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Electromagnetism I
Review of Electrodynamics

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REVIEW OF ELECTRODYNAMICS

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Layout

- Electrostatic : Revisited
- Magneto- static : Revisited
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Nomenclature

- *E* = Electric field
- *D* = Electric displacement
- *B* = Magnetic flux density
- *H* = Auxiliary field
- ρ= Charge density
- *j* = Current density
- μ_0 (permeability of free space) = $4\pi\times10^{-7}$T-m/A
- ε_0 (permittivity of free space) = 8.854×10^{-12}N-m^2/ C^2
- c (speed of light) = 2.99792458×10^{8} m/s

Introduction

- **Electrostatics**

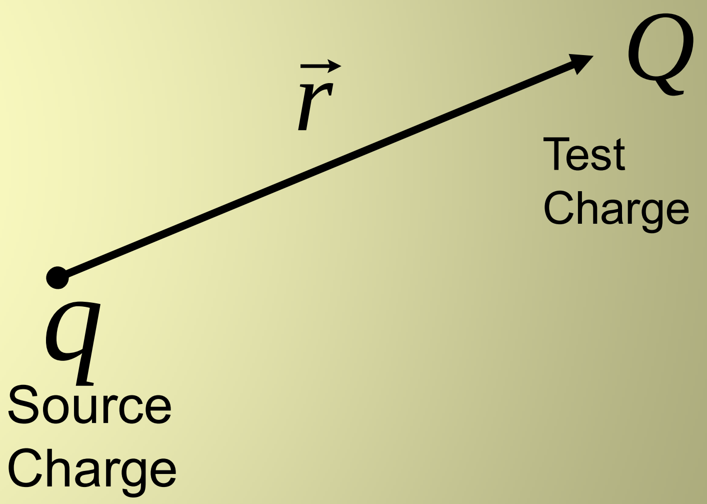
  - **Electrostatic field : Stationary charges produce electric fields that are constant in time. The theory of static charges is called electrostatics.**

Stationary charges ⟹ Constant Electric field;

Electrostatic :Revisited

Coulombs Law

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r}$$



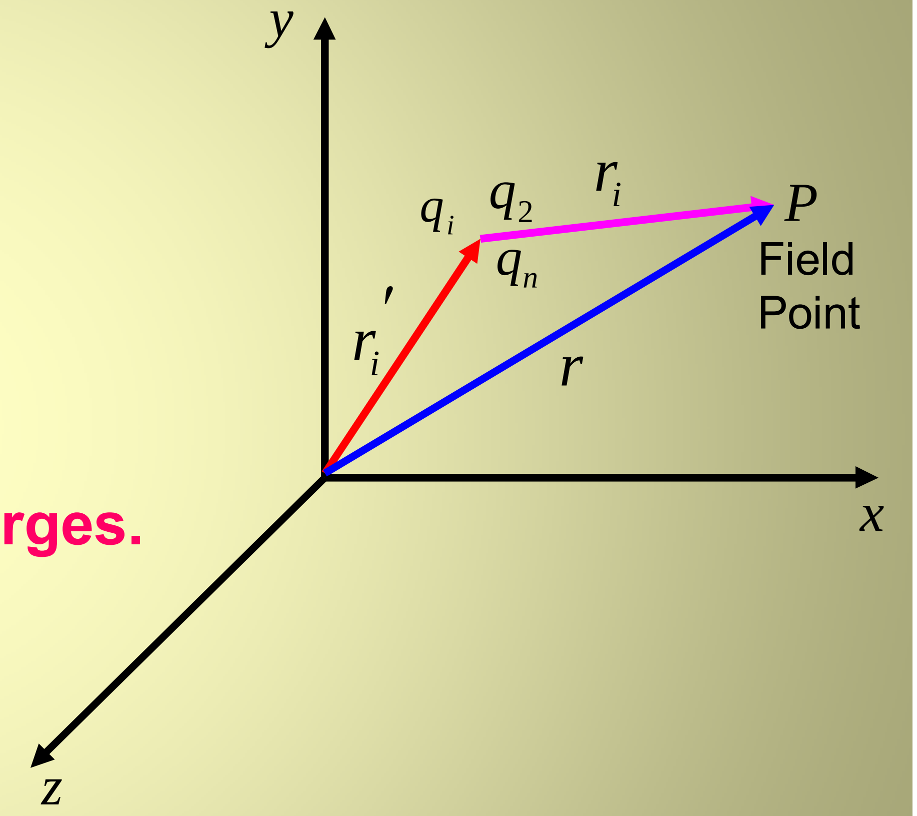
$$\varepsilon_0 = 8.85\times10^{-12}\,\frac{C^2}{N-m^2}$$

Permittivity of free space

The Electric Field

$$\vec{F} = Q\vec{E}$$

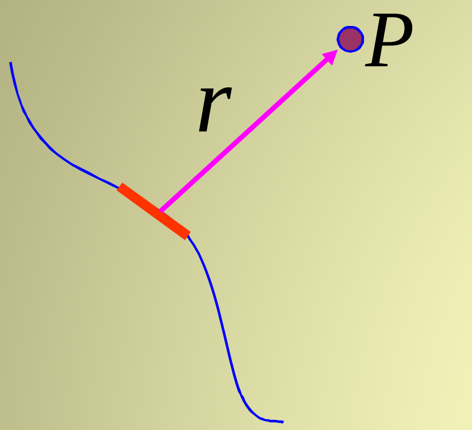
$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^{2}}\hat{r}_i$$



$\vec{E}$ - the electric field of the source charges.

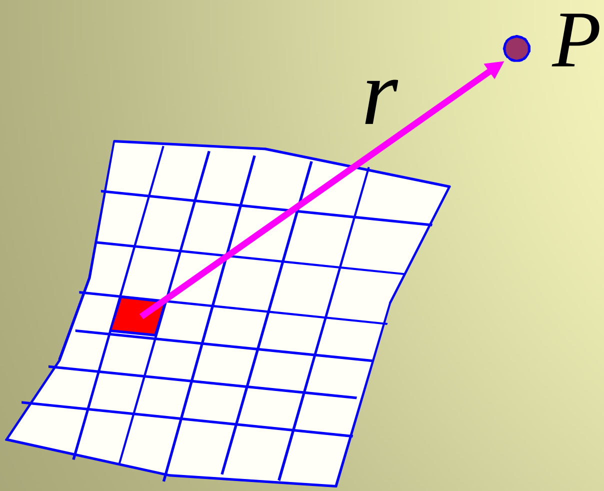
Physically E(P) Is force per unit charge exerted on a test charge placed at P.

The Electric Field: cont'd



$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0} \int_{Line} \frac{\hat{r}}{r^2} \lambda dl$$

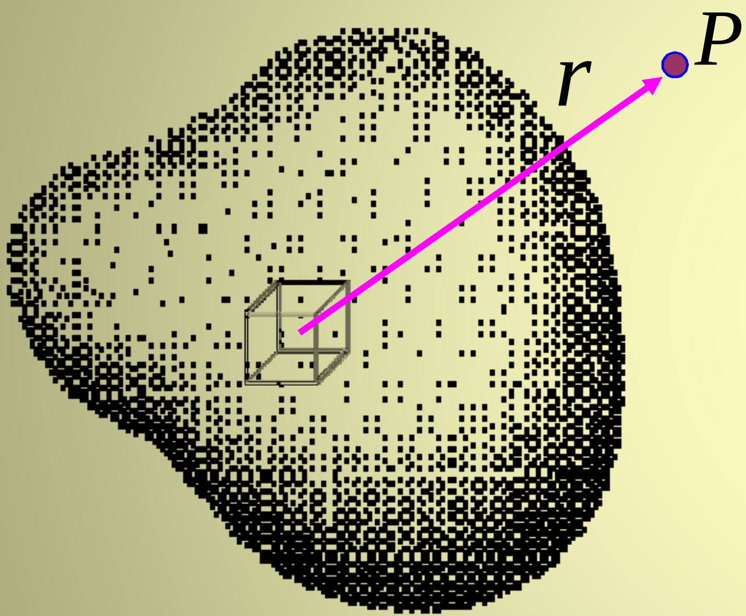
λ **is the line charge density**



$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0} \int_{Surface} \frac{\hat{r}}{r^2} \sigma da$$

σ **is the surface charge density**

The Electric Field: cont'd



$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0} \int_{Volume} \frac{\hat{r}}{r^2} \rho \, d\tau$$

ρ **is the volume charge density**

Electric Potential

The work done in moving a test charge Q in an electric field from point P_1 to P_2 with a constant speed.

$$W = Force \bullet dis\tan ce$$

$$W = -\int_{p_1}^{P_2} Q\vec{E} \bullet d\vec{l}$$

negative sign - work done is against the field.

For any distribution of fixed charges.

$$\oint \vec{E} \bullet d\vec{l} = 0$$

The electrostatic field is conservative

Electric Potential: cont'd

Stokes's Theorem gives

$$\vec{\nabla} \times \vec{E} = 0$$

$$\vec{E} = -\vec{\nabla} V$$

where *V* is Scalar Potential

The work done in moving a charge Q from infinity to a point P_2 where potential is *V*

$$W = QV$$

V = Work per unit charge

= Volts = joules/Coulomb

Electric Potential : cont'd

Field due to a single point charge q at origin

$$V = \int_r^{\infty} \frac{q\,dr}{4\pi\varepsilon_0 r^2} = \frac{q}{4\pi\varepsilon_0 r}$$

$$F \propto \frac{1}{r^2}$$

$$E \propto \frac{1}{r^2}$$

$$V \propto \frac{1}{r}$$

Gauss's Law

$$\oint \vec{E} \bullet d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

Differential form of Gauss's Law

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_0}$$

Poisson's Equation

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

Laplace's Equation

$$\nabla^2 V = 0$$

Electrostatic Fields in Matter

Matter: Solids, liquids, gases, metal, wood and glasses - behave differently in electric field.

Two Large Classes of Matter

(i) Conductors

(ii) Dielectric

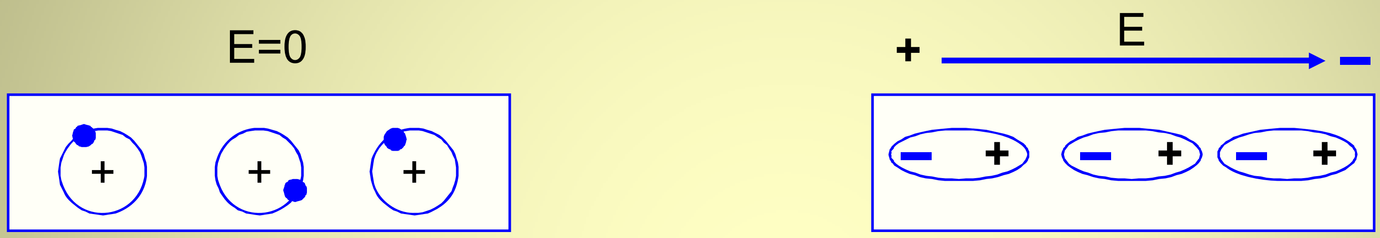
Conductors: Unlimited supply of free charges.

Dielectrics:

- Charges are attached to specific atoms or molecules- No free charges.
- Only possible motion - minute displacement of positive and negative charges in opposite direction.
- Large fields- pull the atom apart completely (ionizing it).

Polarization

A dielectric with charge displacements or induced dipole moment is said to be polarized.



