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ELECTROMAGNETIC WAVES IN VACUUM

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ELECTROMAGNETIC WAVES IN VACUUM

➢ THE WAVE EQUATION

- In regions of free space (i.e. the vacuum) - where no electric charges - no electric currents and no matter of any kind are present - Maxwell's equations (in differential form) are:

1) $\vec{\nabla}\bullet\vec{E}(\vec{r},t)=0$

2) $\vec{\nabla}\bullet\vec{B}(\vec{r},t)=0$

3) $\vec{\nabla}\times\vec{E}(\vec{r},t)=-\dfrac{\partial\vec{B}(\vec{r},t)}{\partial t}$

4) $\vec{\nabla}\times\vec{B}(\vec{r},t)=\mu_o\varepsilon_o\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}=\dfrac{1}{c^2}\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}$

$\left(c^2=1/\varepsilon_o\mu_o\right)$

Set of coupled first-order partial differential equations

ELECTROMAGNETIC WAVES IN VACUUM . . .

- We can de-couple Maxwell's equations -by applying the curl operator to equations 3) and 4):

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)=\vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$=\vec{\nabla}\left(\overbrace{\vec{\nabla}\bullet\vec{E}}^{=0}\right)-\nabla^2\vec{E}=-\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right)$$

$$=-\nabla^2\vec{E}=-\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\right)$$

$$=\boxed{\nabla^2\vec{E}=\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}}$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right)=\vec{\nabla}\times\left(\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\right)$$

$$=\vec{\nabla}\left(\overbrace{\vec{\nabla}\bullet\vec{B}}^{=0}\right)-\nabla^2\vec{B}=\frac{1}{c^2}\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{E}\right)$$

$$=-\nabla^2\vec{B}=\frac{1}{c^2}\frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$=\boxed{\nabla^2\vec{B}=\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}}$$

ELECTROMAGNETIC WAVES IN VACUUM . . .

- These are three-dimensional de-coupled wave equations.
- Have exactly the same structure – both are linear, homogeneous, 2nd order differential equations.
- Remember that each of the above equations is explicitly dependent on space and time,

$$i.e.\ \vec{E} = \vec{E}(\vec{r},t) \text{ and } \vec{B} = \vec{B}(\vec{r},t):$$

$$\nabla^2 \vec{E}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} = 0$$

$$\nabla^2 \vec{B}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \vec{B}(\vec{r},t)}{\partial t^2} = 0$$

ELECTROMAGNETIC WAVES IN VACUUM . . .

- Thus, Maxwell's equations implies that empty space – the vacuum {which is not empty, at the microscopic scale} – supports the propagation of {macroscopic} electromagnetic waves - which propagate at the speed of light {in vacuum}:

$$c = 1/\sqrt{\varepsilon_o \mu_o} = 3\times 10^8 \text{ m/s}$$

MONOCHROMATIC EM PLANE WAVES

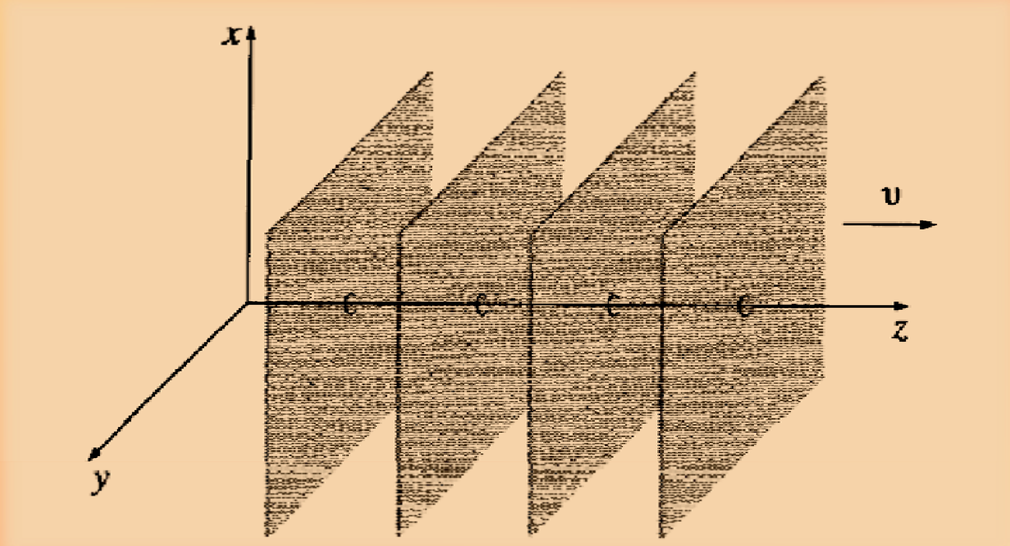
Monochromatic EM plane waves propagating in free space/the vacuum are sinusoidal EM plane waves consisting of a single frequency f , wavelength $\lambda = c\ f$, angular frequency $\omega = 2\pi\ f$ and wave-number $k = 2\pi /\lambda$. They propagate with speed $c = f\lambda = \omega\ k$.

In the visible region of the EM spectrum *{~380 nm (violet)* $\leq \lambda \leq$ ~ *780 nm (red)}-* EM light waves (consisting of real photons) of a given frequency / wavelength are perceived by the human eye as having a specific, single colour.

Single- frequency sinusoidal EM waves are called mono-chromatic.

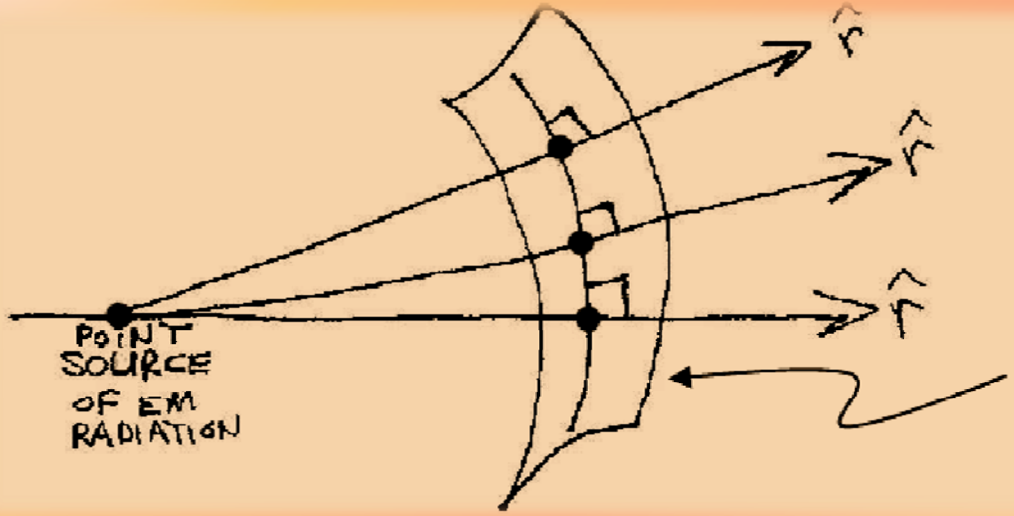
MONOCHROMATIC EM PLANE WAVES

EM waves that propagate e.g. in the $+\hat{z}$ direction but which additionally have no explicit *x*- or *y*-dependence are known as plane waves, because for a given time, t the wave front(s) of the EM wave lie in a plane which is ⊥ to the $\hat{z}$ *-axis*,



MONOCHROMATIC EM PLANE WAVES

There also exist spherical EM waves – emitted from a point source – the wave-fronts associated with these EM waves are spherical - and thus do not lie in a plane ⊥ to the direction of propagation of the EM wave



Portion of a spherical wavefront associated with a spherical wave

MONOCHROMATIC EM PLANE WAVES

If the point source is infinitely far away from observer- then a spherical wave → plane wave in this limit, (the radius of curvature → ∞); a spherical surface becomes planar as $R_C \to \infty$.

Criterion for a plane wave: $\lambda \ll R_c$

Monochromatic plane waves associated with $\vec{E}$ and $\vec{B}$

$$\vec{\tilde{B}}(z,t) = \vec{\tilde{B}}_o e^{i(kz-\omega t)}$$

$$\vec{\tilde{E}}(z,t) = \vec{\tilde{E}}_o e^{i(kz-\omega t)}$$

MONOCHROMATIC EM PLANE WAVES

$$\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_o e^{\overbrace{i(kz-\omega t)}} \qquad \tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_o e^{\overbrace{i(kz-\omega t)}}$$

Propagating in $+\hat{z}$ direction

Propagating in $+\hat{z}$ direction

n.b. complex vectors:

e.g. $\tilde{\vec{E}}_o = E_o e^{i\delta}\hat{x}$

n.b. complex vectors:

e.g. $\tilde{\vec{B}}_o = B_o e^{i\delta}\hat{y}$

n.b. The *real*, *physical* (instantaneous) fields are:

$$\left\{ \begin{array}{l} \vec{E}(\vec{r},t) \equiv \mathrm{Re}\left(\tilde{\vec{E}}(\vec{r},t)\right) \\ \vec{B}(\vec{r},t) \equiv \mathrm{Re}\left(\tilde{\vec{B}}(\vec{r},t)\right) \end{array} \right\}$$

Very important to keep in mind!!

MONOCHROMATIC EM PLANE WAVES

Maxwell's equations for free space impose additional constraints on $\vec{\tilde{E}}_o$ and $\vec{\tilde{B}}_o$

$$\text{Since: } \vec{\nabla}\bullet\vec{E} = 0 \qquad \text{and:} \qquad \vec{\nabla}\bullet\vec{B} = 0$$

$$= \text{Re}\left(\vec{\nabla}\bullet\vec{\tilde{E}}\right) = 0 \qquad\qquad = \text{Re}\left(\vec{\nabla}\bullet\vec{\tilde{B}}\right) = 0$$

These two relations can only be satisfied

$$\forall(\vec{r},t) \text{ if } \vec{\nabla}\bullet\vec{\tilde{E}} = 0 \;\; \forall(\vec{r},t) \text{ and } \vec{\nabla}\bullet\vec{\tilde{B}} = 0 \;\; \forall(\vec{r},t)$$

In Cartesian coordinates: $\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$

<u>Thus</u>: $\left(\vec{\nabla}\bullet\vec{\tilde{E}}\right) = 0$ and $\left(\vec{\nabla}\bullet\vec{\tilde{B}}\right) = 0$ become:

MONOCHROMATIC EM PLANE WAVES

$$\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\bullet\left(\vec{\tilde{E}}_o e^{i(kz-\omega t)}\right)=0 \quad \text{and} \quad \left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\bullet\left(\vec{\tilde{B}}_o e^{i(kz-\omega t)}\right)=0$$

Now suppose we do allow:

$$\vec{\tilde{E}}_o=\underbrace{\left(E_{ox}\hat{x}+E_{oy}\hat{y}+E_{oz}\hat{z}\right)}_{\text{polarization in } \hat{x}-\hat{y}-\hat{z}\ (3-D)}e^{i\delta}\equiv\vec{E}_o e^{i\delta}$$

$$\vec{\tilde{B}}_o=\underbrace{\left(B_{ox}\hat{x}+B_{oy}\hat{y}+B_{oz}\hat{z}\right)}_{\text{polarization in } \hat{x}-\hat{y}-\hat{z}\ (3-D)}e^{i\delta}\equiv\vec{B}_o e^{i\delta}$$

Then

$$\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\bullet\left(E_{ox}\hat{x}+E_{oy}\hat{y}+E_{oz}\hat{z}\right)e^{i\delta}e^{i(kz-\omega t)}=0$$

$$\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\bullet\left(B_{ox}\hat{x}+B_{oy}\hat{y}+B_{oz}\hat{z}\right)e^{i\delta}e^{i(kz-\omega t)}=0$$

MONOCHROMATIC EM PLANE WAVES

E_{ox}, E_{oy}, E_{oz} = Amplitudes (constants) of the electric field components in x, y, z directions respectively.

B_{ox}, B_{oy}, B_{oz} =Amplitudes (constants) of the magnetic field components in *x, y, z* directions respectively.

$$\frac{\partial}{\partial x}\hat{x}\bullet E_{ox}\hat{x}e^{i(kz-\omega t)}e^{i\delta}=0$$

$$\frac{\partial}{\partial y}\hat{y}\bullet E_{oy}\hat{y}e^{i(kz-\omega t)}e^{i\delta}=0$$

$$\frac{\partial}{\partial x}\hat{x}\bullet B_{ox}\hat{x}e^{i(kz-\omega t)}e^{i\delta}=0$$

$$\frac{\partial}{\partial y}\hat{y}\bullet B_{oy}\hat{y}e^{i(kz-\omega t)}e^{i\delta}=0$$

$$\frac{\partial}{\partial z}\left(e^{az}\right)=ae^{az}$$

MONOCHROMATIC EM PLANE WAVES . . .

$$\frac{\partial}{\partial z}\hat{z}\bullet E_{oz}\hat{z}e^{i(kz-\omega t)}e^{i\delta} = ikE_{oz}e^{i(kz-\omega t)}e^{i\delta} = 0 \quad \Leftarrow \text{ true } \textbf{\textit{iff}} \ \boxed{E_{oz} \equiv 0} \ !!!$$

$$\frac{\partial}{\partial z}\hat{z}\bullet B_{oz}\hat{z}e^{i(kz-\omega t)}e^{i\delta} = ik\mathrm{E}_{oz}e^{i(kz-\omega t)}e^{i\delta} = 0 \quad \Leftarrow \text{ true } \textbf{\textit{iff}} \ \boxed{B_{oz} \equiv 0} \ !!!$$

- Maxwell's equations additionally impose the restriction that an electromagnetic plane wave cannot have any component of **E** or **B** ‖ to (or anti- ‖ to) the propagation direction (in this case here, the z -direction)
- Another way of stating this is that an EM wave cannot have any longitudinal components of **E** and **B** (i.e. components of **E** and **B** lying along the propagation direction).

MONOCHROMATIC EM PLANE WAVES . . .

- Thus, Maxwell's equations additionally tell us that an EM wave is a purely transverse wave (at least for propagation in free space) – the components of **E** and **B** must be ⊥ to propagation direction.
- The plane of polarization of an EM wave is defined (by convention) to be parallel to **E.**

MONOCHROMATIC EM PLANE WAVES . . .

Maxwell's equations impose another restriction on the allowed form of E and B for an EM wave:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{and/or:} \quad \vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

$$= \mathrm{Re}\left(\vec{\nabla}\times\vec{\tilde{E}}\right) = \mathrm{Re}\left(-\frac{\partial \vec{\tilde{B}}}{\partial t}\right) \qquad = \mathrm{Re}\left(\vec{\nabla}\times\vec{\tilde{B}}\right) = \mathrm{Re}\left(\frac{1}{c^2}\frac{\partial \vec{\tilde{E}}}{\partial t}\right)$$

Can *only* be satisfied $\forall\ (\vec{r}, t)$ ***iff***:

$$\vec{\nabla}\times\vec{\tilde{E}} = -\frac{\partial \vec{\tilde{B}}}{\partial t} \quad \text{and/or:} \quad \vec{\nabla}\times\vec{\tilde{B}} = \frac{1}{c^2}\frac{\partial \vec{\tilde{E}}}{\partial t}$$

MONOCHROMATIC EM PLANE WAVES . . .

$$\vec{\nabla}\times\vec{\tilde{E}} = \left(\overbrace{\frac{\partial \tilde{E}_z}{\partial y}}^{=0} - \frac{\partial \tilde{E}_y}{\partial z}\right)\hat{x} + \left(\frac{\partial \tilde{E}_x}{\partial z} - \overbrace{\frac{\partial \tilde{E}_y}{\partial x}}^{=0}\right)\hat{y} + \left(\overbrace{\frac{\partial \tilde{E}_y}{\partial x}}^{=0} - \overbrace{\frac{\partial \tilde{E}_x}{\partial y}}^{=0}\right)\hat{z} = -\frac{\partial \tilde{B}_x}{\partial t}\hat{x} - \frac{\partial \tilde{B}_y}{\partial t}\hat{y} - \overbrace{\frac{\partial \tilde{B}_z}{\partial t}}^{=0}\hat{z}$$

$$\vec{\nabla}\times\tilde{B} = \left(\overbrace{\frac{\partial \tilde{B}_z}{\partial y}}^{=0} - \frac{\partial \tilde{B}_y}{\partial z}\right)\hat{x} + \left(\frac{\partial \tilde{B}_x}{\partial z} - \overbrace{\frac{\partial \tilde{B}_y}{\partial x}}^{=0}\right)\hat{y} + \left(\overbrace{\frac{\partial \tilde{B}_y}{\partial x}}^{=0} - \overbrace{\frac{\partial \tilde{B}_x}{\partial y}}^{=0}\right)\hat{z} = \frac{1}{c^2}\frac{\partial \tilde{E}_x}{\partial t}\hat{x} + \frac{1}{c^2}\frac{\partial \tilde{E}_y}{\partial t}\hat{y} + \frac{1}{c^2}\overbrace{\frac{\partial \tilde{E}_z}{\partial t}}^{=0}\hat{z}$$

$$\vec{\tilde{E}} = \tilde{E}_x\hat{x} + \tilde{E}_y\hat{y} + \overbrace{\tilde{E}_z}^{=0}\hat{z} = \left(E_{ox}\hat{x} + E_{oy}\hat{y} + \overbrace{E_{oz}}^{=0}\hat{z}\right)e^{i(kz-\omega t)}e^{i\delta}$$

$$\vec{\tilde{B}} = \tilde{B}_x\hat{x} + \tilde{B}_y\hat{y} + \overbrace{\tilde{B}_z}^{=0}\hat{z} = \left(B_{ox}\hat{x} + B_{oy}\hat{y} + \overbrace{B_{oz}}^{=0}\hat{z}\right)e^{i(kz-\omega t)}e^{i\delta}$$

MONOCHROMATIC EM PLANE WAVES . . .

$$\vec{\tilde{E}} = \tilde{E}_x \hat{x} + \tilde{E}_y \hat{y} = \left(E_{ox} \hat{x} + E_{oy} \hat{y} \right) e^{i(kz-\omega t)} e^{i\delta}$$

$$\vec{\tilde{B}} = \tilde{B}_x \hat{x} + \tilde{B}_y \hat{y} = \left(B_{ox} \hat{x} + B_{oy} \hat{y} \right) e^{i(kz-\omega t)} e^{i\delta}$$

$$\vec{\nabla} \times \vec{\tilde{E}} = -\frac{\partial \tilde{E}_y}{\partial z} \hat{x} + \frac{\partial \tilde{E}_x}{\partial z} \hat{y} = -\frac{\partial \tilde{B}_x}{\partial t} \hat{x} - \frac{\partial \tilde{B}_y}{\partial t} \hat{y}$$

$$\vec{\nabla} \times \vec{\tilde{B}} = -\frac{\partial \tilde{B}_y}{\partial z} \hat{x} + \frac{\partial \tilde{B}_x}{\partial z} \hat{y} = \frac{1}{c^2} \frac{\partial \tilde{E}_x}{\partial t} \hat{x} + \frac{1}{c^2} \frac{\partial \tilde{E}_y}{\partial t} \hat{y}$$

Can only be satisfied / can only be true *iff* the $\hat{x}$ and $\hat{y}$ relations are separately / independently satisfied $\forall \ (\vec{r}, t)$!

MONOCHROMATIC EM PLANE WAVES . . .

$\vec{\nabla} \times \tilde{\vec{E}}$:

$$-\frac{\partial \tilde{E}_y}{\partial z}\hat{x} = -\frac{\partial \tilde{B}_x}{\partial t}\hat{x} \Rightarrow \frac{\partial \tilde{E}_y}{\partial z} = \frac{\partial \tilde{B}_x}{\partial t} \Rightarrow ikE_{oy} = -i\omega B_{ox} \quad (1)$$

$$+\frac{\partial \tilde{E}_x}{\partial z}\hat{y} = -\frac{\partial \tilde{B}_y}{\partial t}\hat{y} \Rightarrow \frac{\partial \tilde{E}_x}{\partial z} = -\frac{\partial \tilde{B}_y}{\partial t} \Rightarrow ikE_{ox} = +i\omega B_{oy} \quad (2)$$

$\vec{\nabla} \times \tilde{B}$:

$$-\frac{\partial \tilde{B}_y}{\partial z}\hat{x} = \frac{1}{c^2}\frac{\partial \tilde{E}_x}{\partial t}\hat{x} \Rightarrow -\frac{\partial \tilde{B}_y}{\partial z} = \frac{1}{c^2}\frac{\partial \tilde{E}_x}{\partial t} \Rightarrow -ikB_{oy} = -\frac{1}{c^2}i\omega E_{ox} \quad (3)$$

$$+\frac{\partial \tilde{B}_x}{\partial z}\hat{y} = \frac{1}{c^2}\frac{\partial \tilde{E}_y}{\partial t}\hat{y} \Rightarrow \frac{\partial \tilde{B}_x}{\partial z} = \frac{1}{c^2}\frac{\partial \tilde{E}_y}{\partial t} \Rightarrow ikB_{ox} = -\frac{1}{c^2}i\omega E_{oy} \quad (4)$$

From (1): $ik\tilde{E}_{oy} = -i\omega B_{ox} \Rightarrow E_{oy} = -\left(\frac{\omega}{k}\right)B_{ox}$ or: $B_{ox} = -\left(\frac{k}{\omega}\right)E_{oy}$

MONOCHROMATIC EM PLANE WAVES . . .

