 2\theta + \sqrt{3}\sin 2\theta} \stackrel{\theta=0.34}{\approx} 0.61 \quad \text{limit } \theta = 0: \quad \frac{4}{5}$$

$$\sin^2\theta_{12} = \frac{2}{4 + \cos 2\theta + \sqrt{3}\sin 2\theta} \stackrel{\theta=0.34}{\approx} 0.34 \quad \text{limit } \theta = 0: \quad \frac{2}{5}$$

and J_{CP} still vanishes.

Conclusions

- Combination of flavor symmetry and CP seems interesting
- Showcase: S_4 and CP
  - Results of "$\mu\tau$ reflection symmetry" are found
  - Study of all possible combinations
  - Extension to analysis of $Z_2 \times$ CP in neutrino sector
  - Peculiarity: always one free parameter in mixing which is independent of $S_4 \Rightarrow$ model building
- Generalization of this study to any discrete group