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Diffraction

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DIFFRACTION

Diffraction requires wave treatment

Light $\rightarrow$ electromagnetic waves

Optics approximation: one Cartesian component of the e.m. field

$$v(p,t)$$

is representative of the entire field
Energy is proportional to the square of this component (power flux α Poynting vector)

Complex form (coherent monocromatic)

$$v(P,t) = u(P)e^{-i\omega t}$$

time dependence:

oscillation with frequency $\nu = \omega / 2\pi$

$\omega \rightarrow$ source

u(P) complex amplitude

complex amplitude

$$u(P) = A(P)\, e^{i\varphi}$$

A amplitude

φ phase

wavefronts: surfaces φ = constant

plane wave

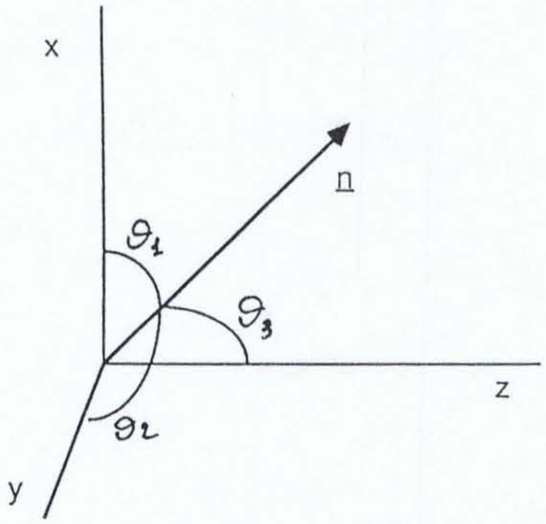
$$u(P) = Ae^{ik(\alpha x + \beta y + \gamma z)}$$

A is constant

$k = \dfrac{\omega}{c}$ is wavenumber

$k = 2\pi/\lambda$ $\quad\lambda$ wavelength

α, β, γ direction cosines of the normal to the wavefront, i. e. of the propagation direction



$$\alpha = \cos\theta_1$$
$$\beta = \cos\theta_2$$
$$\gamma = \cos\theta_3$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

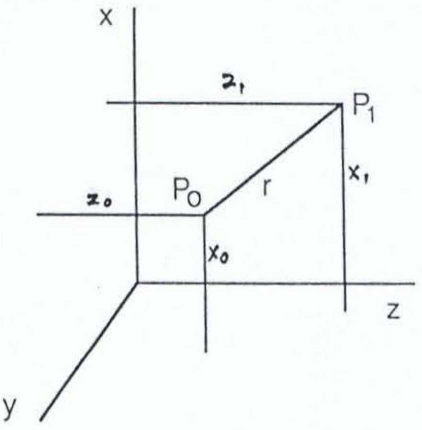
spherical wave

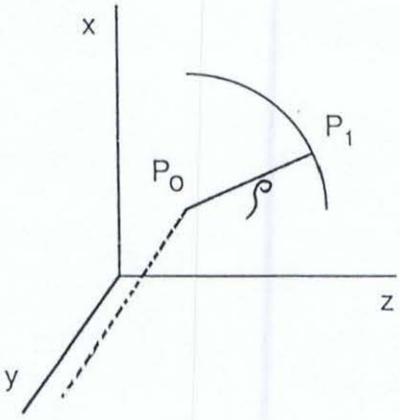
$$u(P) = \frac{A}{r} e^{ikr - i\pi/2}$$

$$r = |P_1 - P_0| = \left[(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2\right]^{1/2}$$

$P_0(x_0, y_0, z_0)$ source

wavefronts: spheres with center in P_0





cylindrical wave

$$u(P) = \frac{A}{\sqrt{\rho}} e^{ik\rho - i\pi/4}$$

$$\rho = \left[(x_0 - x_1)^2 + (z_0 - z_1)^2\right]^{1/2}$$

wavefronts: cylinders with axis parallel to y through P_0 useful for bidimensional cases

GRIMALDI 1660

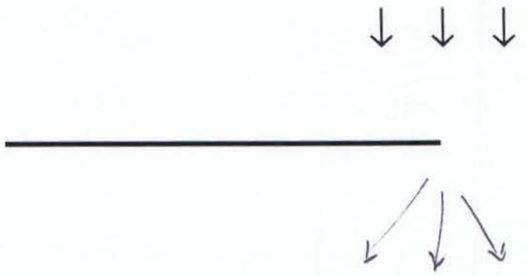
LIGHT propagates

1 - Straight

2 - Reflection

3 - Refraction

4 - Diffraction (go round obstacles)



DIFFRACTION

Occurs where there is an abrupt discontinuity in amplitude[1]

Examples

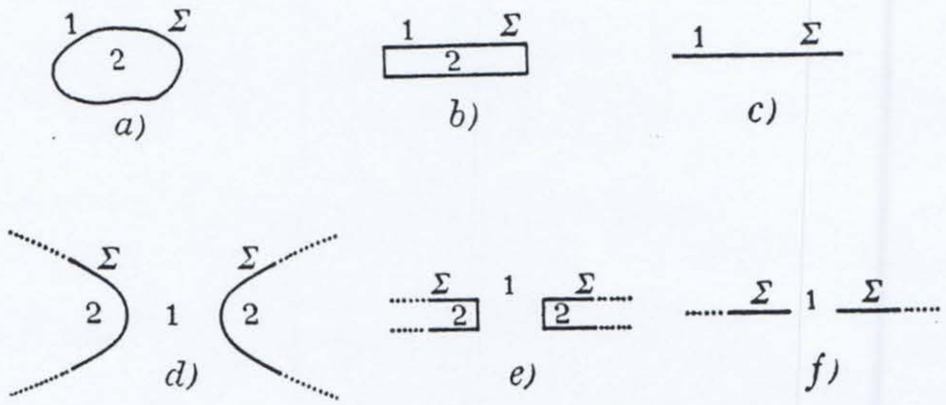


Fig. XII. 1

from: G.Toraldo di Francia and P.Bruscaglioni "Onde Elettromagnetiche" seconda edizione. Zanichelli 1988.

First observation: Grimaldi (1660)

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[1]The key point in the theory is that diffraction takes place <u>where</u> the term $\nabla^2 A$ (∇^2 laplacian) is not negligible with respect to $A\, n^2 k_0^2$ (n refractive index, k_0 wave number in the free space). This implies that amplitude variations (second difference) taking place in the space of a wavelength must be negligible in order to neglect diffraction.

Due to diffraction, light reaches regions behind the screen or the obstacle, that are expected to be in the shade according to ray theory (geometrical optics).

Rigorous theory of diffraction requires:

solutions of Maxwell's equations with appropriate boundary conditions.

