First ICO-ICTP-TWAS Central American Workshop in Lasers, Laser Applications and Laser Safety Regulations

Laser Resonators

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Contents

The Fundamental Tranverse Mode or Gaussian Beam.

The *complex* beam parameter:

$$\frac{1}{q(z)} = \frac{1}{R(z)} - j\frac{\lambda}{\pi \cdot w^2(z)}$$

By inserting $q(z)$ in ψ:

$$\psi(x, y, z) = \exp\left\{-j\left[P(z) + \frac{k}{2 \cdot q(z)} r^2\right]\right\}$$

$$\Rightarrow \psi(x, y, z) = \exp\left\{-j\left[P(z) + \frac{kr^2}{2} \cdot \left(\frac{1}{R(z)} - j\frac{\lambda}{\pi \cdot w^2(z)}\right)\right]\right\}$$

$$\psi(x,y,z) = \exp\left\{ -j\cdot P(z) - j\frac{kr^2}{2\cdot R(z)} + j^2\frac{kr^2\lambda}{2\pi\cdot w^2(z)} \right\}$$

$$\psi(x,y,z) = \exp\left\{ -j\frac{kr^2}{2\cdot R(z)} - \frac{kr^2\lambda}{2\pi\cdot w^2(z)} - j\cdot P(z) \right\}$$

Since: $k = \dfrac{2\pi}{\lambda}$

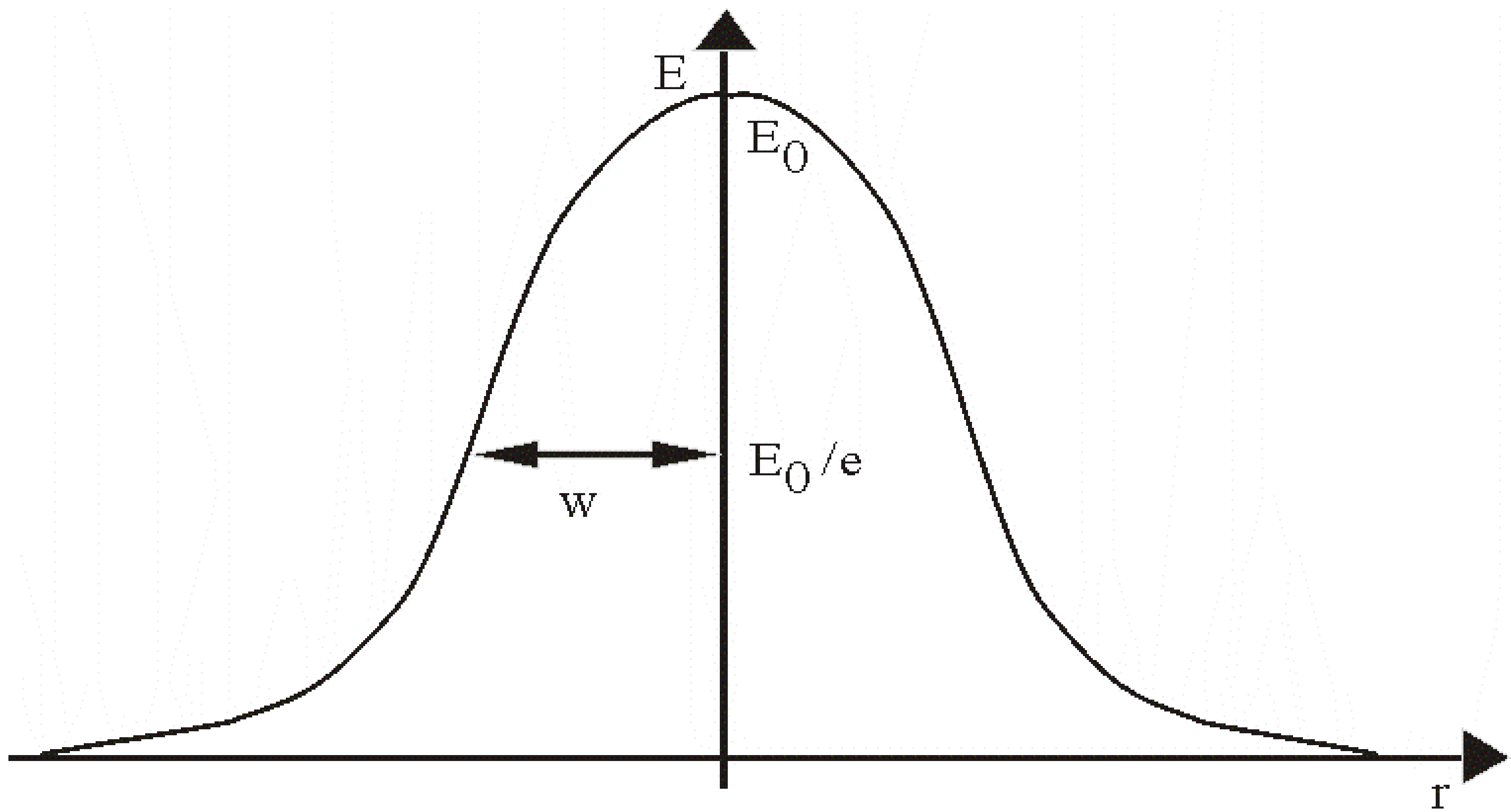
$$\boxed{\psi(x,y,z) = \exp\left\{ -j\frac{kr^2}{2\cdot R(z)} - \frac{r^2}{w^2(z)} - j\cdot P(z) \right\}}$$

