Workshop on Strongly Coupled Physics Beyond the Standard Model

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Scale without Conformal Invariance

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Scale without Conformal Invariance

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Beyond the Standard Model

based on
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with Benjamín Grinstein and Andreas Stergiou
Historical review

Scale versus conformal invariance

Scale-invariant trajectories

Discussion and conclusion

Historical review

Scale and conformal invariance in two dimensions

Does scale invariance imply conformal invariance?

- Polchinski following Zamolodchikov  
  - Unitarity
  - Finiteness of EM tensor correlation functions
  ⇒ Scale invariance implies conformal invariance from conservation of EM tensor

“Counter” examples

- Non-linear $\sigma$ model  
  Hull, Townsend (1986)
  - Non-existence of EM tensor two-point correlation functions

- Theory of elasticity  
  Cardy, Riva (2005)
  - Non-reflection-positive

Scale invariance implies conformal invariance
Does scale invariance imply conformal invariance?

- No proof à la Polchinski
  - Conservation of EM tensor ⇒ Not enough information
  ⇒ Scale invariance does not necessarily imply conformal invariance

No relevant counterexamples

- AdS/CFT Kerr-AdS black holes in $d = 5, 7$ dimensions Awad, Johnson (1999)
  - Conformal invariance broken to scale invariance by black hole rotation

- Maxwell theory in $d \neq 4$ dimensions Jackiw, Pi (2011) & El-Showk, Nakayama, Rychkov (2011)
  - Free field theory
  - Scale invariance broken by interactions
Study of non-conformal scale-invariant QFTs

Scale invariance does not necessarily imply conformal invariance but no proper counterexamples ⇒ Possible proof!

- Without proper counterexamples
  - Physical implications of non-conformal scale-invariant QFTs (correlation functions in non-conformal scale-invariant QFTs versus CFTs ?)

- With proper counterexamples
  - Scale invariance conditions weaker than conformal invariance conditions (plentiful examples ?)

Uncharted territory!
Historical review

Scale versus conformal invariance
- Preliminaries
- Scale invariance and new improved energy-momentum tensor
- RG flows along scale-invariant trajectories
- Scale invariance and recurrent behaviors
- Scale invariance, gradient flows and $a$-theorem
- Why dilatation generators generate dilatations

Scale-invariant trajectories
- Systematic approach
- Examples

Discussion and conclusion
- Features and future work
Preliminaries \((d > 2)\)

- **Dilatation current** \(^{\text{Polchinski (1988)}}\)
  - \(D^\mu(x) = x^\nu T_\nu{}^\mu(x) - V^\mu(x)\)
  - \(T_\nu{}^\mu(x)\) any symmetric EM tensor following from spacetime nature of scale transformations
  - \(V^\mu(x)\) local operator (virial current) contributing to scale dimensions of fields
  - Freedom in choice of \(T_\nu{}^\mu(x)\) compensated by freedom in choice of \(V^\mu(x)\)

- **Scale invariance** \(\Rightarrow T_\mu{}^\mu(x) = \partial_\mu V^\mu(x)\)
- **Conformal current**  \[ \text{Polchinski (1988)} \]
  - \[ C_{\nu}^{\mu}(x) = v^{\nu}(x) T_{\nu}^{\mu}(x) - \partial \cdot v(x) V''^{\mu}(x) + \partial_{\nu} \partial \cdot v(x) L^{\nu \mu}(x) \]
  - \[ T_{\nu}^{\mu}(x) \] any symmetric EM tensor following from spacetime nature of conformal transformations
  - \[ V''^{\mu}(x) \] local operator corresponding to ambiguity in choice of dilatation current
  - \[ L^{\nu \mu}(x) \] local symmetric operator correcting position dependence of scale factor
  - \[ \partial \cdot v(x) \] scale factor (general linear function of \( x^{\mu} \))
  - Freedom in choice of \[ T_{\nu}^{\mu}(x) \] compensated by freedom in choice of \[ V''^{\mu}(x) \] and \[ L^{\nu \mu}(x) \]