From (2): $ik\tilde{E}_{ox} = +i\omega B_{oy}$ $\Rightarrow$ $E_{ox} = +\left(\frac{\omega}{k}\right) B_{oy}$ or: $B_{oy} = +\left(\frac{k}{\omega}\right) E_{ox}$

From (3): $-ikB_{oy} = -\frac{1}{c^2} i\omega E_{ox}$ $\Rightarrow$ $B_{oy} = +\frac{1}{c^2}\left(\frac{\omega}{k}\right) E_{ox}$

From (4): $ikB_{ox} = -\frac{1}{c^2} i\omega E_{oy}$ $\Rightarrow$ $B_{ox} = -\frac{1}{c^2}\left(\frac{\omega}{k}\right) E_{oy}$

$$c = f\lambda = (2\pi f)\left(\frac{\lambda}{2\pi}\right) = \left(\frac{\omega}{k}\right) \qquad \frac{1}{c} = \left(k/\omega\right) \qquad \left(k = 2\pi/\lambda\right)$$

MONOCHROMATIC EM PLANE WAVES . . .

$\underline{\vec{\nabla} \times \tilde{\vec{E}}:}$

(1) $B_{ox} = -\frac{1}{c} E_{oy}$

(2) $B_{oy} = +\frac{1}{c} E_{ox}$

$\underline{\vec{\nabla} \times \tilde{\vec{B}}:}$

(3) $B_{oy} = +\frac{1}{c} E_{ox}$

(4) $B_{ox} = -\frac{1}{c} E_{oy}$

Maxwell's Equations also have some redundancy encrypted into them!

Actually we have only two independent relations:

$B_{ox} = -\frac{1}{c} E_{oy}$ and $B_{oy} = +\frac{1}{c} E_{ox}$

But: $\hat{z} \times \hat{y} = -\hat{x}$ $\qquad \hat{z} \times \hat{x} = +\hat{y}$

MONOCHROMATIC EM PLANE WAVES . . .

Very Useful Table:

$\hat{x}\times\hat{y}=\hat{z}$	$\hat{y}\times\hat{x}=-\hat{z}$
$\hat{y}\times\hat{z}=\hat{x}$	$\hat{z}\times\hat{y}=-\hat{x}$
$\hat{z}\times\hat{x}=\hat{y}$	$\hat{x}\times\hat{z}=-\hat{y}$

Two relations can be written compactly into one relation:

$$\vec{\tilde{B}}_o=\frac{1}{c}\left(\hat{z}\times\vec{\tilde{E}}_o\right)$$

Physically this relation states that E and B are:

- in phase with each other.
- mutually perpendicular to each other - (**E**⊥**B**)⊥ $\hat{z}$

MONOCHROMATIC EM PLANE WAVES . . .

The **E** and **B** fields associated with this monochromatic plane EM wave are purely transverse { n.b. this is as also required by relativity at the microscopic level – for the extreme relativistic particles – the (massless) real photons travelling at the speed of light c that make up the macroscopic monochromatic plane EM wave.}

The real amplitudes of E and B are related to each other by:

$$B_o = \frac{1}{c} E_o$$

with

$$B_o = \sqrt{B_{ox}^2 + B_{oy}^2}$$

and

$$E_o = \sqrt{E_{ox}^2 + E_{oy}^2}$$

Instantaneous Poynting's Vector for a linearly polarized *EM wave*

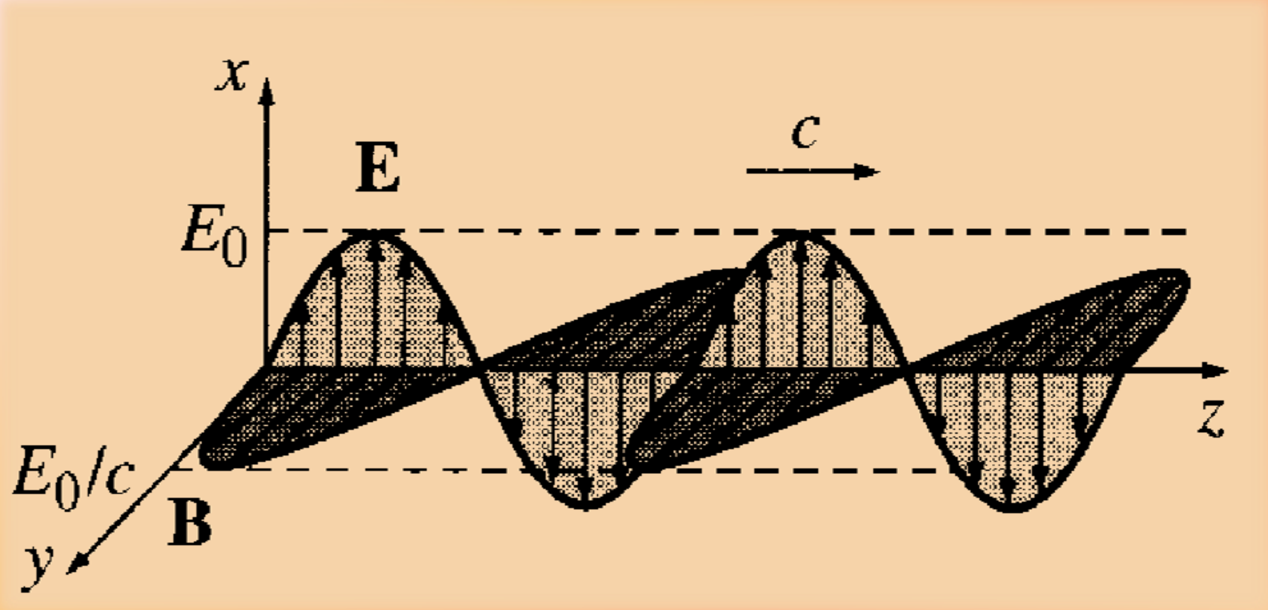
$$\vec{S}(z,t) = \frac{1}{\mu_o}\vec{E}(z,t)\times\vec{B}(z,t) = \frac{1}{\mu_o}\mathrm{Re}\left\{\tilde{\vec{E}}(z,t)\right\}\times\mathrm{Re}\left\{\tilde{\vec{B}}(z,t)\right\}$$

$$\vec{S}(z,t) = \frac{1}{\mu_o}E_oB_o\cos^2(kz-\omega t+\delta)\underbrace{(\hat{x}\times\hat{y})}_{=\hat{z}}$$

$$\vec{S}(z,t) = \frac{1}{\mu_o}E_oB_o\cos^2(kz-\omega t+\delta)\hat{z} \qquad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

⇒EM Power flows in the direction of propagation of the EM wave (here, the +zˆ direction)

Instantaneous Poynting's Vector for a linearly polarized *EM wave*



This is the paradigm for a monochromatic plane wave. The wave as a whole is said to be polarized in the x direction (by convention, we use the direction of E to specify the polarization of an electromagnetic wave).

Instantaneous Energy & Linear Momentum & Angular Momentum in *EM Waves*

Instantaneous Energy Density Associated with an *EM Wave:*

$$u_{EM}(\vec{r},t)=\frac{1}{2}\left(\varepsilon_o E^2(\vec{r},t)+\frac{1}{\mu_o}B^2(\vec{r},t)\right)=u_{elect}(\vec{r},t)+u_{mag}(\vec{r},t)$$

where $$u_{elect}(\vec{r},t)=\frac{1}{2}\varepsilon_o E^2(\vec{r},t)$$

and $$u_{mag}(\vec{r},t)=\frac{1}{2\mu_o}B^2(\vec{r},t)=\frac{1}{2}\varepsilon_o E^2(\vec{r},t)$$

Instantaneous Energy & Linear Momentum & Angular Momentum in *EM* Waves

But $\boxed{B^2 = \frac{1}{c^2}E^2}$ - EM waves in vacuum, and $\boxed{\frac{1}{c^2} = \varepsilon_o \mu_o}$

$$u_{EM}(\vec{r},t) = \frac{1}{2}\left(\varepsilon_o E^2(\vec{r},t) + \frac{\varepsilon_o \cancel{\mu_o}}{\cancel{\mu_o}} E^2(\vec{r},t)\right) = \frac{1}{2}\left(\varepsilon_o E^2(\vec{r},t) + \varepsilon_o E^2(\vec{r},t)\right)$$

$$\boxed{u_{EM}(\vec{r},t) = \varepsilon_o E^2(\vec{r},t) = \varepsilon_o E_o^2 \cos^2\left(\vec{k}\bullet\vec{r} - \omega t + \delta\right)} \quad \left(\frac{\text{Joules}}{\text{m}^3}\right)$$

$\boxed{u_{elect}(\vec{r},t) = u_{mag}(\vec{r},t)}$ - EM waves propagating in the vacuum !!!!

Instantaneous Poynting's Vector Associated with an *EM Wave*

$$\vec{S}(\vec{r},t) = \frac{1}{\mu_o}\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t) = \frac{1}{\mu_o}\mathrm{Re}\left\{\tilde{\vec{E}}(z,t)\right\}\times\mathrm{Re}\left\{\tilde{\vec{B}}(z,t)\right\} \quad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

For a linearly polarized monochromatic plane EM wave propagating in the vacuum,

$$\vec{S}(\vec{r},t) = c\left(\frac{\varepsilon_o \mu_o}{\mu_o}\right)E_o^2\cos^2(kz-\omega t+\delta)\hat{z} = c\varepsilon_o E_o^2\cos^2(kz-\omega t+\delta)\hat{z}$$

But $$u_{EM}(\vec{r},t) = \varepsilon_o E^2(\vec{r},t) = \varepsilon_o E_o^2\cos^2(kz-\omega t+\delta)$$

$$\vec{S}(\vec{r},t) = c u_{EM}(\vec{r},t)\hat{z}$$

Instantaneous Poynting's Vector Associated with an *EM Wave*

The propagation velocity of energy $\vec{v}_{prop} = c\hat{z}$

Poynting's Vector = Energy Density * Propagation Velocity

$$\vec{S}(\vec{r},t) = u_{EM}(\vec{r},t)\vec{v}_{prop}$$

Instantaneous Linear Momentum Density Associated with an EM Wave:

$$\vec{\wp}_{EM}(\vec{r},t) = \varepsilon_o\mu_o\vec{S}(\vec{r},t) = \frac{1}{c^2}\vec{S}(\vec{r},t) \quad \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

Instantaneous Linear Momentum Density Associated with an *EM Wave*

For linearly polarized monochromatic plane EM waves propagating in the vacuum:

$$\vec{\wp}_{EM} = \frac{1}{c^{\not 2}} \not c\, \varepsilon_o E_o^2 \cos^2\left(kz - \omega t + \delta\right)\hat{z} = \frac{1}{c} \underbrace{\varepsilon_o E_o^2 \cos^2\left(kz - \omega t + \delta\right)}_{= u_{EM}} \hat{z}$$

But: $$u_{EM}\left(\vec{r},t\right) = \varepsilon_o E^2\left(\vec{r},t\right) = \varepsilon_o E_o^2 \cos^2\left(kz - \omega t + \delta\right)$$

$$\boxed{\vec{\wp}_{EM}\left(\vec{r},t\right) = \varepsilon_o \mu_o \vec{S}\left(\vec{r},t\right) = \frac{1}{c^2}\vec{S}\left(\vec{r},t\right) = \frac{1}{c} u_{EM}\left(\vec{r},t\right)\hat{z}} \quad \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

Instantaneous Angular Momentum Density Associated with an *EM wave*

$$\boxed{\vec{\ell}_{EM}(\vec{r},t) = \vec{r} \times \vec{\wp}_{EM}(\vec{r},t)} \quad \left(\frac{\text{kg}}{\text{m-sec}}\right)$$

But: $$\boxed{\vec{\wp}_{EM}(\vec{r},t) = \varepsilon_o \mu_o \vec{S}(\vec{r},t) = \frac{1}{c^2}\vec{S}(\vec{r},t) = \frac{1}{c}u_{EM}(\vec{r},t)\hat{z}} \quad \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

For an EM wave propagating in the +z^ direction:

$$\boxed{\vec{\ell}_{EM}(\vec{r},t) = \frac{1}{c^2}\vec{r} \times \vec{S}(\vec{r},t) = \frac{1}{c}u_{EM}(\vec{r},t)(\vec{r} \times \hat{z})} \quad \left(\frac{\text{kg}}{\text{m-sec}}\right)$$

↑ Depends on the choice of origin

Instantaneous Power Associated with an *EM wave*

The instantaneous EM power flowing into/out of volume v with bounding surface S enclosing volume v (containing EM fields in the volume v) is:

$$P_{EM}(t) = \frac{\partial U_{EM}(t)}{\partial t} = \int_v \frac{\partial u_{EM}(\vec{r},t)}{\partial t} d\tau = -\oint_S \vec{S}(\vec{r},t) \bullet d\vec{a}$$

The instantaneous EM power crossing (imaginary) surface is:

$$P_{EM}(t) = -\int_S \vec{S}(\vec{r},t) \bullet d\vec{a}_\perp$$

The instantaneous total *E*M energy contained in volume v

$$U_{EM}(t) = \int_v u_{EM}(\vec{r},t)\, d\tau \quad \text{(Joules)}$$

Instantaneous Angular Momentum Density Associated with an *EM wave*

The instantaneous total EM linear momentum contained in the volume v is:

$$\boxed{\vec{p}_{EM}(t) = \int_v \vec{\wp}_{EM}(\vec{r},t)\, d\tau} \qquad \left(\frac{\text{kg-m}}{\text{sec}}\right)$$

The instantaneous total EM angular momentum contained in the volume v is:

$$\boxed{\vec{\mathcal{L}}_{EM}(t) = \int_v \vec{\ell}_{EM}(\vec{r},t)\, d\tau} \qquad \left(\frac{\text{kg-m}^2}{\text{sec}}\right)$$

Time-Averaged Quantities Associated with EM Waves

Usually we are not interested in knowing the instantaneous power P(t), energy / energy density, Poynting's vector, linear and angular momentum, *etc.*- because experimental measurements of these quantities are very often averages over many extremely fast cycles of oscillation. For example period of oscillation of light wave

$$\tau_{light} = 1/f_{light} \simeq \frac{1}{10^{15}\,\text{cps}} = 10^{-15}\,\text{sec/cycle} = 1\,\text{femto-sec}\,)$$

We need time averaged expressions for each of these quantities - in order to compare directly with experimental data- for monochromatic plane EM light waves:

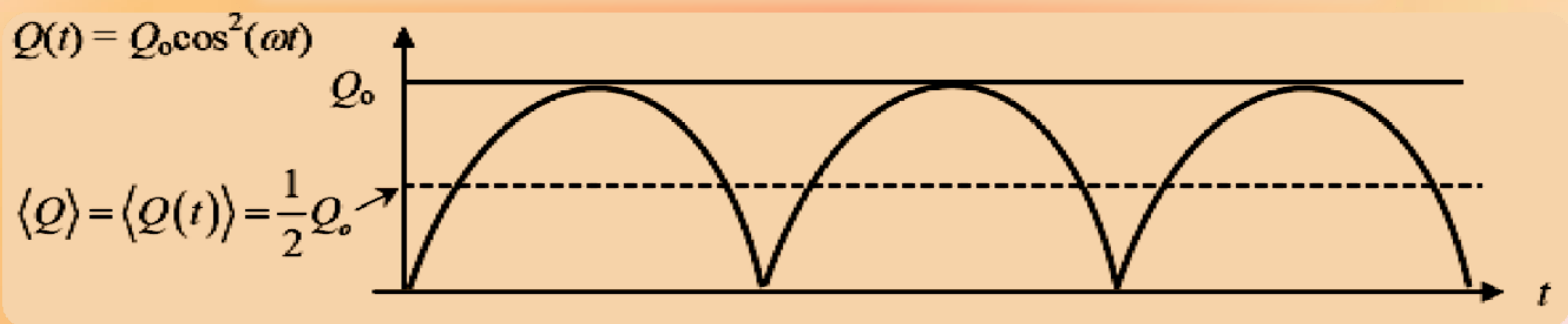
Time-Averaged Quantities Associated with EM Waves

If we have *a "generic"* instantaneous physical quantity of the form:

$$Q(t) = Q_o \cos^2(\omega t)$$

The time-average of *Q(t)* is defined as:

$$\langle Q(t) \rangle \equiv \langle Q \rangle = \frac{1}{\tau}\int_{t=0}^{t=\tau} Q(t)\,dt = \frac{Q_o}{\tau}\int_{t=0}^{t=\tau} \cos^2(\omega t)\,dt$$



Time-Averaged Quantities Associated with EM Waves

The time average of the $\cos^2(\omega t)$ function:

$$\frac{1}{\tau}\int_0^\tau \cos^2(\omega t)\,dt = \frac{1}{\tau}\left[\frac{t}{2}+\frac{\sin 2\omega t}{4\omega}\right]_{t=0}^{t=\tau} = \frac{1}{2\tau}\left[(\tau-0)+\left(\frac{\sin 2\omega\tau}{2\omega}-0\right)\right] = \frac{1}{2\tau}\left[\tau+\frac{\sin 2\omega\tau}{2\omega}\right]$$

$\omega\tau = 2\pi f\tau$ $\quad f = 1/\tau$ $\quad \omega\tau = 2\pi(\tau/\tau) = 2\pi$ $\quad \sin(\omega\tau) = \sin(2\pi) = 0$

$$\frac{1}{\tau}\int_0^\tau \cos^2(\omega t)\,dt = \frac{1}{2\not{t}}\left[\not{t}\right] = \frac{1}{2}$$

$$\langle Q(t)\rangle = \langle Q\rangle = \frac{1}{2}Q_o$$

Thus, the time-averaged quantities associated with an EM wave propagating in free space are:

Time-Averaged Quantities Associated with EM Waves

EM Energy Density:	$u_{EM}(\vec{r},t) \Rightarrow \langle u_{EM}(\vec{r},t) \rangle$
Total EM Energy:	$U_{EM}(t) \Rightarrow \langle U_{EM}(t) \rangle$
Poynting's Vector:	$\vec{S}(\vec{r},t) \Rightarrow \langle \vec{S}_{EM}(\vec{r},t) \rangle$
EM Power:	$P_{EM}(t) \Rightarrow \langle P_{EM}(t) \rangle$

Time-Averaged Quantities Associated with EM Waves

Linear Momentum Density: $\vec{\wp}_{EM}(\vec{r},t) \Rightarrow \langle \vec{\wp}_{EM}(\vec{r},t) \rangle$

Linear Momentum: $\vec{p}_{EM}(t) \Rightarrow \langle \vec{p}_{EM}(t) \rangle$

Angular Momentum Density: $\vec{\ell}_{EM}(\vec{r},t) \Rightarrow \langle \vec{\ell}_{EM}(\vec{r},t) \rangle$

Angular Momentum: $\vec{\mathcal{L}}_{EM}(t) \Rightarrow \langle \vec{\mathcal{L}}_{EM}(t) \rangle$

Time-Averaged Quantities Associated with EM Waves

For a monochromatic EM plane wave propagating in free space / vacuum in ˆz direction:

$$\left\langle u_{EM}(\vec{r},t)\right\rangle = \frac{1}{2}\varepsilon_o E_o^2 \quad \left(\frac{\text{Joules}}{\text{m}^3}\right)$$

$$\left\langle \vec{S}(\vec{r},t)\right\rangle = \frac{1}{2}c\varepsilon_o E_o^2 \hat{z} = c\left\langle u_{EM}(\vec{r},t)\right\rangle \hat{z} \quad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

$$\left\langle \vec{\wp}_{EM}(\vec{r},t)\right\rangle = \frac{1}{2c}\varepsilon_o E_o^2 \hat{z} = \frac{1}{c^2}\left\langle \vec{S}(\vec{r},t)\right\rangle = \frac{1}{c}\left\langle u_{EM}(\vec{r},t)\right\rangle \hat{z} \quad \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

$$\left\langle \vec{\ell}_{EM}(\vec{r},t)\right\rangle = \left(\vec{r}\times\left\langle \vec{\wp}_{EM}(\vec{r},t)\right\rangle\right) = \frac{1}{c^2}\left(\vec{r}\times\left\langle \vec{S}(\vec{r},t)\right\rangle\right) = \frac{1}{c}\left\langle u_{EM}(\vec{r},t)\right\rangle(\hat{r}\times\hat{z}) \quad \left(\frac{\text{kg}}{\text{m-sec}}\right)$$

Time-Averaged Quantities Associated with EM Waves

Intensity of an *EM* wave:

$$I(\vec{r}) \equiv \langle S(\vec{r},t) \rangle = \langle |\vec{S}(\vec{r},t)| \rangle = c \langle u_{EM}(\vec{r},t) \rangle = \frac{1}{2} c \varepsilon_o E_o^2 \quad \left(\frac{\text{Watts}}{\text{m}^2} \right)$$

The intensity of an EM wave is also known as the irradiance of the EM wave – it is the radiant power incident per unit area upon a surface.

