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Basic Optics: Waves and Interference

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Basic Optics

In a region free from charges and currents, and ferromagnetic materials, Maxwell Equations reduce to

1)
$$\mathrm{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\mathrm{curl}\mathbf{B} = \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\mathrm{curl} = \nabla \times \quad (\times \text{ or } \wedge \text{ vectorial product})$$
$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$$

By using well known relation: $\mathrm{curl}\ \mathrm{curl}\ \mathbf{E} = \mathrm{grad}\ \mathrm{div}\ \mathbf{E} - \nabla^2 \mathbf{E}$

Where Laplacian ∇^2

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

One obtains

2) $$\nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$ D'Alembert Equation

Scalar Approximation

A transverse Cartesian component, v = v(P,t) of **E** or **B** is representative of the complete e.m. field. Scalar approximation is also called optics approximation.

$v^2 \propto |\mathbf{S}|$ v square is proportional to modulus of Poynting vector, **S**. It is denoted by I, intensity, and is proportional to power flux.

Monochromatic radiation, frequency of the order of $0.5\ 10^{+15}$ Hertz

Linearity,
Complete systems.

For component v

3) $$\nabla^2 v - \frac{1}{V^2}\frac{\partial^2 v}{\partial t^2} = 0 \qquad \text{where} \qquad \mu\varepsilon = 1/V^2$$

Choice of coordinate system.
Separation of variables.

Method of complex exponentials

Let us remember that the e.m. field is real quantity. For instance one solution of Eq.3 is

4) $$v_1(P,t) = A(P)\cos\left[\omega t - \varphi(P)\right]$$

Use of complex exponentials helps with mathematics and allows one to find two simultaneous solutions. Let us write:

5) $$v(P,t) = A(P)\, e^{i\varphi(P)} e^{-i\omega t} = u(P) e^{-i\omega t}$$
$$u(P) = A(P)\, e^{i\varphi(P)}$$

where u(P) is called complex amplitude.
It is immediately verified that real part of v(P,t) gives above solution v_1(P,t) and the coefficient of imaginary part gives another solution v_2 (P,t)

6) $$v_2(P,t) = A(P)\sin\left[\omega t - \varphi(P)\right]$$

Conclusion: one can use complex exponentials method by taking into account that the **real part** and the **coefficient of the imaginary** part only **have physical meaning**.
Introduction of Eq.s 5) in D'Alembert Equation gives

7) $$\nabla^2 u(P) + k^2 u(P) = 0$$

Helmholtz Equation or

Wave Equation

Quantity u(P) is called complex amplitude

Some notations :
k = ω/ V Wavenumber
ω = 2π/ν ν frequency
T =1/ν T period

In Eq. 5)

8) $\Phi(P,t) = \varphi(P) - \omega(t)$

is the total phase phase, and $\varphi(P)$ simply phase.
A surfaces where, at a given time, the total phase is constant or, what is the same:

$\varphi(P) = \text{Constant}$ is "equiphase surface" called **WAVEFRONT**

two wavefronts differing by an entire number of 2π are said to be "in phase",

$$\varphi(P_1) - \varphi(P_2) = m\,2\pi \qquad \text{m entire number}$$

If difference is $(2m+1)\pi$, that is an odd number of π, one has opposite phase.
Intensity I(P)

9) $I(P) = v(P,t).\ v^*(P,t) = u(P).u^*(P) = A^2(P)$

Values in Optics:
$\nu \sim 0.75 \div 0.37\ 10^{15}$ Hertz
$k \sim 1.6 \div 0.8\ 10^{7}\ m^{-1}$
$\lambda \sim 4 \div 8\ 10^{-7}\ m = 400 \div 8oo$ nm
$T \sim 1.3 \div 2.7\ 10^{-15}$ s

NOTE: Laser has reached large part of the spectrum outside optics and now speaking of Optics one often includes infrared and also ultraviolet.