Induced Dipole Moment $\mathrm{p} = \alpha E$

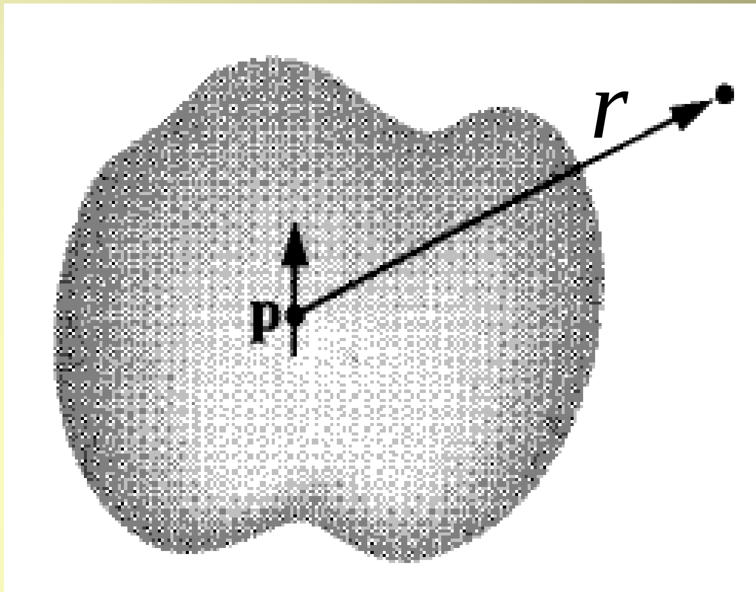
The constant of proportionality α is called the atomic polarizability

P ≡ dipole moment per unit volume

The Field of a Polarized Object

Potential of single dipole p is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{\hat{r}\bullet\vec{p}}{r^2}$$



$$V = \frac{1}{4\pi\varepsilon_0}\int_{volume}\frac{\vec{P}\bullet\hat{r}}{r^2}d\tau$$

$$V = \frac{1}{4\pi\varepsilon_0}\left[\int_{surface}\frac{1}{r}\vec{P}\bullet da - \int_{volume}\frac{1}{r}\left(\vec{\nabla}\bullet\vec{P}\right)d\tau\right]$$

Potential due to dipoles in the dielectric

The Field of a Polarized Object: cont'd

$$\sigma_b = \vec{P} \bullet \hat{n}$$

Bound charges at surface

$$\rho_b = -\vec{\nabla} \bullet \vec{P}$$

Bound charges in volume

$$V = \frac{1}{4\pi\varepsilon_0}\left[\int_{surface} \frac{1}{r}\sigma_b \, da + \int_{volume} \frac{1}{r}\rho_b \, d\tau \right]$$

The total field is field due to bound charges plus due to free charges

Gauss's law in Dielectric

- Effect of polarization is to produce accumulations of bound charges.
- The total charge density

$$\rho = \rho_f + \rho_b$$

$$\oint \vec{D} \bullet d\vec{a} = Q_{fenc}$$

From Gauss's law

Q_{fenc} -Free charges enclosed

$$\varepsilon_0 \vec{\nabla} \bullet \vec{E} = \rho = \rho_b + \rho_f$$

Displacement vector

$$\vec{\nabla} \bullet \vec{D} = \rho_f$$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

Magnetostatics : Revisited

- **Magnetostatics**
  - **Steady current produce magnetic fields that are constant in time. The theory of constant current is called magnetostatics.**

Steady currents → Constant Magnetic field;

Magnetic Forces

Lorentz Force

$$\vec{F} = q\left[\vec{E} + \left(\vec{v} \times \vec{B}\right)\right]$$

- The magnetic force on a segment of current carrying wire is

$$F_{mag} = \int \left(\vec{I} \times \vec{B}\right) dl$$

$$F_{mag} = \int I\left(d\vec{l} \times \vec{B}\right)$$

Equation of Continuity

The current crossing a surface s can be written as

$$I = \int_s \vec{J} \bullet d\vec{a} = \int_v \left(\vec{\nabla} \bullet \vec{J}\right) d\tau$$

$$\int_v \left(\vec{\nabla} \bullet \vec{J}\right) d\tau = -\frac{d}{dt}\int \rho d\tau = -\int \left(\frac{\partial \rho}{\partial t}\right) d\tau$$

Charge is conserved whatever flows out must come at the expense of that remaining inside - outward flow decreases the charge left in v

$$\vec{\nabla} \bullet \vec{J} = -\frac{\partial \rho}{\partial t}$$

This is called equation of continuity

Equation of Continuity 1

In Magnetostatic steady currents flow in the wire and its magnitude I must be the same along the line- otherwise charge would be pilling up some where and current can not be maintained indefinitely.

$$\frac{\partial \rho}{\partial t} = 0$$

In Magnetostatic and equation of continuity

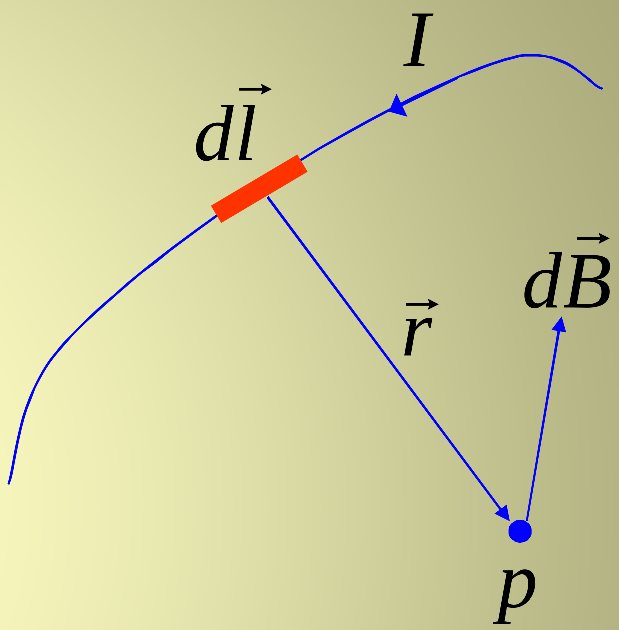
$$\vec{\nabla} \bullet \vec{J} = 0$$

Steady Currents: The flow of charges that has been going on forever - never increasing - never decreasing.

Magnetostatic and Current Distributions

Biot and Savart Law

$$\vec{B}(p) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\vec{r}}{|\vec{r}|^3}\,dl$$



dl is an element of length.

$\vec{r}$ vector from source to point p.

μ_0 Permeability of free space.

Unit of B = N/Am = Tesla (T)

Biot and Savart Law for Surface and Volume Currents

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{K} \times \vec{r}}{|\vec{r}|^3} da$$

For Surface Currents

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \vec{r}}{|\vec{r}|^3} d\tau$$

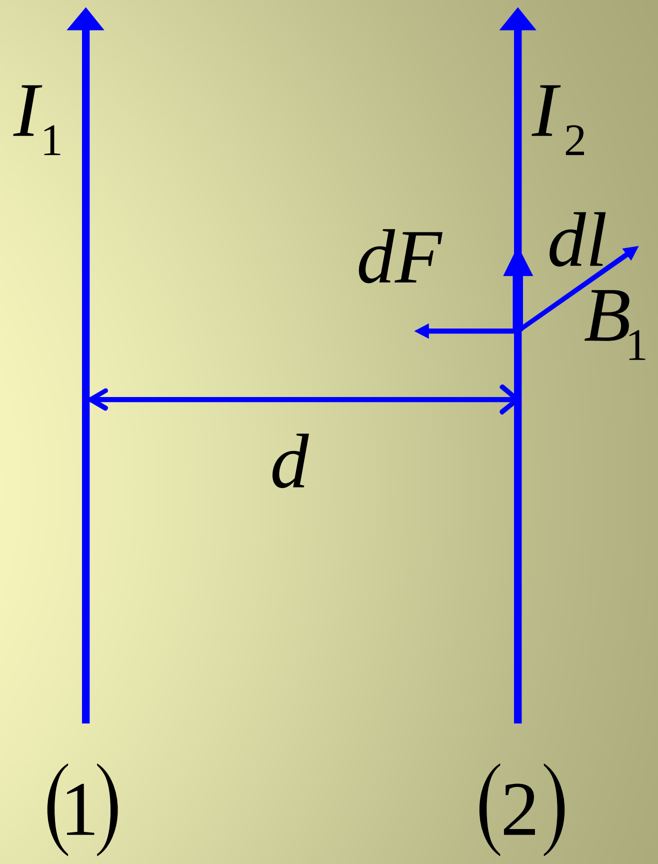
For Volume Currents

Force between two parallel wires

The magnetic field at (2) due to current I_1 is

$$B_1 = \frac{\mu_0 I_1}{2\pi d}$$

Points inside



Magnetic force law

$$dF = \int I_2 \left(d\vec{l}_2 \times \vec{B}_1 \right)$$

$$dF = \int I_2 \left(d\vec{l}_2 \times \frac{\mu_0 I_1}{2\pi d} \hat{k} \right)$$

Force between two parallel wires

$$dF = \frac{\mu_0 I_1 I_2}{2\pi d} dl_2$$

The total force is infinite but force per unit length is

$$\frac{dF}{dl_2} = \frac{\mu_0 I_1 I_2}{2\pi d}$$

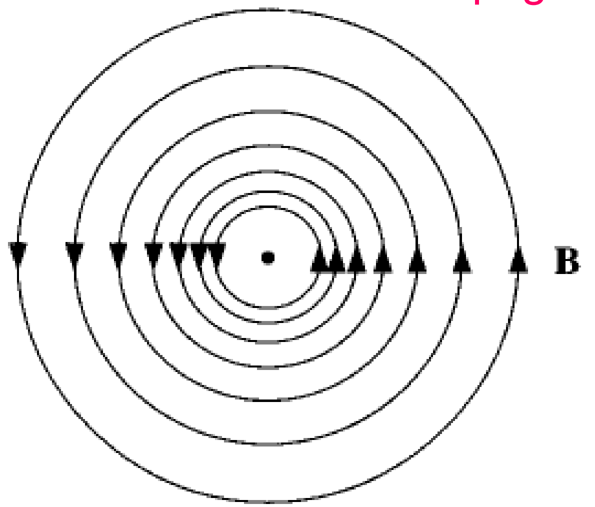
If currents are anti-parallel the force is repulsive.

Straight line currents

The integral of B around a circular path of radius s, centered at the wire is

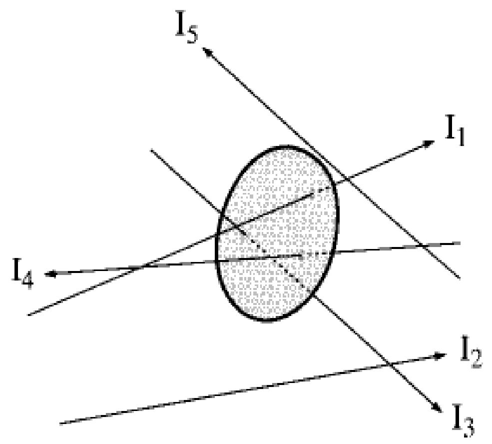
$$\oint \vec{B} \bullet d\vec{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \mu_0 I$$

The current is out of the page



For bundle of straight wires. Wire that passes through loop contributes only.

$$\oint \vec{B} \bullet d\vec{l} = \mu_0 I_{enc}$$



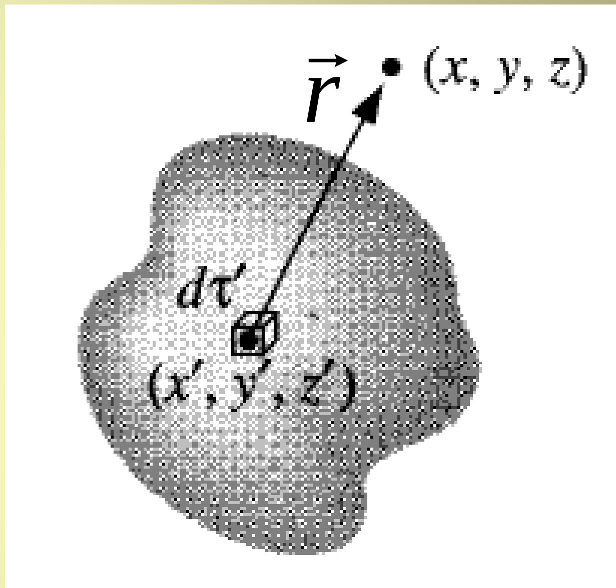
Applying Stokes' theorem

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

Divergence and Curl of B

Biot-Savart law for the general case of a volume current reads

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(r')\times\vec{r}}{r^3}\,d\tau'$$



B is a function of (x, y, z),

J is a function of (x', y', z'),

$$\vec{r} = (x - x')\,\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} + (z - z')\,\hat{\mathbf{z}},$$

$$d\tau' = dx'\,dy'\,dz'.$$

$\vec{\nabla}\bullet\vec{B} = 0$ **and** $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

Ampere's Law

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

Ampere's law

Integral form of Ampere's law

Using Stokes' theorem

$$\int \left(\vec{\nabla} \times \vec{B}\right) \bullet d\vec{a} = \oint \vec{B} \bullet d\vec{l} = \mu_0 \int \vec{J} \bullet d\vec{a}$$

$$\oint \vec{B} \bullet d\vec{l} = \mu_0 I_{enc}$$

Vector Potential

The basic differential law of Magnetostatics

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

B is curl of some vector field called vector potential $A(P)$

$$\vec{B}(P) = \vec{\nabla} \times \vec{A}(P)$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \mu_0 J$$

Coulomb's gauge

$$\vec{\nabla} \bullet \vec{A} = 0$$

$$\nabla^2 A = -\mu_0 J$$

Magnetostatic Field in Matter

- Magnetic fields- due to electrical charges in motion.
- Examine a magnet on atomic scale we would find tiny currents.
- Two reasons for atomic currents.
  - Electrons orbiting around nuclei.
  - Electrons spinning on their axes.
- Current loops form magnetic dipoles - they cancel each other due to random orientation of the atoms.
- Under an applied Magnetic field- a net alignment of - magnetic dipole occurs- and medium becomes magnetically polarized or magnetized

Magnetization

If m is the average magnetic dipole moment per unit atom and N is the number of atoms per unit volume, the magnetization is define as

$$\overrightarrow{M} = N\overrightarrow{m}$$

$$\overrightarrow{m} = I\overrightarrow{a} = Am^2$$

or

$$m = Md\tau$$

$$M = \frac{Am^2}{m^3} = \frac{A}{m}$$

Magnetic Materials

Paramagnetic Materials

The materials having magnetization parallel to B are called paramagnets.