Approximate solutions

1) Huygens-Fresnel principle

2) Helmholtz-Kirchhoff theory and formula

3) Plane wave expansion (Toraldo-Duffieux)

Huygens-Fresnel principle $\begin{cases} \text{historically first} \\ \text{simplest} \end{cases}$

Helmholtz-Kirchhoff : general formula, classic

Plane wave expansion: introduction to Fourier optics and holography

1 - Huygens-Fresnel principle

Huygens (1678)
Each point of a wavefront can be considered a source of a spherical wave, "wavelet", propagating in the same direction with the same velocity. The wavefront at a later time is the geometrical "envelope" of the secondary waves.
Fresnel (1818): the different wavelets interfere at each point.

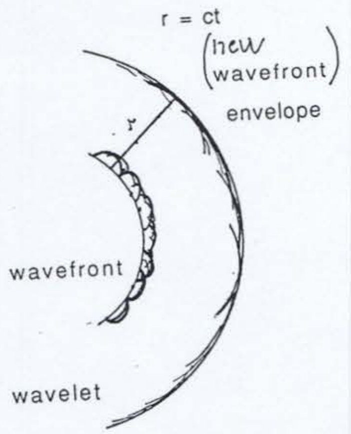


Fig 1

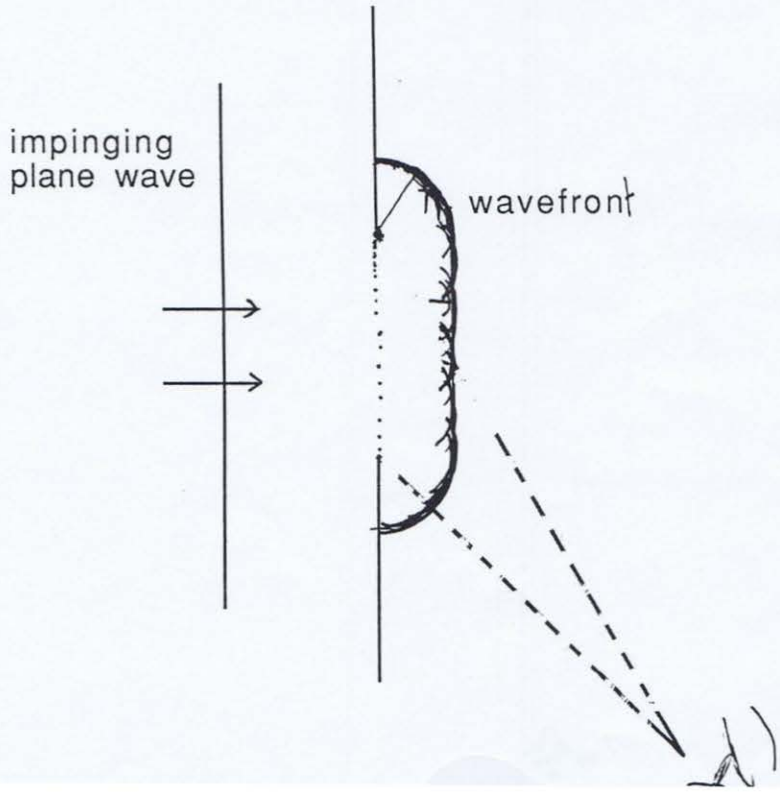
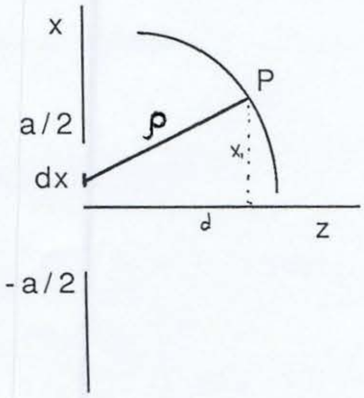


FIG. 2
diffraction: slit in absorbing screen

Use of Huygens-Fresnel principle:

pattern of a slit illuminated by a normally impinging plane wave.
From each point x a cylindrical wave. From each element dx a cylindrical wave of amplitude a(x)dx; a(x) is a linear density. Let us assume $a(x) = \bar{a}$ on the aperture. Its contribution to the field u(P) at point $P = (x_1, d)$ is (apart from a multiplicative factor, see Kirchhoff)

$$\frac{\bar{a}}{\sqrt{\rho}} e^{ik\rho} dx \qquad \rho = \left[(x - x_1)^2 + d^2\right]^{1/2}$$



$$u(P) = \int_{-a/2}^{a/2} \frac{\bar{a}}{\sqrt{\rho}} e^{ik\rho} dx$$

If point P is at a distance d large with respect to both x and x_1 one can develop ρ in a series and stop at the second term in the exponent:

$$k\rho = kd\left[1 + \frac{(x - x_1)^2}{2d^2}\right]^{1/2} \approx kd + \frac{kx^2}{2d} - \frac{kxx_1}{d} + \frac{kx_1^2}{2d}$$

a more accurate analysis could be carried out by considering higher order terms, but it is beyond the scope of the present lessons.

The region where this approximation holds is called Fresnel region.

$$u(P) = \int_{-a/2}^{a/2} \frac{\bar{a}}{\sqrt{\rho}} e^{ik\rho} dx \approx$$

$$\approx \frac{\bar{a}\, e^{\frac{ikx_1^2}{2d}}}{\sqrt{d}}\; e^{ikd} \int_{-a/2}^{a/2} \bar{e}^{ik\frac{xx_1}{d}}\; e^{ik\frac{x^2}{2d}} dx$$

If d is large enough for the maximum value of the exponent (at the borders) to be near zero one can write:

$$e^{i\frac{kx^2}{2d}} \approx 1 \qquad\qquad -\frac{a}{2} \leq x \leq \frac{a}{2}$$

this requires condition

$$\frac{ka^2}{8d} << 1 \qquad\qquad \frac{\pi a^2}{4\lambda d} << 1$$

or, what amounts to the same

$$\frac{a^2}{\lambda d} << 1 \qquad \text{Fraunhofer condition}$$

the region where

$$\frac{a^2}{\lambda d} << 1 \quad \text{Fraunhofer region}^{(2)}$$

Therefore

$$u(P) = K \int_{-a/2}^{a/2} e^{-ik\frac{xx_1}{d}} dx$$

where K is a complex quantity including all terms multiplying integral.
Evaluation of integral:

$$u(P) = K\left[\frac{e^{-ik\frac{xx_1}{d}}}{-\frac{ikx_1}{d}}\right]_{x=-\frac{a}{2}}^{x=a/2} = aK\left[\frac{e^{i\frac{kax_1}{2d}} - e^{-i\frac{kax_1}{2d}}}{2i\frac{kx_1a}{2d}}\right]$$

$$= aK\frac{\sin\frac{kax_1}{2d}}{\frac{kax_1}{2d}} = aK\frac{\sin\left(\frac{\pi a}{\lambda}\frac{x_1}{d}\right)}{\frac{\pi a}{\lambda}\frac{x_1}{d}}$$

$$= aK\,\mathrm{Sinc}\left(\pi\frac{a}{\lambda}\frac{x_1}{d}\right)$$

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[2]note: Fraunhofer condition is opposite to the condition required for geometrical optics to hold $\frac{\lambda}{a}d << a \rightarrow \frac{a^2}{d1} >> 1$

