The Abdus Salam
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Workshop on GNSS Data Application to Low Latitude Ionospheric Research

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Calculation of GPS PNT Solution

John F. Raquet
Air Force Institute of Technology
USA

Calculation of GPS PNT Solution

Dr. John F. Raquet
Air Force Institute of Technology
Advanced Navigation Technology (ANT) Center

Overview

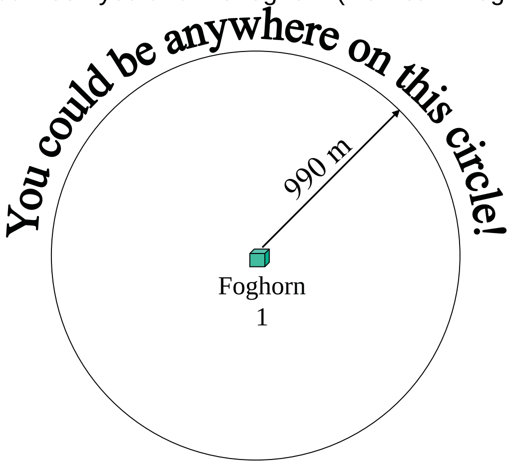
1. Positioning Using Time-of-Arrival
2. GPS Receiver Measurements
3. Coordinate Frames
4. Calculation of Satellite Position
5. GPS Navigation Solution
6. Dilution of Precision

Ranging Using Time-Of-Arrival

- Time-of-arrival (TOA) is one method that can be used to perform positioning
- Basic concept
  - You must know
    - When a signal was transmitted
    - How fast the signal travels
    - Time that the signal was received
  - Then you can determine how far away you are from the signal emitter
- Foghorn example
  - Assume there is a foghorn that goes off at exactly 12:00:00 noon every day
  - You know that the velocity of sound around the foghorn is 330 m/sec
  - You have a device that measures the time when the foghorn blast is received, and it says it heard a foghorn blast at 12:00:03
  - What is the distance between the foghorn and the foghorn "receiver"?
  - Now that you know how far you are from the foghorn, the question is, "Where are you?"

Two-Dimensional Positioning Using Single Range Measurement

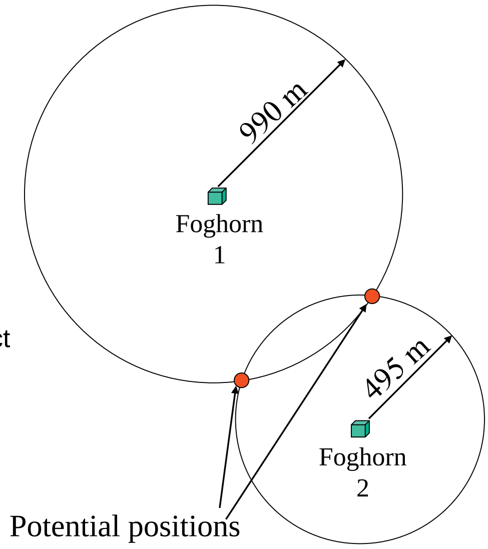
- Range between you and the foghorn (we'll call it foghorn #1) is 990m



- Unable to determine exact position in this case

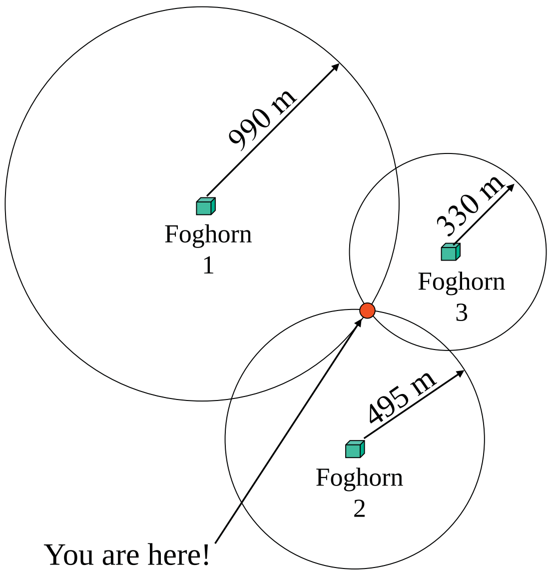
Two-Dimensional Ranging Using Two Measurements

- Now, you take a measurement from foghorn #2 at 12:00:01.5 (for a range of 495 m)
- Yields two potential solutions
  - How would you determine the correct solution?