PLANE WAVES

Solution of wave Equation

10) $u(P) = A\,e^{ik(\alpha x+\beta y+\gamma z)}$, where $\alpha^2 + \beta^2 + \gamma^2 = 1$

and A constant (real or complex), is called "plane wave" and can also be written as:

11) $u(P) = A\ e^{i\mathbf{k}\cdot\mathbf{r}}$ where $\mathbf{k} = k\,\mathbf{n} = k\,(\alpha\,\mathbf{i} + \beta\,\mathbf{j} + \gamma\,\boldsymbol{k})$
$\mathbf{n} = \alpha\,\mathbf{i} + \beta\,\mathbf{j} + \gamma\,\boldsymbol{k}$
$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\boldsymbol{k}$ vector from origin to a wave point P

Wavefronts are the planes:

12) $$\mathbf{k} \cdot \mathbf{r} = k(\alpha x + \beta y + \gamma z) = \text{Const}$$

here α, β, γ are real quantities and represent the "cosine directors" of the normal to the wavefront from the origin; $p = \mathbf{n} \cdot \mathbf{r}$ is distance of wavefront from origin.

13) $$u(P) = A\ e^{ik\mathbf{n}\cdot\mathbf{r}} = A\ e^{ikp}$$

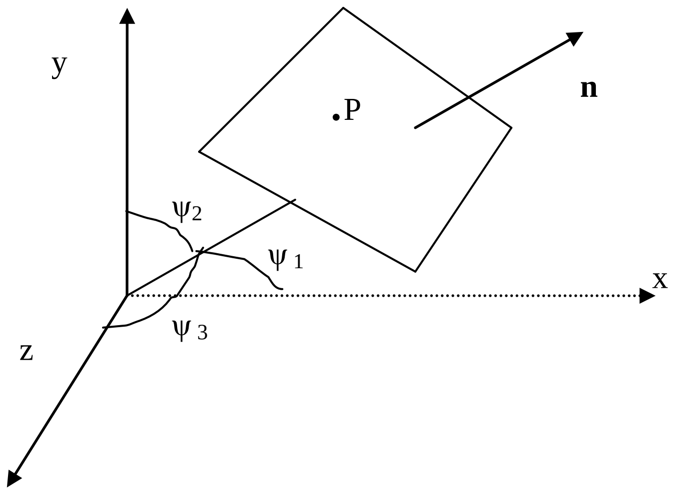


Fig. 1

At time t_1 the wavefront Φ_1 is at distance p_1 from origin:

$$\Phi_1 = k\,p_1 - \omega\,t_1$$

after a time dt position of Φ_1 is

$$\Phi_1 = k\,(p_1 + dp) - \omega\,(t_1 + dt)$$

from which

$$k\,dp - \omega\,dt = 0; \qquad dp = (\omega/k)\,dt$$

p increases. Linear motion. Velocity

14) $$V_f = \frac{\omega}{k}$$

Wavefront moves with velocity V_f "phase velocity"

Wavelength: distance between two subsequent equiphase planes. At time t_1 :

$$[k(p_1+\lambda)-\omega t_1]-(kp_1-\omega t_1) = 2\pi$$

Therefore

15) $$\lambda = \frac{2\pi}{k} = 2\pi\frac{V_f}{\omega} = \frac{V_f}{\nu} = T\,V_f$$ useful relations

Important consequence: Frequency ν is from source and does not change in linear media, the effect of a medium is to **change propagation velocity and wavelength**.

Amplitude, A, can be complex $A = A_0 e^{i\varphi_0}$ and constant φ_0 represents initial phase. A real plane wave solution, e.g. the real part of the complex solution, of the wave equation, written in complete explicit, form is

16) $$v_1(P,t) = A_0\cos[k(\alpha x+\beta y+\gamma z)+\varphi_0-\omega t]$$

Quantity φ_0 represents the initial phase in the origin (t=0, p=0). Generally one has to deal with phase differences where it does not play a role, often one assumes φ_0=0.

A_0^2 is **Intensity** (proportional to the power density flux) on a surface normal to propagation direction **n.**

IMPORTANCE OF PLANE WAVES:

- plane wave is an approximation to describe a wave in limited regions,
 - e.g. the beam from a lens due to a point source in the focus
 - the field from a distant source; distance much larger than wavelength and limited region

- basically plane waves are the elements for representing any e.m. field in terms of Fourier Series or Fourier Integrals.