- **Conformal invariance** \( \Rightarrow \)
  - \[ \partial_{\nu} V''^{\mu}(x) = \partial_{\nu} L^{\nu \mu}(x) \]

- **Conformal invariance** \( \Rightarrow \)
  - Existence of symmetric traceless energy-momentum tensor
Historical review

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Discussion and conclusion

Scale without conformal invariance

Non-conformal scale-invariant QFTs  Polchinski (1988)

- Scale invariance  $\Rightarrow \ T_\mu^\mu(x) = \partial_\mu V^\mu(x)$
- Conformal invariance  $\Rightarrow \ T_\mu^\mu(x) = \partial_\mu \partial_\nu L^{\nu\mu}(x)$
- Scale without conformal invariance
  
  $$\Rightarrow \ T_\mu^\mu(x) = \partial_\mu V^\mu(x) \text{ where } V^\mu(x) \neq J^\mu(x) + \partial_\nu L^{\nu\mu}(x) \text{ with } \partial_\mu J^\mu(x) = 0$$

- Constraints on possible virial current candidates
  - Gauge-invariant spatial integral
  - Fixed $d-1$ scale dimension in $d$ spacetime dimensions

- No suitable virial current  $\Rightarrow$ Scale invariance implies conformal invariance (examples: $\phi^p$ in $d = n - \epsilon$ for $(p, n) = (6, 3), (4, 4) \text{ and } (3, 6)$)
Virial current candidates

Most general classically scale-invariant renormalizable theory in $d = 4$ spacetime dimensions Jack, Osborn (1985)

$$
L = - \mu^{-\epsilon} Z_A \frac{1}{4 g_A^2} F^A_{\mu \nu} F^{A \mu \nu} + \frac{1}{2} Z_{ab}^2 Z_{ac}^2 D_\mu \phi_b D^\mu \phi_c
$$

$$
+ \frac{1}{2} Z_{ij}^2 Z_{ik}^2 \bar{\psi}_j i \sigma^\mu D_\mu \psi_k - \frac{1}{2} Z_{ij}^2 Z_{ik}^2 D_\mu \bar{\psi}_j i \sigma^\mu \psi_k
$$

$$
- \frac{1}{4!} \mu^\epsilon (\lambda Z^\lambda)_{abcd} \phi_a \phi_b \phi_c \phi_d
$$

$$
- \frac{1}{2} \mu^\frac{\epsilon}{2} (y Z^y)_{a|ij} \phi_a \psi_i \psi_j - \frac{1}{2} \mu^\frac{\epsilon}{2} (y Z^y)_{a|ij} \phi_a \bar{\psi}_i \bar{\psi}_j
$$

- $A^A_\mu(x)$ gauge fields
- $\phi_a(x)$ real scalar fields
- $\psi^{\alpha}_i(x)$ Weyl fermions
- Dimensional regularization ($d = 4 - \epsilon$)
Virial current candidates and new improved EM tensor

- Virial current \( V^\mu(x) = Q_{ab} \phi_a D^\mu \phi_b - P_{ij} \bar{\psi}_i i \bar{\sigma}^\mu \psi_j \)
  - \( Q_{ba} = -Q_{ab} \)
  - \( P_{ji}^* = -P_{ij} \)