Diamagnetic Materials

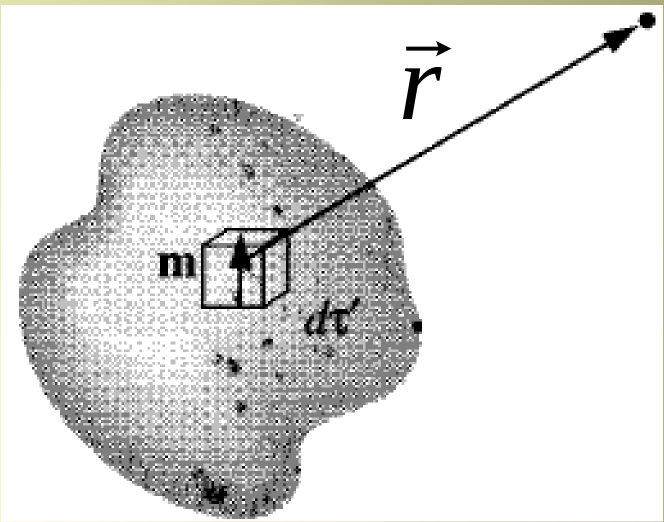
The elementary moment are not permanent but are induced according to Faraday's law of induction. In these materials magnetization is opposite to B.

Ferromagnetic Materials

Have large magnetization due to electron spin. Elementary moments are aligned in form of groups called domain

The Field of Magnetized Object

Using the vector potential of current loop



$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{M}\times\hat{n}}{r}da + \frac{\mu_0}{4\pi}\int\frac{\vec{\nabla}\times\vec{M}}{r}d\tau$$

$$\vec{K_b} = \vec{M}\times\hat{n}$$ Bound Surface Current

$$\vec{J_b} = \vec{\nabla}\times\vec{M}$$ Bound Volume Current

Ampere's Law in Magnetized Material

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{J} = \vec{J}_b + \vec{J}_f$$

$$\frac{1}{\mu_0}\left(\vec{\nabla} \times \vec{B}\right) = \vec{J}_b + \vec{J}_f = \vec{J}_f + \left(\vec{\nabla} \times \vec{M}\right)$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f$$

where

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

Integral form

$$\oint \vec{H}.d\vec{l} = I_{fenc}$$

Faraday's Law of Induction

- Faraday's Law - a changing -magnetic flux through circuit induces an electromotive force around the circuit.

$$\varepsilon = \oint \vec{E}.\vec{dl} = -\frac{d\phi}{dt} = -\frac{d}{dt}\int \vec{B}.\vec{da}$$

ϵ – Induced emf

E – Induced electric field intensity

Differential form of Faraday's law

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Faraday's Law of Induction

Induced Electric field intensity in terms of vector potential

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V$$

For steady currents

$$\vec{E} = -\vec{\nabla} V$$

V – Scalar potential

Induced emf in a system moving in a changing magnetic field

$$\varepsilon = \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \left(\vec{v} \times \vec{B} \right)$$

Maxwell's Equations

Introduction to Maxwell's Equation

- In electrodynamics Maxwell's equations are a set of four equations, that describes the behavior of both the electric and magnetic fields as well as their interaction with matter
- Maxwell's four equations express
  - How electric charges produce electric field (Gauss's law)
  - The absence of magnetic monopoles
  - How currents and changing electric fields produces magnetic fields (Ampere's law)
  - How changing magnetic fields produces electric fields (Faraday's law of induction)

Historical Background

- **1864 Maxwell in his paper "A Dynamical Theory of the Electromagnetic Field" collected all four equations**

- **1884 Oliver Heaviside and Willard Gibbs gave the modern mathematical formulation using vector calculus.**

- **The change to vector notation produced a symmetric mathematical representation, that reinforced the perception of physical symmetries between the various fields.**

Electrodynamics Before Maxwell

Gauss's Law

$$(i)\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_o}$$

No name

$$(ii)\vec{\nabla} \bullet \vec{B} = 0$$

Faraday's Law

$$(iii)\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Ampere's Law

$$(iv)\vec{\nabla} \times \vec{B} = \mu_o \vec{J}$$

$$\vec{E} = -\vec{\nabla} W - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Electrodynamics Before Maxwell (Cont'd)

Apply divergence to (iii)

$$\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{E}\right) = \vec{\nabla} \bullet \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\vec{\nabla} \bullet \vec{B}\right)$$

The left hand side is zero, because divergence of a curl is zero.

The right hand side is zero because $\vec{\nabla} \bullet \vec{B} = 0$.

Apply divergence to (iv)

$$\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{B}\right) = \mu_o \left(\vec{\nabla} \bullet \vec{J}\right)$$

Electrodynamics Before Maxwell (Cont'd)

- The left hand side is zero, because divergence of a curl is zero.
- The right hand side is zero for steady currents i.e.,

$$\vec{\nabla} \bullet \vec{J} = 0$$

- In electrodynamics from conservation of charge

$$\vec{\nabla} \bullet \vec{J} = -\frac{\partial \rho}{\partial t}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = 0$$

ρ is constant at any point in space which is wrong.

Maxwell's Correction to Ampere's Law

Consider Gauss's Law

$$\vec{\nabla} \bullet \varepsilon_o \vec{E} = \rho$$

$$\frac{\partial}{\partial t}\left(\vec{\nabla} \bullet \varepsilon_o \vec{E}\right) = \frac{\partial \rho}{\partial t}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = \vec{\nabla} \bullet \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

$$\frac{\partial \vec{D}}{\partial t} = \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

Displacement current

This result along with Ampere's law and the conservation of charge equation suggest that there are actually two sources of magnetic field.
The current density and displacement current.

Maxwell's Correction to Ampere's Law (Cont'd)

Amperes law with Maxwell's correction

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

General Form of Maxwell's Equations

Differential Form

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_o}$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

Integral Form

$$\oint_S \vec{E} \bullet d\vec{S} = \frac{1}{\varepsilon_o} \int_V \rho dV$$

$$\oint_S \vec{B} \bullet d\vec{S} = 0$$

$$\oint_C \vec{E} \bullet d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \bullet d\vec{S}$$

$$\oint_C \vec{B} \bullet d\vec{l} = \mu_o I_{enc} + \mu_o \varepsilon_o \frac{d}{dt} \oint_S \vec{E} \bullet d\vec{S}$$

Maxwell's Equations in vacuum

- The vacuum is a linear, homogeneous, isotropic and dispersion less medium
- Since there is no current or electric charge is present in the vacuum, hence Maxwell's equations reads as
- These equations have a simple solution interms of traveling sinusoidal waves, with the electric and magnetic fields direction orthogonal to each other and the direction of travel

$$\vec{\nabla} \bullet \vec{E} = 0$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

Maxwell's Equations Inside Matter

Maxwell's equations are modified for polarized and magnetized materials. For linear materials the polarization *P* and magnetization *M* is given by

$$\vec{P} = \varepsilon_o \chi_e \vec{E}$$

$$\vec{M} = \chi_m \vec{H}$$

And the *D* and *B* fields are related to *E* and *H* by

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P} = (1 + \chi_e)\varepsilon_o \vec{E} = \varepsilon \vec{E}$$

$$\vec{B} = \mu_o \left(\vec{H} + \vec{M}\right) = (1 + \chi_m)\mu_o \vec{H} = \mu \vec{H}$$

Where χ_e is the electric susceptibility of material,

χ_m is the magnetic susceptibility of material.

Maxwell's Equations Inside Matter (Cont'd)

- **For polarized materials we have bound charges in addition to free charges**

$$\sigma_b = \vec{P} \bullet \hat{n}$$

$$\rho_b = -\vec{\nabla} \bullet \vec{P}$$

- **For magnetized materials we have bound currents**

$$\overrightarrow{K_b} = \vec{M} \times \hat{n}$$

$$\overrightarrow{J_b} = \vec{\nabla} \times \vec{M}$$

Maxwell's Equations Inside Matter (Cont'd)

- In electrodynamics any change in the electric polarization involves a flow of bound charges resulting in polarization current J_P

$$J_p = \frac{\partial \vec{P}}{\partial t}$$

Polarization current density is due to linear motion of charge when the Electric polarization changes

Total charge density

$$\rho_t = \rho_f + \rho_b$$

Total current density

$$J_t = J_f + J_b + J_p$$

Maxwell's Equations Inside Matter (Cont'd)

- **Maxwell's equations inside matter are written as**

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho_t}{\varepsilon_o}$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J}_f + \mu_o \vec{J}_p + \mu_o \vec{J}_b + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \frac{\vec{B}}{\mu_o} = \vec{J}_f + \frac{\partial \vec{P}}{\partial t} + \vec{\nabla} \times \vec{M} + \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \left(\frac{\vec{B}}{\mu_o} - \vec{M} \right) = \vec{J}_f + \frac{\partial}{\partial t}\left(\varepsilon_o \vec{E} + \vec{P} \right)$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial}{\partial t} \vec{D}$$

Maxwell's Equations Inside Matter (Cont'd)

- **In non-dispersive, isotropic media ε and μ are time-independent scalars, and Maxwell's equations reduces to**

$$\vec{\nabla} \bullet \varepsilon \vec{E} = \rho$$

$$\vec{\nabla} \bullet \mu \vec{H} = 0$$

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

Maxwell's Equations Inside Matter (Cont'd)

- In uniform (homogeneous) medium ε and μ are independent of position, hence Maxwell's equations reads as

$$\vec{\nabla} \bullet \vec{D} = \rho_f \qquad \oint_S \vec{D} \bullet d\vec{S} = Q_{f\ enc}$$

$$\vec{\nabla} \bullet \vec{H} = 0 \qquad \oint_S \vec{H} \bullet d\vec{S} = 0$$

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \qquad \oint_C \vec{E} \bullet d\vec{l} = -\mu \frac{d}{dt} \int_S \vec{H} \bullet d\vec{S}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \varepsilon \frac{\partial \vec{E}}{\partial t} \qquad \oint_C \vec{H} \bullet d\vec{l} = I_{f\ enc} + \frac{d}{dt} \int_S \vec{D} \bullet d\vec{S}$$

Generally, ε and μ can be rank-2 tensor (3X3 matrices) describing birefringent anisotropic materials.

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

Maxwell's equations in differential form:

Gauss' Law:

$$\int_v \vec{\nabla}\cdot\vec{E}(\vec{r},t)\,d\tau' = \frac{1}{\varepsilon_0}\int_v \rho_{Tot}^{E}(\vec{r},t)\,d\tau' = \frac{1}{\varepsilon_0}\int_v\left(\rho_{free}^{E}(\vec{r},t)+\rho_{bound}^{E}(\vec{r},t)\right)d\tau'$$

$$= \oint_S \vec{E}(\vec{r},t)\cdot d\vec{a} = \frac{1}{\varepsilon_0}Q_{ToT}^{enclosed}(t) = \frac{1}{\varepsilon_0}\left(Q_{free}^{enclosed}(t)+Q_{bound}^{enclosed}(t)\right)$$

$$\oint_S \vec{D}(\vec{r},t)\,d\vec{a} = Q_{free}^{enclosed}(t)$$

$$\oint_S \vec{\mathrm{P}}(\vec{r},t)\cdot d\vec{a} \equiv -Q_{bound}^{enclosed}(t)$$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

Auxiliary Relation:

$$\vec{D}(\vec{r},t) = \varepsilon_o \vec{E}(\vec{r},t) + \vec{P}(\vec{r},t)$$

$$\rho_{Bound}(\vec{r},t) \equiv -\vec{\nabla} \cdot \vec{P}(\vec{r},t) \qquad \sigma_{Bound}(\vec{r},t) \equiv \vec{P}(\vec{r},t) \cdot \hat{n}\Big|_{intf}$$

No Magnetic Monopoles:

$$\int_v \vec{\nabla} \cdot \vec{B}(\vec{r},t)\, d\tau' = \oint_S \vec{B}(\vec{r},t) \cdot d\vec{a} = 0$$

Faraday's Law:

$$\int_S \vec{\nabla} \times \vec{E}(\vec{r},t) \cdot d\vec{a} = \oint_C \vec{E}(\vec{r},t) \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{B}(\vec{r},t)}{\partial t} \cdot d\vec{a} = -\frac{d}{dt}\left[\int_S \vec{B}(\vec{r},t) \cdot d\vec{a}\right]$$

$$emf \;\; \varepsilon(t) \equiv \oint_C \vec{E}(\vec{r},t) \cdot d\vec{\ell} = -\frac{d}{dt}\left[\int_S \vec{B}(\vec{r},t) \cdot d\vec{a}\right] = -\frac{d\Phi_M^{enclosed}(t)}{dt}$$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

Ampere's Law:

$$\int_S \vec{\nabla}\times\vec{B}(\vec{r},t)\cdot d\vec{a} = \oint_C \vec{B}(\vec{r},t)\cdot d\vec{\ell} = \mu_0 \int_S \left(\vec{J}_{ToT}(\vec{r},t)+\vec{J}_D(\vec{r},t)\right)\cdot d\vec{a}$$

$$= \oint_C \vec{B}(\vec{r},t)\cdot d\vec{\ell} = \mu_0\left(I_{ToT}^{encl}(t)+I_D^{encl}(t)\right) = \mu_0\left(I_{free}^{encl}(t)+I_{bound}^{m\,encl}(t)+I_{P_{bound}}^{encl}(t)+I_D^{encl}(t)\right)$$

