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Fourier transforms

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Two simple calculations:

1) $I = \int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}} dx \quad , \quad a > 0$

$$I^2 = \int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}} dx \int_{-\infty}^{\infty} e^{-\frac{ay^2}{2}} dy = \iint_{-\infty}^{\infty} e^{-\frac{a(x^2+y^2)}{2}} dx\,dy$$

change to polars

$$= \int_0^{\infty}\int_0^{2\pi} e^{-\frac{ar^2}{2}} r\,d\theta\,dr = 2\pi \int_0^{\infty} e^{-\frac{ar^2}{2}} r\,dr = 2\pi \int_0^{\infty} e^{-au} du$$

$u = \frac{r^2}{2}$, $du = r\,dr$

$$= 2\pi \left. \frac{e^{-au}}{-a} \right|_0^{\infty} = \frac{2\pi}{a} \qquad \therefore I = \int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}} dx = \sqrt{\frac{2\pi}{a}}$$

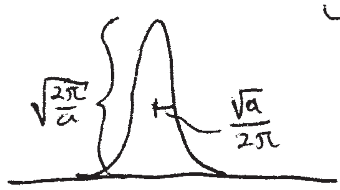
2) $D(\nu) = \int_{-\infty}^{\infty} e^{i2\pi x\nu} dx = \int_{-\infty}^{\infty} \underbrace{\left[\lim_{a\to 0} e^{-\frac{ax^2}{2}}\right]}_{1} e^{i2\pi x\nu} dx$

$$= \lim_{a\to 0} \int_{-\infty}^{\infty} e^{-\frac{ax^2}{2} + i2\pi x\nu} dx = \lim_{a\to 0} \int_{-\infty}^{\infty} e^{-\frac{a}{2}\left[x^2 - 2x\left(\frac{2\pi i \nu}{a}\right)\right]} dx$$

$$= \lim_{a\to 0} \int_{-\infty}^{\infty} e^{-\frac{a}{2}\left[\underbrace{x - \frac{2\pi i\nu}{a}}_{x'}\right]^2} dx\; e^{\frac{a}{2}\left(\frac{2\pi i\nu}{a}\right)^2} = \lim_{a\to 0} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{a}{2}x'^2} dx'}_{\sqrt{\frac{2\pi}{a}}}\; e^{-\frac{2\pi^2\nu^2}{a}}$$

$dx' = dx$

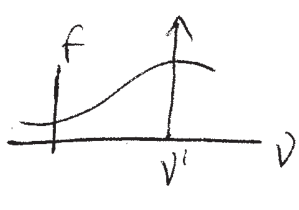
$$= \lim_{a\to 0} \sqrt{\frac{2\pi}{a}}\, e^{-\frac{2\pi^2\nu^2}{a}}$$



$$\int_{-\infty}^{\infty} D(\nu)\,d\nu = \lim_{a\to 0} \sqrt{\frac{2\pi}{a}} \sqrt{\frac{2\pi a}{(2\pi)^2}} = 1$$

so $D(\nu) = \int_{-\infty}^{\infty} e^{i2\pi x\nu} dx = \delta(\nu)$

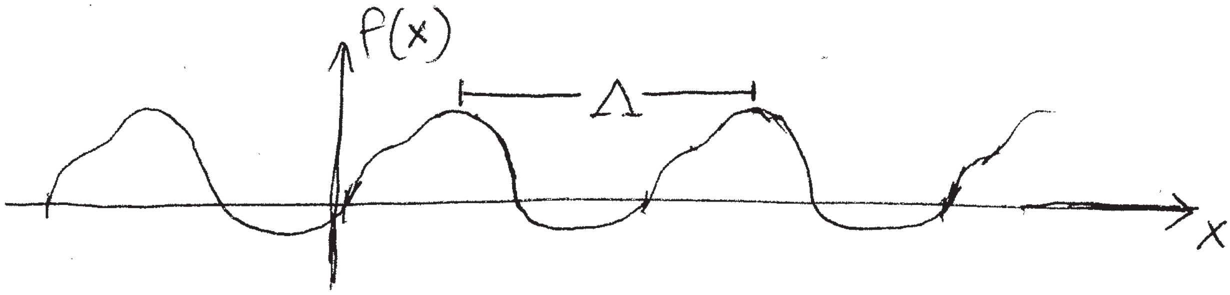
Property: $\int_{-\infty}^{\infty} f(\nu)\,\delta(\nu - \nu')\,d\nu = f(\nu')$



$[\delta(\nu)] = [\frac{1}{\nu}]$, $\quad \delta(a\nu) = \frac{\delta(\nu)}{\;}$

Fourier Series

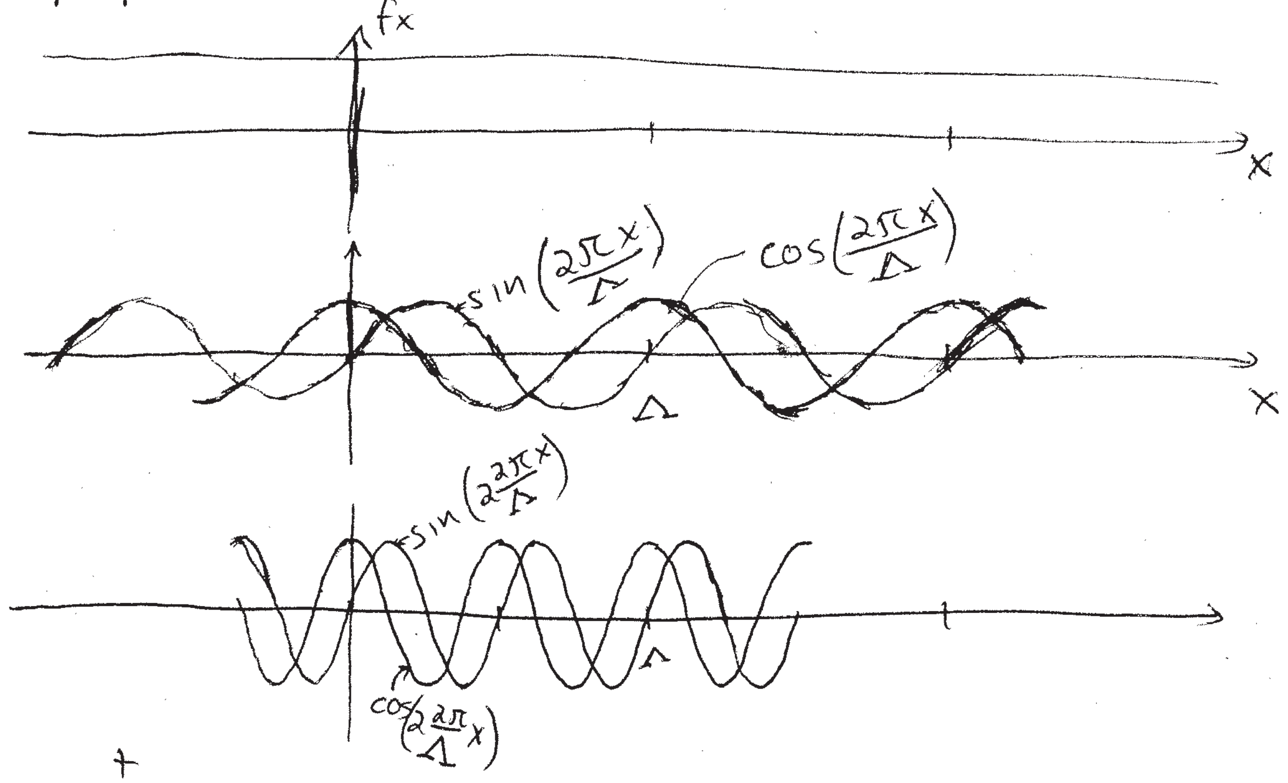
Periodic function:



$$f(x + m\Delta) = f(x), \quad m = \text{integer}$$

Fourier theorem:

Any periodic function can be expressed as a superposition of sinusoidal functions:



$+$

$\vdots$

Propose

$$f(x) = a_0 + \sum_{m=1}^{\infty} a_m \cos\left(\frac{2\pi m x}{\Delta}\right) + \sum_{m=1}^{\infty} b_m \sin\left(\frac{2\pi m x}{\Delta}\right)$$

Recall:

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

so we can rewrite

$$f(x) = \underbrace{a_0}_{C_0} + \sum_{m=1}^{\infty} \underbrace{\left(\frac{a_m}{2} + \frac{b_m}{2i}\right)}_{C_m} e^{\frac{i2\pi mx}{\Lambda}} + \sum_{m=1}^{\infty} \underbrace{\left(\frac{a_m}{2} - \frac{b_m}{2i}\right)}_{C_{-m}} e^{-\frac{i2\pi mx}{\Lambda}}$$

$$\boxed{F(x) = \sum_{m=-\infty}^{\infty} C_m e^{\frac{i2\pi mx}{\Lambda}}}$$

Fourier series
or
Fourier Synthesis.

Notice that, if $f(x)$ is real a_m & b_m are real,

so $\underline{C_{-m} = C_m^*}$

Finding the Coefficients

Consider $\displaystyle\int_{x_0-\frac{\Lambda}{2}}^{x_0+\Lambda/2} f(x)\,dx =$

$= \Lambda C_0$

so $\displaystyle C_0 = \frac{1}{\Lambda}\int_{x_0-\frac{\Lambda}{2}}^{x_0+\frac{\Lambda}{2}} f(x)\,dx$ = average of the function.

Now try

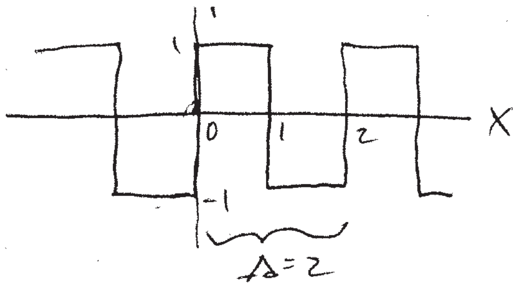
$$\frac{1}{\Lambda}\int_{x_0-\frac{\Lambda}{2}}^{x_0+\frac{\Lambda}{2}} f(x)\, e^{-i\frac{2\pi m' x}{\Lambda}}\,dx =$$

$= C_{m'}$

so $$\boxed{C_m = \frac{1}{\Lambda}\int_{x_0-\frac{\Lambda}{2}}^{x_0+\frac{\Lambda}{2}} f(x)\, e^{-\frac{i2\pi mx}{\Lambda}}\,dx}$$

Fourier analysis

Example:



$C_0 = 0$

$$C_m = \frac{1}{2}\int_{-1}^{1} f(x)\, e^{-i\cancel{2}\pi m x/\cancel{2}}\, dx$$

$$= \frac{1}{2}\int_{-1}^{0} (-1)\, e^{-i\pi m x}\, dx + \frac{1}{2}\int_{0}^{1} (1)\, e^{-i\pi m x}\, dx$$

$$= \frac{1}{2}\left[-\frac{e^{-i\pi m x}}{-i\pi m}\Bigg|_{-1}^{0} + \frac{e^{-i\pi m x}}{-i\pi m}\Bigg|_{0}^{1} \right]$$

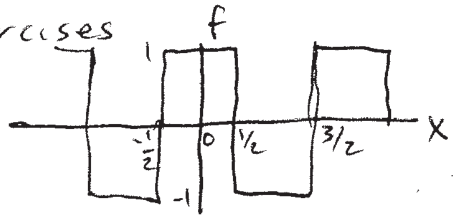
$$= \frac{i}{2\pi m}\left[e^{-i\pi m} - 1 - 1 + e^{i\pi m} \right]$$

$$= \frac{i}{2\pi m}\left[\underbrace{\cos(\pi m)}_{\substack{1 \text{ for even } m \\ -1 \text{ for odd } m}} - 1 \right] = \begin{cases} 0, & m = \text{even} \\ \dfrac{-i}{\pi m}, & m \text{ odd} \end{cases}$$

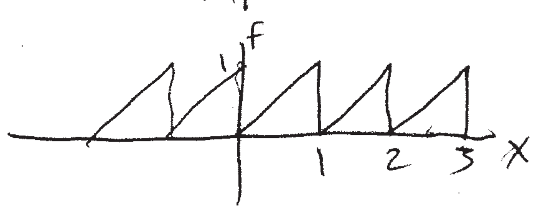
Plot series from $-M$ to M. Play.