ELECTROMAGNETIC WAVES IN MATTER

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Electromagnetic Wave Propagation in Linear Media

Consider EM wave propagation inside matter - in regions where there are NO free charges and/or free currents (the medium is an insulator/non-conductor).

For this situation, Maxwell's equations become:

1) $\vec{\nabla} \cdot \vec{D}(\vec{r},t) = 0$

2) $\vec{\nabla} \cdot \vec{B}(\vec{r},t) = 0$

3) $\vec{\nabla} \times \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t}$

4) $\vec{\nabla} \times \vec{H}(\vec{r},t) = \frac{\partial \vec{D}(\vec{r},t)}{\partial t}$

Electromagnetic Wave Propagation in Linear Media

The medium is assumed to be linear, homogeneous and isotropic- thus the following relations are valid in this medium:

$$\vec{D}(\vec{r},t) = \varepsilon \vec{E}(\vec{r},t) \quad \text{and} \quad \vec{H}(\vec{r},t) = \frac{1}{\mu}\vec{B}(\vec{r},t)$$

- ε = electric permittivity of the medium.
- $\varepsilon = \varepsilon_o (1 + \chi_e)$, χ_e = electric susceptibility of the medium.
- μ = magnetic permeability of the medium.
- $\mu = \mu_o (1 + \chi_m)$, χ_m = magnetic susceptibility of the medium.
- ε_o = electric permittivity of free space = 8.85×10^{-12} Farads/m.
- μ_o = magnetic permeability of free space = $4\pi \times 10^{-7}$ Henrys/m.

Electromagnetic Wave Propagation in Linear Media

Maxwell's equations inside the linear, homogeneous and isotropic non-conducting medium become:

1) $\vec{\nabla} \bullet \vec{E}(\vec{r},t) = 0$

2) $\vec{\nabla} \bullet \vec{B}(\vec{r},t) = 0$

3) $\vec{\nabla} \times \vec{E}(\vec{r},t) = -\dfrac{\partial \vec{B}(\vec{r},t)}{\partial t}$

4) $\vec{\nabla} \times \vec{B}(\vec{r},t) = \mu\varepsilon \dfrac{\partial \vec{E}(\vec{r},t)}{\partial t}$

In a linear /homogeneous/isotropic medium, the speed of propagation of EM waves is:

$$v'_{prop} = \frac{1}{\sqrt{\varepsilon\mu}}$$

Electromagnetic Wave Propagation in Linear Media

The ***E*** and ***B*** fields in the medium obey the following wave equation:

$$\nabla^2 \vec{E}(\vec{r},t) = \varepsilon\mu \frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} = \frac{1}{v'^2_{prop}} \frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2}$$

$$\nabla^2 \vec{B}(\vec{r},t) = \varepsilon\mu \frac{\partial \vec{B}(\vec{r},t)}{\partial t} = \frac{1}{v'^2_{prop}} \frac{\partial^2 \vec{B}(\vec{r},t)}{\partial t^2}$$

Electromagnetic Wave Propagation in Linear Media

For linear / homogeneous / isotropic media:

$$\varepsilon = K_e \varepsilon_o = (1+\chi_e)\varepsilon_o \qquad K_e = \frac{\varepsilon}{\varepsilon_o} = (1+\chi_e) = \text{relative electric permittivity}$$

$$\mu = K_m \mu_0 = (1+\chi_m)\mu_o \qquad K_m = \frac{\mu}{\mu_o} = (1+\chi_m) = \text{relative magnetic permeability}$$

$$v'_{prop} = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{1}{\sqrt{K_e \varepsilon_o K_m \mu_o}} = \frac{1}{\sqrt{K_e K_m}} \frac{1}{\sqrt{\varepsilon_o \mu_o}} = \frac{1}{\sqrt{K_e K_m}} c$$

If $K_e K_m \geq 1$ thus $\frac{1}{\sqrt{K_e K_m}} \leq 1 \Rightarrow v'_{prop} = \frac{1}{\sqrt{K_e K_m}} c \leq c$

Electromagnetic Wave Propagation in Linear Media

Note also that since $K_e = \dfrac{\varepsilon}{\varepsilon_o}$ and $K_m = \dfrac{\mu}{\mu_o}$ are dimensionless quantities, then so is

$$\frac{1}{\sqrt{K_e K_m}}$$

Define the index of refraction { *a dimensionless quantity*} of the linear / homogeneous / isotropic medium as:

$$n \equiv \sqrt{K_e K_m} = \sqrt{\frac{\varepsilon\mu}{\varepsilon_o \mu_o}}$$

Electromagnetic Wave Propagation in Linear Media

Thus, for linear / homogeneous / isotropic media:

$$v'_{prop} = c/n \ (\leq c)$$ because $$n \geq 1$$

Now for many (but not all) linear/homogeneous/isotropic materials:

$$\mu = \mu_o (1 + \chi_m) \simeq \mu_o$$

(*True for many paramagnetic and diamagnetic-type materials*)

$$|\chi_m| \sim \vartheta(10^{-8}) \sim 0$$

Thus $$K_m = \frac{\mu}{\mu_o} = (1 + \chi_m) \simeq 1 \Rightarrow n \simeq \sqrt{K_e}$$ and $$v'_{prop} = \frac{c}{n} \simeq \frac{c}{\sqrt{K_e}}.$$

Electromagnetic Wave Propagation in Linear Media

The instantaneous EM energy density associated with a linear/homogeneous/isotropic material

$$u_{EM}(\vec{r},t) = \frac{1}{2}\left(\varepsilon E^2(\vec{r},t) + \frac{1}{\mu}B^2(\vec{r},t)\right) = \frac{1}{2}\left(\vec{E}(\vec{r},t)\cdot\vec{D}(\vec{r},t) + \vec{B}(\vec{r},t)\cdot\vec{H}(\vec{r},t)\right) \quad \left(\frac{\text{Joules}}{\text{m}^3}\right)$$

with $\vec{D}(\vec{r},t) = \varepsilon\vec{E}(\vec{r},t)$ and $\vec{H}(\vec{r},t) = \frac{1}{\mu}\vec{B}(\vec{r},t)$

Electromagnetic Wave Propagation in Linear Media

The instantaneous Poynting's vector associated with a linear/homogeneous/isotropic material

$$\boxed{\vec{S}(\vec{r},t)=\frac{1}{\mu}\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)=\left(\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)\right)}\left(\frac{\text{Watts}}{\text{m}^2}\right)$$

The intensity of an EM wave propagating in a linear/homogeneous /isotropic medium is:

$$\boxed{I(\vec{r})\equiv\left\langle\left|\vec{S}(\vec{r},t)\right|\right\rangle=v'_{prop}\left\langle u_{EM}(\vec{r},t)\right\rangle=\frac{1}{2}v'_{prop}\varepsilon E_o^2(\vec{r})=\frac{1}{2}\left(\frac{c}{n}\right)\varepsilon E_o^2(\vec{r})=\left(\frac{c}{n}\right)\varepsilon E_{o_{rms}}^2(\vec{r})}\left(\frac{\text{Watts}}{\text{m}^2}\right)$$

Where

$$\boxed{E_{o_{rms}}\equiv\frac{1}{\sqrt{2}}E_o}$$

Electromagnetic Wave Propagation in Linear Media

The instantaneous linear momentum density associated with an EM wave propagating in a linear/homogeneous/isotropic medium is:

$$\boxed{\vec{\wp}_{EM}(\vec{r},t) = \varepsilon\mu\vec{S}(\vec{r},t) = \frac{1}{v_{prop}^{2}}\vec{S}(\vec{r},t) = \varepsilon\cancel{\mu}\frac{1}{\cancel{\mu}}\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right) = \varepsilon\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)}\left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

The instantaneous angular momentum density associated with an EM wave propagating in a linear/homogeneous/isotropic medium is:

$$\boxed{\vec{\ell}_{EM}(\vec{r},t) = \vec{r}\times\vec{\wp}_{EM}(\vec{r},t) = \varepsilon\,\vec{r}\times\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)}\left(\frac{\text{kg}}{\text{m-sec}}\right)$$

Electromagnetic Wave Propagation in Linear Media

Total instantaneous EM energy: $U_{EM}(t) = \int_v u_{EM}(\vec{r},t)\,d\tau$ (Joules)

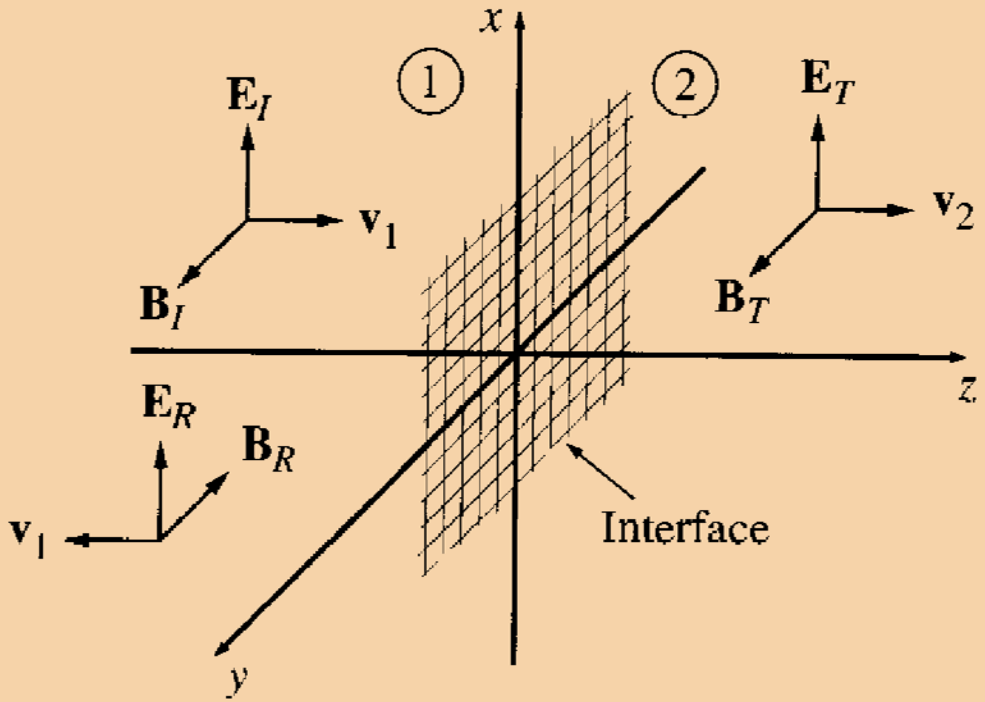
Total instantaneous linear momentum: $\vec{p}_{EM}(t) = \int_v \vec{\wp}_{EM}(\vec{r},t)\,d\tau$ $\left(\frac{\text{kg-m}}{\text{sec}}\right)$

Instantaneous *EM* Power: $P_{EM}(t) = \frac{\partial U_{EM}(t)}{\partial t} = -\oint_S \vec{S}(\vec{r},t)\bullet d\vec{a}$ (Watts)

Total instantaneous angular momentum: $\vec{\mathcal{L}}_{EM}(t) = \int_v \vec{\ell}_{EM}(\vec{r},t)\,d\tau$ $\left(\frac{\text{kg-m}^2}{\text{sec}}\right)$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Suppose the x-y plane forms the boundary between two linear media. A plane wave of frequency ω- travelling in the z- direction and polarized in the x- direction- approaches the interface from the left



Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Incident EM plane wave (in medium 1):

Propagates in the $+\hat{z}$-direction (*i.e.* $\hat{k}_{inc} = +\hat{k}_1 = +\hat{z}$), with polarization $\hat{n}_{inc} = +\hat{x}$

$$\tilde{\vec{E}}_{inc}(z,t) = \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)} \hat{x} \quad \text{with:} \quad k_{inc} = \left|\vec{k}_{inc}\right| = k_1 = \left|\vec{k}_1\right| = 2\pi/\lambda_1 = \omega/v_1$$

$$\tilde{\vec{B}}_{inc}(z,t) = \frac{1}{v_1}\hat{k}_{inc} \times \tilde{\vec{E}}_{inc}(z,t) = \frac{1}{v_1}\tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)} \hat{y} \quad \underline{\text{since}}: \quad \hat{k}_{inc} \times \hat{n}_{inc} = +\hat{z} \times \hat{x} = +\hat{y}$$

Reflected EM plane wave (in medium 1):

Propagates in the $-\hat{z}$-direction (*i.e.* $\hat{k}_{refl} = -\hat{k}_1 = -\hat{z}$), with polarization $\hat{n}_{refl} = +\hat{x}$

$$\tilde{\vec{E}}_{refl}(z,t) = \tilde{E}_{o_{refl}} e^{i(-k_1 z - \omega t)} \hat{x} \quad \text{with:} \quad k_{refl} = \left|\vec{k}_{refl}\right| = k_1 = \left|\vec{k}_1\right| = 2\pi/\lambda_1 = \omega/v_1$$

$$\tilde{B}_{refl}(z,t) = \frac{1}{v_1}\hat{k}_{refl} \times \tilde{\vec{E}}_{refl}(z,t) = -\frac{1}{v_1}\tilde{E}_{o_{refl}} e^{i(-k_1 z - \omega t)} \hat{y} \quad \underline{\text{since}}: \quad \hat{k}_{refl} \times \hat{n}_{refl} = -\hat{z} \times \hat{x} = -\hat{y}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Transmitted EM plane wave (in medium 2):

Propagates in the $+\hat{z}$ -direction (*i.e.* $\hat{k}_{trans} = +\hat{k}_2 = +\hat{z}$), with polarization $\boxed{\hat{n}_{trans} = +\hat{x}}$

$$\boxed{\tilde{\vec{E}}_{trans}(z,t) = \tilde{E}_{o_{trans}} e^{i(k_2 z - \omega t)}\hat{x}} \quad \text{with:} \quad \boxed{k_{trans} = \left|\vec{k}_{trans}\right| = k_2 = \left|\vec{k}_2\right| = 2\pi/\lambda_2 = \omega/v_2}$$

$$\boxed{\tilde{\vec{B}}_{trans}(z,t) = \frac{1}{v_2}\hat{k}_{trans} \times \tilde{\vec{E}}_{trans}(z,t) = \frac{1}{v_2}\tilde{E}_{o_{trans}} e^{i(k_2 z - \omega t)}\hat{y}} \quad \underline{\text{since:}} \quad \boxed{\hat{k}_{trans} \times \hat{n}_{trans} = +\hat{z} \times \hat{x} = +\hat{y}}$$

Note that {*here, in this situation*} *the E* -field / polarization vectors are all oriented in the same direction, i.e.

$$\boxed{\hat{n}_{inc} = \hat{n}_{refl} = \hat{n}_{trans} = +\hat{x}}$$

or equivalently:

$$\boxed{\vec{E}_{inc}(\vec{r},t) \parallel \vec{E}_{refl}(\vec{r},t) \parallel \vec{E}_{trans}(\vec{r},t)}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

At the interface between the two linear / homogeneous / isotropic media -at z = 0 {in the x-y plane} the boundary conditions 1) – 4) must be satisfied for the total E and B -fields immediately present on either side of the interface:

BC 1) Normal $\vec{D}$ continuous: $\boxed{\varepsilon_1 E_{1_{Tot}}^{\perp} = \varepsilon_2 E_{2_{Tot}}^{\perp}}$

(*n.b.* $\perp$ refers to the *x*-*y* boundary, *i.e.* in the $+\hat{z}$ direction)

BC 2) Tangential $\vec{E}$ continuous: $\boxed{E_{1_{Tot}}^{\parallel} = E_{2_{Tot}}^{\parallel}}$

(*n.b.* $\parallel$ refers to the *x*-*y* boundary, *i.e.* in the *x*-*y* plane)

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

BC 3) Normal $\vec{B}$ continuous: $$B_{1_{Tot}}^{\perp} = B_{2_{Tot}}^{\perp}$$

(⊥ to x-y boundary, i.e. in the +z^ direction)

BC 4) Tangential $\vec{H}$ continuous: $$\frac{1}{\mu_1} B_{1_{Tot}}^{\parallel} = \frac{1}{\mu_2} B_{2_{Tot}}^{\parallel}$$

(∥ to x-y boundary, i.e. in x-y plane)

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

For plane EM waves at normal incidence on the boundary at z = 0 lying in the x-y plane- no components of *E or B* (incident, reflected or transmitted waves) - allowed to be along the ±zˆ propagation direction(s) - the E and *B*-field are transverse fields {*constraints imposed by Maxwell's equations*}.

BC 1) and BC 3) impose no restrictions on such *EM* waves since:

$$\{ E^{\perp}_{1_{Tot}} = \mathrm{E}^{z}_{1_{Tot}} = 0 \,;\; E^{\perp}_{2_{Tot}} = E^{z}_{2_{Tot}} = 0 \} \text{ and } \{ B^{\perp}_{1_{Tot}} = B^{z}_{1_{Tot}} = 0 \,;\; B^{\perp}_{2_{Tot}} = B^{z}_{2_{Tot}} = 0 \}$$

⇒ The only restrictions on plane EM waves propagating with normal incidence on the boundary at z = 0 are imposed by BC 2) and BC 4).

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

At z = 0 in medium 1) (i.e. z ≤ 0) we must have:

$$\boxed{\vec{\tilde{E}}^{\parallel}_{1_{Tot}}(z=0,t)=\vec{\tilde{E}}_{inc}(z=0,t)+\vec{\tilde{E}}_{refl}(z=0,t)} \text{ and}$$

$$\boxed{\frac{1}{\mu_1}\vec{\tilde{B}}^{\parallel}_{1_{Tot}}(z=0,t)=\frac{1}{\mu_1}\vec{\tilde{B}}_{inc}(z=0,t)+\frac{1}{\mu_1}\vec{\tilde{B}}_{refl}(z=0,t)}$$

While at z = 0 in medium 2) (i.e. z ≥ 0) we must have:

$$\boxed{\vec{\tilde{E}}^{\parallel}_{2_{Tot}}(z=0,t)=\vec{\tilde{E}}_{trans}(z=0,t)} \text{ and}$$

$$\boxed{\frac{1}{\mu_2}\vec{\tilde{B}}^{\parallel}_{2_{Tot}}(z=0,t)=\frac{1}{\mu_2}\vec{\tilde{B}}_{trans}(z=0,t)}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

BC 2) (Tangential *E* is continuous @ z = 0) requires that:

$$\vec{\tilde{E}}^{\parallel}_{1_{Tot}}\Big|_{z=0} = \vec{\tilde{E}}^{\parallel}_{2_{Tot}}\Big|_{z=0}$$ or: $$\vec{\tilde{E}}_{inc}(z=0,t)+\vec{\tilde{E}}_{refl}(z=0,t)=\vec{\tilde{E}}_{trans}(z=0,t)$$

BC 4) (Tangential *H* is continuous @ z = 0) requires that:

$$\frac{1}{\mu_1}\vec{\tilde{B}}^{\parallel}_{1_{Tot}}\Big|_{z=0} = \frac{1}{\mu_2}\vec{\tilde{B}}^{\parallel}_{2_{Tot}}\Big|_{z=0}$$

or: $$\frac{1}{\mu_1}\vec{\tilde{B}}_{inc}(z=0,t)+\frac{1}{\mu_1}\vec{\tilde{B}}_{refl}(z=0,t)=\frac{1}{\mu_2}\vec{\tilde{B}}_{trans}(z=0,t)$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Using explicit expressions for the complex **E** and **B** fields

$$\vec{\tilde{E}}_{inc}(z,t) = \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)} \hat{x}$$

$$\vec{\tilde{B}}_{inc}(z,t) = \frac{1}{v_1} \hat{k}_{inc} \times \vec{\tilde{E}}_{inc}(z,t) = \frac{1}{v_1} \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)} \hat{y}$$

$$\vec{\tilde{E}}_{refl}(z,t) = \tilde{E}_{o_{refl}} e^{i(-k_1 z - \omega t)} \hat{x}$$

$$\vec{\tilde{B}}_{refl}(z,t) = \frac{1}{v_1} \hat{k}_{refl} \times \vec{\tilde{E}}_{refl}(z,t) = -\frac{1}{v_1} \tilde{E}_{o_{refl}} e^{i(-k_1 z - \omega t)} \hat{y}$$

$$\vec{\tilde{E}}_{trans}(z,t) = \tilde{E}_{o_{trans}} e^{i(k_2 z - \omega t)} \hat{x}$$

$$\vec{\tilde{B}}_{trans}(z,t) = \frac{1}{v_2} \hat{k}_{trans} \times \vec{\tilde{E}}_{trans}(z,t) = \frac{1}{v_2} \tilde{E}_{o_{trans}} e^{i(k_2 z - \omega t)} \hat{y}$$

into the above boundary condition relations- equations become

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

BC 2) (Tangential $\vec{E}$ continuous @ $z = 0$): $\tilde{E}_{o_{inc}} \cancel{e^{-i\omega t}} + \tilde{E}_{o_{refl}} \cancel{e^{-i\omega t}} = \tilde{E}_{o_{trans}} \cancel{e^{-i\omega t}}$

BC 4) (Tangential $\vec{H}$ continuous @ $z = 0$): $\frac{1}{\mu_1 v_1} \tilde{E}_{o_{inc}} \cancel{e^{-i\omega t}} - \frac{1}{\mu_1 v_1} \tilde{E}_{o_{refl}} \cancel{e^{-i\omega t}} = \frac{1}{\mu_2 v_2} \tilde{E}_{o_{trans}} \cancel{e^{-i\omega t}}$

Cancelling the common $e^{-i\omega t}$ factors on the LHS & RHS of above equations - we have at z = 0 { everywhere in the x-y plane- must be independent of any time t}:

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BC 2) (Tangential $\vec{E}$ continuous @ $z = 0$): $\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}$

BC 4) (Tangential $\vec{H}$ continuous @ $z = 0$): $\dfrac{1}{\mu_1 v_1}\tilde{E}_{o_{inc}} - \dfrac{1}{\mu_1 v_1}\tilde{E}_{o_{refl}} = \dfrac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}}$

Assuming that $\{\mu_1$ and $\mu_2\}$ and $\{v_1$ and $v_2\}$ are known / given for the two media, we have <u>two</u> equations {from BC 2) and BC 4)} and <u>*three*</u> unknowns $\{\tilde{E}_{o_{inc}}, \tilde{E}_{o_{refl}}, \tilde{E}_{o_{trans}}\}$

→ Solve above equations simultaneously for

$\{\tilde{E}_{o_{refl}}$ and $\tilde{E}_{o_{trans}}\}$ in terms of / scaled to $\tilde{E}_{o_{inc}}$.