Auxiliary Relation: $\vec{H}(\vec{r},t) = \frac{1}{\mu_0}\vec{B}(\vec{r},t) - \vec{M}(\vec{r},t)$

$$\vec{J}_{bound}^{m}(\vec{r},t) \equiv \vec{\nabla}\times\vec{M}(\vec{r},t)$$

$$\vec{K}_{bound}^{m}(\vec{r},t) \equiv \vec{M}(\vec{r},t)\times\hat{n}\Big|_{intf}$$

$$\vec{J}_{P_{bound}}(\vec{r},t) \equiv \frac{\partial\vec{P}(\vec{r},t)}{\partial t}$$

$$\rho_m^{Bound}(\vec{r},t) \equiv -\vec{\nabla}\cdot\vec{M}(\vec{r},t)$$

$$\sigma_m^{Bound}(\vec{r},t) \equiv \vec{M}(\vec{r},t)\cdot\hat{n}\Big|_{intf}$$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

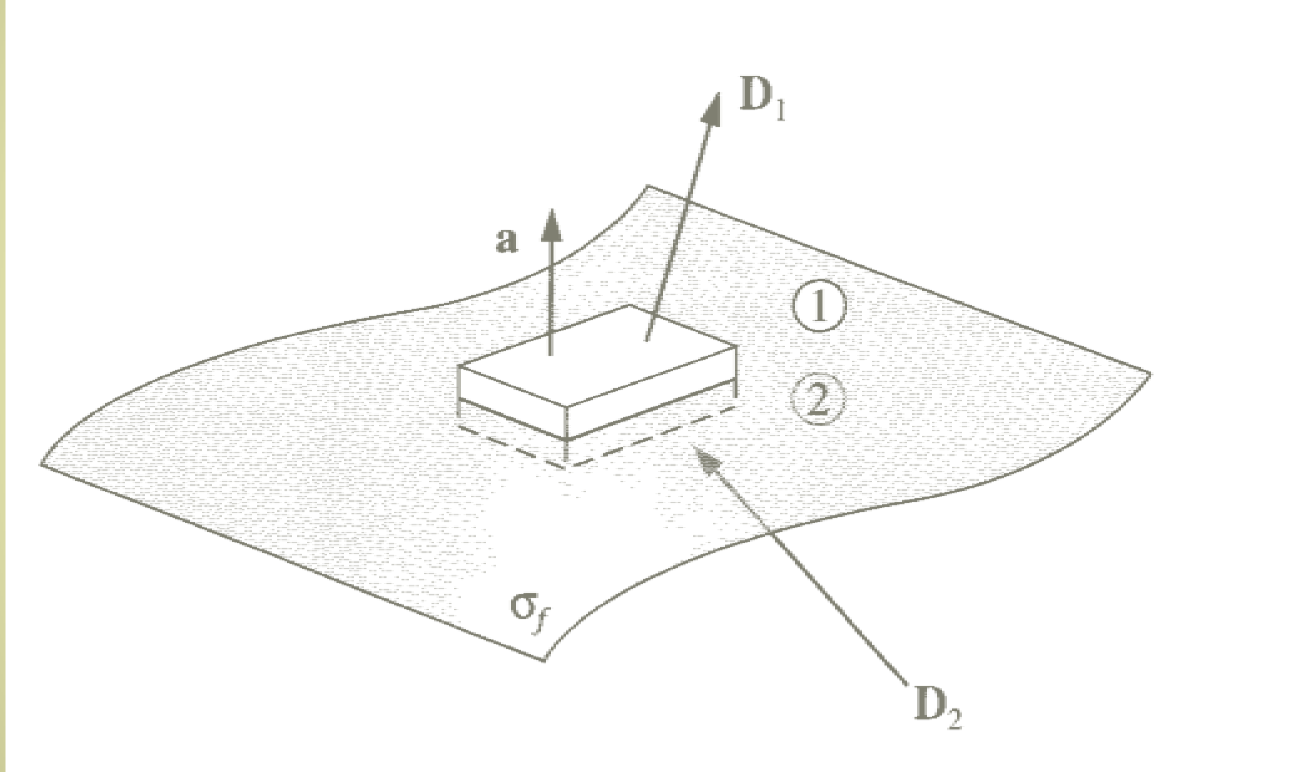
$$\int_S \vec{\nabla}\times\vec{H}(\vec{r},t)\bullet d\vec{a} = \oint_C \vec{H}(\vec{r},t)\bullet d\vec{\ell} = I_{free}^{enclosed}(t) + \int_S \frac{\partial\vec{D}(\vec{r},t)}{\partial t}\bullet d\vec{a} = I_{free}^{enclosed}(t) + \frac{d}{dt}\left[\int_S \vec{D}(\vec{r},t)\bullet d\vec{a}\right]$$

1) Apply the integral form of Gauss' Law at a dielectric interface/boundary using infinitesimally thin Gaussian pillbox extending slightly into dielectric material on either side of interface:

$$\oint_S \vec{E}\bullet d\vec{a} = \frac{1}{\varepsilon_0}Q_{ToT}^{enclosed} = \frac{1}{\varepsilon_0}Q_{free}^{enclosed} + \frac{1}{\varepsilon_0}Q_{bound}^{enclosed} = \frac{1}{\varepsilon_0}\oint_S \sigma_{free}\,da + \frac{1}{\varepsilon_0}\oint_S \sigma_{bound}\,da$$

Gives: $$\underset{above}{E_2^{\perp}} - \underset{below}{E_1^{\perp}} = \frac{1}{\varepsilon_0}\sigma_{ToT} = \frac{1}{\varepsilon_0}\left(\sigma_{free} + \sigma_{bound}\right)$$ (at interface)

Maxwell's Equations and Boundary Conditions at Interfaces in Matter



Maxwell's Equations and Boundary Conditions at Interfaces in Matter

The positive direction is from medium 2 (below) to medium 1 (above)

$$\oint_S \vec{D} \cdot d\vec{a} = Q_{free}^{enclosed} = \oint_S \sigma_{free} da \Rightarrow \boxed{D_{\underset{above}{2}}^{\perp} - D_{\underset{below}{1}}^{\perp} = \sigma_{free}}$$ (at interface)

Likewise: $$\oint_S \vec{\mathrm{P}} \cdot d\vec{a} = Q_{bound}^{enclosed} = -\oint_S \sigma_{bound} da \Rightarrow \boxed{P_{\underset{above}{2}}^{\perp} - P_{\underset{below}{1}}^{\perp} = \sigma_{bound}}$$ (at interface)

$$\vec{E} \equiv -\vec{\nabla} V$$

Since: $$\boxed{\left. \left(\frac{\partial V_2^{above}}{\partial n} - \frac{\partial V_1^{below}}{\partial n} \right) \right|_{interface} = -\frac{1}{\varepsilon_0} \sigma_{ToT} = -\frac{1}{\varepsilon_0} \left(\sigma_{free} + \sigma_{bound} \right)}$$ (at interface)

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

$$\vec{D} = \varepsilon\vec{E} = -\varepsilon\vec{\nabla}V$$

Since:

$$\left.\left(\varepsilon_2 \frac{\partial V_2^{above}}{\partial n} - \varepsilon_1 \frac{\partial V_1^{below}}{\partial n}\right)\right|_{interface} = -\sigma_{free}$$

(at interface)

Similarly, for $\int_v \vec{\nabla}\bullet\vec{B}\,d\tau' = \oint_S \vec{B}\bullet d\vec{a} = 0$ (no magnetic monopoles), then at an interface:

$$\vec{B}_2^{above}\bullet\vec{a} - \vec{B}_1^{above}\bullet\vec{a} = 0 \Rightarrow \boxed{B_2^{\perp}\Big|_{above} - B_1^{\perp}\Big|_{below} = 0} \text{ or: } \boxed{B_2^{\perp}\Big|_{above} = B_1^{\perp}\Big|_{below}}$$

(at interface)

Since:

$$\vec{H} = \left(\frac{1}{\mu_o}\right)\vec{B} - \vec{M} \qquad \underline{\text{Then:}} \qquad \vec{B} = \mu_o\left(\vec{H} + \vec{M}\right)$$

$$\oint_S \vec{B}\bullet d\vec{a} = \mu_o \oint_S \left(\vec{H} + \vec{M}\right)\bullet d\vec{a} = 0 \qquad \underline{\text{or:}} \qquad \oint_S \vec{H}\bullet d\vec{a} = -\oint_S \vec{M}\bullet d\vec{a}$$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

Then: $\vec{H}_2^{above} \bullet \vec{a} - \vec{H}_1^{below} \bullet \vec{a} = -\left(\vec{M}_2^{above} \bullet \vec{a} - \vec{M}_1^{below} \bullet \vec{a}\right)$ (at interface)

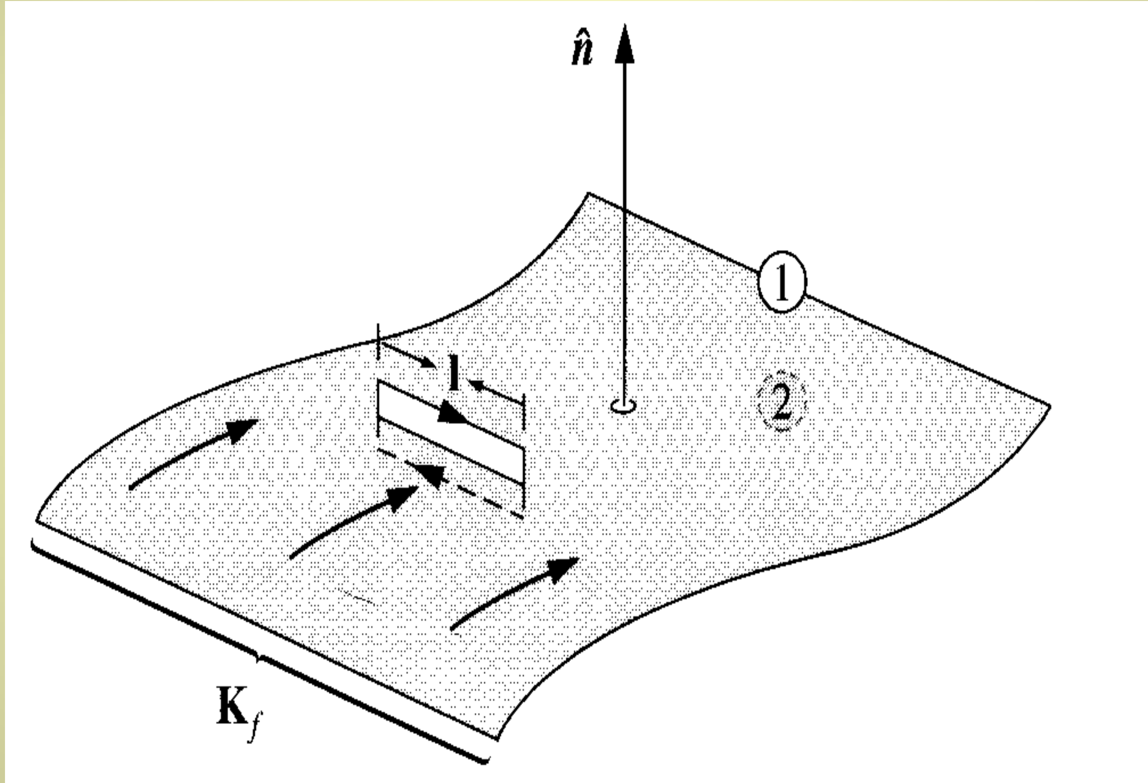
Or: $$\left(H^{\perp}_{\underset{above}{2}} - H^{\perp}_{\underset{below}{1}}\right) = -\left(M^{\perp}_{\underset{above}{2}} - M^{\perp}_{\underset{below}{1}}\right) = -\sigma^{bound}_{magnetic}$$ (at interface)

Effective bound magnetic charge at interface

3) For Faraday's Law: EMF, $\varepsilon = \oint_C \vec{E} \bullet d\vec{\ell} = -\frac{d}{dt}\left(\oint_S \vec{B} \bullet d\vec{a}\right) = -\frac{d\Phi_m}{dt}$

At interface between two different media, taking a closed contour C of width l extending slightly into the material on either side of interface.