$R(z)$ is the radius of curvature of the wavefront that intersects the axis at z.

$w(z)$ is beam radius or spot size, a measure of the Gaussian decrease of the field amplitude E with the distance from the axis; such that $w(z)$ is the distance at which the amplitude is $1/\mathrm{e}$ times that on the axis.

The intensity distribution is Gaussian in every beam cross section, and the width of the Gaussian intensity profile changes along the axis.

A coherent light beam with Gaussian intensity profile.



E. Rosas, Ph. D Thesis, CIO, *Universidad de Guanajuato*, (1998).

$2w(z)$ is beam diameter, which contracts to a minimum value $2w_0$ at the beam waist.

$$\frac{1}{q(z_0)} = \frac{1}{R(z_0)} - j\frac{\lambda}{\pi \cdot w^2(z_0)}$$

$$\frac{1}{R(z_0) \to \infty} \to 0 \quad \Rightarrow \quad \frac{1}{q(z_0)} \equiv \frac{1}{q_0} \to -j\frac{\lambda}{\pi \cdot w_0^2}$$

$$\Rightarrow \quad q_0 = j\frac{\pi \cdot w_0^2}{\lambda}$$

and:

$$q(z) = q_0 + z = j\frac{\pi\, w_0^2}{\lambda} + z$$

Since: $q(z) = q(z_0) + z = q_0 + z$

$$\frac{1}{q(z)} = \frac{1}{q_0 + z} = \frac{1}{q_0 + z} \cdot \frac{(1/q_0)}{(1/q_0)}$$

$$\frac{1}{q(z)} = \frac{1/q_0}{1 + z \cdot (1/q_0)}$$

Substituting: $1/q_0 = j\dfrac{\lambda}{\pi \cdot w_0^2}$

$$\frac{1}{q(z)} = \frac{j\lambda/\pi w_0^2}{1 + zj\lambda/\pi w_0^2}$$

$$= \frac{j\lambda/\pi w_0^2}{1+zj\lambda/\pi w_0^2} \cdot \frac{1-zj\lambda/\pi w_0^2}{1-zj\lambda/\pi w_0^2} = \frac{j\lambda/\pi w_0^2 - zj^2\lambda^2/\pi^2 w_0^4}{1+(z\lambda/\pi w_0^2)^2}$$

$$= \frac{j\lambda/\pi w_0^2 + (1/z)\cdot(\lambda z/\pi w_0^2)^2}{1+(z\lambda/\pi w_0^2)^2}$$

$$\Rightarrow \quad \frac{1}{q(z)} \equiv \frac{(1/z)\cdot(\lambda z/\pi w_0^2)^2}{1+(z\lambda/\pi w_0^2)^2} + \frac{j\lambda/\pi w_0^2}{1+(z\lambda/\pi w_0^2)^2}$$

