First ICO-ICTP-TWAS Central American Workshop in Lasers, Laser Applications and Laser Safety Regulations

Laser Resonators

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Contents

Wave Nature of Laser Beams.

(Without considering diffraction.)

A field component u of the coherent light satisfies the scalar wave equation:

$$\nabla^2 u + k^2 u = 0$$

where:

$$k = \frac{2\pi}{\lambda}$$

k is named the propagation constant in the medium.

For light travelling in the z direction,

$$u = \psi(x, y, z)\exp(-jkz)$$

ψ is a slowly varying complex function; which for a laser beam (alike a plane wave):

- nonuniform intensity distribution,
- expansion of the beam with distance propagation,
- curvature of the phase front,
- other.

Solving the Scalar Wave Equation.

$$\nabla^2 u + k^2 u = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) u + k^2 u = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0$$

Explicitly subsituting u:

$$u = \psi(x, y, z) \cdot \exp(-jkz) \equiv \psi \cdot \mathrm{e}^{-jkz}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (\psi \cdot \mathrm{e}^{-jkz}) + k^2 (\psi \cdot \mathrm{e}^{-jkz}) = 0$$

$$\frac{\partial^2 (\psi \cdot \mathrm{e}^{-jkz})}{\partial x^2} + \frac{\partial^2 (\psi \cdot \mathrm{e}^{-jkz})}{\partial y^2} + \frac{\partial^2 (\psi \cdot \mathrm{e}^{-jkz})}{\partial z^2} + k^2 \cdot \psi \cdot \mathrm{e}^{-jkz} = 0$$

Calculating the derivatives for x and y:

$$\frac{\partial u}{\partial x} = \frac{\partial(\psi \cdot \mathrm{e}^{-jkz})}{\partial x} = \frac{\partial(\psi)}{\partial x} \cdot \mathrm{e}^{-jkz} + \psi \cdot \frac{\partial(\mathrm{e}^{-jkz})}{\partial x} = \frac{\partial \psi}{\partial x} \cdot \mathrm{e}^{-jkz}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x} \cdot \mathrm{e}^{-jkz}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) \cdot \mathrm{e}^{-jkz} + \left(\frac{\partial \psi}{\partial x}\right) \cdot \frac{\partial(\mathrm{e}^{-jkz})}{\partial x} = \frac{\partial^2 \psi}{\partial x^2} \cdot \mathrm{e}^{-jkz}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 \psi}{\partial x^2} \cdot \mathrm{e}^{-jkz} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 \psi}{\partial y^2} \cdot \mathrm{e}^{-jkz}$$

Calculating the derivatives for *z*:

$$\frac{\partial u}{\partial z} = \frac{\partial(\psi \cdot \mathrm{e}^{-jkz})}{\partial z} = \frac{\partial(\psi)}{\partial z} \cdot \mathrm{e}^{-jkz} + \psi \cdot \frac{\partial(\mathrm{e}^{-jkz})}{\partial z}$$

$$= \frac{\partial \psi}{\partial z} \cdot \mathrm{e}^{-jkz} + \psi \cdot \mathrm{e}^{-jkz} \cdot \frac{\partial(-jkz)}{\partial z}$$

$$= \frac{\partial \psi}{\partial z} \cdot \mathrm{e}^{-jkz} + \psi \cdot \mathrm{e}^{-jkz} \cdot (-jk)$$

$$= \frac{\partial \psi}{\partial z} \cdot \mathrm{e}^{-jkz} - jk\psi \cdot \mathrm{e}^{-jkz}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial z}\cdot \mathrm{e}^{-jkz} - jk\psi\cdot \mathrm{e}^{-jkz}\right)$$

$$= \frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial z}\cdot \mathrm{e}^{-jkz}\right) - \frac{\partial(jk\psi\cdot \mathrm{e}^{-jkz})}{\partial z}$$

$$= \frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial z}\right)\cdot \mathrm{e}^{-jkz} + \left(\frac{\partial \psi}{\partial z}\right)\cdot\frac{\partial(\mathrm{e}^{-jkz})}{\partial z} - jk\left[\frac{\partial(\psi)}{\partial z}\cdot \mathrm{e}^{-jkz} + \psi\cdot\frac{\partial(\mathrm{e}^{-jkz})}{\partial z}\right]$$

$$= \frac{\partial^2 \psi}{\partial z^2}\cdot \mathrm{e}^{-jkz} - jk\cdot\frac{\partial \psi}{\partial z}\cdot \mathrm{e}^{-jkz} + \left(\frac{\partial \psi}{\partial z} - jk\psi\right)\cdot\frac{\partial(\mathrm{e}^{-jkz})}{\partial z}$$