- New improved energy-momentum tensor \( \Theta^{\mu}_\nu(x) \)  
  Callan, Coleman, Jackiw (1970)
  - Finite
  - Not renormalized
  - Anomalous trace  
  Robertson (1991)

\[
\Theta^{\mu}_\nu(x) = \frac{\beta_A}{2g_A^3} F^A_{\mu\nu} F_{A\mu\nu} + \gamma_{aa'} D^2 \phi_a \phi_{a'} \\
- \gamma^{*}_{ii'} \bar{\psi}_i i \bar{\sigma}^\mu D_\mu \psi_{i'} + \gamma_{ii'} D_\mu \bar{\psi}_i i \bar{\sigma}^\mu \psi_{i'} \\
- \frac{1}{4!} (\beta_{abcd} - \gamma_{a'a} \lambda_{a'bcd} - \gamma_{b'b} \lambda_{ab'cd} \\
\quad - \gamma_{c'c} \lambda_{abc'd} - \gamma_{d'd} \lambda_{abcd'}) \phi_{a} \phi_{b} \phi_{c} \phi_{d} \\
- \frac{1}{2} (\beta_{a|ij} - \gamma_{a'a} Y_{a'|ij} - \gamma_{i'i} Y_{a|i'i'} - \gamma_{j'j} Y_{a|ij'}) \phi_{a} \psi_{i} \psi_{j} + h.c.
\]
- $\beta$-functions from vertex corrections and wavefunction renormalizations ($d = 4$ spacetime dimensions)
  
  - RG time $t = \ln(\mu_0/\mu)$

  \[
  \beta_A = -\frac{d g_A}{dt} = \gamma_A g_A \quad \text{(no sum)}
  \]

  \[
  \beta_{abcd} = -\frac{d \lambda_{abcd}}{dt} = - (\lambda \gamma^\lambda)_{abcd} + \gamma_{a\prime} a \lambda_{a\prime bcd} + \gamma_{b\prime} b \lambda_{ab' cd} + \gamma_{c\prime} c \lambda_{abc'd} + \gamma_{d\prime} d \lambda_{abcd'}
  \]

  \[
  \beta_{a|ij} = -\frac{d y_{a|ij}}{dt} = -(y \gamma^y)_{a|ij} + \gamma_{a' a} Y_{a'|ij} + \gamma_{i' i} Y_{a|i'j} + \gamma_{j' j} Y_{a|ij'}
  \]
Divergence of dilatation current

\[ \partial_\mu D^\mu(x) = \frac{\beta_A}{2g_A^2} F^A_{\mu\nu} F^{A\mu\nu} + (\gamma_{aa'} + Q_{aa'}) D^2 \phi_a \phi_{a'} - (\gamma_{i'i} + P_{i'i}) \bar{\psi}_i i\bar{\sigma}^\mu D_\mu \psi_{i'} + (\gamma_{ii'} + P_{ii'}) D_\mu \bar{\psi}_i i\bar{\sigma}^\mu \psi_{i'} - \frac{1}{4!} (\beta_{abcd} - \gamma_{a'a'b'c'd'} - \gamma_{b'b'\lambda_{ab'c'd'}} - \gamma_{c'c'\lambda_{ab'c'd'}} - \gamma_{d'd'\lambda_{ab'c'd'}}) \phi_a \phi_b \phi_c \phi_d - \frac{1}{2} (\beta_a|i - \gamma_{a'a} Y_{a|i} - \gamma_{i'i} Y_{a|i'j} - \gamma_{j'j} Y_{a|ij} + h.c.) \phi_a \psi_i \psi_j + \text{h.c.} \]

Conserved dilatation current \( \partial_\mu D^\mu(x) = 0 \) (up to EOMs)

\[ \beta_A = 0 \]
\[ \beta_{abcd} = -Q_{a'a'b'c'd'} - Q_{b'b'\lambda_{ab'c'd'}} - Q_{c'c'\lambda_{ab'c'd'}} - Q_{d'd'\lambda_{ab'c'd'}} \]
\[ \beta_{a|i} = -Q_{a'a} Y_{a|i} - P_{i'i} Y_{a|i'j} - P_{j'j} Y_{a|ij} \]

Conserved conformal current \( \partial_\mu C^\mu_{\nu}(x) = 0 \) (up to EOMs)

\[ \beta_A = \beta_{abcd} = \beta_{a|i} = 0 \]
Historical review

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Scale-invariant trajectories