Maxwell's Equations and Boundary Conditions at Interfaces in Matter



Maxwell's Equations and Boundary Conditions at Interfaces in Matter

$$\vec{E}_2^{above} \cdot \vec{\ell} - \vec{E}_1^{below} \cdot \vec{\ell} = -\frac{d}{dt} \oint_S \vec{B} \cdot d\vec{a} = 0$$

(in limit area of contour loop → 0, magnetic flux enclosed → 0)

$$\boxed{\underset{above}{E_2^{\parallel}} - \underset{below}{E_1^{\parallel}} = 0}$$ (<u>at interface</u>) <u>or:</u> $$\boxed{\underset{above}{E_2^{\parallel}} = \underset{below}{E_1^{\parallel}}}$$ (<u>at interface</u>)

<u>Since:</u> $\vec{D} = \varepsilon_o \vec{E} + \vec{P}$ And: $\varepsilon_o \vec{E} = \vec{D} - \vec{P}$

<u>Thus:</u> $\left(\vec{E}_2^{above} \cdot \vec{\ell} - \vec{E}_1^{below} \cdot \vec{\ell}\right) = \left(\vec{D}_2^{above} \cdot \vec{\ell} - \vec{D}_1^{below} \cdot \vec{\ell}\right) - \left(\vec{P}_2^{above} \cdot \vec{\ell} - \vec{P}_1^{below} \cdot \vec{\ell}\right) = 0$

In limit area of contour loop → 0 magnetic flux enclosed → 0

$$\Rightarrow \left(\underset{above}{\vec{D}_2^{\parallel}} - \underset{below}{\vec{D}_1^{\parallel}}\right) = \left(\underset{above}{\vec{P}_2^{\parallel}} - \underset{below}{\vec{P}_1^{\parallel}}\right)$$ (<u>at interface</u>)

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

4) Finally, for Ampere's Law: $\oint_C \vec{B} \bullet d\vec{\ell} = \mu_o \left(I_{TOT}^{encl} + I_D^{encl} \right)$

$$\vec{B}_2^{above} \bullet \vec{\ell} - \vec{B}_1^{below} \bullet \vec{\ell} = \mu_o I_{TOT}^{encl} + \mu_o I_D^{encl}$$

$$I_D^{encl} = \int_S \vec{J}_D \bullet d\vec{a} = \varepsilon_o \int_S \frac{\partial \vec{E}}{\partial t} \bullet d\vec{a}$$

$$I_{TOT}^{encl} = I_{free}^{encl} + I_{bound}^{encl} + I_{P_{bound}}^{encl}$$

$$I_{P_{bound}}^{encl} = \int_S \vec{J}_{P_{bound}} \bullet d\vec{a} = \int_S \frac{\partial \vec{\mathrm{P}}}{\partial t} \bullet d\vec{a}$$

$$I_{bound}^{encl} = \int_S \vec{J}_m^{bound} \bullet d\vec{a} = \int_S \vec{\nabla} \times \vec{\mathrm{M}} \bullet d\vec{a}$$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

Where I_{TOT}^{encl} = TOTAL current (free + bound + polarization) passing through enclosing Amperian loop contour C

No volume current density $\vec{J}_{TOT}, \vec{J}_{free}, \vec{J}_{bound}^{m}$ or $\vec{J}_p$ contributes to I_{TOT}^{encl} in the limit area of contour loop $\rightarrow 0$, however a surface current $\vec{K}_{TOT}, \vec{K}_{free}, \vec{K}_{bound}^{m} = \vec{M} \times \hat{n}$ <u>can</u> contribute!

In the limit that the enclosing Amperian loop contour C shrinks to zero height above/below interface- the enclosed area of loop contour → 0,

Then: $$I_D^{encl} = \varepsilon_o \int_S \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} = \varepsilon_o \frac{d}{dt}\left[\int_S \vec{E} \cdot d\vec{a}\right] = \varepsilon \frac{d\Phi_E}{dt} \rightarrow 0$$

$(\Phi_E \equiv \int_S \vec{E} \cdot d\vec{a}$ = enclosed flux of electric field lines)

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

Similarly:

$$I_{P_{bound}}^{encl} = \int_S \frac{\partial \vec{P}}{\partial t} \cdot d\vec{a} = \frac{d}{dt}\left[\int_S \vec{P} \cdot d\vec{a}\right] = \frac{d\Phi_P}{dt} \to 0$$

$$\left(\Phi_P \equiv \int_S \vec{P} \cdot d\vec{a} = \text{enclosed flux of electric polarization field lines}\right)$$

If $\hat{n}$ *is unit normal/perpendicular to interface*, note that $\left(\hat{n} \times \vec{\ell}\right)$ is normal/perpendicular to plane of the Amperian loop contour.

$$I_{TOT}^{encl} = \vec{K}_{TOT} \cdot \left(\hat{n} \times \vec{\ell}\right) = \left(\vec{K}_{TOT} \times \hat{n}\right) \cdot \vec{\ell}$$

$$I_{free}^{encl} = \vec{K}_{free} \cdot \left(\hat{n} \times \vec{\ell}\right) = \left(\vec{K}_{free} \times \hat{n}\right) \cdot \vec{\ell}$$

$$I_{bound}^{encl} = \vec{K}_{bound} \cdot \left(\hat{n} \times \vec{\ell}\right) = \left(\vec{K}_{bound}^{m} \times \hat{n}\right) \cdot \vec{\ell}$$

Using: $\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \vec{B} \cdot \left(\vec{C} \times \vec{A}\right)$
$= \vec{C} \cdot \left(\vec{A} \times \vec{B}\right)$
$\left\{= \left(\vec{A} \times \vec{B}\right) \cdot \vec{C}\right\}$

$$I_{TOT} = I_{free} - I_{bound} \qquad \vec{K}_{TOT} = \vec{K}_{free} + \vec{K}_{bound}$$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

In the limit that the enclosing Amperian loop contour C (of width l) shrinks to zero height above/below interface, causing area of enclosed loop contour → 0, then:

$$\vec{B}_2^{above}\cdot\vec{\ell}-\vec{B}_1^{below}\cdot\vec{\ell}=\mu_o I_{TOT}^{encl}+\overbrace{\cancel{\mu_o I_D^{encl}}}^{=0}=\mu_o I_{TOT}^{encl}=\left(\vec{K}_{TOT}\times\hat{n}\right)\cdot\vec{\ell}$$

$$\boxed{B^{\parallel}_{2\,above}-B^{\parallel}_{1\,below}=\mu_o\vec{K}_{TOT}\times\hat{n}=\mu_o\left(\vec{K}_{free}+\vec{K}^{m}_{bound}\right)\times\hat{n}}\quad \text{(at interface)}$$

Since: $\vec{H}=\frac{1}{\mu_o}\vec{B}-\vec{M}$ and: $\frac{1}{\mu_o}\vec{B}=\vec{H}+\vec{M}$ then:

$$\boxed{\frac{1}{\mu_o}\left(\vec{B}_2^{above}\cdot\vec{\ell}-\vec{B}_1^{below}\cdot\vec{\ell}\right)=\left(\vec{H}_2^{above}\cdot\vec{\ell}-\vec{H}_1^{below}\cdot\vec{\ell}\right)+\left(\vec{M}_2^{above}\cdot\vec{\ell}-\vec{M}_1^{below}\cdot\vec{\ell}\right)=\left[\left(\vec{K}_{free}\times\hat{n}\right)+\left(\vec{K}_{bound}\times\hat{n}\right)\right]}$$

(at interface)

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

We also see that: $$\boxed{H^{\parallel}_{\substack{2\\above}} - H^{\parallel}_{\substack{1\\below}} = \vec{K}_{free} \times \hat{n}}$$ (at interface)

and: $$\boxed{M^{\parallel}_{\substack{2\\above}} - M^{\parallel}_{\substack{1\\below}} = \vec{K}^{m}_{bound} \times \hat{n}}$$ (at interface)

- ‖- components of ***B*** are discontinuous at interface by $\mu_o \vec{K}_{TOT} \times \hat{n}$
- ‖- components of ***H*** are discontinuous at interface by $\vec{K}_{free} \times \hat{n}$
- ‖- components of **M** are discontinuous at interface by $\vec{K}^{m}_{bound} \times \hat{n}$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

If $\vec{B} = \vec{\nabla} \times \vec{A}$

where ***A*** is the magnetic vector potential, then:

$$\left(\frac{1}{\mu_0}\right)\left[B_{2\,above}^{\parallel} - B_{1\,below}^{\parallel}\right] = \vec{K}_{TOT} \times \hat{n}$$

(at interface) is equivalent to:

$$\left(\frac{1}{\mu_0}\right)\left(\frac{\partial \vec{A}_2^{above}}{\partial n} - \frac{\partial \vec{A}_1^{below}}{\partial n}\right)\Bigg|_{interface} = -\vec{K}_{TOT}$$

(at interface)

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

For linear magnetic media:

$$\vec{B} = \mu\vec{H} \quad \text{or:} \quad \vec{H} = \frac{1}{\mu}\vec{B}$$

$$\boxed{\left[\underset{above}{H_2^{\|}} - \underset{below}{H_1^{\|}}\right] = \vec{K}_{free} \times \hat{n}}$$ (<u>at interface</u>) is <u>equivalent</u> to:

$$\boxed{\left(\frac{1}{\underset{above}{\mu_2}}\right)\frac{\partial \vec{A}_2^{above}}{\partial n}\bigg|_{interface} - \left(\frac{1}{\underset{below}{\mu_1}}\right)\frac{\partial \vec{A}_1^{below}}{\partial n}\bigg|_{interface} = -\vec{K}_{free}}$$ (<u>at interface</u>)

Summary of General Boundary Conditions Obtained from Integral Form(s) of Maxwell's Equations

Then at the interface:

Gauss' Law:

$$E^{\perp}_{2\,\text{above}} - E^{\perp}_{1\,\text{below}} = \frac{1}{\varepsilon_o}\sigma_{TOT} = \frac{1}{\varepsilon_o}\left(\sigma_{free} + \sigma_{bound}\right)$$

$$\vec{D} = \varepsilon_o\vec{E} + \vec{\mathrm{P}}$$

$$D^{\perp}_{2\,\text{above}} - D^{\perp}_{1\,\text{below}} = \sigma_{free}$$

$$\mathrm{P}^{\perp}_{2\,\text{above}} - \mathrm{P}^{\perp}_{1\,\text{below}} = -\sigma_{bound}$$

$$\left(\frac{\partial V_2^{above}}{\partial n} - \frac{\partial V_1^{below}}{\partial n}\right)\Bigg|_{\text{interface}} = -\frac{1}{\varepsilon_o}\sigma_{TOT} = -\frac{1}{\varepsilon_o}\left(\sigma_{free} + \sigma_{bound}\right)$$

$$\left(\varepsilon_2\frac{\partial V_2^{above}}{\partial n} - \varepsilon_1\frac{\partial V_1^{below}}{\partial n}\right)\Bigg|_{\text{interface}} = -\sigma_{free}$$

No Magnetic Monopoles:

$$B^{\perp}_{2\,\text{above}} - B^{\perp}_{1\,\text{below}} = 0 \Rightarrow \left(H^{\perp}_{2\,\text{above}} - H^{\perp}_{1\,\text{below}}\right) = -\left(\mathrm{M}^{\perp}_{2\,\text{above}} - \mathrm{M}^{\perp}_{1\,\text{below}}\right) = -\sigma^{bound}_{magnetic}$$

$$\vec{H} = \left(\frac{1}{\mu_o}\right)\vec{B} - \vec{\mathrm{M}}$$

Summary of General Boundary Conditions Obtained from Integral Form(s) of Maxwell's Equations

Faraday's Law:

$$E^{\parallel}_{\underset{above}{2}} - E^{\parallel}_{\underset{below}{1}} = 0 \Rightarrow \left(D^{\parallel}_{\underset{above}{2}} - D^{\parallel}_{\underset{below}{1}} \right) = \left(\mathbf{P}^{\parallel}_{\underset{above}{2}} - \mathbf{P}^{\parallel}_{\underset{below}{1}} \right)$$

Ampere's Law:

$$B^{\parallel}_{\underset{above}{2}} - B^{\parallel}_{\underset{below}{1}} = \mu_o \vec{K}_{TOT} \times \hat{n} = \mu_o \left(\vec{K}_{free} + \vec{K}^{m}_{bound} \right) \times \hat{n}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\Rightarrow \left(\frac{1}{\mu_o} \right) \left(\frac{\partial \vec{A}_2^{above}}{\partial n} - \frac{\partial \vec{A}_1^{below}}{\partial n} \right) \Bigg|_{interface} = -\vec{K}_{TOT}$$

$$H^{\parallel}_{\underset{above}{2}} - H^{\parallel}_{\underset{below}{1}} = \vec{K}_{free} \times \hat{n}$$

$$\mathbf{M}^{\parallel}_{\underset{above}{2}} - \mathbf{M}^{\parallel}_{\underset{below}{1}} = \vec{K}^{m}_{bound} \times \hat{n}$$

$$\Rightarrow \left(\frac{1}{\mu_2} \right) \frac{\partial \vec{A}_2}{\partial n} \Bigg|_{\substack{above \\ interface}} - \left(\frac{1}{\mu_1} \right) \frac{\partial \vec{A}_1}{\partial n} \Bigg|_{\substack{below \\ interface}} = -\vec{K}_{free}$$

Summary of General Boundary Conditions Obtained from Integral Form(s) of Maxwell's Equations