Excercises

a)



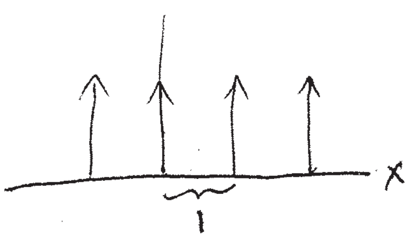
b)



c) $\cos(ax)$, $\sin(ax)$

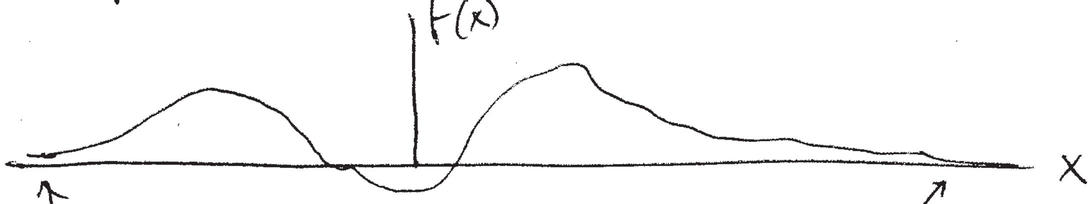
d) $\cos^2(ax)$

e) Dirac comb



Fourier transforms

Nonperiodic function:



must go to zero at infinity, such that

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \text{finite.}$$

Fourier theorem is still valid, but there is no periodicity to constrain the allowed frequencies, so in principle we need all of them.

Propose

$$\boxed{f(x) = \int_{-\infty}^{\infty} \tilde{f}(\nu)\, e^{i2\pi\nu x} d\nu}$$

Fourier Synthesis or Inverse Fourier Transformation

To find $\tilde{f}$, as with series, consider

$$\int_{-\infty}^{\infty} f(x)\, e^{-i2\pi\nu' x} dx = \iint_{-\infty}^{\infty} \tilde{f}(\nu)\, e^{i2\pi\nu x} d\nu\, e^{-i2\pi\nu' x} dx$$

$$= \int_{-\infty}^{\infty} \tilde{f}(\nu) \underbrace{\int_{-\infty}^{\infty} e^{i2\pi(\nu - \nu')x} dx}_{\delta(\nu-\nu')} d\nu = \tilde{f}(\nu')$$

so

$$\boxed{\tilde{f}(\nu) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi\nu x} dx}$$

Fourier analysis or Fourier transformation

Other conventions:

$$\hat{f}(p) = \sqrt{\frac{k}{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikxp}\, dx$$

$$f(x) = \sqrt{\frac{k}{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(p)\, e^{ikxp}\, dp$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{-i\omega t}\, d\omega$$

Properties 1 Let $\hat{\mathfrak{f}}_{x\to\nu}[f(x)] = \tilde{f}(\nu)$

• Shift $\hat{\mathfrak{f}}_{x\to\nu}[f(x-a)] = \int_{-\infty}^{\infty} f(\underbrace{x-a}_{x'})\, e^{-i2\pi x\nu}\, dx$

$$= \int_{-\infty}^{\infty} f(x')\, e^{-i2\pi(x'+a)\nu}\, dx' = e^{-i2\pi a\nu} \int_{-\infty}^{\infty} f(x')\, e^{-i2\pi x'\nu}\, dx'$$

$$= \tilde{f}(\nu)\, e^{-i2\pi a\nu}$$

• Phase $\hat{\mathfrak{f}}_{x\to\nu}[f(x)\, e^{iax}] = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi x(\nu - \frac{a}{2\pi})}\, dx$

$$= \tilde{f}\left(\nu - \frac{a}{2\pi}\right)$$

• Scale $\hat{\mathfrak{f}}_{x\to\nu}[f(ax)] = \int_{-\infty}^{\infty} f(\underbrace{ax}_{x'})\, e^{-i2\pi x\nu}\, dx$

$$= \int_{-\infty\,\mathrm{sgn}(a)}^{\infty\,\mathrm{sgn}(a)} f(x')\, e^{-i2\pi x'\frac{\nu}{a}}\, \frac{dx'}{a}$$

$$= \frac{\mathrm{sgn}(a)}{a}\, \tilde{f}\left(\frac{\nu}{a}\right) = \frac{\tilde{f}\left(\frac{\nu}{a}\right)}{|a|}$$

• Derivative $\hat{\mathcal{F}}_{x\to\nu}[f'(x)] = \int_{-\infty}^{\infty} f'(x)\, e^{-i2\pi x\nu}\, dx$

integrate by parts

$u = e^{-i2\pi x\nu}$ $\quad dv = f'\,dx$

$du = -i2\pi\nu e^{-i2\pi x\nu}\,dx$, $\quad v = f$

$$= f(x)e^{-i2\pi x\nu}\Big|_{-\infty}^{\infty} \to 0 \quad -(-i2\pi)\nu \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi\nu x}\,dx$$

$$= 2\pi i\nu\, \tilde{f}(\nu)$$

By using this repeatedly, can show

$$\hat{\mathcal{F}}_{x\to\nu}\left[f^{(n)}(x)\right] = (2\pi i\nu)^n\, \tilde{f}(\nu)$$

• Powers of x: $\hat{\mathcal{F}}_{x\to\nu}[x^n f(x)] = \int_{-\infty}^{\infty} f(x)\, \underbrace{x^n e^{-i2\pi x\nu}}_{\frac{1}{(2\pi i)^n}\frac{d^n}{d\nu^n} e^{-i2\pi x\nu}}\, dx$

$$= \left(\frac{i}{2\pi}\right)^n \frac{d^n}{d\nu^n}\int_{-\infty}^{\infty} f(x)e^{-i2\pi x\nu}\,dx = \left(\frac{i}{2\pi}\right)^n \tilde{f}^{(n)}(\nu)$$

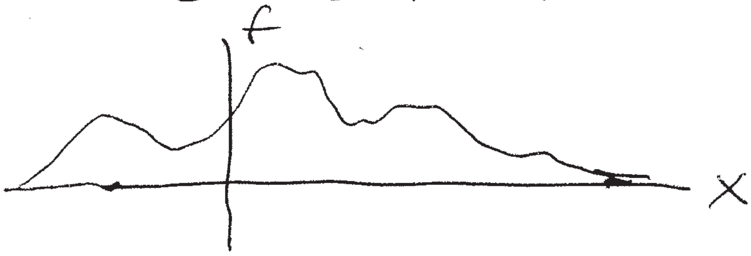
• Real functions: if $f(x) = f^*(x)$, then