Discussion and conclusion

Interlude: Current conservation

- Divergence of current \( J_\mu(x) \) without use of EOMs \cite{Collins1984}

\[
\partial_\mu J^\mu(x) = \Delta_{\text{EOM}} + \Delta_{\text{Classical}} + \Delta_{\text{Anomaly}}
\]

- Green’s function of elementary fields with current \( J_\mu(x) \) and Ward identity
  - \( \Delta_{\text{EOM}} \Rightarrow \) Expected contact terms from Ward identity
  - \( \Delta_{\text{Classical}} \Rightarrow \) Usual non-anomalous classical violation
  - \( \Delta_{\text{Anomaly}} \Rightarrow \) Possible anomalous violation in divergent Green’s function

- Example: \( \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^A F^{A\mu\nu} + \bar{\psi}_i (i \gamma_\mu D_\mu \delta_{ij} - M_{ij}) \psi_j \)

  - Vector current \( J^{\mu a}_V(x) = \bar{\psi} \gamma^\mu t^a \psi \) with \( \Delta_{\text{EOM}} \neq 0 \),
    \( \Delta_{\text{Classical}} = i \bar{\psi} [M, t^a] \psi \) and \( \Delta_{\text{Anomaly}} = 0 \)
  - Axial current \( J^{\mu a}_A(x) = \bar{\psi} \frac{1}{2} [\gamma^\mu, \gamma^5] t^a \psi \) with \( \Delta_{\text{EOM}} \neq 0 \),
    \( \Delta_{\text{Classical}} = i \bar{\psi} \gamma^5 \{M, t^a\} \psi \) and
    \( \Delta_{\text{Anomaly}} = \frac{1}{2} \bar{\psi} \{\gamma^\mu, \gamma^5\} t^a D_\mu \psi - \frac{1}{2} D_\mu \bar{\psi} \{\gamma^\mu, \gamma^5\} t^a \psi \)
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Virial current and unitarity bounds

- New improved energy-momentum tensor $\Rightarrow$ Finite and not renormalized. Callan, Coleman, Jackiw (1970)


- Virial current $\Rightarrow$ Finite and not renormalized
  - Unconserved current with scale dimension exactly 3


- Non-trivial virial current $\Rightarrow$ Non-conformal scale-invariant QFTs
RG flows along scale-invariant trajectories

Scale-invariant solution \((g_A, \lambda_{abcd}, y_{a|ij})\) \Rightarrow RG trajectory

\[
\begin{align*}
\bar{g}_A(t) &= g_A \\
\bar{\lambda}_{abcd}(t) &= \hat{Z}_{a'a}(t)\hat{Z}_{b'b}(t)\hat{Z}_{c'c}(t)\hat{Z}_{d'd}(t)\lambda_{a'b'c'd'} \\
\bar{y}_{a|ij}(t) &= \hat{Z}_{a'a}(t)\hat{Z}_{i'i}(t)\hat{Z}_{j'j}(t)y_{a'|i'j'} \\
\hat{Z}_{aa'}(t) &= (e^{Q_t})_{aa'} \\
\hat{Z}_{ii'}(t) &= (e^{P_t})_{ii'}
\end{align*}
\]

- \((\bar{g}_A(t, g, \lambda, y), \bar{\lambda}_{abcd}(t, g, \lambda, y), \bar{y}_{a|ij}(t, g, \lambda, y))\) also scale-invariant solution

- \(Q_{ab}\) and \(P_{ij}\) constant along RG trajectory

- \(\hat{Z}_{ab}(t)\) orthogonal and \(\hat{Z}_{ij}(t)\) unitary \Rightarrow Always non-vanishing \(\beta\)-functions along scale-invariant trajectory
Scale invariance and recurrent behaviors

RG flows along scale-invariant trajectories ⇒ Recurrent behaviors !