First (for convenience) let us define: $\beta \equiv \dfrac{\mu_1 v_1}{\mu_2 v_2}$

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BC 4) (Tangential ***H*** continuous @ z = 0) relation becomes:

$$\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \beta\, \tilde{E}_{o_{trans}}$$

BC 2) (Tangential **E** continuous @ *z = 0):*

$$\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}$$

BC 4) (Tangential ***H*** continuous @ *z = 0):*

$$\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \beta\, \tilde{E}_{o_{trans}}$$ with $$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2}$$

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Add and Subtract BC 2) and BC 4) relations:

$$2\tilde{E}_{o_{inc}} = (1+\beta)\,\tilde{E}_{o_{trans}} \quad \Rightarrow \quad \tilde{E}_{o_{trans}} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{o_{inc}} \qquad (2+4)$$

$$2\tilde{E}_{o_{refl}} = (1-\beta)\,\tilde{E}_{o_{trans}} \quad \Rightarrow \quad \tilde{E}_{o_{refl}} = \left(\frac{1-\beta}{2}\right)\tilde{E}_{o_{trans}} \qquad (2-4)$$

Insert the result of eqn. (2+4) into eqn. (2−4):

$$\tilde{E}_{o_{refl}} = \left(\frac{1-\beta}{\not{2}}\right)\left(\frac{\not{2}}{1+\beta}\right)\tilde{E}_{o_{inc}} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{o_{inc}}$$

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$$\tilde{E}_{o_{refl}} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{o_{inc}} \quad \text{and} \quad \tilde{E}_{o_{trans}} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{o_{inc}}$$

<u>Now</u>: $\beta \equiv \dfrac{\mu_1 v_1}{\mu_2 v_2}$ and: $v_1 = \dfrac{c}{n_1}$, $v_2 = \dfrac{c}{n_2}$ where: $n_1 = \sqrt{\dfrac{\varepsilon_1\mu_1}{\varepsilon_o\mu_o}}$ and $n_2 = \sqrt{\dfrac{\varepsilon_2\mu_2}{\varepsilon_o\mu_o}}$

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 (c/n_1)}{\mu_2 (c/n_2)} = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{\mu_1\sqrt{\varepsilon_2\mu_2/\varepsilon_o\mu_o}}{\mu_2\sqrt{\varepsilon_1\mu_1/\varepsilon_o\mu_o}} = \frac{\mu_1}{\mu_2}\frac{\sqrt{\varepsilon_2\mu_2}}{\sqrt{\varepsilon_1\mu_1}} = \sqrt{\left(\frac{\varepsilon_2}{\mu_2}\right)\Big/\left(\frac{\varepsilon_1}{\mu_1}\right)} = \sqrt{\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}}$$

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Now if the two media are both paramagnetic and/or diamagnetic, such that $\left|\chi_{m_{1,2}}\right| \ll 1$

$$i.e. \quad \mu_1 = \mu_o\left(1+\chi_{m_1}\right) \approx \mu_o \quad \text{and:} \quad \mu_2 = \mu_o\left(1+\chi_{m_2}\right) \approx \mu_o$$

Very common for many (but not all) non-conducting linear/homogeneous/isotropic media

Then $$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} \simeq \left(\frac{v_1}{v_2}\right) = \left(\frac{n_2}{n_1}\right) \quad \text{for } \mu_1 \approx \mu_2 \approx \mu_o \quad \text{or } \left|\chi_{m_{1,2}}\right| \ll 1$$

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Then

$$\tilde{E}_{o_{refl}} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{o_{inc}} \simeq \left(\frac{1-(v_1/v_2)}{1+(v_1/v_2)}\right)\tilde{E}_{o_{inc}} = \left(\frac{v_2-v_1}{v_2+v_1}\right)\tilde{E}_{o_{inc}}$$

$$\tilde{E}_{o_{trans}} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{o_{inc}} \simeq \left(\frac{2}{1+(v_1/v_2)}\right)\tilde{E}_{o_{nc}} = \left(\frac{2v_2}{v_2+v_1}\right)\tilde{E}_{o_{inc}}$$

We can alternatively express these relations in terms of the indices of refraction n_1 *&* n_2*:*

$$\tilde{E}_{o_{refl}} = \left(\frac{n_1-n_2}{n_1+n_2}\right)\tilde{E}_{o_{inc}} \quad \text{and} \quad \tilde{E}_{o_{trans}} = \left(\frac{2n_1}{n_1+n_2}\right)\tilde{E}_{o_{inc}}$$

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Now since:

$$\tilde{E}_{o_{inc}} = E_{o_{inc}} e^{i\delta}$$

$$\tilde{E}_{o_{refl}} = E_{o_{refl}} e^{i\delta}$$

$$\tilde{E}_{o_{trans}} = E_{o_{trans}} e^{i\delta}$$

δ = phase angle (in radians) defined at the zero of time - $t = 0$

Then for the purely real amplitudes $(E_{o_{inc}}, E_{o_{refl}}, E_{o_{trans}})$

these relations become:

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

for $\mu_1 \simeq \mu_2 \simeq \mu_o$

$$E_{o_{refl}} = \left(\frac{1-\beta}{1+\beta}\right) E_{o_{inc}} \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right) E_{o_{inc}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) E_{o_{inc}}$$

$$\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)$$

$$E_{o_{trans}} = \left(\frac{2}{1+\beta}\right) E_{o_{inc}} \simeq \left(\frac{2v_2}{v_1 + v_1}\right) E_{o_{inc}} = \left(\frac{2n_1}{n_1 + n_2}\right) E_{o_{inc}}$$

for $\mu_1 \simeq \mu_2 \simeq \mu_o$

Monochromatic plane *EM wave at normal* incidence on a boundary between two linear / homogeneous / isotropic media

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

For a monochromatic plane EM wave at normal incidence on a boundary between two linear / homogeneous / isotropic media, $\mu_1 \simeq \mu_2 \simeq \mu_o$ note the following points:

If $v_2 > v_1$ (*i.e.* $n_2 < n_1$) {*e.g.* medium 1) = glass $\Rightarrow$ medium 2) = air}:

$$\boxed{E_{o_{refl}} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right) E_{o_{inc}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) E_{o_{inc}}} \Rightarrow$$

$E_{o_{refl}}$ is precisely in-phase with $E_{o_{inc}}$ because $(v_2 - v_1) > 0$.

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If $v_2 < v_1$ $(i.e.\ n_2 > n_1)$ $\{e.g.$ medium 1) = air $\Rightarrow$ medium 2) = glass$\}$:

$$E_{0_{refl}} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right) E_{0_{inc}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) E_{0_{inc}} \Rightarrow$$

$E_{0_{refl}}$ is 180° out-of-phase with $E_{0_{inc}}$ because $(v_2 - v_1) < 0$.

i.e. $$E_{0_{refl}} = -\left|\frac{v_2 - v_1}{v_2 + v_1}\right| E_{0_{inc}} = -\left|\frac{n_1 - n_2}{n_1 + n_2}\right| E_{0_{inc}} \Rightarrow$$

The minus sign indicates a 180° phase shift occurs upon reflection for $v_2 < v_1$ (i.e. $n_2 > n_1$) !!!

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

$E_{o_{trans}}$ is always in-phase with $E_{o_{inc}}$ for all possible v_1 & v_2 $(n_1 \& n_2)$ because:

$$E_{o_{trans}} = \left(\frac{2}{1+\beta}\right) E_{o_{inc}} \simeq \left(\frac{2v_2}{v_1+v_1}\right) E_{o_{inc}} = \left(\frac{2n_1}{n_1+n_2}\right) E_{o_{inc}}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

What fraction of the incident *EM* wave energy is reflected?

What fraction of the incident *EM* wave energy is transmitted?

In a given linear/homogeneous/isotropic medium with

$$v = \sqrt{\frac{\varepsilon_o \mu_o}{\varepsilon \mu}}\, c = c/n$$

The time-averaged energy density in the EM wave is:

$$\langle u_{EM}(\vec{r},t)\rangle = \frac{1}{2}\varepsilon E_o^2(\vec{r}) = \varepsilon E_{o_{rms}}^2(\vec{r}) \quad \left(\frac{\text{Joules}}{\text{m}^3}\right)$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

The time-averaged Poynting's vector is:

$$\left\langle \vec{S}(\vec{r},t) \right\rangle = \frac{1}{\mu} \left\langle \vec{E}(\vec{r},t) \times \vec{B}(\vec{r},t) \right\rangle \quad \left(\frac{\text{Watts}}{\text{m}^2} \right)$$

The intensity of the EM wave is:

$$I(\vec{r}) \equiv \left\langle \left| \vec{S}(\vec{r},t) \right| \right\rangle = v \left\langle u_{EM}(\vec{r},t) \right\rangle = v \left(\frac{1}{2} \varepsilon E_o^2(\vec{r}) \right) = \frac{1}{2} \varepsilon v E_o^2(\vec{r}) = \varepsilon v E_{o_{rms}}^2(\vec{r}) \quad \left(\frac{\text{Watts}}{\text{m}^2} \right)$$

Note that the three Poynting's vectors associated with this problem are such that

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

$$\vec{S}_{inc} \parallel (+\hat{z}), \quad \vec{S}_{refl} \parallel (-\hat{z}) \text{ and } \vec{S}_{trans} \parallel (+\hat{z})$$

For a monochromatic plane *EM wave at normal incidence on a boundary between two linear* /homogeneous / isotropic media, with $\mu_1 \simeq \mu_2 \simeq \mu_o$

$$E_{o_{refl}} = \left(\frac{1-\beta}{1+\beta}\right) E_{o_{inc}} \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right) E_{o_{inc}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) E_{o_{inc}}$$

$$\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)$$

$$E_{o_{trans}} = \left(\frac{2}{1+\beta}\right) E_{o_{inc}} \simeq \left(\frac{2v_2}{v_1 + v_1}\right) E_{o_{inc}} = \left(\frac{2n_1}{n_1 + n_2}\right) E_{o_{inc}}$$

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Take the ratios $\left(E_{o_{refl}}/E_{o_{inc}}\right)$ and $\left(E_{o_{trans}}/E_{o_{inc}}\right)$ - then square them:

$$\left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right)^2 = \left(\frac{1-\beta}{1+\beta}\right)^2 \approx \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

and

$$\left(\frac{E_{o_{trans}}}{E_{o_{inc}}}\right)^2 = \left(\frac{2}{1+\beta}\right)^2 \approx \left(\frac{2v_2}{v_2 + v_1}\right)^2 = \left(\frac{2n_1}{n_1 + n_2}\right)^2$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Define the reflection coefficient as:

$$R(\vec{r}) \equiv \left(\frac{I_{refl}(\vec{r})}{I_{inc}(\vec{r})}\right) = \frac{\left\langle \left|\vec{S}_{refl}(\vec{r},t)\right|\right\rangle}{\left\langle \left|\vec{S}_{inc}(\vec{r},t)\right|\right\rangle} = \frac{v_1 \left\langle u_{EM}^{refl}(\vec{r},t)\right\rangle}{v_1 \left\langle u_{EM}^{inc}(\vec{r},t)\right\rangle} = \frac{\left\langle u_{EM}^{refl}(\vec{r},t)\right\rangle}{\left\langle u_{EM}^{inc}(\vec{r},t)\right\rangle} = \frac{\frac{1}{2}\varepsilon_1 v_1 E_{o_{refl}}^2(\vec{r})}{\frac{1}{2}\varepsilon_1 v_1 E_{o_{inc}}^2(\vec{r})} = \frac{E_{o_{refl}}^2(\vec{r})}{E_{o_{inc}}^2(\vec{r})}$$

Define the transmission coefficient as:

$$T(\vec{r}) \equiv \left(\frac{I_{trans}(\vec{r})}{I_{inc}(\vec{r})}\right) = \frac{\left\langle \left|\vec{S}_{trans}(\vec{r},t)\right|\right\rangle}{\left\langle \left|\vec{S}_{inc}(\vec{r},t)\right|\right\rangle} = \frac{v_2 \left\langle u_{EM}^{trans}(\vec{r},t)\right\rangle}{v_1 \left\langle u_{EM}^{inc}(\vec{r},t)\right\rangle} = \frac{\left(\frac{1}{2}\varepsilon_2 v_2 E_{o_{trans}}^2(\vec{r})\right)}{\left(\frac{1}{2}\varepsilon_1 v_1 E_{o_{inc}}^2(\vec{r})\right)} = \frac{\varepsilon_2 v_2 E_{o_{trans}}^2(\vec{r})}{\varepsilon_1 v_1 E_{o_{inc}}^2(\vec{r})}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

For a linearly-polarized monochromatic plane EM wave at normal incidence on a boundary between two linear / homogeneous / isotropic media, with $\mu_1 \simeq \mu_2 \simeq \mu_o$

Reflection coefficient:

$$R(\vec{r}) \equiv \left(\frac{I_{refl}(\vec{r})}{I_{inc}(\vec{r})}\right) = \left(\frac{E_{o_{refl}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2$$

Transmission coefficient:

$$T(\vec{r}) \equiv \left(\frac{I_{trans}(\vec{r})}{I_{inc}(\vec{r})}\right) = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{E_{o_{trans}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2$$

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But:

$$\left(\frac{E_{o_{refl}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2 = \left(\frac{1-\beta}{1+\beta}\right)^2 \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \quad \&$$

$$\left(\frac{E_{o_{trans}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2 = \left(\frac{2}{1+\beta}\right)^2 \simeq \left(\frac{2v_2}{v_2 + v_1}\right)^2 = \left(\frac{2n_1}{n_1 + n_2}\right)^2$$

Thus Reflection and Transmission coefficient:

$$R(\vec{r}) \equiv \left(\frac{1-\beta}{1+\beta}\right)^2 \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

$$\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)$$

$$T(\vec{r}) \equiv \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{2}{1+\beta}\right)^2 \simeq \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\frac{2v_2}{v_2 + v_1}\right)^2 = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\frac{2n_1}{n_1 + n_2}\right)^2$$

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Now: $\dfrac{\varepsilon_2 v_2}{\varepsilon_1 v_1} = \dfrac{\dfrac{\varepsilon_2 \mu_2 v_2}{\mu_2}}{\dfrac{\varepsilon_1 \mu_1 v_1}{\mu_1}}$ **but:** $v_2^2 = \dfrac{1}{\varepsilon_2 \mu_2} \Rightarrow \varepsilon_2 \mu_2 = \dfrac{1}{v_2^2}$, $v_1^2 = \dfrac{1}{\varepsilon_1 \mu_1} \Rightarrow \varepsilon_1 \mu_1 = \dfrac{1}{v_1^2}$

$$\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} = \frac{\left(\frac{1}{v_2{}^2} \cdot v_2\right) \Big/ \mu_2}{\left(\frac{1}{v_1{}^2} \cdot v_1\right) \Big/ \mu_2} = \frac{1 \big/ \mu_2 v_2}{1 \big/ \mu_1 v_1} = \frac{\mu_1 v_1}{\mu_2 v_2} \equiv \beta$$

!!! *i.e.*

$$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

for $\mu_1 \approx \mu_2 \approx \mu_o$

$$T(\vec{r}) = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{2}{1+\beta}\right)^2 = \beta\left(\frac{2}{1+\beta}\right)^2 = \frac{4\beta}{(1+\beta)^2} \simeq \frac{4v_2 v_1}{(v_2+v_1)^2} = \frac{4n_1 n_2}{(n_1+n_2)^2}$$

Thus:

$$R(\vec{r})+T(\vec{r}) = \frac{(1-\beta)^2}{(1+\beta)^2} + \frac{4\beta}{(1+\beta)^2} = \frac{(1-\beta)^2+4\beta}{(1+\beta)^2} = \frac{1-2\beta+\beta^2+4\beta}{(1+\beta)^2} = \frac{1+2\beta+\beta^2}{(1+\beta)^2} = \frac{(1+\beta)^2}{(1+\beta)^2} = 1$$

$$\boxed{R(\vec{r})+T(\vec{r})=1}$$

$\Rightarrow$EM energy is conserved at the interface/boundary between two L/H/I media

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

For a linearly-polarized monochromatic plane EM wave at normal incidence on a boundary between two linear / homogeneous / isotropic media, with $\mu_1 \simeq \mu_2 \simeq \mu_o$

Reflection coefficient:

$\mu_1 \simeq \mu_2 \simeq \mu_o$

$$R(\vec{r}) \equiv \left(\frac{I_{refl}(\vec{r})}{I_{inc}(\vec{r})}\right) = \left(\frac{E_{o_{refl}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2 = \frac{(1-\beta)^2}{(1+\beta)^2} \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

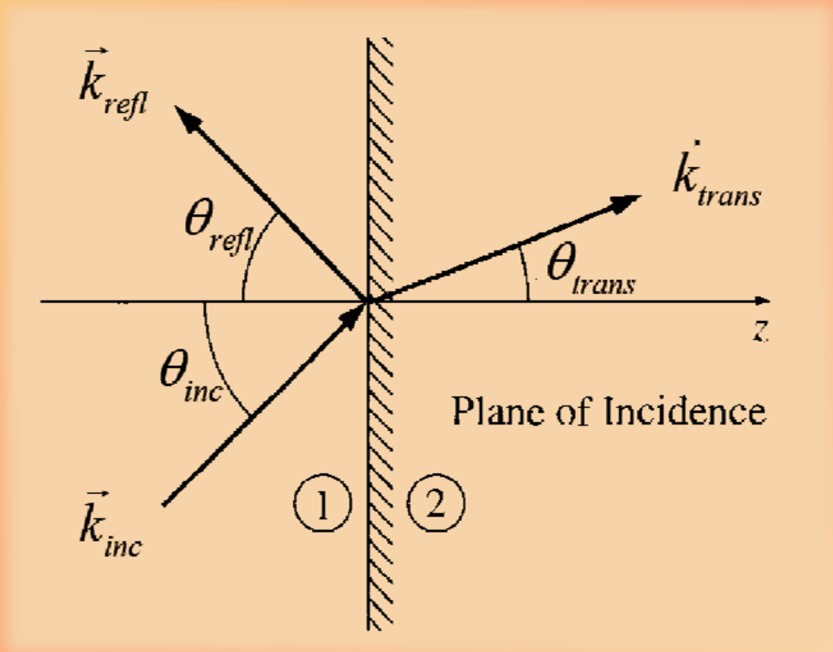
Transmission coefficient:

$$T(\vec{r}) \equiv \left(\frac{I_{trans}(\vec{r})}{I_{inc}(\vec{r})}\right) = \beta\left(\frac{E_{o_{trans}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2 = \frac{4\beta}{(1+\beta)^2} \simeq \frac{4v_2v_1}{(v_2+v_1)^2} = \frac{4n_1n_2}{(n_1+n_2)^2}$$