EVANESCENT WAVES

Evanescent waves are also called surface waves and inhomogeneous or dissociated waves, see later.
Solution of wave equation for plane waves requires condition

17) $$\alpha^2+\beta^2+\gamma^2=1$$

for the three cosine directors of the normal to the wavefront with respect to the three axes. For plane waves they are real quantities.
However Eq. 10) is still solution of the wave equation even if one or more of this quantities are not real, provided that condition 17) is satisfied. To understand the

meaning of this solution let us assume that α is a purely imaginary and β vanishes, so that

18) $\alpha = i\ \alpha_i \quad ; \quad \beta = 0$

and consequently

γ^2 real and >1

19) $\gamma = \gamma_r = \pm\sqrt{1-\alpha^2} = \pm\sqrt{1+\alpha_i^2}$

Let us choose the positive value of the root. Solution of Eq.13 becomes

20) $$u(P) = A\, e^{-k\alpha_i x}\ e^{i\gamma_r kz}$$

let A be complex with initial phase φ_0 , one can write

21) $$u(P) = A(P)\, e^{i\varphi(P)}$$
$$A(P) = A_0 e^{-k\alpha_i x}$$
$$\varphi(P) = k\gamma_r z + \varphi_0$$

It appears that: **Amplitude is not constant and decays in x direction**. Phase contains z only, that is the **wave propagates in the z direction** (remember time dependence exp(iωt)). If the negative sign of the root of Eq. 19 is taken, the wave propagates in the negative z direction.
Wavelength λ_e of the evanescent wave, that is the distance between two wavefronts, is given by

22) $$k\gamma_r \lambda_e = 2\pi \qquad ; \qquad \lambda_e = \frac{2\pi}{k}\frac{1}{\gamma_r} = \frac{\lambda}{\gamma_r}$$

Result: **Wavelength λ_e of an evanescent wave is smaller** than that λ of a plane wave. Analogously velocity V_e is lower than that of a plane wave

22) $$V_e = \frac{V}{\gamma_r}$$

evanescent waves are also called **“slow waves”**.

Note that the planes of equal amplitude are normal to x and are different from the equiphase planes, normal to z, and for this reason these waves are also called **dissociated waves or inhomogeneous waves**.
Note that evanescent waves cannot exist in the complete space, for instance, if in Eq.19 the positive sign is chosen for the square root, the amplitude diverges when x

tends to negative infinite. Evanescent waves can only exist if there is a surface preventing them to divergence. From here the name of **surface waves**.
Evanescent waves are found in Total Reflection, for example when radiation goes from a transparent medium to another medium with lower refractive index, e.g. glass-air or water-air. They are also present in diffraction, see later.

SPERICAL WAVES

By taking the Laplacian in polar coordinates (r, θ, φ) one can write the wave equation in these coordinates. In general solution u = u(r, θ, φ). First consider a solution depending on r, u = u(r). In this case the wave Equation gives

$$\frac{\partial^2 (ru)}{\partial r^2} + k^2 (ru) = 0$$

whose solution is

$$ru = A e^{\pm ikr} \quad ;$$

where A is, generally, complex constant. One obtains two solutions:

$$\text{23)} \qquad u_1 = A\frac{e^{ikr}}{r} \quad ; \quad u_2 = A\frac{e^{-ikr}}{r}$$

Equiphase surfaces

$$\pm kr + \varphi_0 = \text{Constant} \qquad \varphi_0 \text{ initial phase}$$

are spherical surfaces
At a given instant the total phase

$$\phi = -\omega t \pm kr + \varphi_0$$

that is

$$\text{24)} \qquad \begin{aligned} kr_1 &= \phi + \omega t - \varphi_0 \\ kr_2 &= -\phi - \omega t + \varphi_0 \end{aligned}$$

which represent the phase of a diverging, $u_{1,}$ and converging, $u_{2,}$ spherical wave, respectively.

Wavelength and velocity are equal to those of plane waves.

Dependence of Amplitude on 1/r represents conservation of energy. An element of spherical surface is:

$$d\Sigma = r^2 \sin\vartheta \, d\vartheta \, d\varphi$$

Power across the element is

$$dP = u\,u^{*}\,d\Sigma \;=\; \frac{AA^{*}}{r^{2}}\; r^{2}\;\sin\vartheta\; d\vartheta\; d\varphi$$

Power across an entire sphere is constant, independently of the sphere radius:

$$P \;=\; \iint_{\text{sphere}} dP = 4\pi\, A\, A^{*}$$

Note: spherical waves have singularity for r = 0.
Physical significance: diverging wave u_1 represents radiation emitted by a point source, valid apart from a small volume around r = 0, where the source is.
Converging wave u_2 represent focussing of a wave, for instance by a lens, and is valid everywhere apart from a small region near the focus. The effect of a converging lens can be described by a converging spherical wave before the focus and a diverging one after the focus.