The energy is proportional to u(P) u*(P)

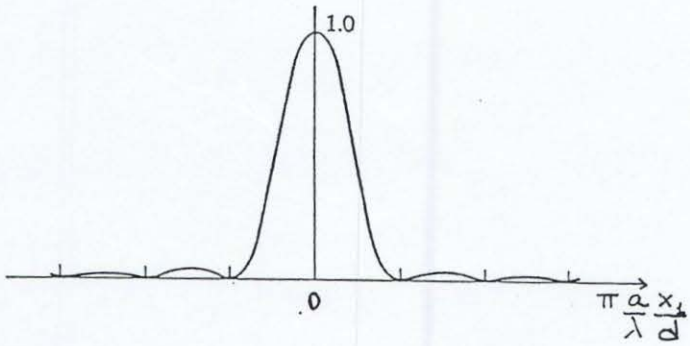
$$u(P)u^*(P) = a^2|K|^2 \frac{\sin^2(\pi \frac{a}{\lambda}\frac{x_1}{d})}{\left[\pi \frac{a}{\lambda}\frac{x_1}{d}\right]^2}$$

Note: $\frac{x_1}{d} = \sin\theta$ angular direction of point P.

This function oscillates, has maximum for x_1=O on the axis, and subsequent zeros and maxima

First zero at

$$\pi \frac{a}{\lambda}\frac{x_1}{d} = \pi$$



$$\theta \approx \frac{x_1}{d} = \frac{\lambda}{a}$$ angular semi width of diffracted beam.

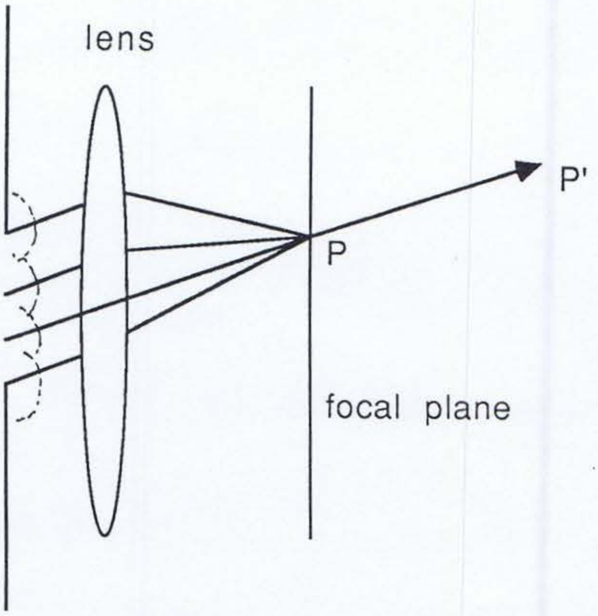
For small aperture, in optics, the Fraunhofer region can be easily reached in a few meters

Example: a=1mm λ=.63 μm

$$d \gg \frac{a^2}{\lambda} \qquad d \gg \frac{10^{-6}m^2}{.63\,10^{-6}m} = 1.6m$$

Note that dependence is with square of aperture: for a= 2 mm , one needs d >> 7 m.

The Fraunhofer diffraction pattern can be easily seen at finite distance by means of a lens (in our case a cylindric lens). The lens transfers at P, in the focal plane, the field of P' at infinity.



If there is no aperture, the border of the lens acts as an aperture, producing diffraction. All instruments present diffraction and give an image of a point source which is a "pattern", not a point. This strongly affects the resolving power of any instrument from microscopes to the larger telescopes. This effect cannot be avoided because it originates from the nature of light.

2 - KIRCHHOFF THEORY

Homogeneous, isotropic, non absorbing medium.

From Maxwell equations any Cartesian component $v(P,t)$ satisfies:

$$\nabla^2 v(P,t) - \frac{1}{v^2}\frac{\partial^2 v(P,t)}{\partial t^2}$$ equation of d'Alembert

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ laplacian operator

v light velocity in the medium

c in empty space

For simplicity empty space

For complex amplitude

$$\nabla^2 u(P) + k^2 u(P) = 0$$ Helmoholtz eq. or wave eq.

$$k = \frac{\omega}{c}$$

Green's Theorem

Let us consider a space V surrounded by a closed surface S.

Let u_1 and u_2 be two scalar functions regular in V and on S.

A regular function is continuous and derivable.

Green's theorem states that:

$$\int_V (u_1 \nabla^2 u_2 - u_2 \nabla^2 u_1)\ dV = \int_S \left(u_1 \frac{\partial u_2}{\partial n} - u_2 \frac{\partial u_1}{\partial n} \right) dS$$

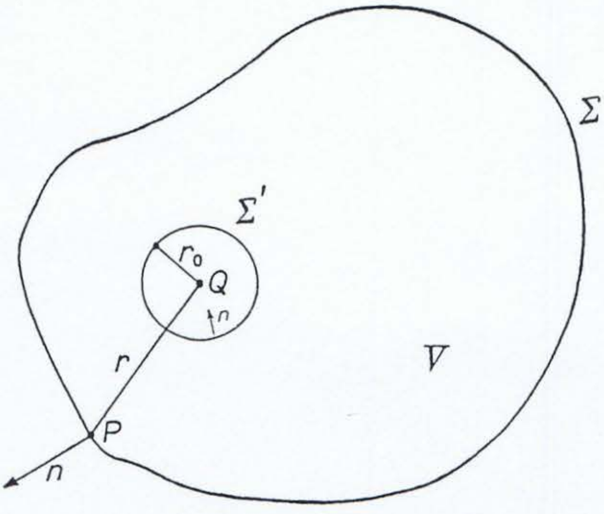
where ∇^2 is the laplacian operator,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and $\partial / \partial n$ denotes partial derivative in normal outward direction at a point on S.

Helmholtz-Kirchhoff formula

Derivation of field at Q from field over Σ.



Σ closed surface

Q point

let Σ' be spherical surface of radius r_0 surrounding Q

$$w = \frac{e^{ikr}}{r}$$

chose a spherical wave centered at Q (Green function)

1) $\nabla^2 w + k^2 w = 0$ wave eq. for w

2) $\nabla^2 u + k^2 u = 0$ wave eq. for complex amplitude

Multiply 1 by u and 2 by w and subtract

$$u\nabla^2 w - w\nabla^2 u = 0$$

Integrate into the space between Σ and Σ'

$$\int_V (u\nabla^2 w - w\nabla^2 u)\, dV = 0$$

from Green's formula

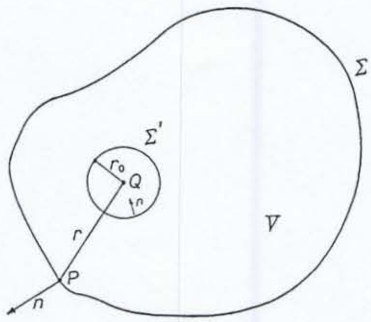
$$1) \quad \int_{\Sigma}\left(w\frac{\partial u}{\partial n}-u\frac{\partial w}{\partial n}\right)d\Sigma+\int_{\Sigma'}\left(w\frac{\partial u}{\partial n}-u\frac{\partial w}{\partial n}\right)d\Sigma'=0$$

one has (over $\Sigma' : \partial w/\partial n = -\partial w/\partial r$)

$$\int_{\Sigma'}\left(w\frac{\partial u}{\partial n}-u\frac{\partial w}{\partial n}\right)d\Sigma=\int_{\Sigma'}\left(-\frac{1}{r}\frac{\partial u}{\partial r}+ik\frac{u}{r}-\frac{u}{r^2}\right)e^{ikr}d\Sigma'=$$

$$=e^{ikr_0}\left(-\frac{1}{r_0}\overline{\frac{\partial u}{\partial r}}+ik\frac{\overline{u}}{r_0}-\frac{\overline{u}}{r_0^2}\right)4\pi r_0^2$$