$\mu_1 \simeq \mu_2 \simeq \mu_o$

Reflection & Transmission of Monochromatic Plane EM Waves at Oblique Incidence

A monochromatic plane EM wave incident at an oblique angle θ_{inc} on a boundary between two linear/ homogeneous/isotropic media, defined with respect to the normal to the interface- as shown in the figure below:



Reflection & Transmission of Monochromatic Plane EM Waves at Oblique Incidence

The incident EM wave is:

$$\tilde{\vec{E}}_{inc}(\vec{r},t) = \tilde{\vec{E}}_{o_{inc}} e^{i(\vec{k}_{inc}\cdot\vec{r}-\omega t)} \quad \text{and} \quad \tilde{\vec{B}}_{inc}(\vec{r},t) = \frac{1}{v_1}\hat{k}_{inc}\times\tilde{\vec{E}}_{inc}(\vec{r},t)$$

The reflected EM wave is:

$$\tilde{\vec{E}}_{refl}(\vec{r},t) = \tilde{\vec{E}}_{o_{refl}} e^{i(\vec{k}_{refl}\cdot\vec{r}-\omega t)} \quad \text{and} \quad \tilde{\vec{B}}_{refl}(\vec{r},t) = \frac{1}{v_1}\hat{k}_{refl}\times\tilde{\vec{E}}_{refl}(\vec{r},t)$$

The transmitted EM wave is:

$$\tilde{\vec{E}}_{trans}(\vec{r},t) = \tilde{\vec{E}}_{o_{trans}} e^{i(\vec{k}_{trans}\cdot\vec{r}-\omega t)} \quad \text{and} \quad \tilde{\vec{B}}_{trans}(\vec{r},t) = \frac{1}{v_2}\hat{k}_{trans}\times\tilde{\vec{E}}_{trans}(\vec{r},t)$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

All three EM waves have the same frequency- $f = \omega/2\pi$

$$\omega = k_{inc} v_1 = k_{refl} v_1 = k_{trans} v_2$$

$$k_{inc} = k_{refl} = k_1 = \left(\frac{v_2}{v_1}\right) k_{trans} = \left(\frac{v_2}{v_1}\right) k_2 = \left(\frac{n_1}{n_2}\right) k_{trans} = \left(\frac{n_1}{n_2}\right) k_2$$

$$v_i = c/n_i \quad i = 1,2$$

The total EM fields in medium 1

$$\tilde{\vec{E}}_{Tot_1}(\vec{r},t) = \tilde{\vec{E}}_{inc}(\vec{r},t) + \tilde{\vec{E}}_{refl}(\vec{r},t) \quad \text{and} \quad \tilde{\vec{B}}_{Tot_1}(\vec{r},t) = \tilde{\vec{B}}_{inc}(\vec{r},t) + \tilde{\vec{B}}_{refl}(\vec{r},t)$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Must match to the total EM fields in medium 2:

$$\vec{\tilde{E}}_{Tot_2}(\vec{r},t) = \vec{\tilde{E}}_{trans}(\vec{r},t) \quad \text{and} \quad \vec{\tilde{B}}_{Tot_2}(\vec{r},t) = \vec{\tilde{B}}_{trans}(\vec{r},t)$$

Using the boundary conditions BC1) → BC4) at z = 0.

At *z* = *0*- four boundary conditions are of the form:

$$(\sim)\, e^{i\left(\vec{k}_{inc}\cdot\vec{r}-\omega t\right)} + (\sim)\, e^{i\left(\vec{k}_{refl}\cdot\vec{r}-\omega t\right)} = (\sim)\, e^{i\left(\vec{k}_{trans}\cdot\vec{r}-\omega t\right)}$$

They must hold for all (x,y) on the interface at z = 0 - and also must hold for all times, t. The above relation is already satisfied for arbitrary time, t - the factor $e^{-i\omega t}$ is common to all terms.

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The following relation must hold for all (x,y) on interface at at z = 0:

$$(\sim\sim)\, e^{i(\vec{k}_{inc}\cdot\vec{r})} + (\sim\sim)\, e^{i(\vec{k}_{refl}\cdot\vec{r})} = (\sim\sim)\, e^{i(\vec{k}_{trans}\cdot\vec{r})}$$

When $z = 0$ - at interface we must have:

$$\vec{k}_{inc} \bullet \vec{r} = \vec{k}_{refl} \bullet \vec{r} = \vec{k}_{trans} \bullet \vec{r}$$

$$k_{inc_x} x + k_{inc_y} y = k_{refl_x} x + k_{refl_y} y = k_{trans_x} x + k_{trans_y} y \qquad @\, z = 0$$

The above relation can only hold for arbitrary (x, y, z = 0) **iff (= if and only if):**

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The above relation can only hold for arbitrary (x, y, z = 0) **iff (= if and only if):**

$$k_{inc_x} x = k_{refl_x} x = k_{trans_x} x \quad \Rightarrow \quad k_{inc_x} = k_{refl_x} = k_{trans_x}$$

$$k_{inc_y} y = k_{refl_y} y = k_{trans_y} y \quad \Rightarrow \quad k_{inc_y} = k_{refl_y} = k_{trans_y}$$

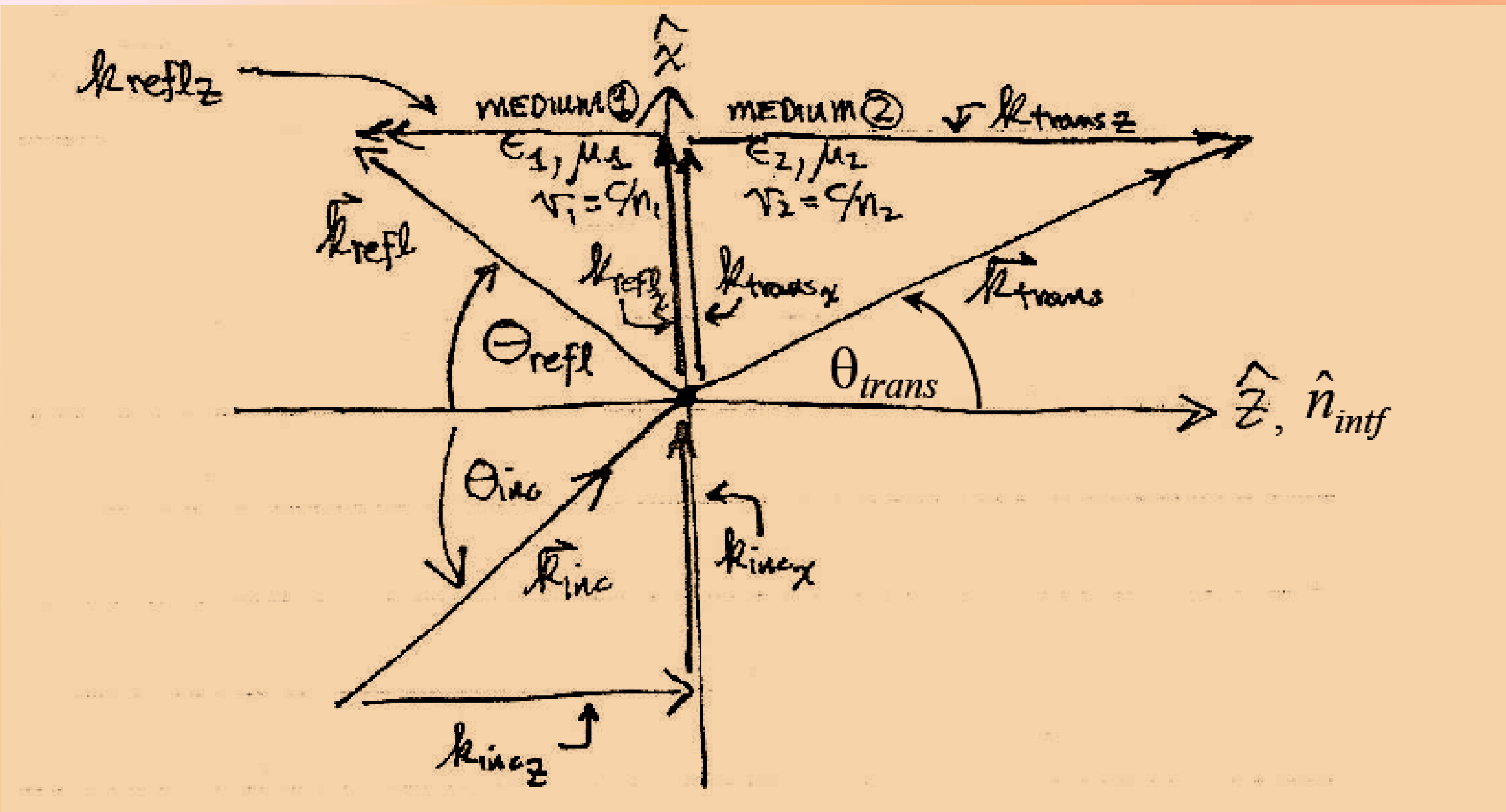
Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The problem has rotational symmetry about the z –axis- then without any loss of generality we can choose *k* to lie entirely within the x-z plane, as shown in the figure

$$k_{inc_y} = k_{refl_y} = k_{trans_y} = 0 \quad \text{and thus:} \quad k_{inc_x} = k_{refl_x} = k_{trans_x}$$

The transverse components of $\vec{k}_{inc}, \vec{k}_{refl}, \vec{k}_{trans}$ are all equal and point in the +x^ direction.

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*



Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The First Law of Geometrical Optics:

The incident, reflected, and transmitted wave vectors form a plane (called the plane of incidence), which also includes the normal to the surface (here, the z axis).

The Second Law of Geometrical Optics (Law of Reflection):

From the figure, we see that:

$$\boxed{k_{inc_x} = k_{inc}\sin\theta_{inc}} = \boxed{k_{refl_x} = k_{refl}\sin\theta_{refl}} = \boxed{k_{trans_x} = k_{trans}\sin\theta_{trans}}$$

$$\boxed{k_{inc} = k_{refl} = k_1} \Rightarrow \boxed{\sin\theta_{inc} = \sin\theta_{refl}}$$

Angle of Incidence = Angle of Reflection $\boxed{\theta_{inc} = \theta_{refl}}$ Law of Reflection!

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The Third Law of Geometrical Optics (Law of Refraction – Snell's Law):

For the transmitted angle, θ_{trans} we see that:

$$k_{inc} \sin\theta_{inc} = k_{trans} \sin\theta_{trans}$$

In medium 1): $k_{inc} = k_1 = \omega/v_1 = n_1\omega/c = n_1 k_o$

where $k_o = $ vacuum wave number $= 2\pi/\lambda_o$

and $\lambda_o = $ vacuum wave length

In medium 2): $k_{trans} = k_2 = \omega/v_2 = n_2\omega/c = n_2 k_o$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

$$k_{inc}\sin\theta_{inc} = k_{trans}\sin\theta_{trans} \Rightarrow k_1\sin\theta_{inc} = k_2\sin\theta_{trans}$$

$$k_{inc} = k_1 = n_1 k_o \quad \text{and} \quad k_{trans} = k_2 = n_2 k_o$$

$$k_1\sin\theta_{inc} = k_2\sin\theta_{trans} \Rightarrow n_1\sin\theta_{inc} = n_2\sin\theta_{trans}$$

Law of Refraction (Snell's Law)

Which can also be written as: $$\frac{\sin\theta_{trans}}{\sin\theta_{inc}} = \frac{n_1}{n_2}$$

Since θ_{trans} refers to medium 2) and θ_{inc} refers to medium 1)

$$n_1\sin\theta_1 = n_2\sin\theta_2 \quad \text{or:} \quad \frac{\sin\theta_2}{\sin\theta_1} = \frac{n_1}{n_2}$$

(θ_1: incident) (θ_2: transmitted)

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Because of the three laws of geometrical optics, we see that:

$$\left.\vec{k}_{inc}\bullet\vec{r}\right|_{z=0}=\left.\vec{k}_{refl}\bullet\vec{r}\right|_{z=0}=\left.\vec{k}_{trans}\bullet\vec{r}\right|_{z=0}$$

everywhere on the interface at $z = 0$ *{in the x-y plane}*

Thus we see that: $$\left.e^{i\left(\vec{k}_{inc}\bullet\vec{r}-\omega t\right)}\right|_{z=0}=\left.e^{i\left(\vec{k}_{refl}\bullet\vec{r}-\omega t\right)}\right|_{z=0}=\left.e^{i\left(\vec{k}_{trans}\bullet\vec{r}-\omega t\right)}\right|_{z=0}$$

everywhere on the interface at z = 0 {in the x-y plane}, valid also for arbitrary/any/all time(s) t, since ω is the same in either medium (1 or 2).

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The BC 1) → BC 4) for a monochromatic plane *EM* wave incident on an interface at an oblique angle between two linear/homogeneous/isotropic media become:

BC 1): Normal (*z-)* component of ***D*** continuous at *z* = *0* (no free surface charges*):*

$$\varepsilon_1\left(\tilde{E}_{o_{inc_z}} + \tilde{E}_{o_{refl_z}}\right) = \varepsilon_2 \tilde{E}_{o_{trans_z}} \qquad \left\{\text{using } \vec{D} = \varepsilon \vec{E}\right\}$$

BC 2): Tangential (*x-, y-)* components of ***E*** continuous at *z* = *0:*

$$\left(\tilde{E}_{o_{inc_{x,y}}} + \tilde{E}_{o_{refl_{x,y}}}\right) = \tilde{E}_{o_{trans_{x,y}}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

BC 3): Normal (z-) component of ***B*** continuous at *z = 0:*

$$\left(\tilde{B}_{o_{inc_z}} + \tilde{B}_{o_{refl_z}}\right) = \tilde{B}_{o_{trans_z}}$$

BC 4): Tangential (x-, y-) components of ***H*** continuous at *z = 0 (*no free surface currents):

$$\frac{1}{\mu_1}\left(\tilde{B}_{o_{inc_{x,y}}} + \tilde{B}_{o_{refl_{x,y}}}\right) = \frac{1}{\mu_2}\tilde{B}_{o_{trans_{x,y}}}$$

Note that in each of the above, we also have the relation

$$\vec{\tilde{B}}_o = \frac{1}{v}\hat{k} \times \vec{\tilde{E}}_o$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

For a monochromatic plane EM wave incident on a boundary between two L / H/ I media at an oblique angle of incidence, there are three possible polarization cases to consider:

Case I): $\vec{E}_{inc} \perp$ plane of incidence

$\{\vec{B}_{inc} \parallel$ plane of incidence$\}$

Transverse Electric (TE) Polarization

Case II): $\vec{E}_{inc} \parallel$ plane of incidence

$\{\vec{B}_{inc} \perp$ plane of incidence$\}$

Transverse Magnetic (TM) Polarization

Case III): The most general case: $\vec{E}_{inc}$ is neither $\perp$ nor $\parallel$ to the plane of incidence.

$\{\Rightarrow \vec{B}_{inc}$ is neither $\parallel$ nor $\perp$ to the plane of incidence$\}$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Case I): Electric Field Vectors Perpendicular to the Plane of Incidence: Transverse Electric (*TE*) *Polarization*

•A monochromatic plane EM wave is incident on a boundary at z = 0 -in the x-y plane between two L/H/I media - at an oblique angle of incidence.

•The polarization of the incident EM wave is transverse (⊥) to the plane of incidence {containing the three wave-vectors and the unit normal to the boundary $\hat{n} = +\hat{z}$ }).

•The three B-field vectors are related to their respective E - field vectors by the right hand rule - all three B-field vectors lie in the x-z plane {the plane of incidence},

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The four boundary conditions on the {complex} *E* and *B* fields on the boundary at z = 0 are:

BC 1) Normal (*z-*) component of D continuous at z = 0 (no free surface charges)

$$\varepsilon_1\left(\cancelto{=0}{\tilde{E}_{o_{inc_z}}}+\cancelto{=0}{\tilde{E}_{o_{refl_z}}}\right)=\varepsilon_2\,\cancelto{=0}{\tilde{E}_{o_{trans_z}}}\quad\Rightarrow\quad\boxed{0+0=0}$$

BC 2) Tangential (*x-*, *y-*) components of E continuous at *z* = *0:*

$$\left(\tilde{E}_{o_{inc_y}}+\tilde{E}_{o_{refl_y}}\right)=\tilde{E}_{o_{trans_y}}\quad\Rightarrow\quad\boxed{\tilde{E}_{o_{inc}}+\tilde{E}_{o_{refl}}=\tilde{E}_{o_{trans}}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

BC 3) Normal (*z-)* component of B continuous at z = 0:

$$\left(\tilde{B}_{o_{inc_z}} + \tilde{B}_{o_{refl_z}}\right) = \tilde{B}_{o_{trans_z}}$$

$$\hat{k}_{inc} = \hat{k}_{inc_x} + \hat{k}_{inc_z} = \sin\theta_{inc}\hat{x} + \cos\theta_{inc}\hat{z}$$

$$\hat{k}_{refl} = \hat{k}_{refl_x} + \hat{k}_{refl_z} = \sin\theta_{refl}\hat{x} - \cos\theta_{refl}\hat{z}$$

$$\hat{k}_{trans} = \hat{k}_{trans_x} + \hat{k}_{trans_z} = \sin\theta_{trans}\hat{x} + \cos\theta_{trans}\hat{z}$$

$$\left(\tilde{B}_{o_{inc_z}}\hat{z} + \tilde{B}_{o_{refl_z}}\hat{z}\right) = \tilde{B}_{o_{trans_z}}\hat{z} = \frac{1}{v_1}\left(\tilde{E}_{o_{inc}}\sin\theta_{inc} + \tilde{E}_{o_{refl}}\sin\theta_{refl}\right)\hat{z} = \frac{1}{v_2}\tilde{E}_{o_{trans}}\sin\theta_{trans}\hat{z}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

BC 4) Tangential (*x*-, *y*-) components of H continuous at z = 0 (no free surface currents):

$$\frac{1}{\mu_1}\left(\tilde{B}_{o_{inc_x}}\hat{x}+\tilde{B}_{o_{refl_x}}\hat{x}\right)=\frac{1}{\mu_2}\tilde{B}_{o_{trans_x}}\hat{x}$$

$$=\frac{1}{\mu_1 v_1}\left(\tilde{E}_{o_{inc}}(-\cos\theta_{inc})+\tilde{E}_{o_{refl}}\cos\theta_{refl}\right)\hat{x}=\frac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}}(-\cos\theta_{trans})\hat{x}$$

$$\tilde{E}_{o_{inc}}+\tilde{E}_{o_{refl}}=\tilde{E}_{o_{trans}} \quad \text{(from BC 2))}$$

Using the Law of Reflection on the BC 3) result:

$$\tilde{E}_{o_{inc}}+\tilde{E}_{o_{refl}}=\left(\frac{v_1}{v_2}\cdot\frac{\sin\theta_{trans}}{\sin\theta_{inc}}\right)\tilde{E}_{o_{trans}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Using Snell's Law / Law of Refraction:

$$\boxed{n_1 \sin\theta_{inc} = n_2 \sin\theta_{trans}} \Rightarrow \boxed{\frac{n_1}{c}\sin\theta_{inc} = \frac{n_2}{c}\sin\theta_{trans}} \Rightarrow \boxed{\frac{1}{v_1}\sin\theta_{inc} = \frac{1}{v_2}\sin\theta_{trans}}$$

or: $\boxed{v_2 \sin\theta_{inc} = v_1 \sin\theta_{trans}}$ or: $\boxed{\left(\frac{v_1}{v_2}\cdot\frac{\sin\theta_{trans}}{\sin\theta_{inc}}\right) = 1}$

From BC 1) → BC 4) actually have only two independent relations for the case of transverse electric (TE) polarization:

1) $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$ 2) $\boxed{\left(\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}}\right) = \left(\frac{\mu_1 v_1}{\mu_2 v_2}\cdot\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)\tilde{E}_{o_{trans}}}$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Now we define: $$\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2} \right)$$ $$\alpha \equiv \left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}} \right)$$

Then eqn. 2) becomes: $$\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \alpha\beta\, \tilde{E}_{o_{trans}}$$