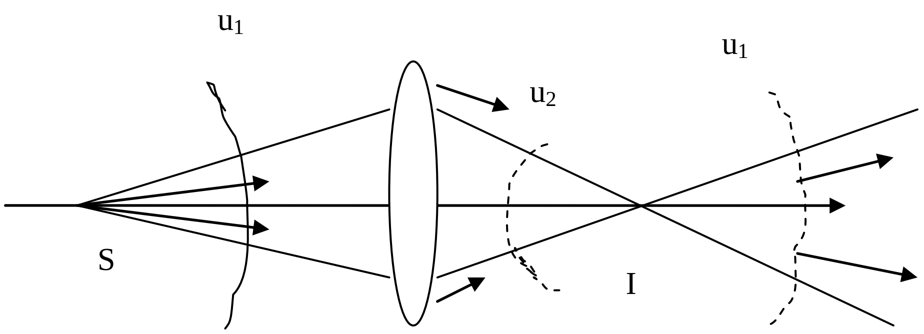


Figure 2. Spherical diverging wave from a source, focussed by a lens at point image I, then diverging again.

Spherical dipole waves

Another solution of the wave Equation in polar coordinate is a dipolar spherical wave where the amplitude also depends on θ. This solution, largely used for radio waves, represents the field irradiated by a dipole antenna, where the maximum radiation is in the direction orthogonal to the dipole axis and vanishes in the axis direction. The two diverging and converging solutions are:

25)

$$u_1 \;=\; A\,\frac{e^{ikr}}{r}\left(1+\frac{i}{k\,r}\right)\cos\vartheta$$

$$u_2 \;=\; A\,\frac{e^{-ikr}}{r}\left(1-\frac{i}{k\,r}\right)\cos\vartheta$$

At optical frequencies the imaginary part in parenthesis can be neglected when r is larger than 0.1mm where 1/(k r) ~ 10^{-3}, obtaining

26)
$$u_1 = A \frac{e^{ikr}}{r} \cos \vartheta$$
$$u_2 = A \frac{e^{-ikr}}{r} \cos \vartheta$$

Wavelength and propagation velocity are the same as plane wave.
Eq.s 26 are useful in diffraction theory, e.g. the diverging one when using Huygens-Fresnel principle to investigate resonant cavities with low Fresnel number, and, apart from a phase constant (in addition to the arbitrary initial one here) correctly represents diffraction by a small aperture as given by Rayleigh-Sommerfeld theory.

Cylindrical waves

Wave Equation, written in cylindrical coordinates, has solutions, called cylindrical wave, of the kind

27)
$$u = A \frac{e^{\pm ik\rho}}{\sqrt{\rho}} f(\rho) g(\vartheta)$$

where f(ρ) and g(θ) denote a separate dependence on ρ and θ.
Solution with g(θ) = 1 is an allowed solution where u only depends on ρ.
Taking f(ρ) = 1 is an approximation valid, as before, when kρ is negligible with respect to unity. In this case the two solutions are:

28)
$$u_1 = A \frac{e^{ik\rho}}{\sqrt{\rho}} \quad \text{and} \quad u_2 = A \frac{e^{-ik\rho}}{\sqrt{\rho}}$$

u_1 represents a diverging cylindrical wave and u_2 a converging one. These expressions are useful when using cylindrical lenses, apart from point ρ=0, source or focus, as spherical waves. The denominator root guarantees conservation of energy.

Cylindrical dipolar waves

When dependence on angle is needed one uses solutions

29) $$u_1 = A\frac{e^{ik\rho}}{\sqrt{\rho}}\cos\vartheta \quad \text{and} \quad u_2 = A\frac{e^{-ik\rho}}{\sqrt{\rho}}\cos\vartheta$$

The first of Eq. 29), as well of Eq. 28, is useful in diffraction and was largely used in the theory of open resonators.