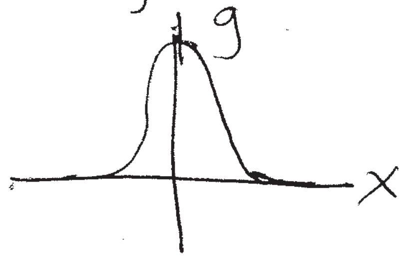
$$\tilde{f}^*(\nu) = \left[\int_{-\infty}^{\infty} f(x)\, e^{-i2\pi x\nu}\,dx\right]^* = \int_{-\infty}^{\infty} \underbrace{f^*(x)}_{f(x)}\, e^{i2\pi x\nu}\,dx$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi x(-\nu)}\,dx = \tilde{f}(-\nu)$$

Convolution:

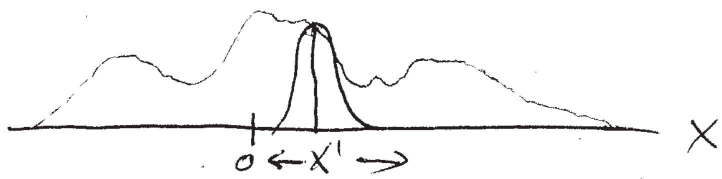
Consider two functions $f(x)$ & $g(x)$.





The convolution of f & g corresponds to a "blurring" of f with g:

$$f * g(x) = \int_{-\infty}^{\infty} f(x')\, g(x-x')\, dx'$$



Notice that this operation is commutative, i.e. that it can also considered as a blurring of g with f:

$$f * g = \int_{-\infty}^{\infty} f(x')\, g(\underbrace{x-x'}_{})\, dx'$$

$x'' = x - x'$, $dx'' = -dx'$

$$= -\int_{\infty}^{-\infty} f(x-x'')\, g(x'')\, dx'' = \int_{-\infty}^{\infty} f(x-x'')\, g(x'')\, dx''.$$

Norm: $\|f\|$

the squared norm of f is defined as

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

For optical fields, it is associated with total power.

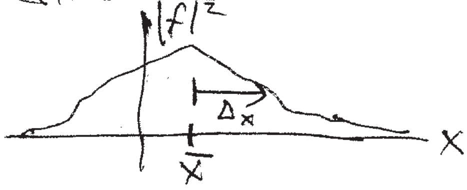
Similarly

$$\|\tilde{f}\|^2 = \int_{-\infty}^{\infty} |\tilde{f}(\nu)|^2 d\nu$$

Centroid :

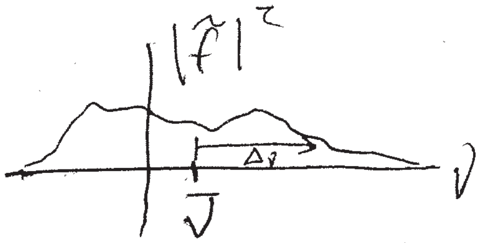
The centroid of a function is defined as

$$\bar{x} = \frac{\int_{-\infty}^{\infty} x |f(x)|^2 dx}{\|f\|^2}$$



similarly

$$\bar{\nu} = \frac{\int_{-\infty}^{\infty} \nu |\tilde{f}(\nu)|^2 d\nu}{\|\tilde{f}\|^2}$$



Standard deviation (measure of spread or width)

$$\Delta_x = \left[\frac{\int_{-\infty}^{\infty} (x-\bar{x})^2 |f(x)|^2 dx}{\|f\|^2} \right]^{1/2}$$

$$\Delta_\nu = \left[\frac{\int_{-\infty}^{\infty} (\nu-\bar{\nu})^2 |\tilde{f}(\nu)|^2 d\nu}{\|\tilde{f}\|^2} \right]^{1/2}$$

Properties 2

- Parseval's theorem

$$\|\tilde{f}\|^2 = \int_{-\infty}^{\infty} \tilde{f}^*(\nu) f(\nu)\, d\nu = \int_{-\infty}^{\infty} \tilde{f}^*(\nu) \int_{-\infty}^{\infty} f(x) e^{-i2\pi x\nu} dx\, d\nu$$

$$= \int_{-\infty}^{\infty} f(x) \left[\int_{-\infty}^{\infty} \tilde{f}(\nu) e^{i2\pi x\nu} d\nu \right] dx = \int_{-\infty}^{\infty} f(x) f^*(x) dx$$

$$= \|f\|^2$$

so the norm is the same for $\tilde{f}$ as for f.

- Product: $\hat{\mathfrak{F}}_{x\to\nu}\left[f(x)g(x)\right] = \int_{-\infty}^{\infty} f(x) g(x) e^{-i2\pi x\nu} dx$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \tilde{f}(\nu') e^{i2\pi x\nu'} d\nu' \right] g(x) e^{-i2\pi x\nu} dx$$

$$= \int_{-\infty}^{\infty} \tilde{f}(\nu') \int_{-\infty}^{\infty} g(x)\, e^{-i2\pi x(\nu-\nu')} dx\, d\nu'$$

$$= \int_{-\infty}^{\infty} \tilde{f}(\nu')\, \tilde{g}(\nu-\nu')\, d\nu' = \tilde{f} * \tilde{g}(\nu)$$

- Convolution: $\hat{\mathfrak{F}}_{x\to\nu}\left[f*g\right] = \iint_{-\infty}^{\infty} f(x') g(\underbrace{x-x'}_{x''}) e^{-i2\pi x\nu} dx\,dx'$ $\quad dx = dx''$

$$= \iint_{-\infty}^{\infty} f(x') g(x'') e^{-i2\pi(x'+x'')\nu} dx' dx'' = \int_{-\infty}^{\infty} f(x') e^{-i2\pi x'\nu} dx' \int_{-\infty}^{\infty} g(x'') e^{-i2\pi x''\nu} dx''$$

$$= \tilde{f}(\nu)\, \tilde{g}(\nu)$$

- Heisenberg uncertainty relation

$$\Delta_x \Delta_\nu \geqslant \frac{1}{4\pi},$$

where the equality holds only for Gaussian functions

Proof of uncertainty relation

Let, for now, $\bar{x}=0$, $\bar{\nu}=0$, so

$$\Delta_x^2 = \frac{\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx}{\|f\|^2}, \quad \Delta_\nu^2 = \frac{\int_{-\infty}^{\infty} \nu^2 |\tilde{f}(\nu)|^2 d\nu}{\|f\|^2}$$