$\overline{u}$ and $\overline{\frac{\partial u}{\partial r}}$ average values over Σ'



Let $r_0 \to 0$

Three terms: first and second terms $\to 0$

last term $\to -\ 4\ \pi\ u(Q)$

From (1)

$$u(Q)=\frac{1}{4\pi}\int_{\Sigma}\left(w\frac{\partial u}{\partial n}-u\frac{\partial w}{n}\right)d\Sigma$$

and finally : Helmholtz-Kirchhoff formula

$$u(Q) = \frac{1}{4\pi}\int_{\Sigma} \frac{e^{ikr}}{r} \left\{ \frac{\partial u}{\partial n} - \left(ik - \frac{1}{r} \right) u \cos(\underline{n}, \underline{r}) \right\} d\Sigma$$

$\cos(\underline{n}, \underline{r})$= cosinus of the angle between $\underline{n}$ and $\underline{r}$

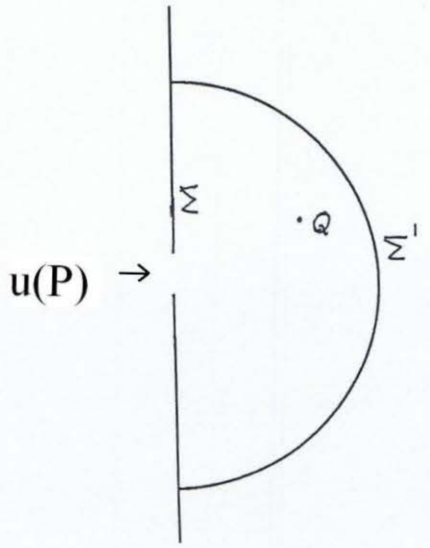
Famous equation derived by Helmholtz (1859)-Kirchhoff gave a more general case (1883).

<u>The value of the field u(Q) at point Q in the volume requires knowledge of the field and its normal derivative on all points of the surface</u> Σ.

This result is not the solution of the problem because it implies knowledge of the field and its normal derivative on Σ, that is solution of the problem on the surface region. Hypothesis for the field on the surface necessary.

Kirchhoff-Diffraction by a plane screen

Aperture in a plane screen illuminated from left



Closed surface: screen Σ+ hemisphere Σ' of large radius R

$$u(Q) = \frac{1}{4\pi} \int_{\Sigma+\Sigma'} \frac{e^{ikr}}{r} \left\{ \frac{\partial u}{\partial n} - \left(ik - \frac{1}{r} \right) u \cos(\underline{n}, \underline{r}) \right\} d\Sigma$$

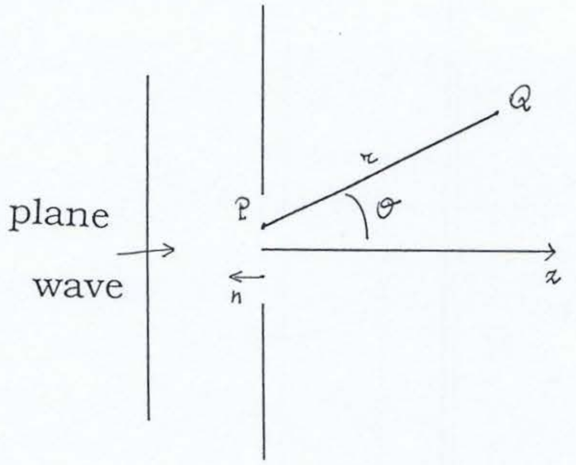
The integral over Σ' requires $(d\Sigma = r^2 d\Omega)$

$$\lim_{r \to \infty} r \left\{ \left(\frac{\partial u}{\partial n} - iku \right) \right\} \to 0$$

This is known as Sommerfeld radiation condition: in practice the field vanishes at infinity at least as a diverging spherical wave.

Assumption over surface Σ:
On the opening the field and its normal derivative have the same values as in the absence of the screen and the values are zero everywhere else.

Example: a plane wave impinging ortogonally on the screen. The aperture is circular



impinging wave

$$u(P) = A\, e^{ikz}$$

$$\frac{\partial u}{\partial n} = -ik\, A\, e^{ikz}$$

$$u(Q) = \frac{1}{4\pi}\int_{\Sigma} \frac{e^{ikr}}{r}\left\{-ik\, A\, e^{ikz} - ikAe^{ikz}\cos(\underline{n},\underline{r})\right\} d\Sigma$$

($1/r << k$, term neglected) Over Σ: $z = 0$

$$= \frac{-ikA}{4\pi}\int_{\Sigma} \frac{e^{ikr}}{r}[1 + \cos(\underline{n},\underline{r})] d\Sigma$$

$$= \frac{-i}{\lambda}\frac{A}{2}\int_{\Sigma} \frac{e^{ikr}}{r}[1 + \cos(\underline{n},\underline{r})] d\Sigma$$

$$u(Q) = \frac{-i}{\lambda}\frac{A}{2}\int_{\Sigma}\frac{e^{ikr}}{r}[1+\cos(\underline{n},\underline{r})]\ d\Sigma$$

express Huygens-Fresnel principle.

- From the aperture elementary spherical waves

- Obliquity factor $(1+\cos(\underline{n},\underline{r}))/2$

- The phase of each spherical wave i s decreased by $\pi/2$ ($e^{-i\pi/2} = -i$) with respect to incident wave

- The amplitude of each elementary wave is smaller by a factor $1/\lambda$ with respect to that, A, of incident wave.

This is a more complete form of Huygens-Fresnel principle valid far from the screen. On screen even at a large distance inconsistency: u(Q) is not zero due to obliquity factor.
The inconsistency was removed in the Rayleigh-Sommerfed theory where a obliquity factor $\cos(\underline{n},\underline{r})$ was found.