- Virial current ⇒ Transformation in symmetry group of kinetic terms \((SO(N_S) \times U(N_F))\)
  
  - \(Q_{ab}\) antisymmetric and \(P_{ij}\) antihermitian ⇒ Purely imaginary eigenvalues
  
  - \(\hat{Z}_{ab}(t)\) and \(\hat{Z}_{ij}(t)\) in \(SO(N_S) \times U(N_F)\)

⇒ Periodic (limit cycle) or quasi-periodic (ergodicity) scale-invariant trajectories
Historical review

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Discussion and conclusion

Recurrent behaviors

Intuition from $D^\mu(x) = x^\nu \Theta_{\nu}^{\mu}(x) - V^\mu(x)$

- RG flow $\Rightarrow$ Generated by scale transformation $(x^\nu \Theta_{\nu}^{\mu}(x))$

- RG flow $\Rightarrow$ Related to virial current through conservation of dilatation current

- Virial current $\Rightarrow$ Generates internal transformation of the fields
  - Internal transformation in compact group $SO(N_S) \times U(N_F)$
  $\Rightarrow$ Rotate back to or close to identity

- RG flow return back to or close to identity $\Rightarrow$ Recurrent behavior
**Scale-invariant trajectories?**

RG flows $\sim$ Field redefinitions $\Rightarrow$ Scale-invariant trajectories or fixed points?

- **RG-time-dependent** field redefinitions $\Rightarrow$ Generates RG flows
  

  - RG-time-dependent field redefinitions $\Rightarrow$ All exact RG flows
    (Wilson, Wegner, Polchinski, etc.)

$\beta$-function operators $\sim$ Redundant operators $\Rightarrow$ Scale-invariant trajectories or fixed points?

- Wavefunction renormalization operators $\Rightarrow$ Redundant operators
  
  - Redundant $\beta$-function operators necessary for scale invariance

Non-conformal scale-invariant QFTs $\Rightarrow$ Non-trivial RG flows (recurrent behaviors)
Scale invariance, gradient flows and $a$-theorem

- **Gradient flow**

  \[
  \beta_i(g) = -\frac{dg_i}{dt} = G_{ij}(g) \frac{\partial c(g)}{\partial g_j}
  \]

  - $G_{ij}$ positive-definite metric
  - Potential $c(g)$ function of couplings

- **Potential $c(g)$ monotonically decreasing along RG trajectory**

  \[
  \frac{dc(g(t))}{dt} = -G^{ij}(g)\beta_i\beta_j \leq 0
  \]

  - Recurrent behaviors (scale-invariant trajectories) $\nRightarrow$ Gradient flows (scale implies conformal invariance) \textit{Wallace, Zia (1975)}

- **$a$-theorem** \textit{Barnes, Intriligator, Wecht, Wright (2004)}

  - RG flow $\Rightarrow$ Irreversible process (integrating out DOFs)
  - $c(g)$ $\sim$ measure of number of massless DOFs
  - $a$-theorem $\Rightarrow$ weak ($c_{IR} < c_{UV}$), stronger ($\frac{dc}{dt} \leq 0$), $strongest$ (RG flows as gradient flows)
Dilatation generators do not generate dilatations in non-scale-invariant QFTs  \textit{Coleman, Jackiw (1971)}

- **Quantum anomalies at low orders**
  - Anomalous dimensions
  \[ \Rightarrow \text{Possible to absorb into redefinition of scale dimensions of fields} \]
  \[ \checkmark \text{Preserve scale invariance} \]

- **Quantum anomalies at high orders**
  - $\beta$-functions
  \[ \Rightarrow \text{Not possible to absorb} \]
  \[ \xmark \text{Break scale invariance} \]
Why dilatation generators generate dilatations in non-conformal scale-invariant QFTs?