WAVES AND RAYS

IN HOMOGENEOUS AND INHOMOGENEOUS LINEAR MEDIA

Refractive index completely describes a homogeneous medium, however a non homogeneous medium can also be described by a refractive index as function of point, provided that the function is a slowly varying function. Slowly varying function means negligible changes of refractive index over distances of the order of wavelength. One has

30) $$n = n(P)$$

Wave equation can be written as:

$$\nabla^2 u + n^2 k_0^2 u = 0$$

where $k^2 = n^2 k_0^2$ is explicitly written in terms of empty space k_0 and refractive index n of the material.

Let us introduce $u(P)= A(P)\, e^{i\varphi(P)}$ shortened as $u = A\, e^{i\varphi}$ into wave equations

$$\nabla^2 (A e^{i\varphi}) + n^2 k_0^2 A e^{i\varphi} = 0$$

By recalling some mathematical formulas:

$$\nabla^2 u = \nabla \bullet \nabla u = \nabla \bullet \mathrm{grad}\, u \qquad \bullet \text{ Scalar product}$$

$$\nabla^2 (ab) = \nabla \bullet \nabla(ab) = a\nabla^2 b + 2\nabla a \bullet \nabla b + b\nabla^2 a$$

$$\nabla^2 (A e^{i\varphi}) = 2i e^{i\varphi}\mathrm{grad}A \bullet \mathrm{grad}\,\varphi - A e^{i\varphi}|\mathrm{grad}\,\varphi|^2 + Aie^{i\varphi}\nabla^2\varphi + e^{i\varphi}\nabla^2 A$$

Wave equation becomes:

$$2i\, \mathrm{grad}A \bullet \mathrm{grad}\,\varphi - A|\mathrm{grad}\,\varphi|^2 + Ai\nabla^2\varphi + \nabla^2 A + n^2 k_0^2 A = 0$$

Separation of real and imaginary part gives

34) $$\begin{cases} a) & \nabla^2 A + n^2 k_0^2 A - A|\mathrm{grad}\,\varphi|^2 = 0 \\ b) & 2\,\mathrm{grad}\,A \bullet \mathrm{grad}\,\varphi + A\,\nabla^2\varphi = 0 \end{cases}$$

If

35) $$\nabla^2 A << n^2 k_0^2 A$$

Term $\nabla^2 A$ Eq. 34a) can be neglected and Eq. 34a) becomes

36) $$|\mathrm{grad}\,\varphi|^2 = n^2 k_0^2$$

Let ***s*** be a unit vector in the gradient direction, one has

37) $$\mathrm{grad}\,\varphi = n\,k_0\,\boldsymbol{s}$$

This result gives the link between wave optics and geometrical optics and explains that rays are the normal to wavefront, or better: rays are the tubes of flux of gradφ. They are straight in the case of homogeneous media, in general they are curved and the shape depends on the "local value" of the refractive index. It can be shown (see e.g. Born and Wolf book) that in a inhomogeneous medium a ray deflects, with respect to a straight path towards the region of higher refractive index.

Meaning of the approximation.
Eq. 35) requires

38) $$\frac{\nabla^2 A}{A} << n^2 k_0^2$$

Let consider a displacement dP and the corresponding amplitude variation dA. One has

$$\frac{d^2 A}{A\,dP^2} << \frac{4\pi^2}{\lambda^2}$$

$\lambda = \lambda_0 / n$ wavelength in the medium. Let the displacement be of the order of the wavelength, $dP = \lambda/(2\pi)$, Eq. 35) becomes

39) $$d^2 A / A << 1$$

This is the condition under which geometrical optics is valid. Shortly: when there are abrupt changes of amplitude geometrical optics approximation cannot be used and one has to use the complete system of Eq.s 34. From Eq.s 34) one sees that abrupt changes in amplitude influence the phase and the wavefront changes, one has diffraction. Diffraction is not a special phenomenon it is typical of waves when there are abrupt spatial changes in the amplitude. Important: any border!

From Eq.37, by introducing quantity $S = \varphi/k$ called eikonal one obtains the so called "eikonal equation" (from Greek $\varepsilon\iota\kappa\varpi\nu$ = image)

40) $$\text{grad}\, S = n\, \mathbf{s}$$

Integration of Eq. 37) along the ray path l_0 between two points P′ and P″ allows one to obtain the phase difference between the two points :

41) $$\varphi(P') - \varphi(P) = k_0 \int_{l_0} n(P)\, ds$$

here the dependence of n on P is explicitly indicated.
From Eq.37 one can also derive Fermat Principle, Reflection and refraction laws, and the so called Ray Equation.