Thus:

$$\frac{1}{R(z)} + j\cdot\frac{\lambda}{\pi\cdot w^2(z)} \equiv \frac{(1/z)\cdot(\lambda z/\pi w_0^2)^2}{1+(z\lambda/\pi w_0^2)^2} + j\cdot\frac{\lambda/\pi w_0^2}{1+(z\lambda/\pi w_0^2)^2}$$

Equating the real parts:

$$\frac{1}{R(z)} = \frac{(1/z)\cdot(\lambda z/\pi w_0^2)^2}{1+(\lambda z/\pi w_0^2)^2}$$

$$R(z) = \frac{1+(\lambda z/\pi w_0^2)^2}{(1/z)\cdot(\lambda z/\pi w_0^2)^2} = \frac{z\cdot\left[1+(\lambda z/\pi w_0^2)^2\right]}{(\lambda z/\pi w_0^2)^2}$$

$$= z\cdot\left[\frac{1+(\lambda z/\pi w_0^2)^2}{(\lambda z/\pi w_0^2)^2}\right] = z\cdot\left[\frac{1}{(\lambda z/\pi w_0^2)^2} + \frac{(\lambda z/\pi w_0^2)^2}{(\lambda z/\pi w_0^2)^2}\right]$$

$$\boxed{R(z) = z\cdot\left[\left(\frac{\pi w_0^2}{\lambda z}\right)^2 + 1\right]}$$

Equating the imaginary parts:

$$\frac{\lambda}{\pi \cdot w^2(z)} = \frac{\lambda / \pi w_0^2}{1 + (\lambda z / \pi w_0^2)^2}$$

$$w^2(z) = \frac{\lambda}{\pi} \cdot \frac{1 + (\lambda z / \pi w_0^2)^2}{\lambda / \pi w_0^2}$$

$$= \frac{\lambda}{\pi} \cdot \frac{1}{\lambda / \pi w_0^2} \cdot \left[1 + \left(\frac{\lambda z}{\pi w_0^2} \right)^2 \right]$$

$$\boxed{w^2(z) = w_0^2 \cdot \left[1 + \left(\frac{\lambda z}{\pi \, w_0^2} \right)^2 \right]}$$

Thus:

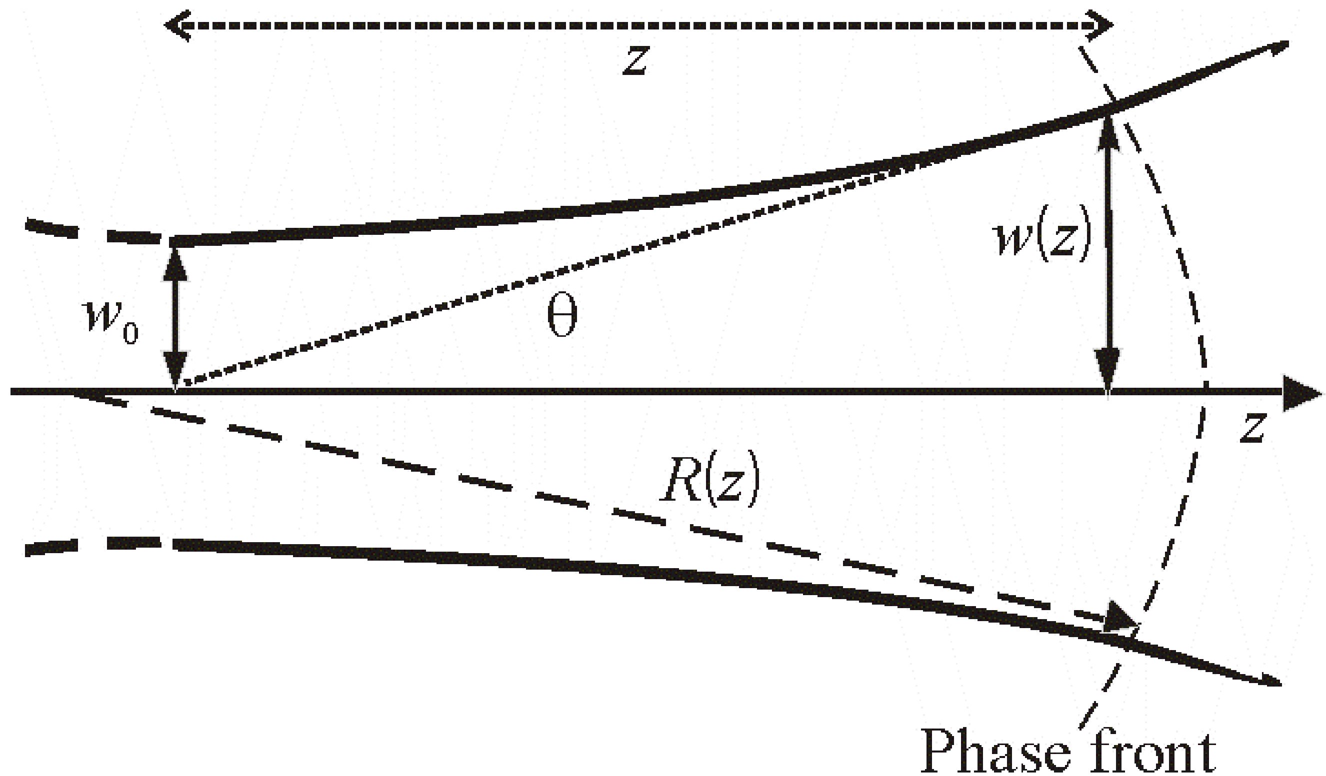
$$R(z) = z \cdot \left[1 + \left(\frac{\pi w_0^2}{\lambda z} \right)^2 \right] \quad \Rightarrow \quad R(z) = z + \frac{z_0^2}{z}$$