Fraunhofer approximation [paraxial rays]

P(x,y) , Q(x_1,y_1,d)

$$r = \left[d^2 + (x - x_1)^2 + (y - y_1)^2\right]^{1/2}$$

$$\approx d\left\{1 + \frac{x^2 - 2xx_1 + x_1^2}{2d^2} + \frac{y^2 - 2yy_1 + y_1^2}{2d^2}\right\}$$

$$\rightarrow d - \frac{xx_1}{d} - \frac{yy_1}{d}$$

$$u(x_1, y_1, d) = \frac{-i}{\lambda}\frac{A}{2d}e^{ikd}\int_\Sigma e^{-ik\frac{xx_1}{d}-ik\frac{yy_1}{d}}[1 + \cos(\underline{n}, \underline{r})]d\Sigma$$

source plane

$$\begin{cases} x = \rho \quad \cos\varphi \\ y = \rho \quad \sin\varphi \end{cases}$$

observation plane

$$\begin{cases} x_1 = \rho_1 \quad \cos\varphi_1 \\ y_1 = \rho_1 \quad \sin\varphi_1 \end{cases}$$

$$\cos(\underline{n}, \underline{r}) \sim 1$$

$$d\Sigma = \rho \quad d\rho \quad d\varphi$$

$$\frac{xx_1}{d} + \frac{yy_1}{d} = \frac{\rho\rho_1}{d}\cos(\varphi - \varphi_1)$$

(discussion about higher order terms)

$$u(Q) = \frac{-i}{\lambda}\frac{Ae^{ikd}}{d}\int_0^a\left\{\int_0^{2\pi} e^{-ik\frac{\rho\rho_1}{d}\cos(\rho-\rho_1)}d\varphi\right\}\rho d\rho$$

Internal integral: Bessel function of zero order *[1]

$$J_0\left(\frac{k}{d}\rho\rho_1\right) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\left[k\frac{\rho\rho_1}{d}\right]\cos(\varphi-\varphi_1)}d\varphi$$

$$u(Q) = \frac{-i}{\lambda}\frac{A\,e^{ikd}}{d}2\pi\int_0^a J_0\left(\frac{k}{d}\rho_1\rho\right)\rho d\rho =$$

Recalling that

$$\int_0^z J_{\nu-1}(t)t^\nu dt = z^\nu J_\nu(z)$$

$$t = \frac{k}{d}\rho_1\rho \qquad dt = \frac{k\rho_1}{d}d\rho$$

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*[1]Bessel function of order n $J_n(z)$ can be defined as

$$J_n(z) = \frac{1}{2\pi}\int_\alpha^{2\pi+\alpha} e^{i(n\theta - z\sin\theta)}d\theta$$

In our case n = 0, change of variable $\varphi = \theta - \alpha$ gives

$$J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{-iz\sin(\varphi+\alpha)}d\varphi$$

choice $\alpha = -\varphi_1 + \frac{\pi}{2}$ gives $\sin(\varphi+\alpha) = \cos(\varphi-\varphi_1)$ and

therefore $J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{-iz\cos(\varphi-\varphi_1)}d\varphi$

one finally has

$$u(Q) = \frac{-iA2\pi a}{\lambda} \frac{e^{ikd}}{d} \frac{J_1(ka \sin\theta)}{(k \sin\theta)}$$

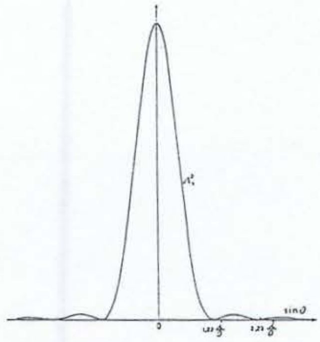
field in Fraunhofer region, distant in the direction θ

$\frac{e^{ikd}}{d}$	spherical wave
$-i = e^{-i\pi/2}$	dephasing factor
$ka = 2\pi \frac{a}{\lambda}$	parameter for angular dependence

energy (intensity) in direction θ apart from unessential constant

$$I = u\,u^{*} \;\propto\; \frac{J_1^2(ka\sin\theta)}{(ka\sin\theta)^2}$$

first four zeros of $J_1(x)$



$x = 3.83\,;\ 7.02\,;\ 10.17\,;\ 13.23$

when $ka\sin\theta = 0$ maximum

when $ka\sin\theta = 3.83$ first zero

$$2\,\frac{a}{\lambda}\sin\theta = \frac{3.83}{3.14} = 1.22$$

Values of subsequent maxima, with respect to the central one

central	1
first	0.0175
second	0.0042
third	0.0016

It can be shown (Rayleigh 1899) that the energy flux the i-th ring is

$$\Phi_i = J_0^2(x_1) - J_0^2(x_{1+1})$$

Through the central disc and subsequent rings
Energy flux (total flux = 1)

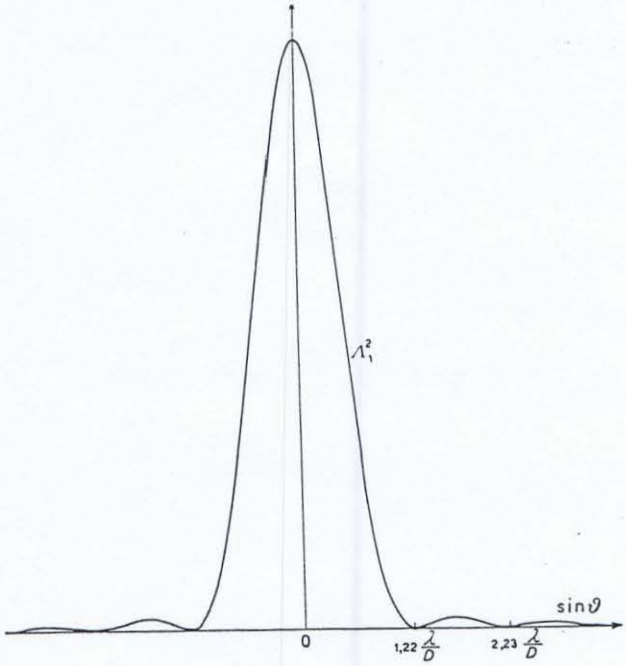
central disc	0.8378
first ring	0.0722
second "	0.0276
third "	0.0147
and so on	

The energy in the central disc of the pattern is ~ 84% of the total.
Energy is mostly concentrated in the central ring, whose total angular with is

$$k\, a \sin\theta = 3.83$$

$$\frac{2a}{\lambda}\sin\theta = \frac{3.83}{3.14} = 1.22$$

$2a = D$ diameter



$$\sin\theta = 1.22\frac{\lambda}{D} \rightarrow \vartheta = 1.22\frac{\lambda}{D}$$

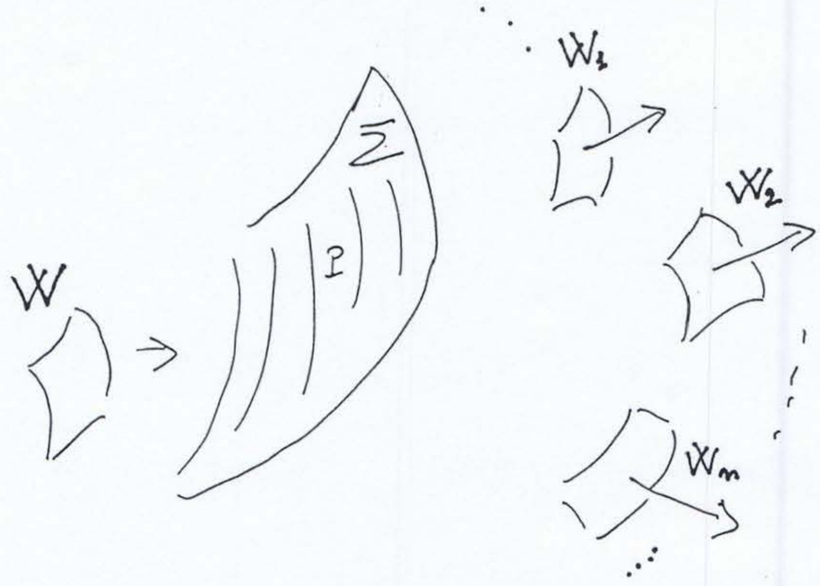
effect on Resolving power of instruments.