INTERFERENCE

Two coherent waves

Two coherent waves (of the same frequency) at point P:

$$v_1 = A\, e^{-i\omega t} \qquad v_2 = A\, e^{-(i\omega t - \varphi)}$$

without loss of generality the amplitudes are assumed equal and real.
Total field $v = v_1 + v_2$. Intensity I:

$$I = |v^2| = (v_1 + v_2)(v_1^* + v_2^*) = 2A^2 + 2A^2 \cos\varphi$$

Therefore the intensity value depends on the phase difference, and can be higher or lower than the total intensity of the sum of the two waves, as expected. If $\varphi = \pi/2$ I=0, and if $\varphi=0$ $I = 4\, A^2$. Of course total energy is conserved.

Two waves of different frequency (time incoherent)

$$\omega \neq \omega_1$$

$$v_1 = A e^{-i\omega t} \qquad\qquad v_2 = A e^{-(i\omega_1 t - \varphi)}$$

$$I = 2A^2 + 2A^2 \,\mathrm{Re}\left(e^{-i\omega t + i\omega_1 t - \varphi}\right) = 2A^2 \left\{1 + \cos\left[(\omega_1 - \omega)t - \varphi\right]\right\}$$

The intensity is an oscillating function, called "beating", with frequency $|\omega_1 - \omega|$, a high frequency in optics. Our eyes (or conventional instruments) cannot follow it, and one sees an average value $\bar{I}$ of I, averaged over a characteristic time τ of the eye (or the instrument):

$$\bar{I} = \frac{1}{\tau} \int_0^{\tau} I(t)\,dt$$

If τ is large with respect to the period of the beating $T_b = 2\pi/(\omega-\omega_1)$ the average intensity vanishes and the total intensity of the interference is the sum of the intensity of the two waves. Sum of energies means incoherent waves.

On the contrary, if τ is of the order of or smaller than the period T_b then time dependence can be revealed. Modern instrumentations can do this.
Each instrument has its characteristic time τ and one has:

if $\quad \tau \le \dfrac{2\pi}{\omega - \omega_1} \quad$ the instrument "sees" **coherent waves**

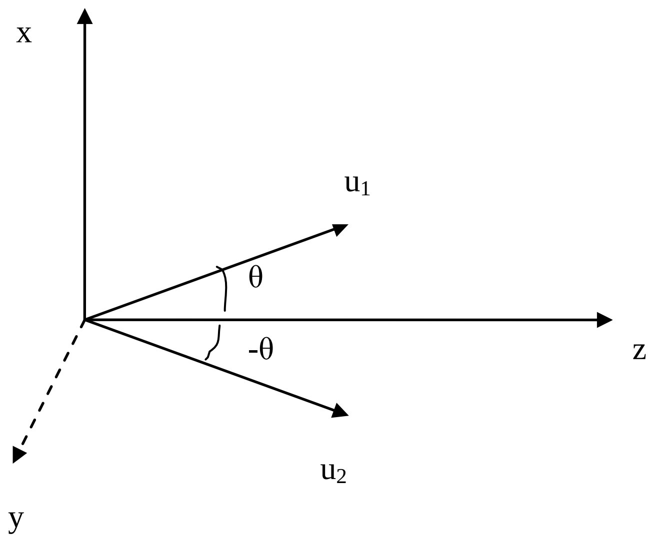
if $\quad \tau > \dfrac{2\pi}{\omega - \omega_1} \quad$ the instrument "sees" **incoherent waves**

Therefore interference from two incoherent waves can be seen as coherent or incoherent depending on the characteristic time of the instrument with respect to the period of the beating.

SPACE INTERFERENCE of COHERENT WAVES

Two plane Waves

Let consider two plane waves, of same frequency and amplitude propagating in two directions symmetric with respect to z axis, in the plane (x,z). (β =0).