- $\beta$-functions on scale-invariant trajectories
  - Both vertex correction and wavefunction renormalization contributions
  - Very specific form for vertex correction contribution
  - Equivalent in form to wavefunction renormalization contribution (redundant operators)

$\Rightarrow$ Also possible to absorb into redefinition of scale dimensions of fields

✓ Preserve scale invariance!
Ward identity for scale invariance

Callan-Symanzik equation for effective action  

\[ \Gamma[\varphi(x), g, M] = 0 \]

- In non-scale-invariant QFTs
  - ✔  Anomalous dimensions
  - ✗  \( \beta \)-functions

- In CFTs
  - ✔  Anomalous dimensions
  - ✔  Vanishing \( \beta \)-functions

\[ \Gamma[\varphi(x), g, M] = 0 \]

- In non-conformal scale-invariant QFTs
  - ✔  Anomalous dimensions
  - ✔  \( \beta \)-functions (redundant operators)
Poincaré algebra augmented with dilatation charge

- **β-functions on scale-invariant trajectories**
  - Quantum-mechanical generation of scale dimensions
  - Appropriate scale dimensions required by virial current
  \[ \Rightarrow \text{Conserved dilatation current } D^\mu(x) \]

- Poincaré algebra with dilatation charge \( D = \int d^3x \mathcal{D}^0(x) \)
  \[
  [M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho}) \\
  [M_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \\
  [D, P_\mu] = -iP_\mu
  \]

- Algebra action on fields \( \mathcal{O}_I(x) \)
  \[
  [M_{\mu\nu}, \mathcal{O}_I(x)] = -i(x_\mu \partial_\nu - x_\nu \partial_\mu + \Sigma_{\mu\nu})\mathcal{O}_I(x) \\
  [P_\mu, \mathcal{O}_I(x)] = -i\partial_\mu \mathcal{O}_I(x) \\
  [D, \mathcal{O}_I(x)] = -i(x \cdot \partial + \Delta)\mathcal{O}_I(x)
  \]
New classical scale dimensions of fields due to virial current

\[ [D, \phi_a(x)] = -i(x \cdot \partial + 1)\phi_a(x) - iQ_{ab}\phi_b(x) \]
\[ [D, \psi_i(x)] = -i(x \cdot \partial + \frac{3}{2})\psi_i(x) - iP_{ij}\psi_j(x) \]

- How do non-conformal scale-invariant QFTs know about new scale dimensions?
  \( \Rightarrow \) Generated by \( \beta \)-functions!

Quantum-mechanical scale dimensions of fields

\[ \Delta_{ab} = \delta_{ab} + \gamma_{ab} + Q_{ab} \]
\[ \Delta_{ij} = \frac{3}{2}\delta_{ij} + \gamma_{ij} + P_{ij} \]
Scale-invariant trajectories ??

\[ \beta \text{-functions} \sim \text{Anomalous dimensions} \Rightarrow \text{Scale-invariant trajectories or fixed points?} \]

- Shift \( \beta \text{-functions away} \Rightarrow \text{Scheme change} \\
  \times \text{Non-conformal scale-invariant QFTs with traceless EM tensor}

- Shift \( \beta \text{-functions away} \Rightarrow \text{Global shift} \\
  \times \text{Conformal fixed points become conformal trajectories}

Non-conformal scale-invariant QFTs \( \Rightarrow \) Non-trivial RG flows
Non-conformal scale-invariant correlation functions

- Scalar fields $O_I(x)$ with scale dimensions $\Delta_I$

$$\langle O_I(x_1)O_J(x_2) \rangle = \frac{g_{IJ}}{(x_1 - x_2)^{\Delta_I + \Delta_J}}$$

$$\langle O_I(x_1)O_J(x_2)O_K(x_3) \rangle = \sum_{\delta_1 + \delta_2 + \delta_3 = \Delta_I + \Delta_J + \Delta_K} \frac{c_{\delta_1 \delta_2 \delta_3}^{IJK}}{(x_1 - x_2)^{\delta_1} (x_2 - x_3)^{\delta_2} (x_3 - x_1)^{\delta_3}}$$