Resolving Position Ambiguity Using Three Measurements

- You get a third measurement from foghorn #3 at 12:00:01 (Range = 330 m)
  - Now there's a unique solution



Receiver Clock Errors

- The foghorn example assumed that the foghorn "receiver" had a perfectly synchronized clock, so the measurements were perfect
- What happens if there is an unknown receiver clock error?
- Effect on range measurement
  - Without clock error

$$R = v_{sound}\Delta t$$

R = range

v_{sound} = velocity of sound

Δt = transmit/receive time difference

  - With clock error δt

$$R' = v_{sound}\left(\Delta t + \delta t\right)$$

where

R' = range with error (pseudo-range)

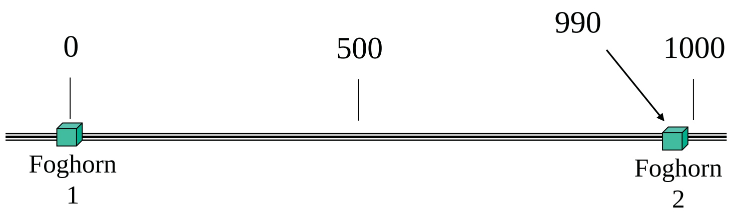
Receiver Clock Errors One-Dimensional Example (1/3)

- Now, we'll look at the foghorn example, except in only one dimension
  - The foghorn(s) and receiver are constrained to be along a line
  - We want to determine the position of the receiver on that line

- If the receiver measured a signal at 12:00:10, where is it on the line?
- Now, assume an unknown clock bias δt in the clock used by the foghorn receiver
- Your foghorn receiver measures a foghorn blast at 12:00:10
- What can you say about where you are?

Receiver Clock Errors
One-Dimensional Example (2/3)

- Clearly, more information is needed
- Assume that there is a second foghorn located 990 m away from the first



- You receive a signal from the second foghorn at 12:00:09
- What can you tell about where you are at this point?

Receiver Clock Errors
One-Dimensional Example (3/3)

- Here are the measurements we have:

$$\text{Pseudorange 1} = 330 \times 10 = 3300 = R_1'$$

$$\text{Pseudorange 2} = 330 \times 9 = 2970 = R_2'$$

- From the pseudorange equation:

$$R_1' = v_{sound}\left(\Delta t_1 + \delta t\right) = \quad x \quad + v_{sound}\delta t = 3300$$

$$R_2' = v_{sound}\left(\Delta t_2 + \delta t\right) = 990 - x + v_{sound}\delta t = 2970$$

- Rearranging terms we get

$$x + v_{sound}\delta t = 3300$$

$$x - v_{sound}\delta t = -1980$$

- We can then solve for the two unknowns

$$\delta t = 8 \text{ seconds}$$

$$x = 660 \text{ m}$$

← Does this work?

Receiver Clock Errors
Extending to Three Dimensions

- In the single-dimensional case
  - We needed two measurements to solve for the two unknowns, x and δt.
  - The quantities x and (990 - x) were the "distances" between the position of the receiver and the two foghorns.
- In three-dimensional case
  - We need four measurements to solve for the four unknowns, x, y, z, and δt.
  - The distances between receiver and satellite are not linear equations (as was case in single-dimensional case).
  - The four equations need to be solved simultaneously, for pseudorange measurements $R_1'\ldots R_4'$ and transmitter positions $(x_1,y_1,z_1)\ldots(x_4,y_4,z_4)$:

$$R_1' = \sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2} + c\delta t$$
$$R_2' = \sqrt{(x-x_2)^2+(y-y_2)^2+(z-z_2)^2} + c\delta t$$
$$R_3' = \sqrt{(x-x_3)^2+(y-y_3)^2+(z-z_3)^2} + c\delta t$$
$$R_4' = \sqrt{(x-x_4)^2+(y-y_4)^2+(z-z_4)^2} + c\delta t$$

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GPS Measurements (Overview)

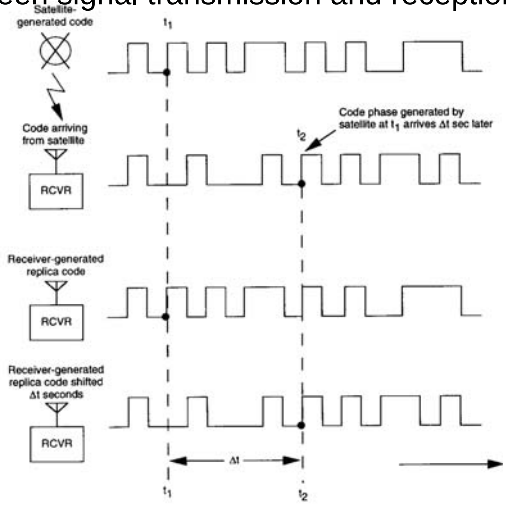
- Each separate tracking loop typically can give 4 different measurement outputs
  - Pseudorange measurement
  - Carrier-phase measurement (sometimes called integrated Doppler)
  - Doppler measurement
  - Carrier-to-noise density C/N_0
- Actual output varies depending upon receiver
  - Ashtech Z-surveyor (or Z-12) gives them all
  - RCVR-3A gives just C/N_0
- Note: We're talking here about *raw measurements*
  - Almost all receivers generate navigation processor outputs (position, velocity, heading, etc.)