3 - Decomposition in plane waves

Diffraction as decomposition in plane waves is the basis of Fourier optics (Duffieux)

The decomposition by the so called "inverse interference principle", Toraldo (1941), and valid for surfaces planar or not.
Inverse interference principle:

A screen is illuminated from the left by a field W, that produces phase $\varphi(P)$ and amplitude $A(P)$ distribution at points P over the output side surface Σ.



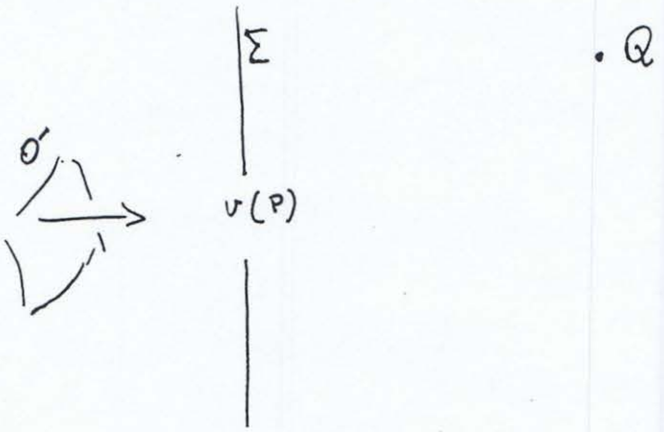
If a system of waves outgoing from Σ is found whose interference produces the field $v(P) = A \exp(i\varphi)$ over Σ, these waves are the true diffracted waves
- uniqueness of the solution

Screen:
-partially transparent: transmitted diffracted waves
-partially refecting: reflected diffracted waves
-both
(eigenfunctions)

Generally: v(P) unknown on the screen
Hypotheses about v(P) necessary.

Examples: amplitude or phase or both
1) A
2) $\varphi_0(P) = \varphi_i(P) + k\Delta(P)$
3) both

Plane screen: amplitude

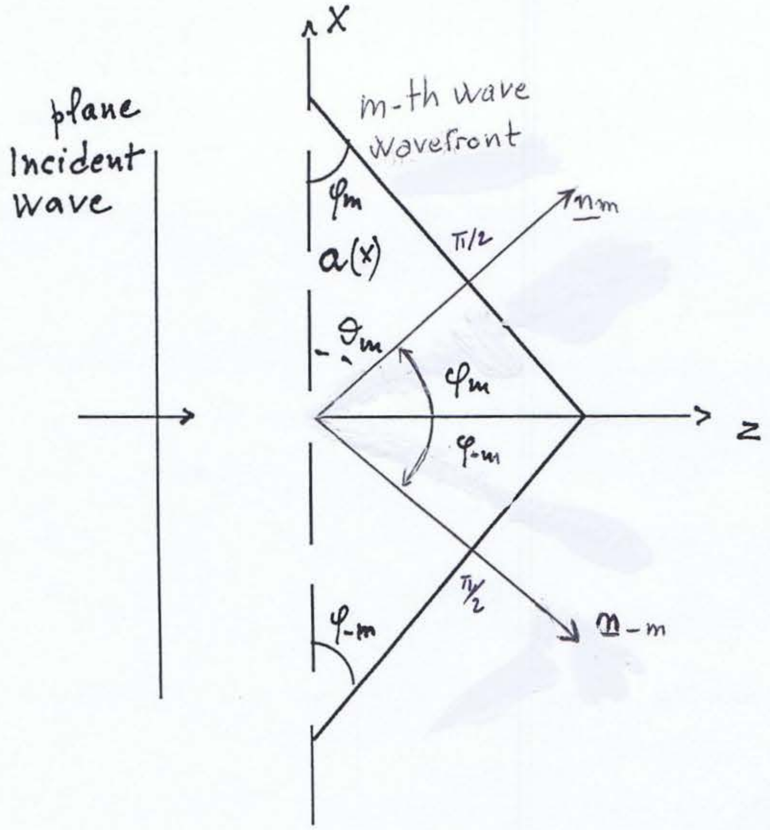


assume: v(P) on the aperture has the same value as in the absence of the screen
no need to know the normal derivative

Periodic aperture: grating

example: unidimensional periodic grating

a(x) periodic, period p



System of plane waves propagating in direction z

Fourier series for a(x)

$$a(x) = \sum_{m=-\infty}^{\infty} A_m \, e^{i2m\pi\frac{x}{p}}$$

$$A_m = \frac{1}{p}\int_{-p/2}^{p/2} a(x)\, e^{-i2m\pi\frac{x}{p}}\, dx$$

A system of plane waves $\alpha_m = \cos\theta_m = \sin\varphi_m$

$$v(x,z) = \sum_{m=-\infty}^{\infty} B_m \; e^{ik(\alpha_m x + \gamma_m z)}$$

Condition $v(x,0) = a(x)$ gives

$$v(x,0) = \sum_{m=-\infty}^{\infty} B_m \; e^{ik\alpha_m x}$$

Comparison with a(x) gives

$$2m\pi\frac{x}{p} = k\alpha_m x \;\rightarrow\; \alpha_m = m\frac{\lambda}{p}$$

$B_m = A_m$ $\gamma_m \geq 0$

$$v(x,z) = \sum_{m=-\infty}^{\infty} A_m \; e^{ikm\frac{\lambda}{p}x} \; e^{ik\sqrt{1-\frac{m^2\lambda^2}{p^2}}\,z}$$

for $\alpha_m = m\frac{\lambda}{p} \leq 1 \;\rightarrow\; m \leq \frac{p}{\lambda}$ <u>real waves</u>

$\alpha_m = \frac{m\lambda}{p} > 1 \;\rightarrow\; m > \frac{p}{\lambda}$ <u>evanescent waves</u>