BC's Specific to *Linear Homogeneous Isotropic Dielectric and/or Magnetic Media:*

$$\vec{P} = \varepsilon_o \chi_e \vec{E}$$

$$\vec{M} = \chi_m \vec{H}$$

$$\vec{D} = \varepsilon \vec{E} = \varepsilon_o \vec{E} + \vec{P}$$

$$\varepsilon = \varepsilon_o (1 + \chi_e)$$

$$\mu = \mu_o (1 + \chi_m)$$

$$\vec{B} = \mu \vec{H} \text{ or } \vec{H} = \frac{1}{\mu} \vec{B}$$

$$K_e = \frac{\varepsilon}{\varepsilon_o} = 1 + \chi_e$$

$$K_m = \frac{\mu}{\mu_o} = (1 + \chi_m)$$

$$\vec{H} = \frac{1}{\mu_o} \vec{B} - \vec{M}$$

Summary of General Boundary Conditions Obtained from Integral Form(s) of Maxwell's Equations

The boundary conditions at the interface between linear dielectric or magnetic media become:

Gauss' Law:

$$\left(E^{\perp}_{\underset{above}{2}} - E^{\perp}_{\underset{below}{1}}\right) - \frac{1}{\varepsilon_o}\sigma_{TOT} - \frac{1}{\varepsilon_o}\left(\sigma_{free} + \sigma_{bound}\right)$$

$$\left(\frac{\partial V_2^{above}}{\partial n} - \frac{\partial V_1^{below}}{\partial n}\right)\Bigg|_{interface} = -\frac{1}{\varepsilon_o}\sigma_{TOT} = -\frac{1}{\varepsilon_o}\left(\sigma_{free} + \sigma_{bound}\right)$$

$$\left(D^{\perp}_{\underset{above}{2}} - D^{\perp}_{\underset{below}{1}}\right) = \sigma_{free} \Rightarrow \left(\varepsilon_2 E^{\perp}_{\underset{above}{2}} - \varepsilon_1 E^{\perp}_{\underset{below}{1}}\right) = \sigma_{free}$$

$$\left(\mathrm{P}^{\perp}_{\underset{above}{2}} - \mathrm{P}^{\perp}_{\underset{below}{1}}\right) = -\sigma_{bound} \Rightarrow \varepsilon_o\left(\chi_{e_2} E^{\perp}_{\underset{above}{2}} - \chi_{e_1} E^{\perp}_{\underset{below}{1}}\right) = -\sigma_{bound}$$

$$\varepsilon_2 \frac{\partial V_2}{\partial n}\Bigg|_{\substack{above\\interface}} - \varepsilon_1 \frac{\partial V_1}{\partial n}\Bigg|_{\substack{below\\interface}} = -\sigma_{free}$$

Summary of General Boundary Conditions Obtained from Integral Form(s) of Maxwell's Equations

No magnetic monopoles:

$$\left(B_{\underset{above}{2}}^{\perp}-B_{\underset{below}{1}}^{\perp}\right)=0 \Rightarrow \left(H_{\underset{above}{2}}^{\perp}-H_{\underset{below}{1}}^{\perp}\right)=-\left(M_{\underset{above}{2}}^{\perp}-M_{\underset{below}{1}}^{\perp}\right)=-\sigma_{magnetic}^{bound}$$

$$=\left(\mu_{\underset{above}{2}}H_2^{\perp}-\mu_{\underset{below}{1}}H_1^{\perp}\right)=0$$

$$=\left(\frac{1}{\mu_{\underset{above}{2}}}B_2^{\perp}-\frac{1}{\mu_{\underset{below}{1}}}B_1^{\perp}\right)$$

Faraday's Law:

$$\left(E_{\underset{above}{2}}^{\parallel}-E_{\underset{below}{1}}^{\parallel}\right)=0 \Rightarrow \left(D_{\underset{above}{2}}^{\parallel}-D_{\underset{below}{1}}^{\parallel}\right)=\left(P_{\underset{above}{2}}^{\parallel}-P_{\underset{below}{1}}^{\parallel}\right)$$

$$=\left(\frac{1}{\varepsilon_{\underset{above}{2}}}D_2^{\parallel}-\frac{1}{\varepsilon_{\underset{below}{1}}}D_1^{\parallel}\right)=0 \Rightarrow \left(\varepsilon_{\underset{above}{2}}E_2^{\parallel}-\varepsilon_{\underset{below}{1}}E_1^{\parallel}\right)=-\varepsilon_o\left(\chi_{e_{\underset{above}{2}}}E_2^{\parallel}-\chi_{e_{\underset{below}{1}}}E_1^{\parallel}\right)$$

Summary of General Boundary Conditions Obtained from Integral Form(s) of Maxwell's Equations

Ampere's Law:

$$B_{2\,above}^{\parallel} - B_{1\,below}^{\parallel} = \mu_o \vec{K}_{TOT} \times \hat{n} = \mu_o \left(\vec{K}_{free} + \vec{K}_{bound}^{m} \right) \times \hat{n} \Rightarrow \left(\frac{1}{\mu_o} \right) \left(\frac{\partial \vec{A}_2^{above}}{\partial n} - \frac{\partial \vec{A}_1^{below}}{\partial n} \right) \Bigg|_{interface} = -\vec{K}_{TOT}$$

$$= \left(\mu_{2\,above} H_2^{\parallel} - \mu_{1\,below} H_1^{\parallel} \right)$$

using $\vec{B} = \vec{\nabla} \times \vec{A}$

$$H_{2\,above}^{\parallel} - H_{1\,below}^{\parallel} = \vec{K}_{free} \times \hat{n}$$

$$= \frac{1}{\mu_{2\,above}} B_{2\,above}^{\parallel} - \frac{1}{\mu_{1\,below}} B_{1\,below}^{\parallel}$$

and

$$\left(M_{2\,above}^{\parallel} - M_{1\,below}^{\parallel} \right) = \vec{K}_{bound}^{m} \times \hat{n}$$

$$= \left(\chi_{m_2\,above} H_{2\,above}^{\parallel} - \chi_{m_1\,below} H_{1\,below}^{\parallel} \right)$$

$$= \left(\frac{\chi_{m_2\,above}}{\mu_{2\,above}} B_{2\,above}^{\parallel} - \frac{\chi_{m_1\,below}}{\mu_{1\,below}} B_{1\,below}^{\parallel} \right)$$

$$\left(\frac{1}{\mu_2} \right) \frac{\partial \vec{A}_2}{\partial n} \Bigg|_{above\ interface} - \left(\frac{1}{\mu_1} \right) \frac{\partial \vec{A}_1}{\partial n} \Bigg|_{below\ interface} = -\vec{K}_{free}$$

using $\vec{B} = \vec{\nabla} \times \vec{A}$

Potential Formulation of Electrodynamics 1

- In electrostatic

$$\nabla \times \vec{E} = 0$$

$$\Rightarrow \vec{E} = -\nabla V$$

In electrodynamics

$$\nabla \times \vec{E} \neq 0$$

But

$$\nabla \cdot \vec{B} = 0$$

$$\vec{B} = \nabla \times \vec{A}$$

Putting this in Faraday's Law

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\left(\nabla \times \vec{A}\right)$$

$$\nabla \times \left(\vec{E} + \frac{\partial}{\partial t}\vec{A} \right) = 0$$

$$\Rightarrow \left(\vec{E} + \frac{\partial}{\partial t}\vec{A} \right) = -\nabla V$$

$$\Rightarrow \vec{E} = -\nabla V - \frac{\partial}{\partial t}\vec{A}$$

Potential Formulation of Electrodynamics 2

$$\nabla . B = \nabla . \left(\nabla \times A \right) = 0$$

and from

$$\nabla \times E = -\nabla \times \nabla V - \frac{\partial}{\partial t} \nabla \times A$$

$$\nabla \times E = 0 - \frac{\partial}{\partial t} B$$

Explain Maxwell's ii and iii equations

Potential Formulation of Electrodynamics 3

Now consider Maxwell's i and iv equations

As

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_o}$$

Gauss's Law

$$\nabla \cdot \left(\nabla V + \frac{\partial A}{\partial t} \right) = -\frac{\rho}{\varepsilon_o}$$

$$\nabla^2 V + \frac{\partial}{\partial t} \nabla \cdot A = -\frac{\rho}{\varepsilon_o}$$

This replaces Poisson's Equation in electrodynamics

Potential Formulation of Electrodynamics 4

Now consider

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

Putting values of E and B we get

$$\nabla \times \left(\nabla \times \vec{A} \right) = \mu_o \vec{J} - \mu_o \varepsilon_o \nabla \left(\frac{\partial V}{\partial t} \right) - \mu_o \varepsilon_o \left(\frac{\partial^2 \vec{A}}{\partial t^2} \right)$$

Potential Formulation of Electrodynamics 5

Using vector identity

$$\nabla \times \left(\nabla \times \vec{A}\right) = \nabla\left(\nabla \cdot A\right) - \nabla^2 A$$

Re-arranging

$$\left(\nabla^2 \vec{A} - \mu_o \varepsilon_o \left(\frac{\partial^2 \vec{A}}{\partial t^2}\right)\right) - \nabla\left(\nabla \cdot \vec{A} + \mu_o \varepsilon_o \left(\frac{\partial V}{\partial t}\right)\right) = -\mu_o \vec{J}$$

These equation carry all information in Maxwell's equations

Potential Formulation of Electrodynamics 6

$$\nabla^2 V + \frac{\partial}{\partial t} \nabla \cdot A = -\frac{\rho}{\varepsilon_o}$$

$$\left(\nabla^2 \vec{A} - \mu_o \varepsilon_o \left(\frac{\partial^2 \vec{A}}{\partial t^2} \right) \right) - \nabla \left(\nabla \cdot \vec{A} + \mu_o \varepsilon_o \left(\frac{\partial V}{\partial t} \right) \right) = -\mu_o \vec{J}$$

Four Maxwell's equations reduced to two equations using potential formulation.
Potentials V and A are not uniquely defined by above equations.

Gauge Transformations

- Two sets of potentials, (V, A) and (V',A')- corresponds to same electric and Magnetic fields.
- Write;

  A'=A+α and V'=V+β- as A's give same B

→ curl of α=0, which implies α= grad. of λ. As the two potentials also give same E, then from

$$\vec{E} = -\nabla V' - \frac{\partial}{\partial t} A'$$

$$\vec{E} = -\nabla V - \nabla \beta - \frac{\partial}{\partial t} A - \frac{\partial}{\partial t} \alpha$$

Gauge Transformations 1

$$\Rightarrow \nabla \beta + \frac{\partial}{\partial t}\alpha = 0$$

Putting value of α we get

$$\nabla \left(\beta + \frac{\partial}{\partial t}\lambda \right) = 0$$

The term in paratheses is independent of position

$$\Rightarrow \beta = -\frac{\partial}{\partial t}\lambda$$

Using this we get

$$A' = A + \nabla \lambda$$

$$V' = V - \frac{\partial \lambda}{\partial t}$$

Such changes in V and A are called Gauge Transformations

Coulomb's and Lorentz Gauges

Coulomb Gauge $\nabla \bullet A = 0$

Using this we get $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$

It is Poisson's equation, setting V=0, we get

$$V = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho}{r} d\tau$$

Scalar potential is easy to calculate in Coulomb's gauge but vector potential is difficult to calculate

Coulomb's Gauge

The differential equations for V and A in Coulombs gauge reads

$$\nabla^2 V = -\frac{\rho}{\varepsilon_o}$$

$$\left(\nabla^2 \vec{A} - \mu_o \varepsilon_o \left(\frac{\partial^2 \vec{A}}{\partial t^2}\right)\right) = -\mu_o \vec{J} + \mu_o \varepsilon_o \nabla \left(\frac{\partial V}{\partial t}\right)$$

Lorentz Gauge

The Lorentz gauge:

$$\nabla\cdot\vec{A}=-\mu_o\varepsilon_o\left(\frac{\partial V}{\partial t}\right)$$

This is design to eliminate the middle term in eqn. for A

$$\left(\nabla^2\vec{A}-\mu_o\varepsilon_o\left(\frac{\partial^2\vec{A}}{\partial t^2}\right)\right)=-\mu_o\vec{J}$$

and equation for V will become

$$\nabla^2 V-\mu_o\varepsilon_o\frac{\partial^2 V}{\partial t^2}=-\frac{\rho}{\varepsilon_o}$$

Lorentz Gauge

The Lorentz gauge treats V and A on equal footing. The same differential operator

$$\nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} = \Box^2$$

called the d'Alembertian

$$\Box^2 A = -\mu_0 j$$

and

$$\Box^2 V = -\frac{1}{\varepsilon_0} \rho$$

The Electromagnetic Waves

Electromagnetic Wave Equation

The electromagnetic wave equation is a second-order partial differential equation that describes the propagation of electromagnetic waves through a medium or in a vacuum.

To obtain the electromagnetic wave equation in a vacuum we begin with the modern 'Heaviside' form of Maxwell's equations.