$$w^2(z) = w_0^2 \cdot \left[1 + \left(\frac{\lambda z}{\pi\, w_0^2} \right)^2 \right] \quad \Rightarrow \quad w(z) = w_0 \cdot \sqrt{1 + \left(\frac{z^2}{z_0^2} \right)}$$

Where z_0, known as the *Rayleight range*, is defined as:

$$z = \frac{\pi\, w_0^2}{\lambda}$$

And $2z_0$ is defined as the *confocal parameter*.



E. Rosas, Ph. D Thesis, CIO, *Universidad de Guanajuato*, (1998).

The beam contour of $w(z)$ is a hyperbola with asymptotes inclined to the axis at an angle θ, the far field diffraction angle for the fundamental transverse mode.

Dividing $w^2(z)$ by $R(z)$:

$$\frac{w^2(z)}{R(z)} = \frac{w_0^2 \cdot \left[1 + \left(\frac{\lambda z}{\pi\, w_0^2}\right)^2\right]}{z \cdot \left[1 + \left(\frac{\pi w_0^2}{\lambda z}\right)^2\right]} = \frac{w_0^2 \cdot \left[1 + \frac{\lambda^2 z^2}{\pi^2 w_0^4}\right]}{z \cdot \left[1 + \frac{\pi^2 w_0^4}{\lambda^2 z^2}\right]}$$

$$= \frac{w_0^2 \cdot \left[\frac{\pi^2 w_0^4 + \lambda^2 z^2}{\pi^2 w_0^4}\right]}{z \cdot \left[\frac{\lambda^2 z^2 + \pi^2 w_0^4}{\lambda^2 z^2}\right]} = \frac{\frac{w_0^2}{\pi^2 w_0^4}}{\frac{z}{\lambda^2 z^2}} \cdot \frac{(\pi^2 w_0^4 + \lambda^2 z^2)}{(\lambda^2 z^2 + \pi^2 w_0^4)}$$

$$= \frac{\dfrac{w_0^2}{\pi^2 w_0^4}}{\dfrac{z}{\lambda^2 z^2}} \cdot \frac{(\lambda^2 z^2 + \pi^2 w_0^4)}{(\lambda^2 z^2 + \pi^2 w_0^4)}$$

$$= \frac{w_0^2}{\pi^2 w_0^4} \cdot \frac{\lambda^2 z^2}{z} = \frac{1}{\pi^2 w_0^2} \cdot \frac{\lambda^2 z}{1} = \frac{\lambda^2 z}{\pi^2 w_0^2} \equiv \frac{w^2(z)}{R(z)}$$

$$\Rightarrow \quad \boxed{\frac{\lambda z}{\pi\, w_0^2} = \frac{\pi \cdot w^2(z)}{\lambda \cdot R(z)}}$$