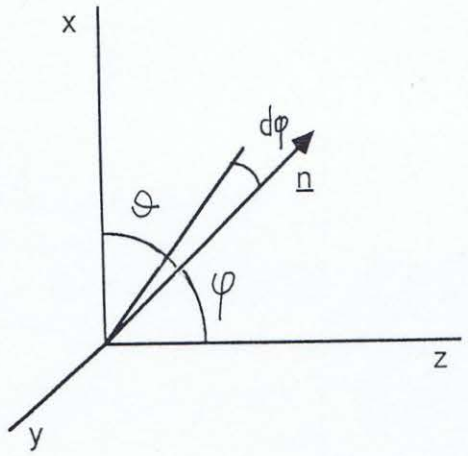
N=2max integer $\left(\frac{p}{\lambda}\right) + 1$ =number of real waves

$\varphi_m \approx m\frac{\lambda}{p}$ for smal φ_m

PLANE SCREEN : UNIDIMENSIONAL CASE

Plane screen xy with transparency or opening; symmetric with respect to y. Complex amplitude on the screen a(x)

Plane wave of unit amplitude



$$e^{ik(\alpha x+\beta y+\gamma z)}$$

$$\alpha^2+\beta^2+\gamma=1$$

$$\alpha=\cos\theta$$
$$\beta=\cos\psi$$
$$\gamma=\cos\varphi$$

Let us construct a continuous system of plane waves. In our case no dependence on y $\rightarrow \beta=0$

Let us consider a $\mathrm{d}\varphi$. Let

$$A\,\mathrm{d}\varphi$$

the amplitude of all waves having propagation direction between φ and $\varphi+\mathrm{d}\varphi$

$$\alpha=\sin\varphi \qquad \mathrm{d}\alpha=\cos\varphi\ \ \mathrm{d}\varphi=\sqrt{1-\alpha^2}\mathrm{d}\varphi$$

$$\mathrm{d}\varphi=\frac{\mathrm{d}\alpha}{\sqrt{1-\alpha^2}}$$

$$v(x,z)=\int\frac{A\,e^{ik(\alpha x+\gamma z)}}{\sqrt{1-\alpha^2}}\mathrm{d}\alpha$$

continuous system of diffracting waves

on the aperture $v(x,0)=a(x)$

Let us write a(x) by its Fourier transform $\overline{A}(f)$

$$\overline{A}(f) = \int_{-\infty}^{\infty} a(s)\; e^{-2\pi i f s} ds$$

$$a(x) = \int_{-\infty}^{\infty} e^{2\pi i f x}\, \overline{A}(f) df = \int_{-\infty}^{\infty} e^{2\pi i f x} \left(\int_{\infty}^{\infty} a(s)\; e^{-2\pi i f s} ds \right) df$$

Let $\boxed{f = \alpha/\lambda}$

$$a(x) = \frac{1}{\lambda} \int_{-\infty}^{\infty} e^{2\pi i \frac{\alpha}{\lambda} x} \left(\int_{-\infty}^{\infty} a(s)\; e^{-\frac{2\pi i}{\lambda} \alpha s} ds \right) d\alpha$$

On the aperture the integrale of plane waves

$$v(x,0) = \int_{-\infty}^{\infty} \frac{A}{\sqrt{1-\alpha^2}}\, e^{ik\alpha x}\, d\alpha$$

must have <u>infinite limits</u> (real and evanescent waves) and $A = A(\alpha)$

$$\frac{A}{\sqrt{1-\alpha^2}} = \frac{1}{\lambda} \int_{-\infty}^{\infty} a(s)\; e^{-\frac{2\pi i}{\lambda} \alpha s} ds$$

or

$$A(\alpha) = \frac{\sqrt{1-\alpha^2}}{\lambda} \int_{-\infty}^{\infty} a(x)\; e^{-ik\alpha x} dx$$

In terms of φ $A(\alpha) \rightarrow A(\varphi)$

$$A(\varphi) = \frac{\cos\varphi}{\lambda} \int_{-\infty}^{\infty} a(x)\, e^{-ik\sin\varphi x} dx$$

by denoting

$$A(f) = \frac{\lambda}{\sqrt{1-\alpha^2}} A(\alpha) = \int_{-\infty}^{\infty} a(x)\, e^{-i2\pi f x} dx$$

and recall that $f = \frac{\alpha}{\lambda} = \frac{\sin\varphi}{\lambda}$

then $A(f) \rightarrow$ Fourier transform of $a(x)$

The amplitude $A(f)$ of the wave diffracted in the direction φ is the component at frequency $f = \frac{\sin\varphi}{\lambda}$ of the Fourier transform of $a(x)$

In other words the system of diffracted waves is the Fourier Transform of the field $a(x)$ on the screen. Diffraction $\rightarrow$ Fourier transform

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Transform relationship

$$a(x) = \int_{-\infty}^{\infty} A(f)\, e^{2\pi i f x} df \qquad A(f) = \int_{-\infty}^{\infty} a(x)\, e^{-2\pi i f x} dx$$

ENERGY FLUX - POYNTING VECTOR

Let us recall Parceval theorem for transforms

$$\int_{-\infty}^{\infty} a(\xi)\, b^*(\xi) d\xi = \int_{-\infty}^{\infty} A(f)\, B^*(f) df$$

where A(f) Fourier transform of $a(\xi)$ and B(f) of $b(\xi)$

if $a(\xi) = b(\xi)$

$$\int_{-\infty}^{\infty} a(x) a^*(x) dx = \int_{-\infty}^{\infty} A(f) A^*(f) df$$

$$A(f) = \frac{\lambda A(\alpha)}{\sqrt{1-\alpha^2}} = \frac{\lambda A(\varphi)}{\cos\varphi} \qquad \alpha = \sin\varphi$$

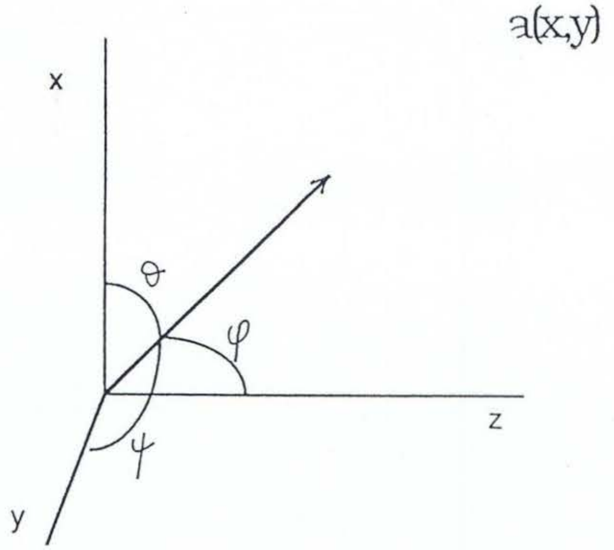
$$\int_{-\infty}^{\infty} a(x) a^*(x) dx = \lambda \int_{\sin\varphi=-\infty}^{\sin\varphi=\infty} \frac{A(\varphi) A^*(\varphi)}{\cos\varphi} d\varphi$$

<u>left side term:</u> energy per unit time transmitted per unit y through the screen

<u>right side term</u>: the energy per unit y carried by each wave is that which a wave of intensity $A(\varphi)A^*(\varphi)d\varphi$ carries through a slit of width $\lambda / \cos\varphi$

PLANE SCREEN Bidimensional case

Analogous to unidimensional case



$$\alpha = \cos \vartheta$$
$$\beta = \cos \psi$$
$$\gamma = \cos \varphi$$

One choses a system of plane waves of any direction. Amplitude in small solid angle $Ad\Omega$, $d\Omega$ solid angle, A amplitude density

$$A(\alpha, \beta) \frac{d\alpha d\beta}{\sqrt{1 - \alpha^2 - \beta^2}}$$

$$v(x, y, z) = \iint \frac{A(\alpha, \beta)}{\sqrt{1 - \alpha^2 - \beta^2}} e^{ik(\alpha x + \beta y + \gamma z)} d\alpha d\beta$$

on the aperture

$$v(x,y,0) = a(x,y)$$

a(x,y) is Fourier transformed

Procedure analogous to previous case gives the same result.