By parseval's theorem

$$\int_{-\infty}^{\infty} \nu^2 |\tilde{f}|^2 d\nu = \int_{-\infty}^{\infty} |\nu \tilde{f}|^2 d\nu = \int_{-\infty}^{\infty} |\hat{\mathfrak{F}}^{-1}(\nu \tilde{f})|^2 d\nu$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} |f'(x)|^2 dx,$$

So

$$\Delta_\nu^2 = \frac{\int_{-\infty}^{\infty} |f'(x)|^2 dx}{(2\pi)^2 \|f\|^2}$$

Now, consider the integral:

$$I = \iint_{-\infty}^{\infty} \left| x_1 f(x_1) f'(x_2) - x_2 f(x_2) f'(x_1) \right|^2 dx_1 dx_2.$$

Because this is the integral of a nonnegative quantity, it must be nonnegative:

$$I \geq 0.$$

Now use the fact that $|a|^2 = a^* a$:

$$I = \iint_{-\infty}^{\infty} \left[x_1 f^*(x_1) f'^*(x_2) - x_2 f^*(x_2) f'^*(x_1)\right]\left[x_1 f(x_1) f'(x_2) - x_2 f(x_2) f'(x_1)\right] dx_1 dx_2$$

$$I = \iint_{-\infty}^{\infty} \Big[x_1^2 |f(x_1)|^2 |f'(x_2)|^2 - x_1 x_2 f^*(x_1) f'(x_1) f(x_2) f'^*(x_2)$$

$$- x_1 x_2 f(x_1) f'^*(x_1) f^*(x_2) f'(x_2) + x_2^2 |f(x_2)|^2 |f'(x_1)|^2 \Big] dx_1 dx_2$$

$$= \int_{-\infty}^{\infty} x_1^2 |f(x_1)|^2 dx_1 \int_{-\infty}^{\infty} |f'(x_2)|^2 dx_2 - \int_{-\infty}^{\infty} x_1 f^*(x_1) f'(x_1) dx_1 \int_{-\infty}^{\infty} x_2 f(x_2) f'^*(x_2) dx_2$$

$$- \int_{-\infty}^{\infty} x_1 f(x_1) f'^*(x_1) dx_1 \int_{-\infty}^{\infty} x_2 f^*(x_2) f'(x_2) dx_2 + \int_{-\infty}^{\infty} |f'(x_1)|^2 dx_1 \int_{-\infty}^{\infty} x_2^2 |f(x_2)|^2 dx_2$$

but $$\int_{-\infty}^{\infty} x_n^2 |f(x_n)|^2 dx_n = ||f||^2 \Delta_x^2$$

$$\int_{-\infty}^{\infty} |f'(x_n)|^2 dx_n = ||f||^2 \Delta_\nu^2 (2\pi)^2$$

Let us define:

$$A = \int_{-\infty}^{\infty} x_n f^*(x_n) f'(x_n) dx_n$$

then

$$I = ||f||^4 \Delta_x^2 \Delta_\nu^2 - AA^* - A^*A + ||f||^4 \Delta_x^2 \Delta_\nu^2$$

$$= 2||f||^4 \Delta_x^2 \Delta_\nu^2 - 2|A|^2 \geq 0$$

so

$$\Delta_x^2 \Delta_\nu^2 \geq \frac{|A|^2}{(2\pi)^2 ||f||^4}$$

Note that

$$A = \int_{-\infty}^{\infty} x f^*(x) f'(x) dx \quad \text{integrate by parts}$$

$u = xf^*$ $\quad$ $dv = f' dx$

$du = (f^* + xf'^*) dx$ $\quad$ $v = f$

$$A = \cancel{x f^*(x) f(x) \Big|_{-\infty}^{\infty}} \to 0 - \int_{-\infty}^{\infty} \left(f^*(x) + x f'^*(x)\right) f(x)\, dx$$

$$= - \int_{-\infty}^{\infty} |f(x)|^2 dx - \int_{-\infty}^{\infty} x f(x) f'^*(x)\, dx$$

$$= - \|f\|^2 - A^*$$

so $\underbrace{A + A^*}_{2\,\mathrm{Re}\{A\}} = - \|f\|^2$

$$\mathrm{Re}\{A\} = \frac{-\|f\|^2}{2}$$

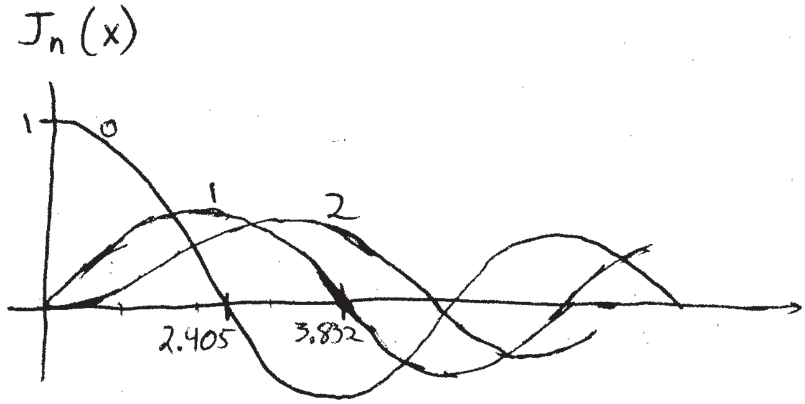
$$\frac{|A|^2}{\|f\|^4} = \frac{(\mathrm{Re}\{A\})^2 + (\mathrm{Im}\{A\})^2}{\|f\|^4} = \frac{1}{4} + \frac{(\mathrm{Im}\{A\})^2}{\|f\|^4} \geq \frac{1}{4}$$

so

$$\Delta_x^2 \Delta_\nu^2 \geq \frac{|A|^2}{(2\pi)^2 \|f\|^4} \geq \frac{1}{4(2\pi)^2}$$

$\therefore$ $\Delta_x^2 \Delta_\nu^2 \geq \frac{1}{4(2\pi)^2}$ and $\boxed{\Delta_x \Delta_\nu > \frac{1}{4\pi}}$

Bessel functions of the first kind



They are solutions of

$$x^2 J_n'' + x J_n' + (x^2 - n^2) J_n = 0$$

- For $x \ll 1$

$$J_n(x) \approx \frac{x^n}{2^n n!}$$

- For $x \gg 1$

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

- Parity:

$$J_{-n}(x) = (-1)^n J_n(x) \quad , \quad J_n(-x) = (-1)^n J_n(x)$$

- Integral form

$$J_n(x) = \frac{1}{2\pi i^n} \int_0^{2\pi} e^{i(x\cos\theta + n\theta)} d\theta$$