Adding and subtracting Eqn's 1 &2 to get:

$$\tilde{E}_{o_{trans}} = \left(\frac{2}{1+\alpha\beta} \right) \tilde{E}_{o_{inc}} \quad \text{eqn. (1+2)}$$

$$\tilde{E}_{o_{refl}} = \left(\frac{1-\alpha\beta}{2} \right) \tilde{E}_{o_{trans}} \quad \text{eqn. (2–1)}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Plug eqn. (2+1) into eqn. (2−1) to obtain:

$$\tilde{E}_{o_{refl}} = \left(\frac{1-\alpha\beta}{2}\right)\left(\frac{2}{1+\alpha\beta}\right)\tilde{E}_{o_{inc}} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)\tilde{E}_{o_{inc}}$$

$$\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right) \quad \text{and} \quad \frac{\tilde{E}_{o_{trans}}}{\tilde{E}_{o_{inc}}} = \left(\frac{2}{1+\alpha\beta}\right)$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The Fresnel Equations for $\vec{E} \parallel$ to Interface

$= \vec{E} \perp$ Plane of Incidence = Transverse Electric (TE) Polarization

$$E_{o_{refl}}^{TE} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right) E_{o_{inc}}^{TE} \quad \text{and} \quad E_{o_{trans}}^{TE} = \left(\frac{2}{1+\alpha\beta}\right) E_{o_{inc}}^{TE}$$

with

$$\alpha \equiv \left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right) \quad \text{and} \quad \beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

For *TE polarization:*

Incident Intensity

$$I_{inc}^{TE} = \left|\left\langle \vec{S}_{inc}^{TE}(t)\right\rangle \bullet \hat{z}\right| = \left(\frac{1}{2} v_1 \varepsilon_1 \left(E_{o_{inc}}^{TE}\right)^2\right) \left|\hat{k}_{inc} \bullet \hat{z}\right| = \left(\frac{1}{2} v_1 \varepsilon_1 \left(E_{o_{inc}}^{TE}\right)^2\right) \cos\theta_{inc} = \frac{1}{2} \varepsilon_1 v_1 \left(E_{o_{inc}}^{TE}\right)^2 \cos\theta_{inc}$$

Reflection Intensity

$$I_{refl}^{TE} = \left|\left\langle \vec{S}_{refl}^{TE}(t)\right\rangle \bullet \hat{z}\right| = \left(\frac{1}{2} v_1 \varepsilon_1 \left(E_{o_{refl}}^{TE}\right)^2\right) \cos\theta_{refl} = \frac{1}{2} \varepsilon_1 v_1 \left(E_{o_{refl}}^{TE}\right)^2 \cos\theta_{inc}$$

Transmission Intensity

$$I_{trans}^{TE} = \left|\left\langle \vec{S}_{trans}^{TE}(t)\right\rangle \bullet \hat{z}\right| = \left(\frac{1}{2} v_2 \varepsilon_2 \left(E_{o_{trans}}^{TE}\right)^2\right) \cos\theta_{trans} = \frac{1}{2} \varepsilon_2 v_2 \left(E_{o_{trans}}^{TE}\right)^2 \cos\theta_{trans}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Reflection and Transmission coefficients for transverse electric (*TE*) *polarization*

$$R_{TE} \equiv \frac{I_{refl}^{TE}}{I_{inc}^{TE}} = \frac{\frac{1}{2}\varepsilon_1 v_1 \left(E_{o_{refl}}^{TE}\right)^2 \cos \overbrace{\theta_{inc}}^{=\theta_{refl}}}{\frac{1}{2}\varepsilon_1 v_1 \left(E_{o_{inc}}^{TE}\right)^2 \cos\theta_{inc}} = \left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2$$

$$T_{TE} \equiv \frac{I_{trans}^{TE}}{I_{inc}^{TE}} = \frac{\frac{1}{2}\varepsilon_2 v_2 \left(E_{o_{trans}}^{TE}\right)^2 \cos\theta_{trans}}{\frac{1}{2}\varepsilon_1 v_1 \left(E_{o_{inc}}^{TE}\right)^2 \cos\theta_{inc}} = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The reflection and transmission coefficients for transverse electric (*TE*) polarization

$$R_{TE} = \left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}} \right)^2 = \left(\frac{1-\alpha\beta}{1+\alpha\beta} \right)^2$$

$$T_{TE} = \alpha\beta \left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}} \right)^2 = \frac{4\alpha\beta}{(1+\alpha\beta)^2}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

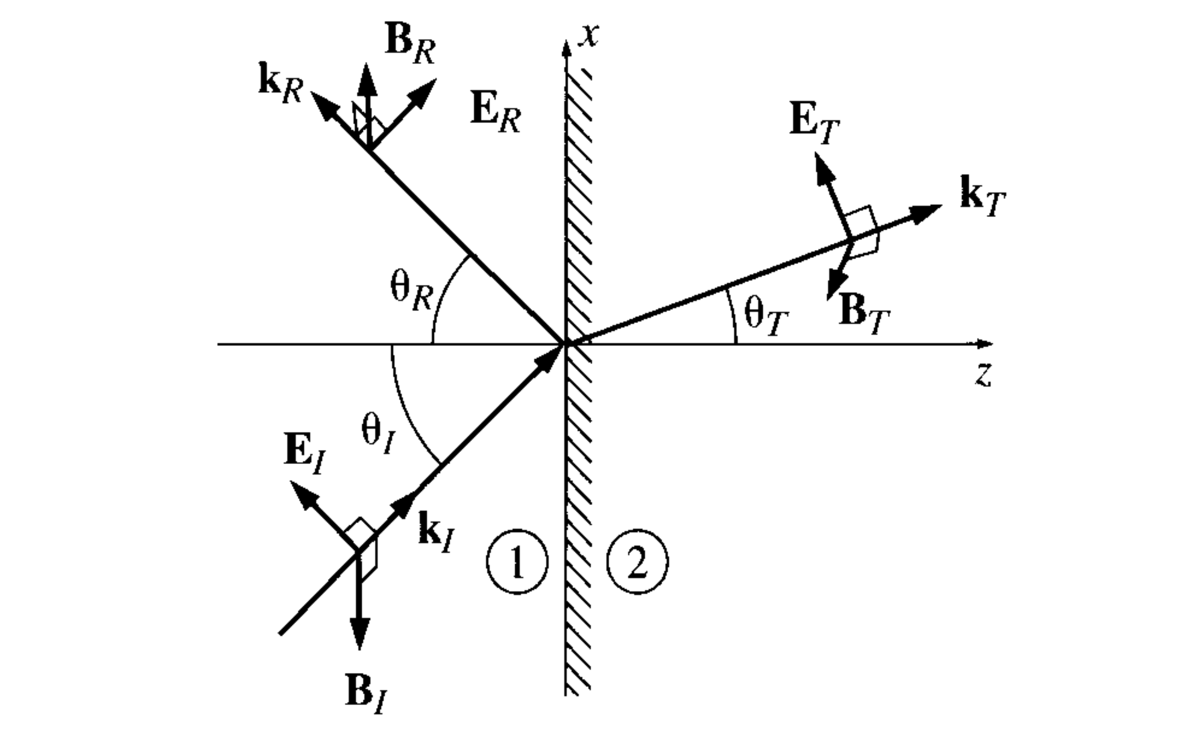
Case II): Electric Field Vectors Parallel to the Plane of Incidence: Transverse Magnetic (*TM*) *Polarization*

•A monochromatic plane EM wave is incident on a boundary at z = 0 in the x-y plane between two L / H/ I media at an oblique angle of incidence.

•The polarization of the incident EM wave is now parallel to the plane of incidence {containing the three wavevectors and the unit normal to the boundary $\hat{n} = +\hat{z}$ }).

• The three B -field vectors are related to E -field vectors by the right hand rule –then all three B-field vectors are ⊥ to the plane of incidence {hence the origin of the name transverse magnetic polarization}.

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*



Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The four boundary conditions on the {complex}E and B-fields on the boundary at z = 0 are:

BC 1) Normal (z-) component of D continuous at z = 0 (no free surface charges)

$$\varepsilon_1\left(\tilde{E}_{o_{inc_z}} + \tilde{E}_{o_{refl_z}}\right) = \varepsilon_2 \tilde{E}_{o_{trans_z}}$$

$$\varepsilon_1\left(-\tilde{E}_{o_{inc}} \sin\theta_{inc} + \tilde{E}_{o_{refl}} \sin\theta_{refl}\right) = \varepsilon_2\left(-\tilde{E}_{o_{trans}} \sin\theta_{trans}\right)$$

BC 2) Tangential (x-, y-) components of E continuous at *z = 0:*

$$\left(\tilde{E}_{o_{inc_x}} + \tilde{E}_{o_{refl_x}}\right) = \tilde{E}_{o_{trans_x}}$$

$$\left(\tilde{E}_{o_{inc}} \cos\theta_{inc} + \tilde{E}_{o_{refl}} \cos\theta_{refl}\right) = \tilde{E}_{o_{trans}} \cos\theta_{trans}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

BC 3) Normal (***z-) component of B* continuous at *z = 0:***

$$\left(\cancelto{=0}{\tilde{B}_{o_{inc_z}}} + \cancelto{=0}{\tilde{B}_{o_{refl_z}}} \right) = \cancelto{=0}{\tilde{B}_{o_{trans_z}}} \quad \Rightarrow \quad \boxed{0+0=0}$$

BC 4) Tangential (x-, y-) components of H continuous at z = 0 (no free surface currents):

$$\boxed{\frac{1}{\mu_1}\left(\tilde{B}_{o_{inc_y}} + \tilde{B}_{o_{refl_y}}\right) = \frac{1}{\mu_2}\left(\tilde{B}_{o_{trans_y}}\right)} \quad \Rightarrow \quad \boxed{\frac{1}{\mu_1 v_1}\left(\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}}\right) = \frac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

From BC 1) at $z = 0$:

$$\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \left(\frac{\varepsilon_2}{\varepsilon_1}\frac{n_1}{n_2}\right)\tilde{E}_{o_{trans}} = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\tilde{E}_{o_{trans}} = \beta\,\tilde{E}_{o_{trans}}$$

From BC 4) at $z = 0$:

$$\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)\tilde{E}_{o_{trans}} = \beta\tilde{E}_{o_{trans}}$$

where:

$$\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right) = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

From BC 2) at $z = 0$:

$$\left(\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}}\right) = \left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)\tilde{E}_{o_{trans}} = \alpha \tilde{E}_{o_{trans}}$$

where:

$$\alpha \equiv \frac{\cos\theta_{trans}}{\cos\theta_{inc}}$$

Thus for the case of transverse magnetic (TM) polarization:

$$\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \beta \tilde{E}_{o_{trans}} \quad \text{and} \quad \tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \alpha \tilde{E}_{o_{trans}}$$

Solving these two above equations simultaneously, we obtain:

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

$$\tilde{E}_{o_{trans}} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{o_{inc}} \qquad \tilde{E}_{o_{refl}} = \left(\frac{\alpha-\beta}{2}\right)\tilde{E}_{o_{trans}} \qquad \tilde{E}_{o_{refl}} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{o_{inc}}$$

The Fresnel Equations for $\vec{B}\parallel$ to Interface

$=\vec{B}\perp$ Plane of Incidence = Transverse Magnetic (*TM*) Polarization

$$\left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right) = \left(\frac{\alpha-\beta}{\alpha+\beta}\right) \quad \text{and} \quad \left(\frac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}}\right) = \left(\frac{2}{\alpha+\beta}\right)$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Reflected & transmitted intensities at oblique incidence for the *TM case*

$$I_{inc}^{TM} = v_1 \left| \left\langle \vec{S}_{inc}^{TM}(t) \right\rangle \bullet \hat{z} \right| = \left(\frac{1}{2} v_1 \varepsilon_1 \left(E_{o_{inc}}^{TM} \right)^2 \right) \cos\theta_{inc} = \frac{1}{2} \varepsilon_1 v_1 \left(E_{o_{inc}}^{TM} \right)^2 \cos\theta_{inc}$$

$$I_{refl}^{TM} = v_1 \left| \left\langle \vec{S}_{refl}^{TM}(t) \right\rangle \bullet \hat{z} \right| = \left(\frac{1}{2} v_1 \varepsilon_1 \left(E_{o_{refl}}^{TM} \right)^2 \right) \cos\theta_{refl} = \frac{1}{2} \varepsilon_1 v_1 \left(E_{o_{refl}}^{TM} \right)^2 \cos\theta_{inc}$$

$$I_{trans}^{TM} = v_2 \left| \left\langle \vec{S}_{trans}^{TM}(t) \right\rangle \bullet \hat{z} \right| = \left(\frac{1}{2} v_2 \varepsilon_2 \left(E_{o_{trans}}^{TM} \right)^2 \right) \cos\theta_{trans} = \frac{1}{2} \varepsilon_2 v_2 \left(E_{o_{trans}}^{TM} \right)^2 \cos\theta_{trans}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Reflection and Transmission coefficients

$$R_{TM} = \left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}} \right)^2 = \left(\frac{\alpha - \beta}{\alpha + \beta} \right)^2$$

$$T_{TM} = \alpha\beta \left(\frac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}} \right)^2 = \frac{4\alpha\beta}{(\alpha + \beta)^2}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The Fresnel Equations

TE Polarization

$$\left(\frac{E^{TE}_{o_{refl}}}{E^{TE}_{o_{inc}}}\right) = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)$$

$$\left(\frac{E^{TE}_{o_{trans}}}{E^{TE}_{o_{inc}}}\right) = \frac{2}{(1+\alpha\beta)}$$

$$\alpha \equiv \frac{\cos\theta_{trans}}{\cos\theta_{inc}}$$

$$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{\varepsilon_2 n_1}{\varepsilon_1 n_2}$$

TM Polarization

$$\left(\frac{E^{TM}_{o_{refl}}}{E^{TM}_{o_{inc}}}\right) = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)$$

$$\left(\frac{E^{TM}_{o_{trans}}}{E^{TM}_{o_{inc}}}\right) = \frac{2}{(\alpha+\beta)}$$

$$v_1 = c/n_1 = 1/\sqrt{\varepsilon_1\mu_1}$$

$$v_2 = c/n_2 = 1/\sqrt{\varepsilon_2\mu_2}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Reflection and Transmission Coefficients *R & T*

$$R + T = 1$$

TE Polarization

$$R_{TE} \equiv \frac{I_{refl}^{TE}}{I_{inc}^{TE}} = \left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2 = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2$$

$$T_{TE} \equiv \left(\frac{I_{trans}^{TE}}{I_{inc}^{TE}}\right) = \alpha\beta\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2 = \frac{4\alpha\beta}{(1+\alpha\beta)^2}$$

$$\alpha \equiv \frac{\cos\theta_{trans}}{\cos\theta_{inc}}$$

$$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{\varepsilon_2 n_1}{\varepsilon_1 n_2}$$

TM Polarization

$$R_{TM} \equiv \frac{I_{refl}^{TM}}{I_{inc}^{TM}} = \left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right)^2 = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2$$

$$T_{TM} \equiv \left(\frac{I_{trans}^{TM}}{I_{inc}^{TM}}\right) = \alpha\beta\left(\frac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}}\right)^2 = \frac{4\alpha\beta}{(\alpha+\beta)^2}$$

$$v_1 = c/n_1 = 1/\sqrt{\varepsilon_1\mu_1}$$

$$v_2 = c/n_2 = 1/\sqrt{\varepsilon_2\mu_2}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Alternate versions of the Fresnel Relations

Fresnel Equations

TE Polarization

$$\left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right) = \frac{\left(\frac{n_1}{\mu_1}\right)\cos\theta_{inc} - \left(\frac{n_2}{\mu_2}\right)\cos\theta_{trans}}{\left(\frac{n_1}{\mu_1}\right)\cos\theta_{inc} + \left(\frac{n_2}{\mu_2}\right)\cos\theta_{trans}}$$

$$\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right) = \frac{2\left(\frac{n_1}{\mu_1}\right)\cos\theta_{inc}}{\left(\frac{n_1}{\mu_1}\right)\cos\theta_{inc} + \left(\frac{n_2}{\mu_2}\right)\cos\theta_{trans}}$$

TM Polarization

$$\left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right) = \frac{\left(\frac{n_2}{\mu_2}\right)\cos\theta_{inc} - \left(\frac{n_1}{\mu_1}\right)\cos\theta_{trans}}{\left(\frac{n_2}{\mu_2}\right)\cos\theta_{inc} + \left(\frac{n_1}{\mu_1}\right)\cos\theta_{trans}}$$

$$\left(\frac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}}\right) = \frac{2\left(\frac{n_1}{\mu_1}\right)\cos\theta_{inc}}{\left(\frac{n_2}{\mu_2}\right)\cos\theta_{inc} + \left(\frac{n_1}{\mu_1}\right)\cos\theta_{trans}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Ignoring the magnetic properties of the two media

$|\chi_m| \ll 1$ then $\mu_1 \simeq \mu_2 \simeq \mu_o$ the Fresnel Relations become:

TE Polarization

$$\left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right) \simeq \frac{n_1 \cos\theta_{inc} - n_2 \cos\theta_{trans}}{n_1 \cos\theta_{inc} + n_2 \cos\theta_{trans}}$$

$$\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right) \simeq \frac{2n_1 \cos\theta_{inc}}{n_1 \cos\theta_{inc} + n_2 \cos\theta_{trans}}$$

TM Polarization

$$\left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right) \simeq \frac{-n_2 \cos\theta_{inc} + n_1 \cos\theta_{trans}}{n_2 \cos\theta_{inc} + n_1 \cos\theta_{trans}}$$

$$\left(\frac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}}\right) \simeq \frac{2n_1 \cos\theta_{inc}}{n_2 \cos\theta_{inc} + n_1 \cos\theta_{trans}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Using Snell's Law and various trigonometric identities

TE Polarization

$$\left(\frac{E^{TE}_{o_{refl}}}{E^{TE}_{o_{inc}}}\right) \simeq -\frac{\sin\left(\theta_{inc}-\theta_{trans}\right)}{\sin\left(\theta_{inc}+\theta_{trans}\right)}$$

$$\left(\frac{E^{TE}_{o_{trans}}}{E^{TE}_{o_{inc}}}\right) \simeq \frac{2\cos\theta_{inc}\bullet\sin\theta_{trans}}{\sin\left(\theta_{inc}+\theta_{trans}\right)}$$

TM Polarization

$$\left(\frac{E^{TM}_{o_{refl}}}{E^{TM}_{o_{inc}}}\right) \simeq -\frac{\tan\left(\theta_{inc}-\theta_{trans}\right)}{\tan\left(\theta_{inc}+\theta_{trans}\right)}$$

$$\left(\frac{E^{TM}_{o_{trans}}}{E^{TM}_{o_{trans}}}\right) \simeq \frac{2\cos\theta_{inc}\bullet\sin\theta_{trans}}{\sin\left(\theta_{inc}+\theta_{trans}\right)\cos\left(\theta_{inc}-\theta_{trans}\right)}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Use Snell's Law $n_{inc}\sin\theta_{inc} = n_{trans}\sin\theta_{trans}$ to eliminate θ_{trans} :

TE Polarization

$$\left(\frac{E^{TE}_{o_{refl}}}{E^{TE}_{o_{inc}}}\right) \simeq \frac{\cos\theta_{inc} - \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}{\cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}$$

$$\left(\frac{E^{TE}_{o_{trans}}}{E^{TE}_{o_{inc}}}\right) \simeq \frac{2\cos\theta_{inc}}{\cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}$$

TM Polarization

$$\left(\frac{E^{TM}_{o_{refl}}}{E^{TM}_{o_{inc}}}\right) \simeq \frac{-\left(\frac{n_2}{n_1}\right)^2\cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}{\left(\frac{n_2}{n_1}\right)^2\cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}$$

$$\left(\frac{E^{TM}_{o_{trans}}}{E^{TM}_{o_{inc}}}\right) \simeq \frac{2\left(\frac{n_2}{n_1}\right)\cos\theta_{inc}}{\left(\frac{n_2}{n_1}\right)^2\cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

- Now explore the physics associated with the Fresnel Equations -the reflection and transmission coefficients.
- Comparing results for TE vs. TM polarization for the cases of external reflection (n1 < n2) and internal reflection n1 > n2)

Comment 1):

- When $(E_{refl}/E_{inc}) < 0$ - E_{orefl} *is* 180^o out-of-phase with E_{oinc} since the numerators of the original Fresnel Equations for TE & TM polarization are $(1-\alpha\beta)$ and $(\alpha - \beta)$ respectively.