Diffracted waves are a continuous system of plane waves and evanescent waves. The diffracted field is the Fourier transform of the field on the aperture, with frequencies

$$f_x = \frac{\alpha}{\lambda} \text{ and } f_y = \frac{\beta}{\lambda}$$

respectively. Therefore

in the space direction θ, ψ

specified by α and β ($\alpha = \cos\theta$, $\beta = \cos\psi$), one has:

frequency components $f_x = \frac{\alpha}{\lambda}$ and $f_y = \frac{\beta}{\lambda}$ respectively.

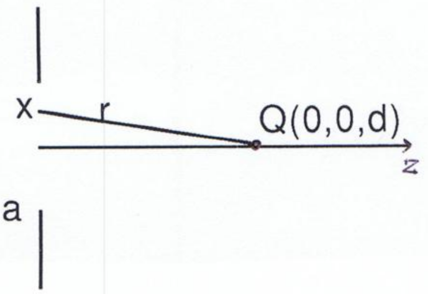
Limited aperture: no upper limit to diffraction angles and always presence of evanescent waves. A well known property of Fourier transform: if the support of the function is finite the support of the transform is infinite.

In systems: loss of information

HOLOGRAPHY easy to explain with expansion in plane waves

FRESNEL ZONES

Bidimensional screen
Circular aperture
Uniformly illuminated: A



Q(0,0,d) on axis

$$r = \left[x^2 + y^2 + d^2\right]^{1/2}$$

$$1 + \cos(\underline{n}, \underline{r}) \approx 2$$

$$\cos 1^\circ = 0.9998$$
$$\cos 5^\circ = 0.996$$
$$\cos 10^\circ = 0.985$$

$$u(Q) = -\frac{iA}{\lambda}\int_\Sigma \frac{e^{ikr}}{r} d\Sigma$$

$$r \approx d\left[1 + \frac{1}{2}\frac{x^2 + y^2}{d^2}\right]$$

Fresnel region

$$u(Q) = -\frac{iA}{\lambda}\frac{e^{ikd}}{d}\int_\Sigma e^{ik\frac{x^2+y^2}{2d}} d\Sigma$$

$$= \frac{-iAe^{ikd}}{\lambda d}\int_{\rho=0}^{a}\int_{\theta=0}^{2\pi} e^{i\frac{k\rho^2}{2d}}\rho d\rho d\theta$$

$$x = \rho \cos\theta$$
$$y = \rho \sin\theta$$
$$d\Sigma = \rho \, d\rho d\theta$$

$$= -\frac{iA}{\lambda}\frac{e^{ikd}}{d} \, 2\pi\left[\frac{e^{i\frac{k\rho^2}{2d}}}{2\frac{ik}{2d}}\right]_0^a$$

$$u(Q) = A\, e^{ikd} \left[1 - e^{i\frac{ka^2}{2d}}\right]$$

$$\underset{\substack{\text{impinging}\\ \text{wave}}}{\Uparrow} \qquad \underset{\substack{\text{multiplicative}\\ \text{factor } T}}{\Uparrow}$$

$$T = 1 - e^{ika^2/2d} = 1 - \cos\frac{ka^2}{2d} - i\sin\frac{ka^2}{2d}$$

For a given d, phase proportional to a^2

$ka^2/2d$	a^2	T
0	0	0 (real)
π	λd	2 (real)
2π	$2\lambda d$	0 (real)
3π	$3\lambda d$	2 (real)

and so on.

$$a^2 = n\lambda d$$

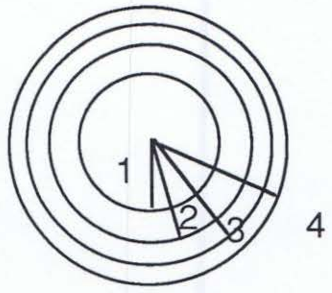
$$a = \sqrt{n\lambda d}$$

Fresnel zone of order n

$a^2 = n\lambda d$ a radius of n-th Fresnel zone

central circle
then rings

contribution from each zone cancels contribution from the preceding one



IF THE AREA CONTAINS

1) an odd number of Fresnel zones, at point Q field is maximum = 2 times the incident wave.
2) an even number: zero field at Q.

Moving along axis (d) field $\rightarrow$ maxima and zeros.

Soret grating - zone grating - is based on removing even zones and has focussing properties.

The Fraunhofer approximation $(a^2 / \lambda d \ll 1)$ requires that a small portion of the first Fresnel zone is seen from point Q, at distance d.

For the case of a slit: see Goodman. Fresnel integrals.

BABINET PRINCIPLE: DIFFRACTION

by COMPLEMENTARY APERTURES

screen with circular aperture	opaque disc

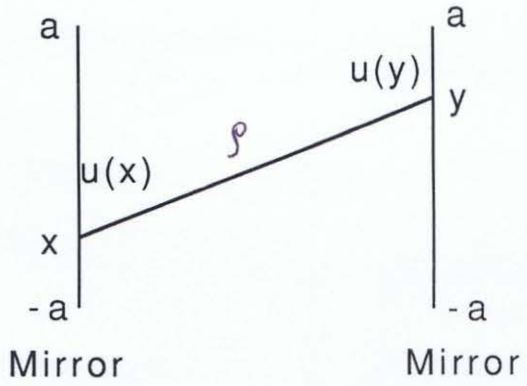
Solution of one problem allows solution of the other one

From the field diffracted from an aperture one derives the field diffracted by the complementary aperture by: adding a phase = π to all diffracted waves and adding to them a wave equal to the incident wave.

Therefore : apart from the phases, the ensemble of the waves diffracted by two complementary screens differ only for the central wave

OPEN CAVITIES FOR LASERS

Devices based on diffraction: Large Fresnel Numbers



example: bidimensional case

$$u(y) = \int_{-a}^{a} e^{-i\pi/4} u(x) \frac{e^{ik\rho}}{\sqrt{\lambda\rho}} dx$$

$u_m(y) = \sigma_m u_m(x)$ modes' eq.

σ_m complex quantity

$1 - |\sigma_m|^2 \rightarrow$ loss of m-th mode

$\arg \sigma_m \rightarrow$ phase shift