- Jacobi-Anger expansion

$$e^{ix\cos\theta} = \sum_{n=-\infty}^{\infty} J_n(x)\, i^n e^{in\theta}$$

- Closure relation

$$\int_0^\infty x\, J_n(ux)\, J_n(vx)\, dx = \frac{1}{u}\, \delta(u-v)$$

- Derivative identity

$$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)$$

- Integral identity (from previous)

$$\int_0^x x'^n J_{n-1}(x')\, dx' = x^n J_n(x)$$

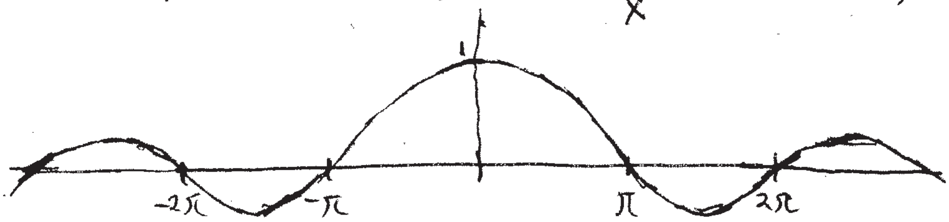
- Recursion relation

$$J_{n+1} + J_{n-1} = \frac{2n J_n}{x}$$

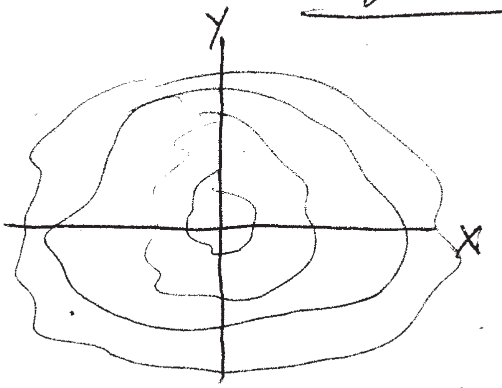
- Spherical Bessel functions

$$j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+\frac{1}{2}}(x)$$

In particular $j_0(x) = \dfrac{\sin x}{x} = \mathrm{sinc}(x)$



2D Fourier transform



$f(x,y)$

let $\underline{x} = (x,y)$

$$\tilde{f}(\underline{\nu}) = \iint_{-\infty}^{\infty} f(\underline{x})\, e^{-i2\pi \underline{x}\cdot\underline{\nu}} \underbrace{dx\,dy}_{d^2x}$$ Fourier Transform

where $\underline{\nu} = (\nu_x, \nu_y)$

$$f(\underline{x}) = \iint_{-\infty}^{\infty} \tilde{f}(\underline{\nu})\, e^{i2\pi \underline{x}\cdot\underline{\nu}} \underbrace{d\nu_x\, d\nu_y}_{d^2\nu}$$ Inverse Fourier Transform

Similar properties:

- Shift $\hat{\mathfrak{f}}_{\underline{x}\to\underline{\nu}}\left[f(\underline{x}-\underline{a})\right] = \tilde{f}(\underline{\nu})\, e^{-i2\pi \underline{a}\cdot\underline{\nu}}$
- Phase: $\hat{\mathfrak{f}}_{\underline{x}\to\underline{\nu}}\left[f(\underline{x})\, e^{i\underline{a}\cdot\underline{x}}\right] = \tilde{f}\left(\underline{\nu} - \frac{\underline{a}}{2\pi}\right)$
- Scale: $\hat{\mathfrak{f}}_{\underline{x}\to\underline{\nu}}\left[f(a\underline{x})\right] = \frac{\tilde{f}\left(\frac{1}{a}\underline{\nu}\right)}{a^2}$
- Gradient: $\hat{\mathfrak{f}}_{\underline{x}\to\underline{\nu}}\left[\nabla f(\underline{x})\right] = 2\pi i\, \underline{\nu}\, \tilde{f}(\underline{\nu})$
- Powers of x, y: $\hat{\mathfrak{f}}_{\underline{x}\to\underline{\nu}}\left[x^m y^n f(\underline{x})\right] = \left(\frac{i}{2\pi}\right)^{m+n} \frac{\partial^m}{\partial \nu_x^m} \frac{\partial^n}{\partial \nu_y^n} \tilde{f}(\underline{\nu})$
- Real Functions: if $f(\underline{x}) = f^*(\underline{x})$ then $\tilde{f}^*(\underline{\nu}) = \tilde{f}(-\underline{\nu})$
- Parseval: $\|f\|^2 = \iint_{-\infty}^{\infty} |f(\underline{x})|^2 d^2x = \iint_{-\infty}^{\infty} |\tilde{f}(\underline{\nu})|^2 d^2\nu = \|\tilde{f}\|^2$

- Product: $\hat{\mathcal{F}}_{\underline{x}\to\underline{\nu}}\left[f(\underline{x})\,g(\underline{x})\right] = \iint_{-\infty} \tilde{f}(\underline{\nu}')\,\tilde{g}(\underline{\nu}-\underline{\nu}')\,d^2\nu' = \tilde{f}*\tilde{g}(\underline{\nu})$
- Convolution $\hat{\mathcal{F}}_{\underline{x}\to\nu}\left[f*g(\underline{x})\right] = \tilde{f}(\underline{\nu})\,\tilde{g}(\underline{\nu})$
- Uncertainty $\quad \Delta_x \Delta_\nu \geq \dfrac{1}{2\pi}$

where $\Delta_x^2 = \dfrac{\iint_{-\infty}^{\infty} (x^2+y^2)\,|f(\underline{x})|^2\,d^2x}{\|f\|^2}$

$$\Delta_\nu^2 = \frac{\iint_{-\infty}^{\infty} (\nu_x^2+\nu_y^2)\,|\tilde{f}(\underline{\nu})|^2\,d^2\nu}{\|f\|^2}$$