$$= \frac{\partial^2 \psi}{\partial z^2} \cdot e^{-jkz} - jk \cdot \frac{\partial \psi}{\partial z} \cdot e^{-jkz} + \left(\frac{\partial \psi}{\partial z} - jk\psi \right) \cdot \left[e^{-jkz} \cdot \frac{\partial(-jkz)}{\partial z} \right]$$

$$= \frac{\partial^2 \psi}{\partial z^2} \cdot e^{-jkz} - jk \cdot \frac{\partial \psi}{\partial z} \cdot e^{-jkz} + \left(\frac{\partial \psi}{\partial z} - jk\psi \right) \cdot \left[e^{-jkz} \cdot (-jk) \right]$$

$$= \frac{\partial^2 \psi}{\partial z^2} \cdot e^{-jkz} - jk \cdot \frac{\partial \psi}{\partial z} \cdot e^{-jkz} + \left(\frac{\partial \psi}{\partial z} - jk\psi \right) \cdot \left(-jk \cdot e^{-jkz} \right)$$

$$= \frac{\partial^2 \psi}{\partial z^2} \cdot e^{-jkz} - jk \cdot \frac{\partial \psi}{\partial z} \cdot e^{-jkz} - jk \cdot \frac{\partial \psi}{\partial z} \cdot e^{-jkz} + j^2 k^2 \psi \cdot e^{-jkz}$$

$$= \frac{\partial^2 \psi}{\partial z^2} \cdot e^{-jkz} - 2jk \cdot \frac{\partial \psi}{\partial z} \cdot e^{-jkz} + j^2 k^2 \psi \cdot e^{-jkz}$$

$$= \frac{\partial^2 \psi}{\partial z^2} \cdot e^{-jkz} - 2jk \cdot \frac{\partial \psi}{\partial z} \cdot e^{-jkz} - k^2 \psi \cdot e^{-jkz}$$

since ψ varies so slowly with z, then: $\frac{\partial^2 \psi}{\partial z^2} \approx 0$

$$\frac{\partial^2 u}{\partial x^2} = -2jk \cdot \frac{\partial \psi}{\partial z} \cdot e^{-jkz} - k^2 \psi \cdot e^{-jkz}$$

Substituting the second derivatives:

$$\frac{\partial^2\psi}{\partial x^2}\cdot e^{-jkz}+\frac{\partial^2\psi}{\partial y^2}\cdot e^{-jkz}-2jk\cdot\frac{\partial\psi}{\partial z}\cdot e^{-jkz}-k^2\psi\cdot e^{-jkz}+k^2\psi\cdot e^{-jkz}=0$$

$$\frac{\partial^2\psi}{\partial x^2}\cdot e^{-jkz}+\frac{\partial^2\psi}{\partial y^2}\cdot e^{-jkz}-2jk\cdot\frac{\partial\psi}{\partial z}\cdot e^{-jkz}=0$$

$$\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}-2jk\cdot\frac{\partial\psi}{\partial z}\right)\cdot e^{-jkz}=0$$

since: $e^{-jkz}\neq 0$ ⇨

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}-2jk\frac{\partial\psi}{\partial z}=0$$

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}-2jk\frac{\partial\psi}{\partial z}=0$$

Similar form to the time dependent Schroedinger equation!, which solution has the form:

$$\psi=\exp\left\{-j\left[P(z)+\frac{k}{2q(z)}r^2\right]\right\}\equiv e^{-j\left[\frac{kx^2}{2q(z)}+\frac{ky^2}{2q(z)}+P(z)\right]}$$

where: $r^2=x^2+y^2$

Solution Physical Meaning.

$P(z)$ Represents a *complex* phase shift associated with the laser beam propagation.

$q(z)$ Represents a *complex* beam parameter describing the Gaussian variation in beam intensity with distance r from the optical axis; as well as the phase front curvature, which is spherical near the axis.

Calculating the first derivative of ψ with respect to x:

$$\frac{\partial \psi}{\partial x} = \frac{\partial\left(\exp\left\{-j\left[\frac{kx^2}{2q(z)} + \frac{ky^2}{2q(z)} + P(z)\right]\right\}\right)}{\partial x}$$

$$= \mathrm{e}^{-j\left[\frac{kx^2}{2q(z)} + \frac{ky^2}{2q(z)} + P(z)\right]} \cdot \frac{\partial\left(-j\left[\frac{kx^2}{2q(z)} + \frac{ky^2}{2q(z)} + P(z)\right]\right)}{\partial x}$$

$$= \mathrm{e}^{-j\left[\frac{kx^2}{2q(z)} + \frac{ky^2}{2q(z)} + P(z)\right]} \cdot \left(\frac{-2jkx}{2q(z)}\right) = \frac{-jkx}{q(z)} \cdot \mathrm{e}^{-j\left[\frac{kx^2}{2q(z)} + \frac{ky^2}{2q(z)} + P(z)\right]}$$