Cosine directors of u_1

$$\alpha = \sin\vartheta$$
$$\beta = 0$$
$$\gamma = \cos\vartheta$$

$$u_1 = A\,e^{ik(\alpha x + \gamma z)} \quad \text{and} \quad u_2 = A\,e^{ik(-\alpha x + \gamma z)}$$

$$u = u_1 + u_2 = Ae^{ik\gamma z}\left(e^{ik\alpha x} + e^{-ik\alpha x}\right) = 2Ae^{ik\gamma z}\cos(kx\sin\vartheta)$$

Intensity

$$I = u\,u^* = 4A^2\cos^2(kx\sin\vartheta)$$

is periodic function of x, period p:

$$k\,p\sin\vartheta = \pi \quad ; \quad p = \frac{\lambda}{2\sin\vartheta}$$

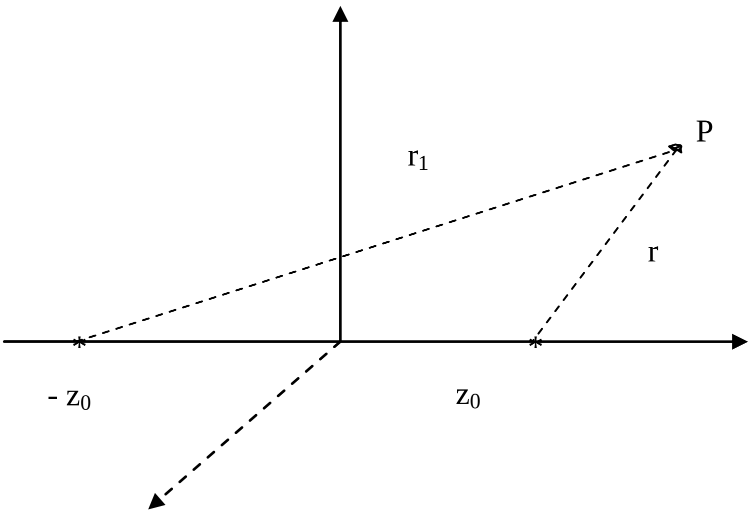
On a screen normal to z axis: interference fringes parallel to y axis.
On the planes parallel to plane (y, z) where one has

$$kx\sin\vartheta = (2n+1)\frac{\pi}{2}$$

field u(P).
One can place metallic plane surfaces on these planes, without disturbing the field in between: this is the basis of **metallic guiding** of waves, e.g in applications of high power lasers.
Michelson interferometer and **Fabry- Perot interferometer** are based on interference between two plane waves

Two spherical waves

Two spherical waves (of the same frequency and equal amplitude coefficient A_1, are centred on $-z_0$ and z_0 , respectively.



$$u_1 = \frac{A_1 \; e^{ikr}}{r} \qquad \text{and} \qquad u_2 = \frac{A_1 \, e^{ikr_1}}{r_1}$$

For small changes of r the phase of the first wave varies strongly, while the denominator of does not change appreciably. We can neglect the dependence of u_1 on r and call A the resulting amplitude. The same can be said for the second wave. When the two distances are not much different from one another, which is true when the distance between the two sources is very small with respect to the distance of point P, one can assume $r = r_1$.

Therefore

$$u = u_1 + u_2 = A\,e^{ikr} + A\,e^{ikr_1} = A\,e^{ik\left(\frac{r+r_1}{2}\right)} \left\{ e^{ik\left(\frac{r-r_1}{2}\right)} + e^{-ik\left(\frac{r-r_1}{2}\right)} \right\} =$$

$$= 2A\,e^{ik\left(\frac{r+r_1}{2}\right)} \cos\left[\frac{k(r-r_1)}{2}\right]$$

Intensity I is:

$$I = 4A^2 \cos^2\left[\frac{k(r-r_1)}{2}\right]$$

Lines where

$$\left[\frac{k(r-r_1)}{2}\right] = n\,\pi \qquad \text{that is} \qquad r - r_1 = n\,\lambda$$

are lines of maximum intensity: they are hyperboloids of rotation, with foci in the two source points. Analogous relation is found for the lines of zero intensity.

Intersections in a plane x =d, d large with respect to all other distances, are hyperbolas. By means of a series development, one can see that the fringes are linear in the central region.

These formulas are the basis for the description of many interferometers, such as **Young interferometer** and **Ronchi test**. When one source goes to infinity one obtains **Newton's rings**.