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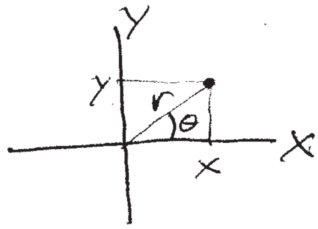
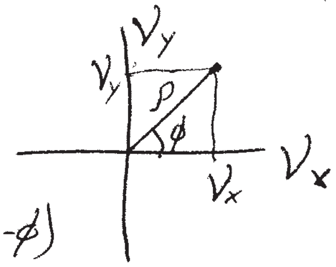
functions with rotational symmetry

$$f(\underline{x}) = f(|\underline{x}|) = f(r)$$



$$\tilde{f}(\underline{\nu}) = \iint_{-\infty}^{\infty} f(|\underline{x}|)\,e^{-i2\pi\underline{x}\cdot\rho}\,d^2x$$



change to polars

$d^2x = r\,dr\,d\theta, \quad \underline{x}\cdot\underline{\nu} = r\rho\cos(\theta-\phi)$

$$\tilde{f}(\underline{\nu}) = \int_0^{2\pi}\int_0^{\infty} f(r)\,e^{-i2\pi r\rho\cos(\underbrace{\theta-\phi}_{\theta'})}\,r\,dr\,d\theta$$

$$= \int_0^{\infty} f(r) \int_{-\phi}^{-\phi+2\pi} e^{-i2\pi r\rho\cos\theta'}\,d\theta'\,r\,dr$$

$$= \int_0^{\infty} f(r) \int_0^{2\pi} e^{-i2\pi r\rho\cos\theta'}\,d\theta'\,r\,dr$$

$$\tilde{F}(\underline{\nu}) = \int_0^\infty f(r) \underbrace{\int_0^{2\pi} e^{-i2\pi r\rho\cos\theta'} d\theta'}_{2\pi J_0(2\pi r\rho)} r\,dr$$

indep. of ϕ

$$\boxed{\tilde{F}(\rho) = 2\pi \int_0^\infty f(r)\, J_0(2\pi r\rho)\, r\,dr}$$

Fourier-Bessel transform
or
Hankel transform

Can show similarly

$$\boxed{f(r) = 2\pi \int_0^\infty \tilde{F}(\rho)\, J_0(2\pi r\rho)\, \rho\,d\rho}$$

Inverse FB transform
or
Inverse Hankel transform

Check:

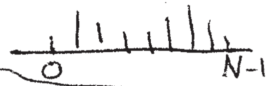
$$f(r) = 2\pi \int_0^\infty 2\pi \int_0^\infty f(r')\, J_0(2\pi r'\rho)\, r'dr'\, J_0(2\pi r\rho)\, \rho\,d\rho$$

$$= \int_0^\infty f(r') \underbrace{\left[(2\pi)^2 \int_0^\infty J_0(2\pi r'\rho)\, J_0(2\pi r\rho)\, \rho\,d\rho\; r'\right]}_{\delta(r-r')} dr'$$

$$= f(r) \checkmark$$

Discrete Fourier transform (DFT)

Instead of $f(x)$ we have f_n, $n=0,\dots,N-1$



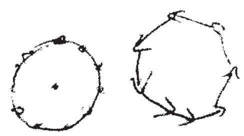
$$\boxed{F_m = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_n \, e^{-i2\pi \frac{mn}{N}}}$$

Discrete Fourier transform

Inverse: try:

$$f_{n'} = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} F_m e^{i2\pi \frac{mn'}{N}}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} f_n \underbrace{\sum_{m=0}^{N-1} e^{i2\pi \frac{(n'-n)m}{N}}}_{N\delta_{n'-n}}$$



So:

$$\boxed{f_n = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} F_m \, e^{i2\pi \frac{mn}{N}}}$$

Inverse Discrete Fourier transform

Approximating FT with DFT

(Notice that the sums can be shifted,) if we define

$$f_{n-N} = f_n$$

$$F_m = \frac{1}{\sqrt{N}} \sum_{n=-\lfloor \frac{N-1}{2} \rfloor}^{\lfloor \frac{N}{2} \rfloor} f_n \, e^{-i2\pi \frac{mn}{N}}$$

Let f_n be a sampling of $f(x)$:

$$f_n = f(n\Delta x)$$

$$F_m = \frac{1}{\sqrt{N}} \sum_{n=-\lfloor \frac{N-1}{2} \rfloor}^{\lfloor \frac{N}{2} \rfloor} f(n\Delta x) \, e^{-i2\pi \frac{mn}{N}}$$

For very large N, and small Δx, can approximate the sum as an integral

$$F_m \approx \frac{1}{\sqrt{N}} \int_{-X_1}^{X_2} f(x)\, e^{-i2\pi m \frac{x}{N\Delta x}} \frac{dx}{\Delta x}$$

where $n\Delta x \rightarrow x$

$$X_1 = \left\lfloor \frac{N-1}{2} \right\rfloor \Delta x, \quad X_2 = \left\lfloor \frac{N}{2} \right\rfloor \Delta x$$

Assume $\underset{\text{big}}{N}\, \underset{\text{small}}{\Delta x} = \text{big} >> \text{width of } f(x)$.

note $X_1 \approx X_2 \approx N\frac{\Delta x}{2} = \text{big}$.

Then

$$F_m \approx \frac{1}{\sqrt{N}\,\Delta x} \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi x \left(\frac{m}{N\Delta x}\right)} dx$$

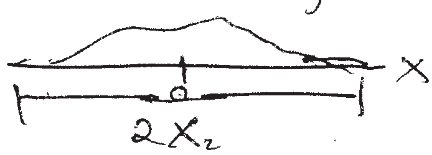
$$= \frac{\tilde{f}\left(\frac{m}{N\Delta x}\right)}{\sqrt{N}\,\Delta x}$$

So the sampling distance in ν is $\frac{1}{N\Delta x} \approx \frac{1}{2X_2}$

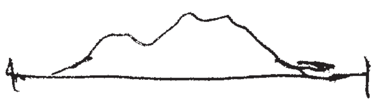
where $2X_2$ is the width over which we're sampling $f(x)$.