From Maxwell's Equations to the Electromagnetic Waves 1

The Wave Equation

Maxwell's equation in free space – no charge or no current are given as

$$\vec{\nabla} \bullet \vec{E} = 0$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

From Maxwell's Equations to the Electromagnetic Waves 2

Take curl of

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla} \times [-\frac{\partial \vec{B}}{\partial t}]$$

Change the order of differentiation on the R.H.S

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}[\vec{\nabla} \times \vec{B}]$$

From Maxwell's Equations to the Electromagnetic Waves 3

As

$$\vec{\nabla} \times \vec{B} = \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

Substituting for $\vec{\nabla} \times \vec{B}$ we have

$$\vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\frac{\partial}{\partial t}[\mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}]$$

$$\vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2}$$

- As μ_o and ε_o are constant in time

From Maxwell's Equations to the Electromagnetic Waves 4

Using the vector identity

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E}=\vec{\nabla}(\vec{\nabla}\bullet\vec{E})-\nabla^2\vec{E}$$

gives,

$$\vec{\nabla}(\vec{\nabla}\bullet\vec{E})-\nabla^2\vec{E}=-\mu_o\varepsilon_o\frac{\partial^2\vec{E}}{\partial t^2}$$

In free space

$$\vec{\nabla}\bullet\vec{E}=0$$

And we are left with the wave equation

$$\nabla^2\vec{E}-\mu_o\varepsilon_o\frac{\partial^2\vec{E}}{\partial t^2}=0$$

From Maxwell's Equations to the Electromagnetic Waves 5

Similarly the wave equation for magnetic field

$$\nabla^2 \vec{B} - \mu_o \varepsilon_o \frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

where,

$$c = \frac{1}{\sqrt{\mu_o \varepsilon_o}}$$

Electromagnetic Wave Equation in Vacuum

$$\nabla^2 \vec{E} - \mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \qquad \nabla^2 \vec{B} - \mu_o \varepsilon_o \frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

The solutions to the wave equations, when there is no source charge present can be plane waves - obtained by method of separation of variables

Solution of Electromagnetic Wave

- Plane electromagnetic waves can be expressed as

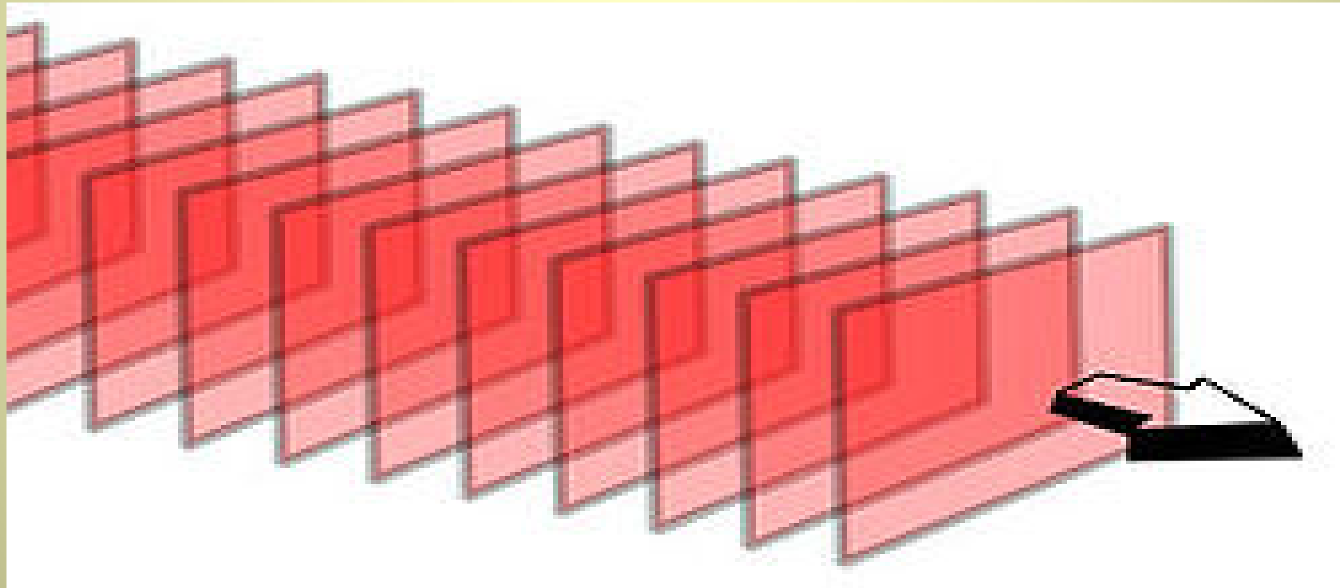
$$\vec{E} = E_o e^{i\left(\omega t - \vec{k}\cdot\vec{r}\right)}\hat{n}$$

$$\vec{B} = \frac{1}{c} E_o e^{i\left(\omega t - \vec{k}\cdot\vec{r}\right)}\left(\hat{k}\times\hat{n}\right) = \frac{1}{c}\left(\hat{k}\times\vec{E}\right)$$

Where $\hat{n}$ is the polarization vector and. $\hat{k}$ is a propagation vector.

Electromagnetic Plane waves

- **Plane wave -** a constant-frequency wave whose wave-fronts (surfaces of constant phase) are infinite parallel planes of constant amplitude normal to the direction of propagation



Real Electromagnetic Plane waves

The real electric and magnetic fields in the form of a mono-chromatic plane wave with propagation vector $\hat{k}$ and polarization $\hat{n}$

$$\vec{E}\left(\vec{r},t\right) = E_o \cos(\vec{k}\cdot\vec{r} - \omega t)\hat{n}$$

$$\vec{B}\left(\vec{r},t\right) = \frac{1}{c} E_o \cos(\vec{k}\cdot\vec{r} - \omega t)\left(\vec{k}\times\hat{n}\right)$$

Homogenous Wave Equations Inside Matter

The homogeneous form of the equation - written in terms of either the electric field **E** or the magnetic field **B** - takes the form:

Vacuum

$$\frac{1}{\mu_o \varepsilon_o}\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\frac{1}{\mu_o \varepsilon_o}\nabla^2 \vec{B} = \frac{\partial^2 \vec{B}}{\partial t^2}$$

Matter

$$\frac{1}{\mu \varepsilon}\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\frac{1}{\mu \varepsilon}\nabla^2 \vec{B} = \frac{\partial^2 \vec{B}}{\partial t^2}$$

Homogenous Wave Equations Inside Matter 1

Permittivity: $\varepsilon=\varepsilon_r\varepsilon_o$ (ε_r is dielectric constant)
Permeability: $\mu=\mu_r\mu_o$ (μ_r is relative permeability ≈1

$$v=\frac{1}{\sqrt{\mu\varepsilon}}=\frac{1}{\sqrt{\mu_r\mu_0\varepsilon_r\varepsilon_0}}=\frac{1}{\sqrt{\mu_0\varepsilon_0}}\frac{1}{\sqrt{\mu_r\varepsilon_r}}$$

=c

= n

$$v=\frac{c}{n}$$

n=Refractive Index

Energy and Momentum of Electromagnetic Waves

The energy per unit volume stored in electromagnetic field is

$$U = \frac{1}{2}\left(\varepsilon_o E^2 + \frac{1}{\mu_o} B^2\right)$$

In the case of monochromatic plane wave

$$B^2 = \frac{1}{c^2} E^2 = \mu_o \varepsilon_o E^2$$

$$\Rightarrow U = \varepsilon_o E^2 = \varepsilon_o E_o{}^2 \cos^2(kx - \omega t)$$

Energy and Momentum of Electromagnetic Waves 1

As the wave propagates, it carries this energy along with it. The energy flux density (energy per unit area per unit time) transported by the field is given by the poynting vector

$$\vec{S} = \frac{1}{\mu_o}\left(\vec{E} \times \vec{B}\right)$$

For monochromatic plane waves

$$\vec{S} = c\varepsilon_o E_o{}^2 \cos^2(kx - \omega t)\hat{i} = cU\hat{i}$$

Polarization of Electromagnetic Waves

- Polarization of electromagnetic waves is very complex- consider the optical (light) part of EM waves.

- Historically, the orientation of a polarized electromagnetic wave has been defined in the optical regime by the orientation of the electric field vector.

- Natural light is generally un-polarized- all planes of propagation being equally probable.

- Light is a transverse electromagnetic wave.

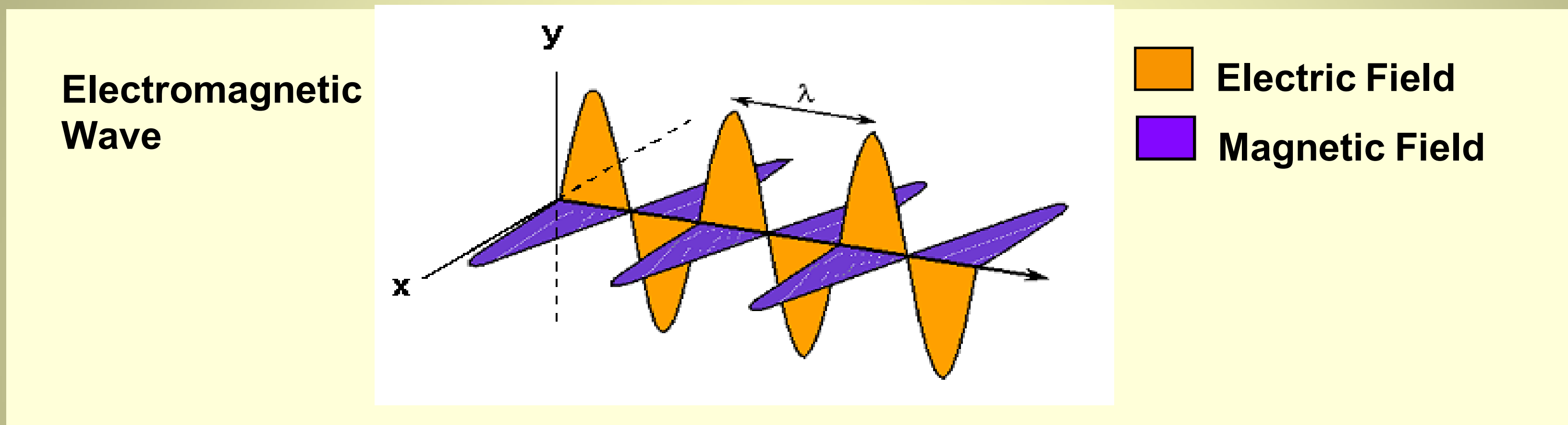
Linear Polarization

- In electrodynamics, linear polarization or plane polarization of electromagnetic radiation is a confinement of the electric field vector to a given plane along the direction of propagation.

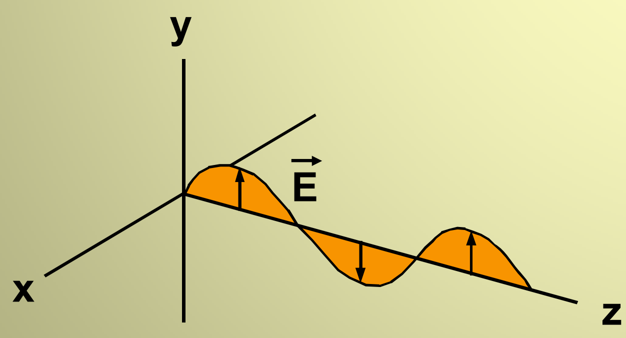
- The plane containing the electric field is called the plane of polarization.

Linear Polarization

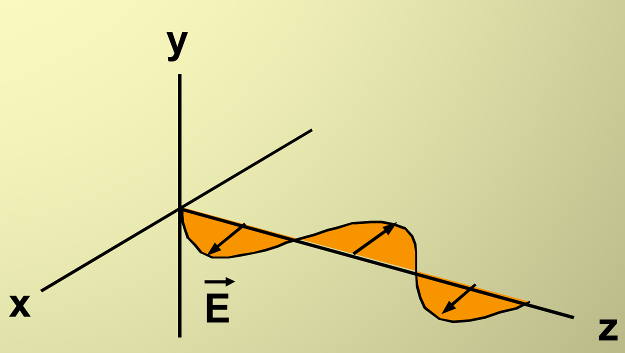
- Linear polarization can be horizontal or vertical



Vertical Polarization



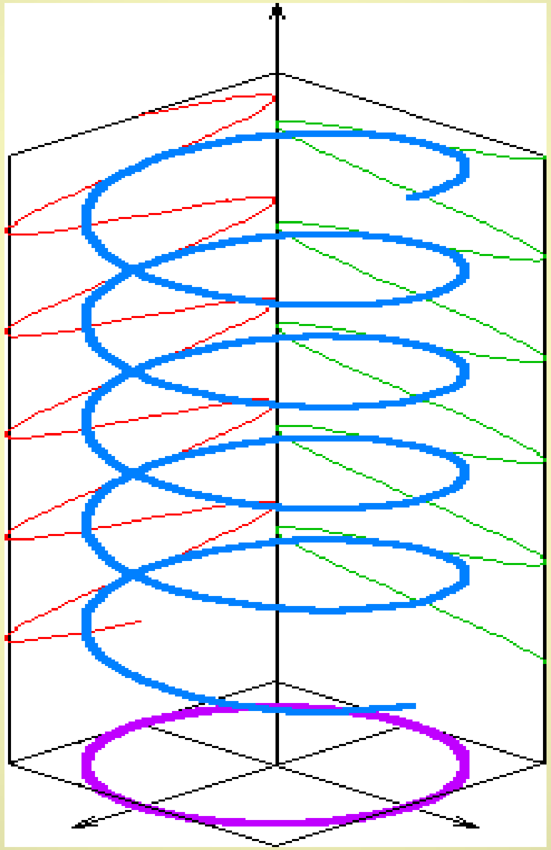
Horizontal Polarization



Circular Polarization

- A polarization in which the tip of the electric field vector- at a fixed point in space - describes a circle as time progresses.
- The electric vector - at one point in time - describes a helix along the direction of wave propagation.
- The magnitude of the electric field vector is constant as it rotates.
- Circular polarization is a limiting case of the more general condition of elliptical polarization.