- Non-vanishing two-point functions with $\Delta_I \neq \Delta_J$ contrary to CFTs

- Two-point correlation functions of fundamental real scalar fields

$$\langle \phi_a(x)\phi_b(0) \rangle = \left[ (x^2)^{-\frac{\Delta}{2}} G^\phi (x^2)^{-\frac{\Delta_T}{2}} \right]_{ab}$$

- $G^\phi$ constant real symmetric matrix
- Two-point correlation functions of scalar operators $O_a(x)$

$$\langle O_a(x)O_b(0) \rangle = \left[ (x^2)^{-\frac{\Delta}{2}} \; \tilde{G} \; (x^2)^{-\frac{\Delta_T}{2}} \right]_{ab}$$

$$= i \int \frac{d^4p}{(2\pi)^4} \; e^{-ip \cdot x} \left[ (-p^2 - i\epsilon)^{\frac{\Delta}{2} - 1} \; \tilde{G} \; (-p^2 - i\epsilon)^{\frac{\Delta_T}{2} - 1} \right]_{ab}$$

- $G$ (and $\tilde{G}$) constant real symmetric matrices

- Two-point correlation functions of vector operators $O_a^\mu(x)$

$$\langle O_a^\mu(x)O_b^\nu(0) \rangle = \left[ (x^2)^{-\frac{\Delta}{2}} \left( g^{\mu\nu} A + \frac{x^\mu x^\nu}{x^2} B \right) \; (x^2)^{-\frac{\Delta_T}{2}} \right]_{ab}$$

$$= -i \int \frac{d^4p}{(2\pi)^4} \; e^{-ip \cdot x} \left[ (-p^2 - i\epsilon)^{\frac{\Delta}{2} - 1} \left( g^{\mu\nu} \tilde{A} + \frac{p^\mu p^\nu}{p^2} \tilde{B} \right) \; (-p^2 - i\epsilon)^{\frac{\Delta_T}{2} - 1} \right]_{ab}$$

- $A$ and $B$ (and $\tilde{A}$ and $\tilde{B}$) constant real symmetric matrices
Coupled QFT/SIT where $\mathcal{L} \supset g_a \chi \mathcal{O}_a + \text{h.c.}$ with external source $\chi$ and scalar operator $\mathcal{O}_a$

$$\mathcal{M} = g_a g_b |\chi|^2 \left[ (-p^2 - i\epsilon)^{-\frac{\Delta}{2} - 1} \tilde{G} (-p^2 - i\epsilon)^{-\frac{\Delta^T}{2} - 1} \right]_{ab}$$

$$\text{Im} \mathcal{M}_{\text{fwd}} = g_a g_b |\chi|^2 \left[ (p^2)^{-\frac{\Delta}{2} - 1} \left\{ \cos \left[ \left( 1 - \frac{\Delta}{2} \right) \pi \right] \tilde{G} \sin \left[ \left( 1 - \frac{\Delta^T}{2} \right) \pi \right] \right. 
+ \sin \left[ \left( 1 - \frac{\Delta}{2} \right) \pi \right] \tilde{G} \cos \left[ \left( 1 - \frac{\Delta^T}{2} \right) \pi \right] \left\} (p^2)^{-\frac{\Delta^T}{2} - 1} \right] \theta(p^0) \theta(p^2)$$
Systematic approach

Scale-invariant trajectories at weak coupling

\[ g_A = \sum_{n \geq 1} g_A^{(n)} \epsilon^{n - \frac{1}{2}} \quad \lambda_{abcd} = \sum_{n \geq 1} \lambda_{abcd}^{(n)} \epsilon^n \quad y_{a|ij} = \sum_{n \geq 1} y_{a|ij}^{(n)} \epsilon^{n - \frac{1}{2}} \]

\[ Q_{ab} = \sum_{n \geq 3} Q_{ab}^{(n)} \epsilon^n \quad P_{ij} = \sum_{n \geq 3} P_{ij}^{(n)} \epsilon^n \]