Therefore:

- To increase resolution in $\tilde{f}(\nu) \longrightarrow$ must increase range in $f(x)$

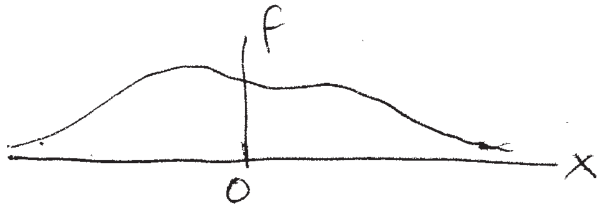


- To increase range in $\tilde{f}(\nu)$ and avoid aliasing $\longrightarrow$ must decrease sampling spacing in $f(x)$

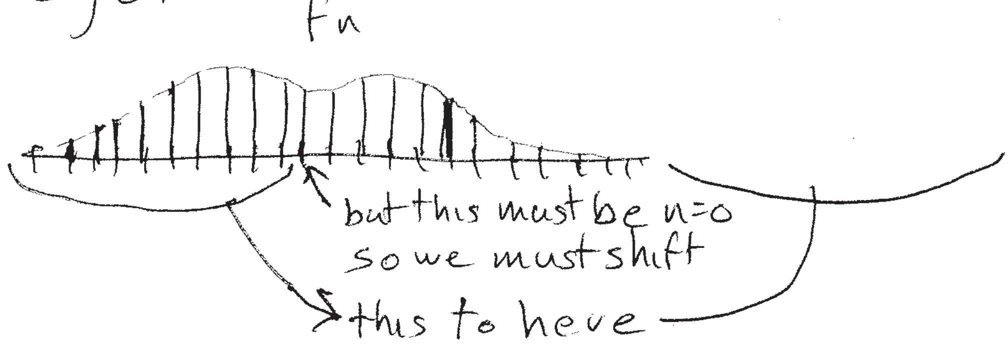


Shifting the functions.

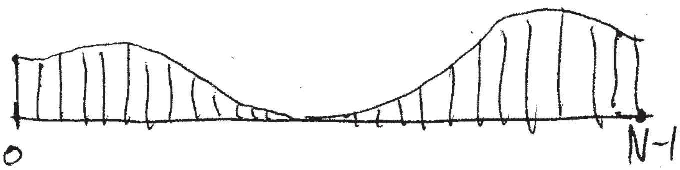
Notice that, if we sample:



we get



so we get



→ this is f_n

Similarly, once we get F_m, it will look like

To reconstruct $\tilde{f}(\nu)$ we must cut the second half and place it before the first. We also need to multiply by $\sqrt{N}\Delta x$.

Fast Fourier transform (FFT)

Notice that the, for each m, the DFT involves the sum of N terms. Since m runs from 0 to $N-1$, then N^2 must be performed. The time of computation can therefore be expected to be proportional to N^2.

The FFT is an algorithm for performing the DFT whose time of computation is proportional to $N\log N$. While it can work for any N, its simplest form can be understood if $N = 2^M$ (so that $M = \log_2 N$):

$$F_m = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_n e^{-\frac{i2\pi nm}{N}} = \frac{1}{\sqrt{N}} \Big[\underbrace{\sum_{n'=0}^{\frac{N}{2}-1} f_{2n'} e^{-\frac{i2\pi(2n')m}{N}}}_{\text{terms with even } n} + \underbrace{\sum_{n'=0}^{\frac{N}{2}-1} f_{(2n'+1)} e^{-\frac{i2\pi(2n'+1)m}{N}}}_{\text{terms with odd } n} \Big]$$

write as $2\frac{N}{2}$

$$= \frac{1}{\sqrt{2}} \Big[\underbrace{\frac{1}{\sqrt{\frac{N}{2}}} \sum_{n'=0}^{\frac{N}{2}-1} f_{2n'} e^{-\frac{i2\pi n'm}{(N/2)}}} + e^{-\frac{i2\pi m}{N}} \underbrace{\frac{1}{\sqrt{\frac{N}{2}}} \sum_{n'=0}^{\frac{N}{2}-1} f_{(2n'+1)} e^{-\frac{i2\pi n'm}{(N/2)}}} \Big]$$

Each of these two sums is itself a DFT of size $\frac{N}{2}$.

They can be joined:

$$F_m = \frac{1}{\sqrt{N}} \sum_{n'=0}^{\frac{N}{2}-1} \left(f_{2n'} + e^{-\frac{i2\pi m}{N}} f_{(2n'+1)} \right) e^{-\frac{i2\pi n'm}{(N/2)}}$$

The same separation can be done M times.

2D DF

$$\boxed{F_{m_1,m_2} = \frac{1}{N} \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} f_{n_1,n_2} \, e^{-i2\pi \frac{(m_1 n_1 + m_2 n_2)}{N}}}$$

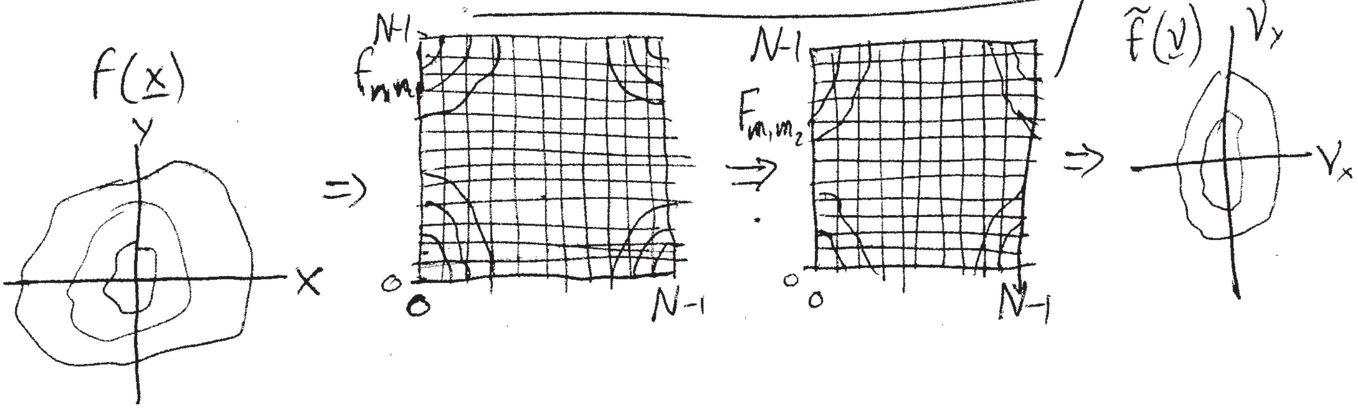
$$\boxed{f_{n_1,n_2} = \frac{1}{N} \sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} F_{m_1,m_2} \, e^{i2\pi \frac{(m_1 n_1 + m_2 n_2)}{N}}}$$

Using 2D DFT to approximate 2D FT:

if $f_{n_1,n_2} = f(n_1 \Delta x, n_2 \Delta x)$,

and $N\Delta x$ is bigger than width of f, then:

$$F_{m_1,m_2} \approx \frac{1}{N\Delta x^2} \tilde{f}\left(\frac{m_1}{N\Delta x}, \frac{m_2}{N\Delta x}\right)$$



Fast Fourier transform: time $\propto N^2 \log N$