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Comment 2):

- For TM Polarization (only)- there exists an angle of incidence where $(E_{refl} / E_{inc}) = 0$ - no reflected wave occurs at this angle for TM polarization!
- This angle is known as Brewster's angle θ_B (also known as the polarizing angle θ_P - because an incident wave which is a linear combination of TE and TM polarizations will have a reflected wave which is 100% pure-TE polarized for an incidence angle $\theta_{inc} = \theta_B = \theta_P$!!).
- Brewster's angle θ_B exists for both external $(n_1 < n_2)$ & internal reflection $(n_1 > n_2)$ for TM polarization (only).

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Brewster's Angle θ_B / *the Polarizing Angle* θ_P *for Transverse Magnetic (TM) Polarization*

From the numerator of $\left(E_{o_{refl}}^{TM} / E_{o_{inc}}^{TM}\right) = \left(\dfrac{\alpha-\beta}{\alpha+\beta}\right)$ -the originally-derived expression for TM polarization- when this ratio = 0 at Brewster's angle θ_B = polarizing angle θ_P - this occurs when $(\alpha - \beta) = 0$, i.e. when $\alpha = \beta$.

$\cos\theta_{trans} = \sqrt{1-\sin^2\theta_{trans}}$ and Snell's Law: $\sin\theta_{trans} = \left(\dfrac{n_1}{n_2}\right)\sin\theta_{inc}$

$$\alpha = \frac{\sqrt{1-\left(\dfrac{n_1}{n_2}\right)^2 \sin^2\theta_{inc}}}{\cos\theta_{inc}} \simeq \left(\frac{n_2}{n_1}\right) = \beta$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Brewster's Angle θ_B / *the Polarizing Angle* θ_P *for Transverse Magnetic (TM) Polarization*

$$1-\frac{1}{\beta^2}\sin^2\theta_{inc} = \beta^2\cos^2\theta_{inc} = \beta^2\left(1-\sin^2\theta_{inc}\right) \leftarrow \text{Solve for } \sin^2\theta_{inc}$$

$$1-\beta^2 = \left(\frac{1}{\beta^2}-\beta^2\right)\sin^2\theta_{inc} \Rightarrow \sin^2\theta_{inc} = \frac{1-\beta^2}{1/\beta^2-\beta^2} = \frac{\left(1-\beta^2\right)\beta^2}{\left(1-\beta^4\right)}$$

$$1-\beta^4 = \left(1-\beta^2\right)\left(1+\beta^2\right)$$

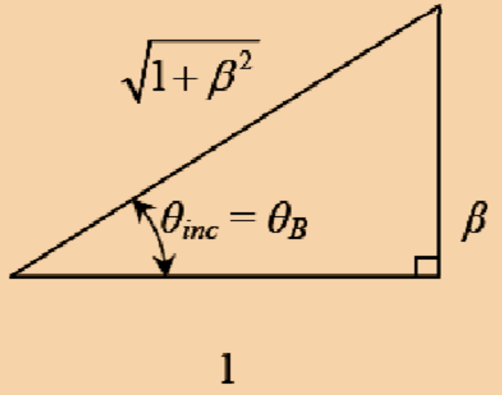
$$\sin^2\theta_{inc} = \frac{\left(1-\beta^2\right)\beta^2}{\left(1-\beta^2\right)\left(1+\beta^2\right)} = \frac{\beta^2}{1+\beta^2} \Rightarrow \sin\theta_{inc} = \frac{\beta}{\sqrt{1+\beta^2}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Geometrically: $\sin\theta_{inc} = \dfrac{\beta}{\sqrt{1+\beta^2}} = \dfrac{\text{opp. side}}{\text{hypotenuse}}$

$\cos\theta_{inc} = \dfrac{1}{\sqrt{1+\beta^2}} = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

$\tan\theta_{inc} = \beta = \dfrac{\text{opp. side}}{\text{adjacent}} \simeq \left(\dfrac{n_2}{n_1}\right)$



Thus, at an angle of incidence $\theta_{inc} = \theta_B^{inc} \equiv \theta_P^{inc}$ = Brewster's angle / the polarizing angle for a *TM* polarized incident wave, where *no reflected* wave exists, we have:

$$\boxed{\tan\theta_B^{inc} \equiv \tan\theta_P^{inc} \simeq \left(\frac{n_2}{n_1}\right)} \quad \text{for} \quad \mu_1 \simeq \mu_2 \simeq \mu_o$$

From Snell's Law: $n_1 \sin\theta_{inc} = n_2 \sin\theta_{trans}$ we also see that: $\tan\theta_B^{inc} = \dfrac{\sin\theta_B^{inc}}{\cos\theta_B^{inc}} \simeq \dfrac{n_2}{n_1}$

or: $n_1 \sin\theta_B^{inc} \simeq n_2 \cos\theta_B^{inc}$ for $\mu_1 \simeq \mu_2 \simeq \mu_o$.

Thus, from Snell's Law we see that: $\cos\theta_B^{inc} = \sin\theta_{trans}$ when $\theta_{inc} = \theta_B^{inc} \equiv \theta_P^{inc}$.

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

So what's so interesting about this???

Well: $\cos\theta_B^{inc} = \sin\left(\frac{\pi}{2} - \theta_B^{inc}\right) = \sin\left(\frac{\pi}{2}\right)\cos\theta_B^{inc} - \underbrace{\cos\left(\frac{\pi}{2}\right)}_{=0}\sin\theta_B^{inc} = \sin\theta_{trans}$ *i.e.* $\sin\left(\frac{\pi}{2} - \theta_B^{inc}\right) = \sin\theta_{trans}$

$\therefore$ When $\theta_{inc} = \theta_B^{inc} \equiv \theta_P^{inc}$ for an incident *TM*-polarized *EM* wave, we see that $\theta_{trans} = \pi/2 - \theta_B^{inc}$

Thus: $\theta_B^{inc} + \theta_{trans} = \pi/2$, *i.e.* $\theta_B^{inc} \equiv \theta_P^{inc}$ and θ_{trans} are *complimentary* angles !!!

Comment 3):

For internal reflection ($n_1 > n_2$) there exists a critical angle of incidence past which no transmitted beam exists for either TE or TM polarization. The critical angle does not depend on polarization – it is actually dictated / defined by Snell's Law:

$$n_1 \sin\theta_{critical}^{inc} = n_2 \sin\theta_{trans}^{\max} = n_2 \sin\left(\frac{\pi}{2}\right) = n_2 \quad \text{or:} \quad \sin\theta_{critical}^{inc} = \left(\frac{n_2}{n_1}\right) \quad \text{or:} \quad \theta_{critical}^{inc} = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

For $\theta_{inc} \geq \theta_{critical}^{inc}$, no transmitted beam exists → incident beam is totally internally reflected.

For $\theta_{inc} > \theta_{critical}^{inc}$, the transmitted wave is actually exponentially damped – becomes a so-called:

Evanescent Wave:

$$\tilde{\vec{E}}_{trans}(\vec{r},t) = \tilde{\vec{E}}_{o_{trans}} \underbrace{e^{-\alpha z}} \underbrace{e^{i\left(k_2 x \sin\theta_{inc}\left(\frac{n_1}{n_2}\right) - \omega t\right)}}$$

$$\alpha = k_2 \sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_{inc} - 1}$$

Exp. damping in z Oscillatory along interface in x-direction

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Brewster's angle for *TE polarization:*

$$\theta^{B}_{\substack{inc\\TE}} = \sin^{-1}\sqrt{\frac{\left(\frac{\varepsilon_2}{\varepsilon_1}\right)-\left(\frac{\mu_2}{\mu_1}\right)}{\left(\frac{\mu_1}{\mu_2}\right)-\left(\frac{\mu_2}{\mu_1}\right)}} = \sin^{-1}\sqrt{A}$$

$$\sin\theta^{B}_{inc} = \sqrt{\frac{\left(\frac{\varepsilon_2}{\varepsilon_1}\right)-\left(\frac{\mu_2}{\mu_1}\right)}{\left(\frac{\mu_1}{\mu_2}\right)-\left(\frac{\mu_2}{\mu_1}\right)}} \equiv \sqrt{A} \quad i.e. \quad A \equiv \left[\frac{\left(\frac{\varepsilon_2}{\varepsilon_1}\right)-\left(\frac{\mu_2}{\mu_1}\right)}{\left(\frac{\mu_1}{\mu_2}\right)-\left(\frac{\mu_2}{\mu_1}\right)}\right]$$

ELECTROMAGNETIC WAVES IN CONDUCTORS

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ELECTROMAGNETIC WAVES IN CONDUCTORS

- Free charge and free currents are zero for propagation through a vacuum or insulating materials such as glass or pure water.
- Inside a conductor, free charges can move around in response to *EM* fields contained therein- free current is not zero.
- Assume that the conductor is linear/homogeneous/ isotropic media.
- From Ohm's Law $\vec{J}_{free}(\vec{r},t) = \sigma_C \vec{E}(\vec{r},t)$

where σ_C = conductivity of the metal conductor (*Ohm*$^{-1}$*/m) and* $\sigma_C = 1/\rho_C$ *where* ρ_C = resistivity of the metal conduct or (*Ohm-m*).

ELECTROMAGNETIC WAVES IN CONDUCTORS

For such a conductor, we can assume that the linear/ homogeneous/isotropic conducting medium has electric permittivity ε and magnetic permeability μ . Maxwell's equations inside such a conductor are thus:

1) $\vec{\nabla} \bullet \vec{E}(\vec{r},t) = \rho_{free}(\vec{r},t)/\varepsilon$

2) $\vec{\nabla} \bullet \vec{B}(\vec{r},t) = 0$

3) $\vec{\nabla} \times \vec{E}(\vec{r},t) = -\dfrac{\partial \vec{B}(\vec{r},t)}{\partial t}$

Using Ohm's Law:
$\vec{J}_{free}(\vec{r},t) = \sigma_C \vec{E}(\vec{r},t)$

4) $\vec{\nabla} \times \vec{B}(\vec{r},t) = \mu \vec{J}_{free}(\vec{r},t) + \mu\varepsilon \dfrac{\partial \vec{E}(\vec{r},t)}{\partial t} = \mu\sigma_C \vec{E}(\vec{r},t) + \mu\varepsilon \dfrac{\partial \vec{E}(\vec{r},t)}{\partial t}$

ELECTROMAGNETIC WAVES IN CONDUCTORS

Electric charge is (always) conserved- thus the continuity equation inside the conductor is:

$$\vec{\nabla}\bullet\vec{J}_{free}(\vec{r},t) = -\frac{\partial\rho_{free}(\vec{r},t)}{\partial t}$$

but: $\vec{J}_{free}(\vec{r},t) = \sigma_C\vec{E}(\vec{r},t)$

$$\sigma_C\left(\vec{\nabla}\bullet\vec{E}(\vec{r},t)\right) = -\frac{\partial\rho_{free}(\vec{r},t)}{\partial t}$$

but: $\vec{\nabla}\bullet\vec{E}(\vec{r},t) = \rho_{free}(\vec{r},t)/\varepsilon$

thus:

$$\frac{\sigma_C\rho_{free}(\vec{r},t)}{\varepsilon} = -\frac{\partial\rho_{free}(\vec{r},t)}{\partial t}$$

or: $\frac{\partial\rho_{free}(\vec{r},t)}{\partial t} + \left(\frac{\sigma_C}{\varepsilon}\right)\rho_{free}(\vec{r},t) = 0$

1st order linear, homogeneous differential equation

ELECTROMAGNETIC WAVES IN CONDUCTORS

The {physical} general solution of this differential equation for the free charge density is of the form:

$$\rho_{free}(\vec{r},t) = \rho_{free}(\vec{r},t=0)\,e^{-\sigma_C t/\varepsilon} = \rho_{free}(\vec{r},t=0)\,e^{-t/\tau_{relax}}$$

A damped exponential!!!

The continuity equation inside a conductor tells us that any free charge density initially present at time t = 0 is exponentially damped in a characteristic time $\tau_{relax} \equiv \varepsilon/\sigma_C$ = charge relaxation time.

ELECTROMAGNETIC WAVES IN CONDUCTORS

Maxwell's equations for a ***charge-equilibrated conductor***

1) $\vec{\nabla}\bullet\vec{E}(\vec{r},t)=0$

2) $\vec{\nabla}\bullet\vec{B}(\vec{r},t)=0$

3) $\vec{\nabla}\times\vec{E}(\vec{r},t)=-\dfrac{\partial\vec{B}(\vec{r},t)}{\partial t}$

4) $\vec{\nabla}\times\vec{B}(\vec{r},t)=\mu\sigma_C\vec{E}(\vec{r},t)+\mu\varepsilon\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}=\mu\left(\sigma_C\vec{E}(\vec{r},t)+\varepsilon\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}\right)$

ELECTROMAGNETIC WAVES IN CONDUCTORS

These equations are different from the previous derivation(s) of monochromatic plane EM waves propagating in free space/vacuum and/or in linear/homogeneous/ isotropic non-conducting materials Re-derive the wave equations for ***E&B*** from scratch. As before, we apply ∇ × ( ) to equations 3) and 4):

We get

$$\nabla^2 \vec{E}(\vec{r},t) = \mu\varepsilon \frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} + \mu\sigma_c \frac{\partial \vec{E}(\vec{r},t)}{\partial t}$$

and

$$\nabla^2 \vec{B}(\vec{r},t) = \mu\varepsilon \frac{\partial^2 \vec{B}(\vec{r},t)}{\partial t^2} + \mu\sigma_c \frac{\partial \vec{B}(\vec{r},t)}{\partial t}$$

ELECTROMAGNETIC WAVES IN CONDUCTORS

General solution(s) - are usually in the form of an oscillatory function times a damping term (*a decaying exponential)* – in the direction of the propagation of the EM wave. A complex plane-wave type solutions for E and B associated with the above wave equation(s) are of the general form:

$$\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_o e^{i(\tilde{k}z - \omega t)}$$

$$\tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_o e^{i(\tilde{k}z - \omega t)} = \left(\frac{\tilde{k}}{\omega}\right)\hat{k} \times \tilde{\vec{E}}(z,t) = \frac{1}{\omega}\tilde{\vec{k}} \times \tilde{\vec{E}}(z,t)$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

With {frequency-dependent} complex wave number:

$$\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)$$

$$k(\omega) = \Re e\left(\tilde{k}(\omega)\right) = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]^{1/2}$$

$$\kappa(\omega) = \Im m\left(\tilde{k}(\omega)\right) = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}-1\right]^{1/2}$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

The imaginary part of k , $\kappa = \mathfrak{Im}(k)$ results in an exponential attenuation/damping of the monochromatic plane *EM* wave with increasing *z:*

$$\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_o e^{-\kappa z} e^{i(kz-\omega t)}$$

$$\tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_o e^{-\kappa z} e^{i(kz-\omega t)} = \frac{1}{\omega}\tilde{\vec{k}} \times \tilde{\vec{E}}(z,t) = \frac{1}{\omega}\tilde{\vec{k}} \times \tilde{\vec{E}}_o e^{-\kappa z} e^{i(kz-\omega t)}$$

These solutions satisfy the above wave equations for any choice $\tilde{\vec{E}}_o$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

The characteristic distance over which E and B are attenuated/reduced to 1/*e*=*0.3679*- of their initial values (at *z* = *0)* is known as the skin depth

$$\delta_{sc}(\omega) \equiv 1/\kappa(\omega)$$

$$\delta_{sc}(\omega) = \frac{1}{\kappa(\omega)} = \frac{1}{\omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}-1\right]^{1/2}} \Rightarrow \begin{array}{l} \tilde{\vec{E}}(z=\delta_{sc},t) = \tilde{\vec{E}}_o e^{-1} e^{i(kz-\omega t)} \\ \tilde{\vec{B}}(z=\delta_{sc},t) = \tilde{\vec{B}}_o e^{-1} e^{i(kz-\omega t)} \end{array}$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

The real part of *k*- determines the spatial wavelength λ (ω)-the propagation speed v(ω) and also the index of refraction

$$\lambda(\omega) = \frac{2\pi}{k(\omega)} = \frac{2\pi}{\Re e\left(\tilde{k}(\omega)\right)}$$

$$v(\omega) = \frac{\omega}{k(\omega)} = \frac{\omega}{\Re e\left(\tilde{k}(\omega)\right)}$$

$$n(\omega) = \frac{c}{v(\omega)} = \frac{ck(\omega)}{\omega} = \frac{c\Re e\left(\tilde{k}(\omega)\right)}{\omega}$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

The above plane wave solutions satisfy the above wave equations(s). Maxwell's equations rue out the presence of any longitudinal i.e, z- component of E and B.

*E and B a*re purely transverse waves (as before), even in a conductor!

If we consider - a linearly polarized monochromatic plane EM wave propagating in the +zˆ -direction in a conducting medium, e.g.

$$\vec{\tilde{E}}(z,t) = \tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)} \hat{x}$$

then

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

$$\boxed{\tilde{\vec{B}}(z,t) = \frac{1}{\omega}\tilde{\vec{k}} \times \tilde{\vec{E}}(z,t) = \left(\frac{\tilde{k}}{\omega}\right)\tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}\hat{y} = \left(\frac{k+i\kappa}{\omega}\right)\tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}\hat{y}}$$

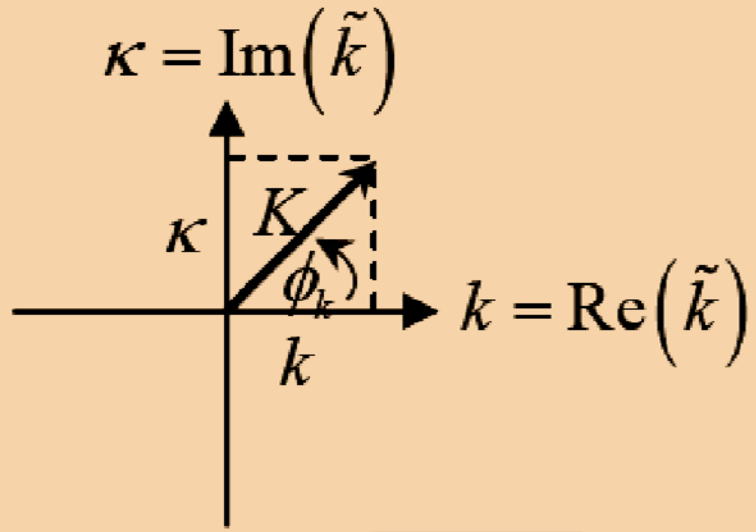
$$\Rightarrow \quad \tilde{\vec{E}}(z,t) \perp \tilde{\vec{B}}(z,t) \perp \hat{z} \quad (+\hat{z} = \text{propagation direction})$$

The complex wave-number $\tilde{k} = k + ik = Ke^{i\phi}$

$$\text{where: } K \equiv \left|\tilde{k}\right| = \sqrt{k^2 + \kappa^2} \text{ and } \phi_k \equiv \tan^{-1}\left(\kappa/k\right)$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

In the complex $\tilde{k}$-plane:



$$K = |\tilde{k}| = \sqrt{k^2 + \kappa^2}$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

Then we see that:

$$\tilde{\vec{E}}(z,t) = \tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}\hat{x}$$

has $\tilde{E}_o = E_o e^{i\delta_E}$

$$\tilde{\vec{B}}(z,t) = \tilde{B}_0 e^{-\kappa z} e^{i(kz-\omega t)}\hat{y} = \frac{\tilde{k}}{\omega}\tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}\hat{y}$$

$$\tilde{k} = K e^{i\phi_k}$$

has $\tilde{B}_o = B_o e^{i\delta_B} = \frac{\tilde{k}}{\omega}\tilde{E}_o = \frac{K e^{i\phi_k}}{\omega} E_o e^{i\delta_E}$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

$$B_o e^{i\delta_B} = \frac{Ke^{i\phi_k}}{\omega} E_o e^{i\delta_B} = \frac{K}{\omega} E_o e^{i(\delta_E + \phi_k)} = \frac{\sqrt{k^2 + \kappa^2}}{\omega} E_o e^{i(\delta_E + \phi_k)}$$

inside a conductor, **E** and **B** are no longer in phase with each other!!!