Which allows to write expressions for w_0 and z in terms of $w(z)$ and $R(z)$ as:

$$R(z) = z \cdot \left[1 + \left(\frac{\pi w_0^2}{\lambda z} \right)^2 \right] = z \cdot \left[1 + \left(\frac{\lambda \cdot R(z)}{\pi \cdot w^2(z)} \right)^2 \right]$$

$$\Rightarrow \quad z = \frac{R(z)}{\left[1 + \left(\frac{\lambda \cdot R(z)}{\pi \cdot w^2(z)} \right)^2 \right]}$$

And:

$$w^2(z) = w_0^2 \cdot \left[1 + \left(\frac{\lambda z}{\pi\, w_0^2} \right)^2 \right] = w_0^2 \cdot \left[1 + \left(\frac{\pi \cdot w^2(z)}{\lambda \cdot R(z)} \right)^2 \right]$$

$$\Rightarrow \quad w_0^2 = \frac{w^2(z)}{\left[1 + \left(\frac{\pi \cdot w^2(z)}{\lambda \cdot R(z)} \right)^2 \right]}$$

In order to calculate the complex phase shift a distance z away from w_0:

Since:

$$P'(z) = -\frac{j}{q(z)}$$

and

$$q(z) = q_0 + z = j\frac{\pi\, w_0^2}{\lambda} + z$$

$$\Rightarrow \quad P'(z) = -\frac{j}{z + j(\pi w_0^2/\lambda)}$$

Integrating:

$$P(z) = \int P'(z)\cdot dz = -j\cdot\int \frac{1}{z + j(\pi w_0^2/\lambda)}\cdot dz$$

$$\Rightarrow \quad \frac{P(z)}{-j} \equiv j\cdot P(z) = \int \frac{1}{z + j(\pi w_0^2/\lambda)}\cdot dz$$

$$= \ln\left[1 - j\cdot\left(\frac{\lambda\, z}{\pi\, w_0^2}\right)\right]$$

$$\Rightarrow \quad \boxed{j\cdot P(z) = \ln\sqrt{1+\left(\frac{\lambda\, z}{\pi\, w_0^2}\right)^2} - j\cdot\arctan\left(\frac{\lambda\, z}{\pi\, w_0^2}\right)}$$

The real part of $P(z)$ represents a phase shift difference Φ between the Gaussian beam and an ideal plane wave.

$$\mathrm{Re}\left\{P(z)\right\} = \ln\sqrt{1+\left(\frac{\lambda z}{\pi w_0^2}\right)^2}$$

The imaginary part of $P(z)$ produces an amplitude factor $w_0/w(z)$ which gives the expected intensity decrease on the axis due to the expansion of the beam.

$$\mathrm{Im}\left\{P(z)\right\} = -\arctan\left(\frac{\lambda z}{\pi w_0^2}\right)$$

Then:

$$u(r,z) = \psi(x,y,z)\exp(-jkz)$$

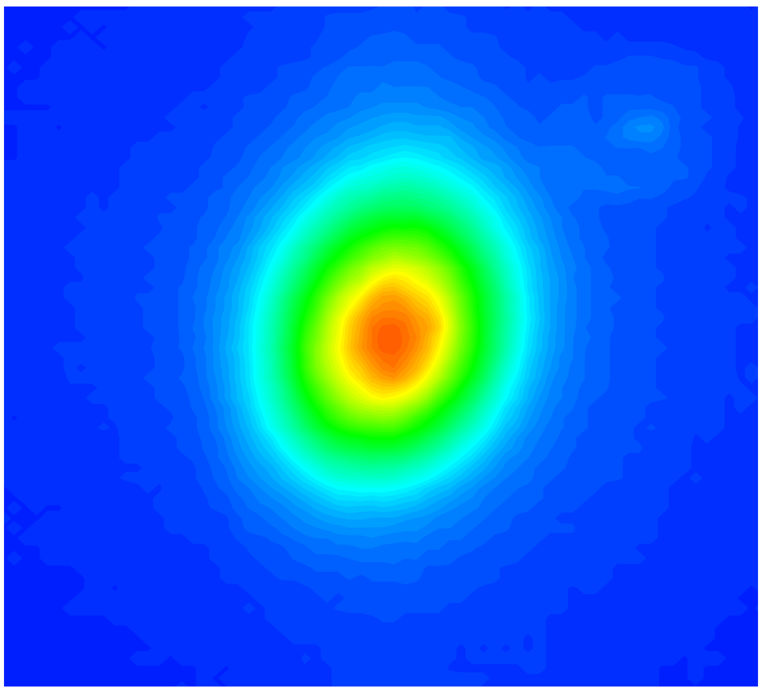
can be written for the fundamental (Gaussian) transverse mode as:

$$u(r,z) = \frac{w_0}{w(z)} \cdot \exp\left\{ -j(kz - \Phi) - r^2 \left[\frac{1}{w^2(z)} + \frac{jk}{2R(z)} \right] \right\}$$

where: $$\Phi = \arctan\left(\frac{\lambda z}{\pi w_0^2} \right)$$

Fundamental (Gaussian) transverse propagation mode.

CCD image
Nd:YVO4 laser.



E. Rosas *et al*, Rev. Mex. Fis. 47(3), (2001).

Photograph
He-Ne laser.

Wikipedia, (2012).

Higher Order Rectangular Modes.

There is a complete orthogonal set of similar soutions, known as the *transverse modes of propagation*:

In cartesian coordinates:

$$\psi = g(x,z)\cdot\left(\frac{x}{w}\right)\cdot h(y,z)\left(\frac{y}{w}\right)\cdot \exp\left\{-j\cdot\left[P+\frac{k}{2q}(x^2+y^2)\right]\right\}$$

Those proposed modes have intensity patterns that scale according to the width $2w(z)$ of a Gaussian beam.

Inserting these proposed solutions for $g(x, z)$ and $h(y, z)$ in the scalar wave equation:

$$\frac{d^2 H_m}{dx^2} - 2x\frac{dH_m}{dx} + 2mH_m = 0$$

and:

$$\frac{d^2 H_n}{dy^2} - 2y\frac{dH_n}{dy} + 2nH_n = 0$$

The differential equations for the Hermite polynomials $H_m(x)$ and $H_n(y)$ of orders m and n, respectively.:

The differential equations for the Hermite polynomials are satisfied by:

$$g(x,z)\cdot h(y,z) = H_m\left(\sqrt{2}\cdot\frac{x}{w}\right)\cdot H_n\left(\sqrt{2}\cdot\frac{y}{w}\right) = 0$$

where m and n are then known as the *transverse mode numbers*.

The same pattern scaling parameter $w(z)$ applies to modes of all orders.