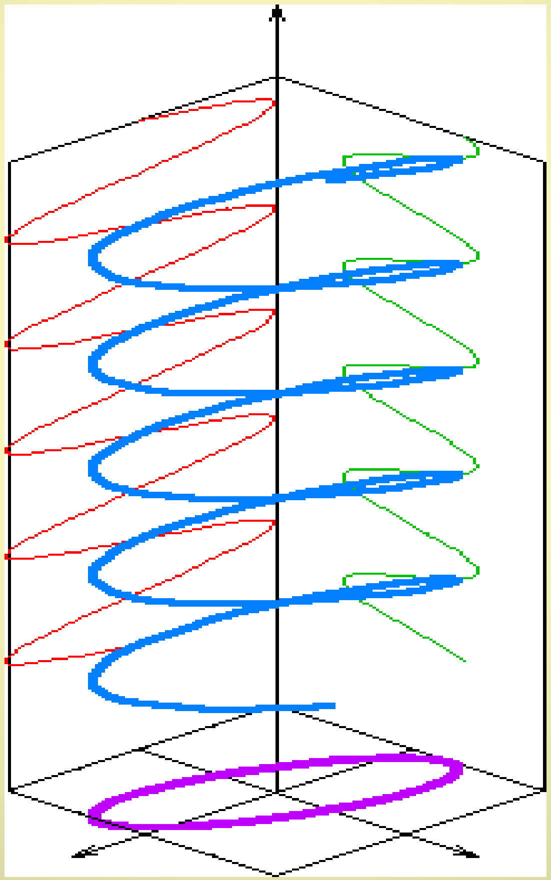
Circular Polarization



Elliptical Polarization

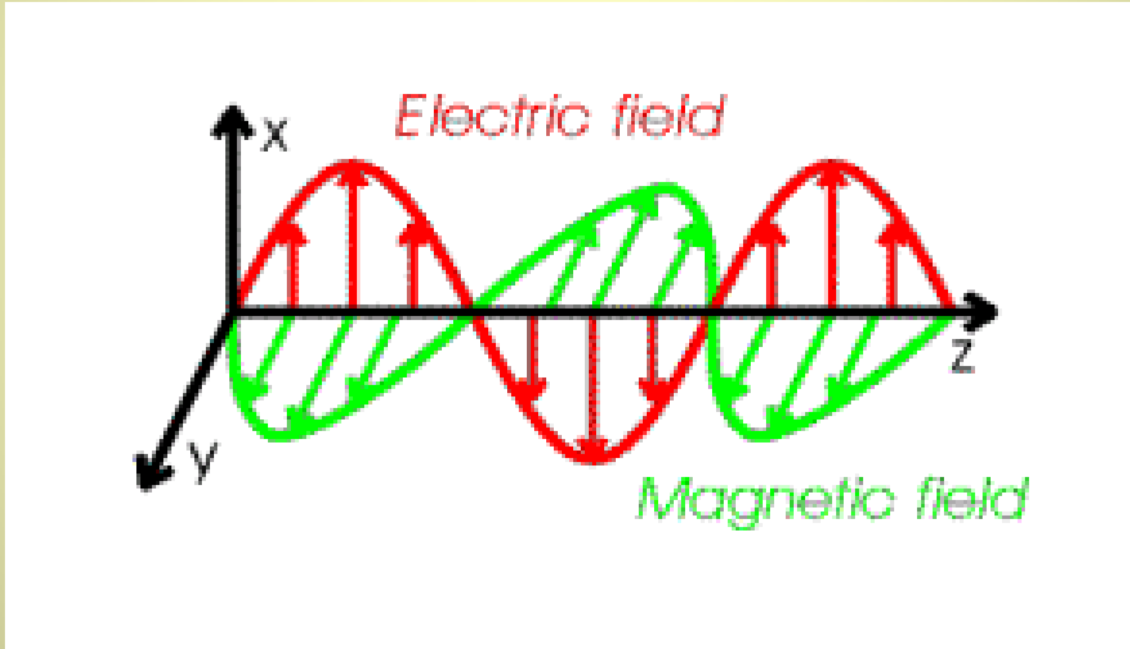
- **Elliptical polarization -** is the polarization of electromagnetic radiation such that the tip of the electric field vector describes an ellipse in any fixed plane intersecting - and normal to - the direction of propagation.
- An elliptically polarized wave may be resolved into two linearly polarized waves in phase quadrature- with their polarization planes at right angles to each other.

Elliptical Polarization



Light as an electromagnetic wave

Light is a transverse wave- an electromagnetic wave



Polarization Ellipse

- The polarization ellipse appeared in early 19th century- remained the only satisfactory way to visualize polarized light for nearly entire century.
- Numerous relations can be derived from the polarization ellipse- like degenerate cases for linearly and circularly polarized light.
- The description of light using polarization ellipse- allows us by means of a single equation – to describe any state of completely polarized light.

The Polarization Ellipse 1

- **States of Polarization**

The optical (light) field in free space is described by the wave equation.

$$\nabla^2 \vec{E}_i(r,t) = \frac{1}{c^2}\frac{\partial^2 \vec{E}_i(r,t)}{\partial^2 t} \qquad i = x, y$$

∇^2 is the Laplacian operator and **c** is the velocity of light.

The Polarization Ellipse 2

- This equation represents two independent wave equations for field components

$$E_x(r,t) \text{ and } E_y(r,t)$$

- Orthogonal to each other- in a plane perpendicular to the direction of propagation -z-axis.
- The simplest solution of wave equation is in terms of sinusoidal functions
- For propagation in positive z-direction- the solution of wave equation can be represented

The Polarization Ellipse 3

$$E(z,t) = E_{0x}\cos(\mathrm{kz} - \omega\mathrm{t} + \delta\mathrm{x})$$

$$E_y(z,t) = E_{0y}\cos(\mathrm{kz} - \omega\mathrm{t} + \delta\mathrm{y})$$

$$E(x,y,z,t) = E_x i + E_y j$$

- E_{0x} *and* E_{0y} **are the maximum amplitudes.**
- $\omega\mathrm{t} - \mathrm{kz}$ **is the propagator that describes the propagation of the component in the positive z-direction.**
- $\delta\mathrm{x}$ **and** $\delta\mathrm{y}$ **are phases of each of the components.**
- **The oscillations of above equations are perpendicular to each other- are called polarized.**

The Instantaneous Optical Field

- **Equations for E_{0x} *and* E_{0y} are said to be "instantaneous" because the time duration for a propagating wave is of the order of10^{-15}.**
- **The time duration of this magnitude cannot be measured or observed.**
- **The instantaneous view of the propagating optical field is**

$$E_x(z,t) = E_{0x}\cos(\tau + \delta x)$$
$$E_y(z,t) = E_{0y}\cos(\tau + \delta y)$$

The Instantaneous Optical Field 1

- This equation describes that the field propagates in the z-direction and the components $E_x(z,t)$ and $E_y(z,t)$ give rise to a resultant vector in the transverse x- y plane. These vectors describe the locus of points.

$$\frac{E_x{}^2(z,t)}{E_{0x}{}^2} + \frac{E_y{}^2(z,t)}{E_{0y}{}^2} - 2\frac{E_x(z,t)E_y(z,t)}{E_{0x}E_{0y}}cos\delta = sin^2\delta$$

- This is an equation of an ellipse.
- The ellipse does not change as the field propagates.

The Instantaneous Optical Field 2

- If $E_y(z,t)$ is set to zero and $\delta = \frac{\pi}{2}$, then

$$E_x{}^2(z,t) = E_{0x}{}^2$$

- If $E_x(z,t)$ is set to zero and $\delta = \frac{\pi}{2}$, a similar equation is obtained for $E_y{}^2(z,t)$
- Elliptically polarized light is also called completely polarized light.
- The special forms of polarization ellipse (elliptically polarized light) are known as degenerate polarization states.

Degenerate States of the Polarization Ellipse

- Consider the equations

$$E_x(z,t) = E_{0x}\cos(\mathrm{kz} - \omega\mathrm{t} + \delta\mathrm{x})$$

$$E_y(z,t) = E_{0y}\cos(\mathrm{kz} - \omega\mathrm{t} + \delta\mathrm{y})$$

1. **If $E_{0y} = 0$, then from**

$$E_x(z,t) = E_{0x}\cos(\mathrm{kz} - \omega\mathrm{t} + \delta\mathrm{x})$$

$$E_y(z,t) = 0$$

- **The light is said to be linearly polarized along the x-direction called as linear horizontal polarized light.**

Degenerate States of the Polarization Ellipse 1

2. If $E_{0x} = 0$, then it implies

$$E_x(z,t) = 0 \text{ and}$$

$$E_y(z,t) = E_{0y}\cos(\mathrm{kz} - \omega\mathrm{t} + \delta\mathrm{y})$$

- The linear oscillation is along the y-axis called as linear vertically polarized light.

3. For $\delta = 0, \pi$. Eq. of polarization ellipse reduces to

$$\frac{{E_x}^2(z,t)}{{E_{0x}}^2} + \frac{{E_y}^2(z,t)}{{E_{0y}}^2} \pm 2\frac{E_x(z,t)E_y(z,t)}{E_{0x}E_{0y}} = 0$$

Degenerate States of the Polarization Ellipse 2

$$\left[\frac{E_x(z,t)}{E_{0x}} \pm \frac{E_y(z,t)}{E_{0y}}\right]^2 = 0$$

$$E_y(z,t) = \pm\frac{E_{0y}E_x(z,t)}{E_{0x}}$$

- It is the equation of straight line with slope $\pm\frac{E_{0y}}{E_{0x}}$ and zero intercept.
- For special case

$$E_{0x} = E_{0y}$$

The equation of straight line reduces to

$$E_y(z,t) = \pm E_x(z,t)$$

Degenerate States of the Polarization Ellipse 3

- The positive value represents linear $+45^{\circ}$ polarized light ; called L $+45^{\circ}$P.
- The negative value represents linear -45° polarized light ; written as L-45°P.

4. For $\delta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, the polarization ellipse eq. reduces to

$$\frac{{E_x}^2(z,t)}{{E_{0x}}^2} + \frac{{E_y}^2(z,t)}{{E_{0y}}^2} = 1$$

- This is the equation of an ellipse in standard form.

Degenerate States of the Polarization Ellipse 4

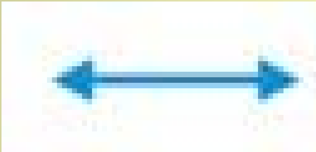
5. For $E_{0x} = E_{0y} = E_0$ and $\delta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, the polarization ellipse equation becomes

$$\frac{{E_x}^2(z,t)}{{E_0}^2} + \frac{{E_y}^2(z,t)}{{E_0}^2} = 1$$

- This describes the equation of a circle in which the light is either right or left circularly polarized.

Degenerate States of the Polarization Ellipse 5

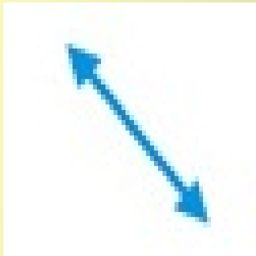
LHP



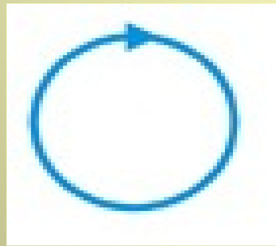
VHP

L+45

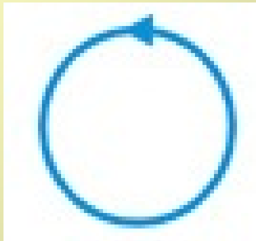
L-45



RCP



LCP



Concluding Polarization Ellipse

- The polarization ellipse can describe any state of completely polarized light. However this representation is inadequate for several reasons.

1. As the beam of light propagates-it traces out an ellipse or some special form of ellipse- circle or st. line- in time interval of the order of 10^{-15} sec. This period is too short to measure.
2. The polarization ellipse cannot describe un-polarized and partial polarized light.

Concluding Polarization Ellipse

3. The polarization ellipse is an amplitude description of polarized light.

The fact that we can only measure the intensity of light and not the amplitudes requires that the polarization ellipse must be transformed so that only intensities are present- observable quantities

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THANK YOU