Phases of ***E*** *and* ***B***

$$\delta_B = \delta_E + \phi_k$$

With phase difference:

$$\Delta\varphi_{B-E} \equiv \delta_B - \delta_E = \phi_k$$

We also see that:

$$\frac{B_o}{E_o} = \frac{K}{\omega} = \left[\varepsilon\mu \sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega} \right)^2} \right]^{1/2} \neq \frac{1}{c}$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

The real/physical ***E*** and ***B*** fields associated with linearly polarized monochromatic plane *EM* waves propagating in a conducting medium are exponentially damped:

$$\vec{E}(z,t) = \Re e\left(\tilde{\vec{E}}(z,t)\right) = E_o e^{-\kappa z}\cos\left(kz - \omega t + \delta_E\right)\hat{x}$$

$$\delta_B = \delta_E + \phi_k$$

$$\vec{B}(z,t) = \Re e\left(\tilde{\vec{B}}(z,t)\right) = B_o e^{-\kappa z}\cos\left(kz - \omega t + \delta_B\right)\hat{y} = B_o e^{-\kappa z}\cos\left(kz - \omega t + \left\{\delta_E + \phi_k\right\}\right)\hat{y}$$

$$\frac{B_o}{E_o} = \frac{K(\omega)}{\omega} = \left[\varepsilon\mu\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right]^{\frac{1}{2}}$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

where $$K(\omega) \equiv \left|\tilde{k}(\omega)\right| = \sqrt{k^2(\omega) + \kappa^2(\omega)} = \omega\left[\varepsilon\mu\sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right]^{1/2}$$

$$\delta_B = \delta_E + \phi_k, \quad \phi_k(\omega) \equiv \tan^{-1}\left(\frac{\kappa(\omega)}{k(\omega)}\right)$$

and $$\tilde{k}(\omega) = \left|\tilde{\vec{k}}(\omega)\right| = k(\omega) + i\kappa(\omega)$$

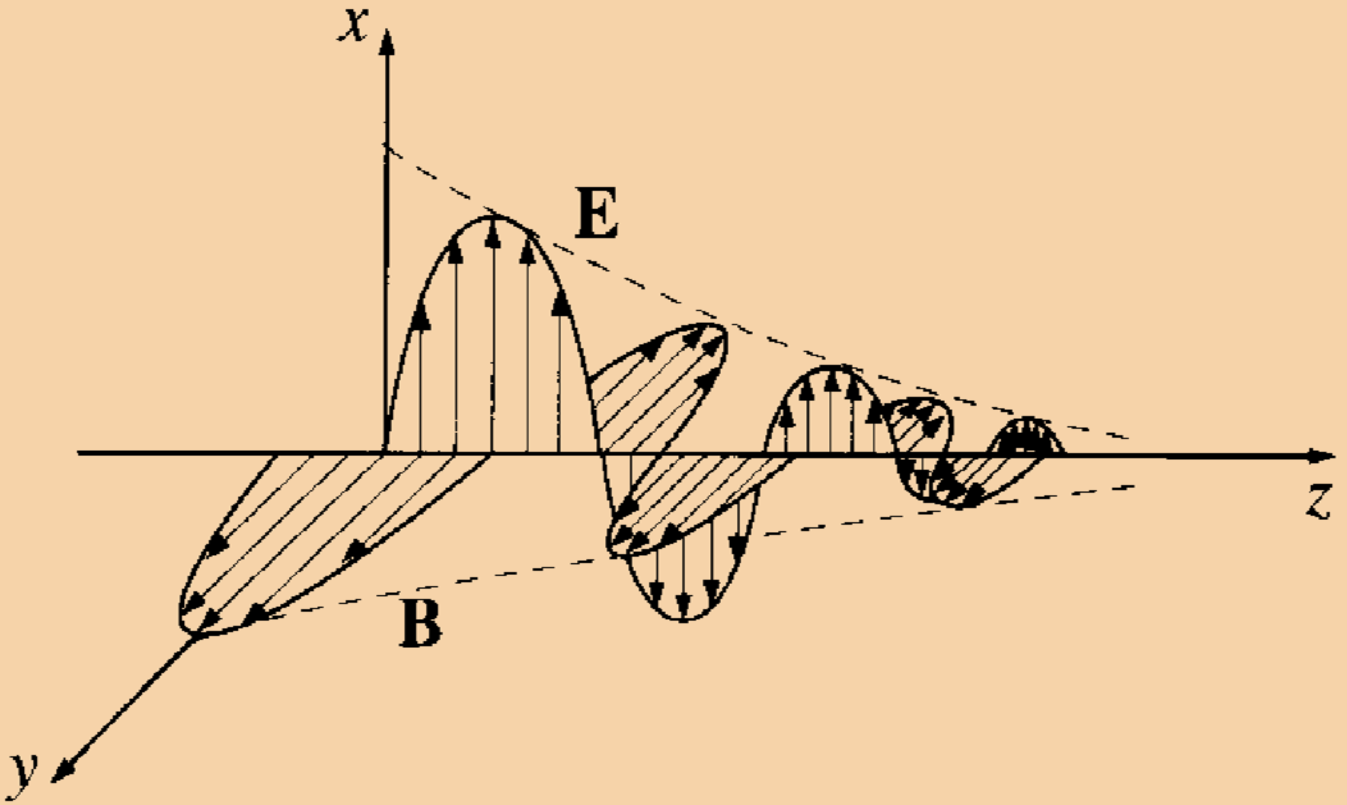
MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

Definition of the ***skin depth in a conductor:***

$$\delta_{sc}(\omega) \equiv \frac{1}{\kappa(\omega)} = \frac{1}{\omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}-1\right]^{1/2}}$$

= Distance over which the $\vec{E}$ and $\vec{B}$ fields fall to $1/e = e^{-1} = 0.3679$ of their initial values.

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA



Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

In the presence of free surface charges σ and free surface currents- the Bc's for reflection and refraction at *e.g.* a dielectric-conductor interface become:

BC 1): (normal ***D*** at interface): $\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_{free}$

BC 2): (tangential ***E*** at interface): $E_1^{\parallel} - E_2^{\parallel} = 0 \Rightarrow E_1^{\parallel} = E_2^{\parallel}$

BC 3): (normal ***B*** at interface): $B_1^{\perp} - B_2^{\perp} = 0 \Rightarrow B_1^{\perp} = B_2^{\perp}$

BC 4): (tangential *H* at interface): $\frac{1}{\mu_1} B_1^{\parallel} - \frac{1}{\mu_2} B_2^{\parallel} = \vec{K}_{free} \times \hat{n}_{\vec{21}}$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

$\perp$ = normal to plane of interface
$||$ = parallel to plane of interface

Where $\vec{n}_{21}$ is a unit vector $\perp$ to the interface, pointing from medium (2) into medium (1).

Incident *EM* wave {medium (1)}:

$$\tilde{\vec{E}}_{inc}(z,t) = \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)} \hat{x}$$

and

$$\tilde{\vec{B}}_{inc}(z,t) = \frac{1}{v_1} \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)} \hat{y}$$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

Reflected *EM* wave {medium (1)}*:*

$$\vec{\tilde{E}}_{refl}(z,t) = \tilde{E}_{o_{refl}}\, e^{i(-k_1 z = \omega t)}\,\hat{x}$$

and

$$\vec{\tilde{B}}_{refl}(z,t) = -\frac{1}{v_1}\tilde{E}_{o_{refl}}\, e^{i(-k_1 z - \omega t)}\,\hat{y}$$

Transmitted *EM* wave {medium (2)}*:*

$$\vec{\tilde{E}}_{trans}(z,t) = \tilde{E}_{o_{trans}}\, e^{i(\tilde{k}_2 z - \omega t)}\,\hat{x}$$

and

$$\vec{\tilde{B}}_{trans}(z,t) = \frac{\tilde{k}_2}{\omega}\tilde{E}_{o_{trans}}\, e^{i(\tilde{k}_2 z - \omega t)}\,\hat{y}$$

complex wave-number in {conducting} medium (2):

$$\tilde{k}_2 = k_2 + i\kappa_2$$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

In medium (1) EM fields are:

$$\tilde{\vec{E}}_{Tot_1}(z,t)=\tilde{\vec{E}}_{inc}(z,t)+\tilde{\vec{E}}_{refl}(z,t)$$

$$\tilde{\vec{B}}_{Tot_1}(z,t)=\tilde{\vec{B}}_{inc}(z,t)+\tilde{\vec{B}}_{refl}(z,t)$$

In medium (2) EM fields are:

$$\tilde{\vec{E}}_{Tot_2}(z,t)=\tilde{\vec{E}}_{trans}(z,t)$$ <u>and</u>: $$\tilde{\vec{B}}_{Tot_2}(z,t)=\tilde{\vec{B}}_{trans}(z,t)$$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

Apply BC's at the z = 0 interface in the x-y plane:

BC 1): $\boxed{\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_{free}}$ but $\boxed{E_1^{\perp} = \tilde{E}_{1_z} = 0}$ and: $\boxed{E_2^{\perp} = \tilde{E}_{2_z} = 0}$

$$\boxed{0 - 0 = \sigma_{free}} \Rightarrow \boxed{\sigma_{free} = 0}$$

BC 2): $\boxed{E_1^{\parallel} = E_2^{\parallel}}$ $\therefore$ $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

BC 3): $\boxed{B_1^\perp = B_2^\perp}$ but: $\boxed{B_1^\perp = B_{1_z} = 0}$ and: $\boxed{B_2^\perp = B_{2_z} = 0}$ $\Rightarrow$ $\boxed{0 = 0}$

BC 4): $\boxed{\frac{1}{\mu_1} B_1^\parallel - \frac{1}{\mu_2} B_2^\parallel = \vec{K}_{free} \times \hat{n}_{\overrightarrow{21}}}$ but: $\boxed{\vec{K}_{free} = 0}$ $\therefore$ $\boxed{\frac{1}{\mu_1 v_1}\left(\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}}\right) - \frac{\tilde{k}_2}{\mu_2 \omega} \tilde{E}_{o_{trans}} = 0}$

or: $\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \tilde{\beta} \tilde{E}_{o_{trans}}}$ with: $\boxed{\tilde{\beta} \equiv \left(\frac{\mu_1 v_1 \tilde{k}_2}{\mu_2 \omega}\right) = \left(\frac{\mu_1 v_1}{\mu_2 \omega}\right) \tilde{k}_2}$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

Thus we obtain:

$$\boxed{\left(\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right)=\left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)}\quad \underline{\text{and}}: \quad \boxed{\left(\frac{\tilde{E}_{o_{trans}}}{\tilde{E}_{o_{inc}}}\right)=\frac{2}{\left(1+\tilde{\beta}\right)}}$$

with $$\boxed{\tilde{\beta}\equiv\left(\frac{\mu_1 v_1 \tilde{k}_2}{\mu_2\omega}\right)=\left(\frac{\mu_1 v_1}{\mu_2\omega}\right)\tilde{k}_2}$$

The relations for reflection/transmission of EMW at normal incidence on a non-conductor/conductor boundary are identical to those obtained for reflection / transmission of EMW at normal incidence on a boundary/interface between two non-conductors- except for the replacement of β with a complex β.

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

For the case of a perfect conductor, the conductivity

$$\sigma_C = \infty \ \{\text{thus resistivity, } \rho_C = 1/\sigma_C = 0\}$$

$$\Rightarrow \ \underline{both} \ \boxed{k_2 \simeq \kappa_2 \simeq \sqrt{\frac{\omega\mu_2\sigma_C}{2}} = \infty} \ \text{and since: } \boxed{\tilde{k}_2 = k_2 + i\kappa_2} \ \text{then: } \boxed{\tilde{k}_2 = \infty + i\infty = \infty(1+i)}$$

$$\text{and since: } \boxed{\tilde{\beta} \equiv \left(\frac{\mu_1 v_1 \tilde{k}_2}{\mu_2\omega}\right) = \left(\frac{\mu_1 v_1}{\mu_2\omega}\right)\tilde{k}_2} \Rightarrow \underline{\underline{\tilde{\beta} = \infty}}$$

Thus, for a perfect conductor, we see that:

$$\boxed{\tilde{E}_{o_{refl}} = -\tilde{E}_{o_{inc}}} \ \text{and} \ \boxed{\tilde{E}_{trans} = 0}$$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

For a perfect conductor the reflection and transmission coefficients are:

$$R \equiv \left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right)^2 = \left|\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right|^2 = \left(\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right)\left(\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right)^* = 1$$

<u>and</u>: $T = 1 - R = 0$

We also see that for a perfect conductor, for normal incidence, the reflected wave undergoes a 180 degree phase shift with respect to the incident wave at the interface at $z = 0$ in the x-y plane. A perfect conductor screens out all *EM* waves from propagating in its interior.

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

For a good conductor- the conductivity is large- but finite. The reflection coefficient R for monochromatic plane EM waves at normal incidence on a good conductor is not unity- but close to it. *{This is why good conductors make good mirrors!}*.

$$R \equiv \left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right)^2 = \left|\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right|^2 = \left(\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right)\left(\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right)^* = \left|\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right|^2 = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)^*$$

Where

$$\tilde{\beta} = \left(\frac{\mu_1 v_1}{\mu_2 \omega}\right)\tilde{k}_2 = \left(\frac{\mu_1 v_1}{\mu_2 \omega}\right)\sqrt{\frac{\omega \mu_2 \sigma_C}{2}}\,(1+i) = \mu_1 v_1 \sqrt{\frac{\sigma_C}{2\mu_2 \omega}}\,(1+i)$$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

Define

$$\gamma \equiv \mu_1 v_1 \sqrt{\frac{\sigma_C}{2\mu_2\omega}}$$

Then: $\tilde{\beta} = \gamma(1+i)$

Thus, the reflection coefficient R for monochromatic plane EM waves at normal incidence on a good conductor is:

$$R = \left|\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right|^2 = \left|\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right|^2 = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)^* = \left(\frac{1-\gamma-i\gamma}{1+\gamma+i\gamma}\right)\left(\frac{1-\gamma+i\gamma}{1+\gamma-i\gamma}\right) = \left[\frac{(1-\gamma)^2+\gamma^2}{(1+\gamma)^2+\gamma^2}\right]$$

with

$$\gamma \equiv \mu_1 v_1 \sqrt{\frac{\sigma_C}{2\mu_2\omega}}$$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

Obviously, only a small fraction of the normally-incident monochromatic plane EM wave *is* transmitted into the good conductor- since $R < 1$ and $\boxed{T = 1 - R}$, *i.e.*:

$$\boxed{T = 1 - R = 1 - \left[\frac{(1-\gamma)^2 + \gamma^2}{(1+\gamma)^2 + \gamma^2}\right] \quad (\ll 1)}$$

Note that the transmitted wave is exponentially attenuated in the z-direction; the E and *B* fields in the good conductor fall to 1/e of their initial {z = 0} values (at/on the interface) after the monochromatic plane *EM* wave propagates a distance of one skin depth in z into the conductor:

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

$$\delta_{sc}(\omega) \equiv \frac{1}{\kappa_2(\omega)} \simeq \sqrt{\frac{2}{\omega\mu_2\sigma_C}}$$

Note also that the energy associated with the transmitted monochromatic plane *EM* wave is ultimately dissipated in the conducting medium as heat.

In {bulk} metals-the transmitted wave is {rapidly} absorbed/attenuated in the metal- we can only study the reflection coefficient *R*.

A full description of the physics of reflection from the surface of a metal conductor as a function of angle of incidence-requires the use of a complex dispersion relation

Full Maxwell Equations in Matter

The electromagnetic state of matter at a given observation point *r* at a given time t is described by four macroscopic quantities:

1.) The volume density of free charge:

$$\rho_{free}(\vec{r},t)$$

2.) The volume density of electric dipoles:

$$\vec{P}(\vec{r},t)$$

⇐ electric polarization

3.) The volume density of magnetic dipoles:

$$\vec{M}(\vec{r},t)$$

⇐ magnetization

4.) The free electric current /unit area:

$$\vec{J}_{free}(\vec{r},t)$$

⇐ {free} current density

Full Maxwell Equations in Matter

These four quantities are related to the macroscopic ***E and B*** fields by the four Maxwell equations for matter

1) <u>Gauss' Law</u>: $\vec{\nabla}\bullet\vec{E}=\frac{\rho_{Tot}}{\varepsilon_o}=\frac{1}{\varepsilon_o}\left(\rho_{free}+\rho_{bound}\right)$, where: $\rho_{bound}=-\vec{\nabla}\bullet\vec{P}$

Auxiliary relation: $\vec{D}=\varepsilon_o\vec{E}+\vec{P}$ & constitutive relation: $\vec{D}=\varepsilon\vec{E}$

Electric polarization $\vec{P}=\left(\varepsilon-\varepsilon_o\right)\vec{E}=\varepsilon_0\chi_e\vec{E}$, electric susceptibility $\chi_e=\left(\frac{\varepsilon}{\varepsilon_o}-1\right)$

$$\vec{\nabla}\bullet\vec{D}=\varepsilon_o\vec{\nabla}\bullet\vec{E}+\vec{\nabla}\bullet\vec{P}=\rho_{free}$$

2) <u>No magnetic charges/monopoles</u>: $\vec{\nabla}\bullet\vec{B}=0$

Auxiliary relation: $\vec{H}=\frac{1}{\mu_o}\vec{B}-\vec{M}$ $\Rightarrow$ $\vec{\nabla}\bullet\vec{H}=-\vec{\nabla}\bullet\vec{M}$ & constitutive relation: $\vec{B}=\mu\vec{H}$

Full Maxwell Equations in Matter

3) <u>Faraday's Law</u>:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\mu_0\frac{\partial\vec{H}}{\partial t} - \mu_0\frac{\partial\vec{\mathrm{M}}}{\partial t}$$

Magnetization: $\vec{\mathrm{M}} - \left(\dfrac{\mu}{\mu_o - 1}\right)\vec{H} = \chi_m\vec{H}$, magnetic susceptibility $\chi_m = \left(\dfrac{\mu}{\mu_o} - 1\right)$

4) <u>Ampere's Law</u>: $\vec{\nabla}\times\vec{B} = \mu_o\vec{J}_{ToT} + \mu_o\vec{J}_D$ with $\vec{J}_D = \varepsilon_o\dfrac{\partial\vec{E}}{\partial t}$

Total current density: $\vec{J}_{ToT} = \vec{J}_{free} + \vec{J}^{mag}_{bound} + \vec{J}^{P}_{bound}$ $\quad \vec{J}^{mag}_{bound} = \vec{\nabla}\times\vec{\mathrm{M}}$ $\quad \vec{J}^{P}_{bound} = \dfrac{\partial\vec{\mathrm{P}}}{\partial t}$

$$\vec{\nabla}\times\vec{B} = \mu_o J_{free} + \mu_o\vec{\nabla}\times\vec{\mathrm{M}} + \mu_o\frac{\partial\vec{\mathrm{P}}}{\partial t} + \mu_o\varepsilon_o\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\times\vec{H} = \mu_o\vec{J}_{free} + \mu_o\frac{\partial\vec{D}}{\partial t}$$

Full Maxwell Equations in Matter

Then Maxwell's equations in matter, for $\rho_{free} = 0$ and $\vec{M} = 0$

1) Gauss' Law: $\vec{\nabla} \bullet \vec{D} = 0$ or: $\vec{\nabla} \bullet \vec{E} = -\frac{1}{\varepsilon_o} \vec{\nabla} \bullet \vec{P} = \rho_{free} / \varepsilon_o$

2) No magnetic charges: $\vec{\nabla} \bullet \vec{B} = 0$

3) Faraday's Law: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

4) Ampere's Law: $\vec{\nabla} \times \vec{B} = \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t} + \mu_o \frac{\partial \vec{P}}{\partial t} + \mu_o \vec{J}_{free}$

Full Maxwell Equations in Matter

We also have Ohm's Law $\vec{J}_{free} = \sigma_c \vec{E}$

and the Continuity eqn. $\vec{\nabla} \bullet \vec{J}_{free} = 0$

Then applying the curl operator to Faraday's Law:

We thus obtain the inhomogeneous wave equation:

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \underbrace{\frac{1}{\varepsilon_o}\nabla \rho_{bound} + \mu_o \frac{\partial^2 \vec{\mathrm{P}}}{\partial t^2} + \mu_o \frac{\partial \vec{J}_{free}}{\partial t}}_{\text{source terms}}$$

{and a similar one for ***B*** }

Full Maxwell Equations in Matter

For non-oronducting/poorly-conducting media, i.e. insulators/ dielectrics- the first two terms on the RHS are important – they explain many optical effects such as dispersion (wavelength/frequency-dependence of the index of refraction), absorption, double – refraction/bi-refringence, optical activity, ...

Note that the $\vec{\nabla}\rho_{bound} = -\vec{\nabla}\left(\vec{\nabla}\bullet\vec{P}\right)$ term is often zero- P uniform

$$\vec{\nabla}\bullet\vec{P} = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z} \text{ and } \vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

$e.g.$ for $\vec{P} \propto \vec{E}$ (i.e. $\vec{P}$ proportional to $\vec{E}$) where: $\vec{E}(z,t) = E_o\cos(kz - \omega t + \delta)\hat{x}$

Full Maxwell Equations in Matter

For good conductors (e.g. metals), the conduction term

$$\mu_o \frac{\partial \vec{J}_{free}}{\partial t} = \mu_o \sigma_C \frac{\partial \vec{E}}{\partial t}$$

is the most important, because it explains the opacity of metals (e.g. in the visible light region) and also explains the high reflectance of metals.