Calculating the second derivative of ψ with respect to x:

$$\frac{\partial^2\psi}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right)=\frac{\partial\left(\frac{-jkx}{q(z)}\cdot\exp\left\{-j\left[\frac{kx^2}{2q(z)}+\frac{ky^2}{2q(z)}+P(z)\right]\right\}\right)}{\partial x}$$

$$=\frac{\partial\left(\frac{-jkx}{q(z)}\right)}{\partial x}\cdot e^{-j\left[\frac{kx^2}{2q(z)}+\frac{ky^2}{2q(z)}+P(z)\right]}+\frac{-jkx}{q(z)}\cdot\frac{\partial\left(\exp\left\{-j\left[\frac{kx^2}{2q(z)}+\frac{ky^2}{2q(z)}+P(z)\right]\right\}\right)}{\partial x}$$

$$= \frac{-jk}{q(z)} \cdot e^{-j\left[\frac{kx^2}{2q(z)} + \frac{ky^2}{2q(z)} + P(z)\right]}$$

$$+ \frac{-jkx}{q(z)} \cdot e^{-j\left[\frac{kx^2}{2q(z)} + \frac{ky^2}{2q(z)} + P(z)\right]} \cdot \frac{\partial\left(-j\left[\frac{kx^2}{2q(z)} + \frac{ky^2}{2q(z)} + P(z)\right]\right)}{\partial x}$$

$$= \frac{-jk}{q(z)} \cdot e^{-j\left[\frac{kx^2}{2q(z)} + \frac{ky^2}{2q(z)} + P(z)\right]} + \frac{-jkx}{q(z)} \cdot e^{-j\left[\frac{kx^2}{2q(z)} + \frac{ky^2}{2q(z)} + P(z)\right]} \cdot \frac{-2jkx}{2q(z)}$$

Simplifying:

$$= \frac{-jk}{q(z)} \cdot e^{-j\left[\frac{kx^2}{2q(z)} + \frac{ky^2}{2q(z)} + P(z)\right]} + \frac{-jkx}{q(z)} \cdot e^{-j\left[\frac{kx^2}{2q(z)} + \frac{ky^2}{2q(z)} + P(z)\right]} \cdot \frac{-jkx}{q(z)}$$

$$= \frac{-jk}{q(z)} \cdot e^{-j\left[\frac{kx^2}{2q(z)} + \frac{ky^2}{2q(z)} + P(z)\right]} + \frac{j^2 k^2 x^2}{q^2(z)} \cdot e^{-j\left[\frac{kx^2}{2q(z)} + \frac{ky^2}{2q(z)} + P(z)\right]}$$

$$= \frac{-jk}{q(z)} \cdot e^{-j\left[\frac{kx^2}{2q(z)} + \frac{ky^2}{2q(z)} + P(z)\right]} - \frac{k^2 x^2}{q^2(z)} \cdot e^{-j\left[\frac{kx^2}{2q(z)} + \frac{ky^2}{2q(z)} + P(z)\right]}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \left[\frac{-jk}{q(z)} - \frac{k^2 x^2}{q^2(z)} \right] \cdot \mathrm{e}^{-j\left[\frac{kx^2}{2q(z)} + \frac{ky^2}{2q(z)} + P(z) \right]}$$

Thus the second derivatives for x and y are:

$$\frac{\partial^2 \psi}{\partial x^2} = \left[\frac{-jk}{q(z)} - \frac{k^2 x^2}{q^2(z)} \right] \cdot \mathrm{e}^{-j\left[\frac{kr^2}{2q(z)} + P(z) \right]}$$

$$\frac{\partial^2 \psi}{\partial y^2} = \left[\frac{-jk}{q(z)} - \frac{k^2 y^2}{q^2(z)} \right] \cdot \mathrm{e}^{-j\left[\frac{kr^2}{2q(z)} + P(z) \right]}$$

Calculating the first derivative of ψ with respect to z:

$$\frac{\partial \psi}{\partial z} = \frac{\partial\left(\exp\left\{-j\left[\frac{kr^2}{2q(z)} + P(z)\right]\right\}\right)}{\partial z}$$

$$= e^{-j\left[\frac{kr^2}{2q(z)} + P(z)\right]} \cdot \frac{\partial\left(\frac{-jkr^2}{2q(z)} - jP(z)\right)}{\partial z}$$

$$= e^{-j\left[\frac{kr^2}{2q(z)} + P(z)\right]} \cdot \left[\frac{\partial}{\partial z}\left(\frac{-jkr^2}{2q(z)}\right) - \frac{\partial\left(jP(z)\right)}{\partial z}\right]$$