- $\epsilon$ small parameter
  - Obvious choice in $d = 4 - \epsilon$
  - One-loop gauge coupling $\beta$-function coefficient in $d = 4$
    Banks, Zaks (1982)

- Form of expansions determined by $\beta$-functions
  - For coupling constants $\Rightarrow$ Lowest-order terms in $\beta$-functions (would-be conformal fixed points)
  - For virial current $\Rightarrow$ Higher-order terms in $\beta$-functions due to Polchinski–Dorigoni–Rychkov argument and gradient flow interpretation
Examples

Physical $d = 4$ case

- No proper example yet $\Rightarrow$ Maybe none?
  - Technically difficult to generate three-loop $\beta$-functions

- $SU(2)$ gauge theory with two real scalars (singlet) and two active flavors of Weyl fermions (fundamental)
  - Unbounded-from-below scalar potential
  - Three-loop $\beta$-functions necessary to conclude

Unphysical $d = 4 - \epsilon$ case

- Two real scalars and two Weyl fermions
  - Limit cycle (bounded-from-below scalar potential, CP conservation, vacuum at origin of field space)
  - Unsatisfactory unless condensed matter example in $\epsilon \to 1$ limit (universality class ?)
$d = 4 - \epsilon$ example with two real scalars and two Weyl fermions

$$V = \frac{1}{24} \lambda_1 \phi_1^4 + \frac{1}{24} \lambda_2 \phi_2^4 + \frac{1}{4} \lambda_3 \phi_1^2 \phi_2^2 + \frac{1}{6} \lambda_4 \phi_1^3 \phi_2 + \frac{1}{6} \lambda_5 \phi_1 \phi_2^3
+ \left( \frac{1}{2} y_1 \phi_1 \psi_1 \psi_1 + \frac{1}{2} y_2 \phi_2 \psi_1 \psi_1 + \frac{1}{2} y_3 \phi_1 \psi_2 \psi_2 + \frac{1}{2} y_4 \phi_2 \psi_2 \psi_2 + \text{h.c.} \right)$$

$$Q = \begin{pmatrix} 0 & q_1 \\ -q_1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} ip_1 & p_3 + ip_4 \\ -p_3 + ip_4 & ip_2 \end{pmatrix}$$

$$\lambda_1 = \frac{8(7087 + 357\sqrt{52953})}{102885} \pi^2 \epsilon$$
$$\lambda_2 = \frac{64(6346 + 9\sqrt{52953})}{102885} \pi^2 \epsilon$$
$$\lambda_3 = -\frac{272(\sqrt{52953} - 57)}{102885} \pi^2 \epsilon$$
$$\lambda_4 = \frac{32\sqrt{\frac{17}{19}} (757 - 3\sqrt{52953})}{5415} \pi^2 \epsilon$$
$$\lambda_5 = \frac{272\sqrt{\frac{17}{19}} (757 - 3\sqrt{52953})}{5415} \pi^2 \epsilon$$
$$y_3 = -y_1$$
$$q_1 = \sqrt{\frac{17}{19}} (757 - 3\sqrt{52953}) \frac{1}{108300} \epsilon^3$$
$$p_4 = \text{undetermined}$$
Features and future work

Physics of non-conformal scale-invariant QFTs

- Less constrained than CFTs
- \( \beta \)-functions \( \sim \) Anomalous dimensions
- Rare RG flows (recurrent behaviors)
  - RG flows \( \neq \) Gradient flows
  - Strongest version of a-theorem violated
- Phenomenological applications
  - Cyclic unparticle physics \( \text{JFF, Grinstein, Stergiou (2011)} \)

Future work

- Generic \( d = 4 \) examples (with \( \beta \)-functions at higher order)?