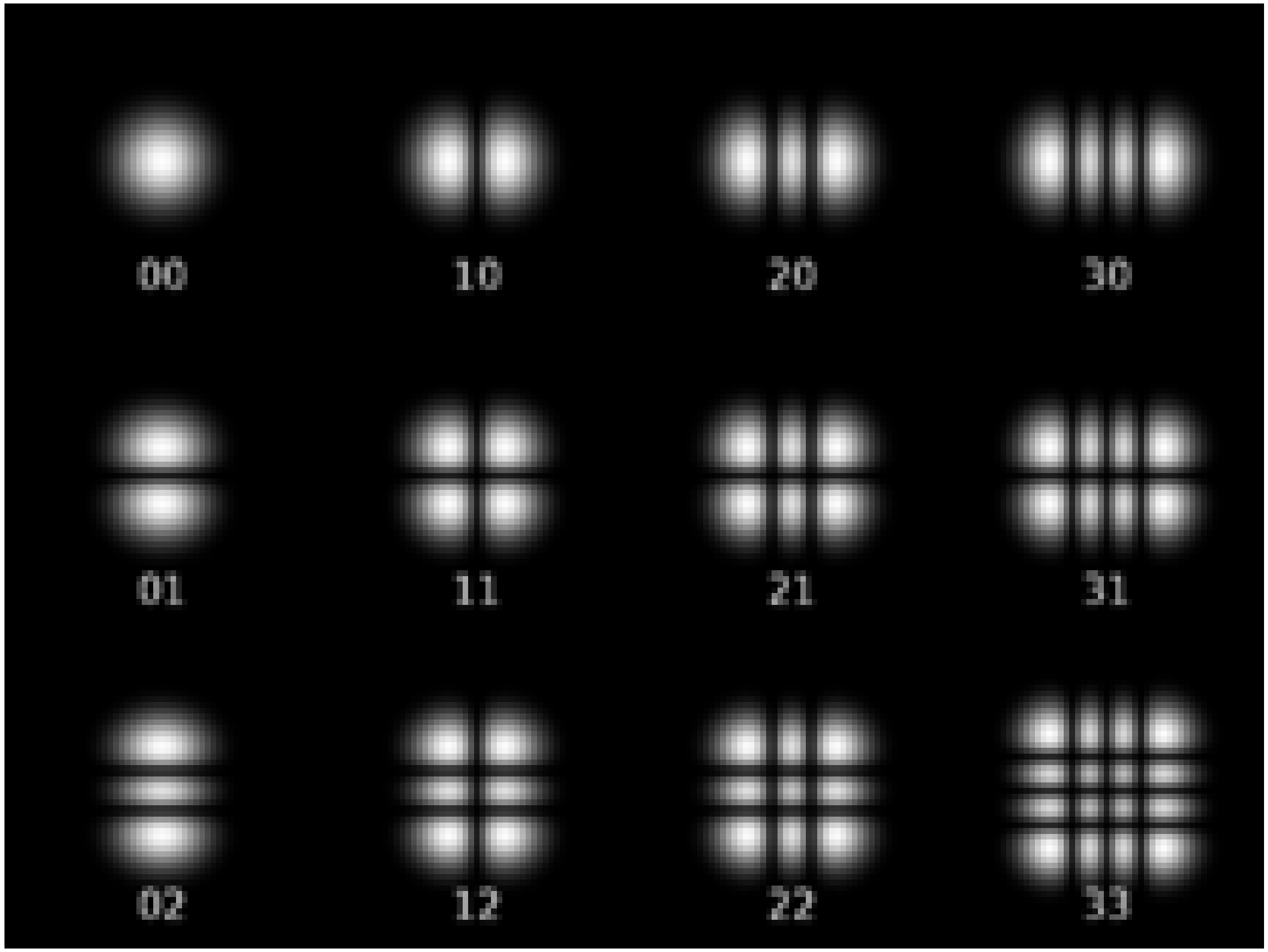
Thus:

$$u(r,z)=\frac{w_0}{w(z)}\cdot g(x,z)\cdot h(y,z)\cdot \mathrm{e}^{\left\{-j(kz-\Phi)-r^2\left[\frac{1}{w^2(z)}+\frac{jk}{2R(z)}\right]\right\}}$$

where: $$\Phi(m,n;z)=(m+n+1)\arctan\left(\frac{\lambda z}{\pi \mathrm{w}_0^2}\right)$$

Is the generalized form of the set of solutions known as the *higher order rectangular transverse modes of propagation*.

Higher order rectangular transverse propagation modes.



Wikipedia (2012).

The intensity pattern in a cross section of a higher order mode of propagation is described by the product of Hermite and Gaussian functions.

The number of zeros in a mode pattern is equal to the corresponding mode number.

The area occupied by a mode increases with the mode number.

The parameter $R(z)$ in the fundamental mode of propagation is the same for all the higher modes.

- The phase front curvature is the same and changes in the same way for modes of all orders.

The phase shift Φ however, is a function of the mode numbers.

- The phase velocity increases with incresing mode number; leading to differences in the resonant frequencies of the variousmodes of oscillation .

Higher Order Cylindrical Modes.

Another set of complete orthogonal soutions can be obtained in cylindrical coordinates (r, ϕ, z):

$$\psi = g \cdot \left(\frac{R}{w} \right) \cdot \exp \left\{ -j \cdot \left[P + \frac{k}{2q} r^2 + l\phi) \right] \right\}$$

where: $$g = \left(\sqrt{2} \cdot \frac{r}{w} \right)^l \cdot L_p^l \left(2 \frac{r^2}{w^2} \right)$$

$L_p^{\,l}$ is a generalized Laguerre polynomial, and p and l are the radial and angular mode numbers.