Simplifying:

$$= \mathrm{e}^{-j\left[\frac{kr^2}{2q(z)}+P(z)\right]} \cdot \left[-\frac{-jkr^2}{2q^2(z)} \cdot \frac{\partial\left(q(z)\right)}{\partial z} - j \cdot \frac{\partial\left(P(z)\right)}{\partial z} \right]$$

$$= \mathrm{e}^{-j\left[\frac{kr^2}{2q(z)}+P(z)\right]} \cdot \left[\frac{jkr^2}{2q^2(z)} \cdot \frac{\partial\left(q(z)\right)}{\partial z} - j \cdot \frac{\partial\left(P(z)\right)}{\partial z} \right]$$

$$= \left[\frac{jkr^2}{2q^2(z)} \cdot \frac{\partial\left(q(z)\right)}{\partial z} - j \cdot \frac{\partial\left(P(z)\right)}{\partial z} \right] \cdot \mathrm{e}^{-j\left[\frac{kr^2}{2q(z)}+P(z)\right]}$$

Thus the first derivative for *z* is:

$$\frac{\partial \psi}{\partial z} = \left[\frac{jkr^2}{2q^2(z)} \cdot q'(z) - j \cdot P'(z) \right] \cdot \mathrm{e}^{-j\left[\frac{kr^2}{2q(z)} + P(z)\right]}$$

where:

$$q'(z) \equiv \frac{\partial \left(q(z)\right)}{\partial z}$$

$$P'(z) \equiv \frac{\partial \left(P(z)\right)}{\partial z}$$

Substituting the involved derivatives of ψ:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}-2jk\frac{\partial\psi}{\partial z}=0$$

$$\left[\frac{-jk}{q(z)}-\frac{k^2x^2}{q^2(z)}\right]\cdot e^{-j\left[\frac{kr^2}{2q(z)}+P(z)\right]}+\left[\frac{-jk}{q(z)}-\frac{k^2y^2}{q^2(z)}\right]\cdot e^{-j\left[\frac{kr^2}{2q(z)}+P(z)\right]}$$

$$-2jk\cdot e^{-j\left[\frac{kr^2}{2q(z)}+P(z)\right]}\cdot\left[\frac{jkr^2}{2q^2(z)}\cdot q'(z)-j\cdot P'(z)\right]=0$$

Simplifying:

$$\left\{\left[\frac{-jk}{q(z)}-\frac{k^2x^2}{q^2(z)}\right]+\left[\frac{-jk}{q(z)}-\frac{k^2y^2}{q^2(z)}\right]-2jk\cdot\left[\frac{jkr^2}{2q^2(z)}\cdot q'(z)-j\cdot P'(z)\right]\right\}\cdot e^{-j\left[\frac{kr^2}{2q(z)}+P(z)\right]}=0$$

$$\left\{-\frac{jk}{q(z)}-\frac{k^2x^2}{q^2(z)}-\frac{jk}{q(z)}-\frac{k^2y^2}{q^2(z)}-\frac{2j^2k^2r^2}{2q^2(z)}\cdot q'(z)+2j^2k\cdot P'(z)\right\}\cdot e^{-j\left[\frac{kr^2}{2q(z)}+P(z)\right]}=0$$

But:

$$e^{-j\left[\frac{kr^2}{2q(z)}+P(z)\right]} \neq 0$$

$$-\frac{jk}{q(z)}-\frac{k^2x^2}{q^2(z)}-\frac{jk}{q(z)}-\frac{k^2y^2}{q^2(z)}-\frac{2j^2k^2r^2}{2q^2(z)}\cdot q'(z)+2j^2k\cdot P'(z)=0$$

$$-\frac{2jk}{q(z)}-\frac{k^2r^2}{q^2(z)}+\frac{k^2r^2}{q^2(z)}\cdot q'(z)-2k\cdot P'(z)=0$$

$$-\frac{2jk}{q(z)}+\frac{k^2r^2}{q^2(z)}\cdot\left[q'(z)-1\right]-2k\cdot P'(z)=0$$

For a polinomial to be zero, all its terms in equal powers in r need to be zero, thus:

$$\frac{-2jk}{q(z)} - 2k \cdot P'(z) = 0 \qquad \frac{k^2 r^2}{q^2(z)} \cdot \left[q'(z) - 1\right] = 0$$

$$-2k\left[\frac{j}{q(z)} + P'(z)\right] = 0$$

$$P'(z) = -\frac{j}{q(z)} \qquad q'(z) = 1$$

The solutions for $q'(z)$ and for $P'(z)$ are:

$$q(z) = \int q'(z) \cdot dz = q_i + z$$

$$P(z) = \int P'(z) \cdot dz = -j \cdot \ln\left(\frac{q_i + z}{q_i}\right)$$

$$q_o = q_i + z$$

q_i is the beam parameter at the initial (input) plane.

q_o is the beam parameter at the final (output) plane.

z is the distance separating the initial and final planes.

End of Session Two.

Thank you very much for your attendance.