$L_p^{\,l}(x)$ obeys the differential equation:

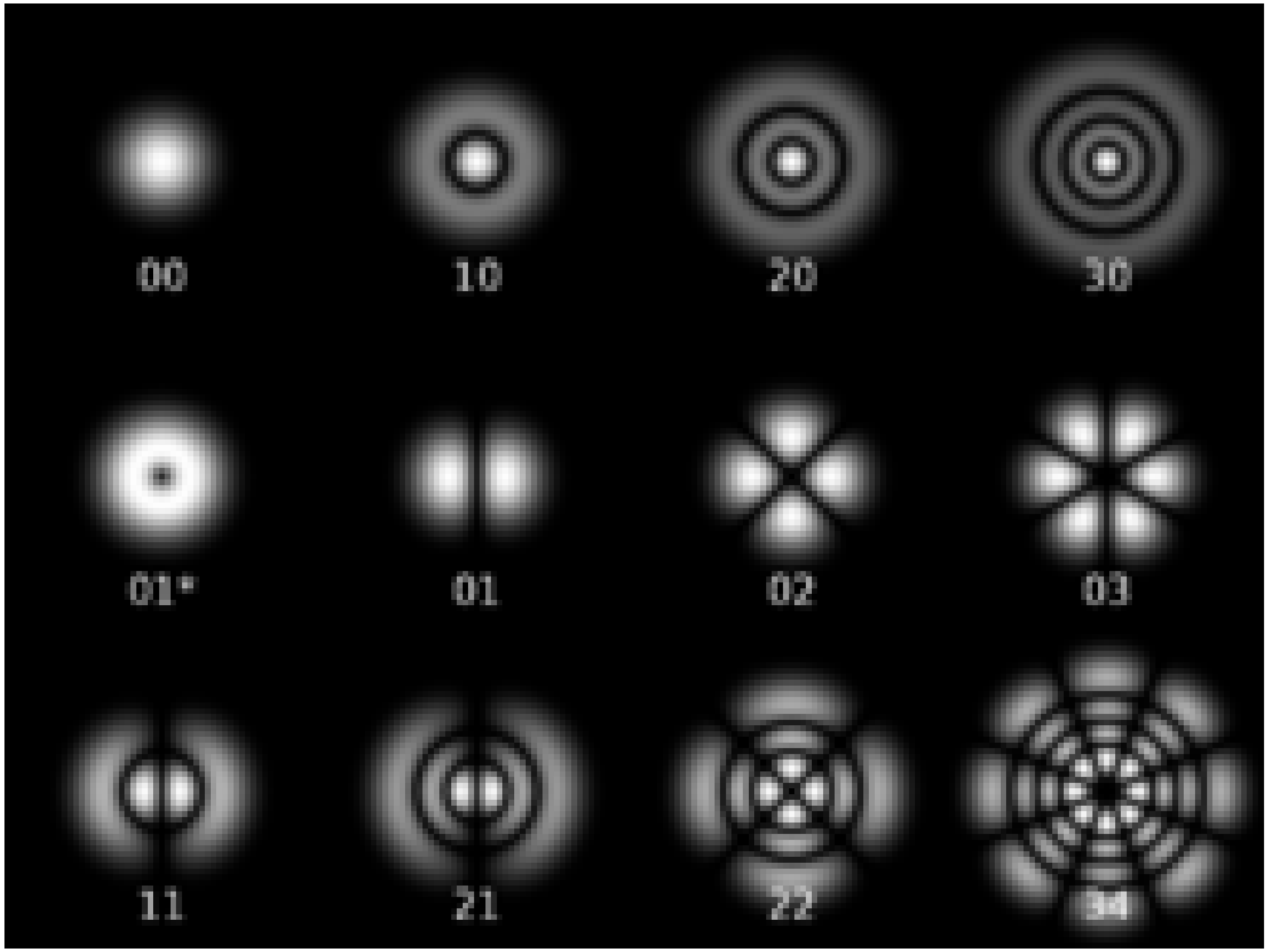
$$x\frac{d^2L_p^l}{dx^2}+(l+1-x)\frac{dL_p^l}{dx}+p\,L_p^l=0$$

which solutions are the Laguerre polynomials.

Again the beam parameters $R(z)$ and $w(z)$ are the same for all the cylindrical higher order modes, but the phase shift Φ is now given by:

$$\Phi(p,l;z)=(2p+l+1)\arctan\left(\frac{\lambda z}{\pi\,\mathrm{w}_0^2}\right)$$

Higher order cylindrical transverse propagation modes.



Wikipedia (2012).

End of Session Three.

Thank you very much for your attendance.