Measurement Rates and Timing

- Most receivers take measurements on all channels/ tracking loops simultaneously
  - Measurements time-tagged with the receiver clock (receiver time)
  - The time at which a set of measurements is made is called a data epoch.
- The data rate varies depending upon receiver/ application. Typical data rates:
  - Static surveying: One measurement every 30 seconds (120 measurements per hour)
  - Typical air, land, and marine navigation: 0.5-2 measurement per second (most common)
  - Specialized high-dynamic applications: Up to 50 measurements per second (recent development)

GPS Pseudorange Measurement

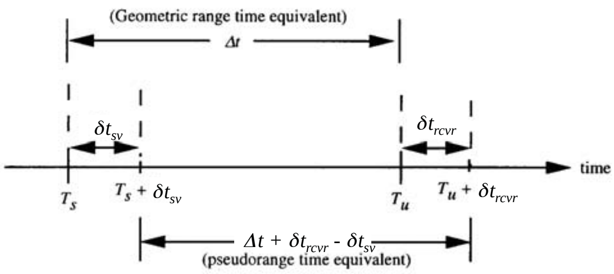
- Pseudorange is a measure of the difference in time between signal transmission and reception



Kaplan (ed.), *Understanding GPS: Principles and Applications*, Artech House, 1996

Effect of Clock Errors on Pseudorange

- Since pseudorange is based on time difference, any clock errors will fold directly into pseudorange



- Small clock errors can result in large pseudorange errors (since clock errors are multiplied by speed of light)
- Satellite clock errors (δt_{sv}) are very small
  - Satellites have atomic time standards
  - Satellite clock corrections transmitted in navigation message
- Receiver clock (δt_{rcvr}) is dominant error

Kaplan (ed.), *Understanding GPS: Principles and Applications*, Artech House, 1996

Doppler Shift

- For electromagnetic waves (which travel at the speed of light), the received frequency f_R is approximated using the standard Doppler equation

$$f_R = f_T\left(1 - \frac{(\boldsymbol{v}_r \cdot \boldsymbol{a})}{c}\right)$$

f_R = received frequency (Hz)

f_T = transmitted frequency (Hz)

$\boldsymbol{v}_r$ = satellite-to-user relative velocity vector (m/s)

$\boldsymbol{a}$ = unit vector pointing along line-of-sight from user to SV

c = speed of light (m/s)

  – Note that $\boldsymbol{v}_r$ is the (vector) velocity difference

$$\boldsymbol{v}_r = \boldsymbol{v} - \dot{\boldsymbol{u}}$$

$\boldsymbol{v}$ = velocity vector for satellite (m/s)

$\dot{\boldsymbol{u}}$ = velocity vector for user (m/s)

- The Doppler shift Δf is then

$$\Delta f = f_R - f_T \quad \text{(Hz)}$$

Doppler Measurement

- The GPS receiver locks onto the carrier of the GPS signal and measures the received signal frequency
  – Relationship between true and measured received signal frequency: $f_{R_{meas}}$

$$f_R = f_{R_{meas}}(1 + \delta\dot{t}_{rcvr})$$

f_R = true received signal frequency (Hz)

$f_{R_{meas}}$ = measured received signal frequency (Hz)

$\delta\dot{t}_{rcvr}$ = receiver clock drift rate (sec/sec)

  – Doppler measurement formed by differencing the measured received frequency and the transmit frequency:

$$\Delta f_{meas} = f_{R_{meas}} - f_T$$

  – Note: transmit frequency is calculated using information about SV clock drift rate given in navigation message

Doppler Measurement Sign Convention

- Sign convention based on Doppler definition
  - A satellite moving away from the receiver (neglecting clock errors) will have a *negative* Doppler shift

$$f_{R_{meas}} < f_T$$

$$\Delta f_{meas} = f_{R_{meas}} - f_T < 0$$

  - Sign convention used for NovAtel (and possibly other) receivers
- Sign convention based on relationship between Doppler and pseudorange
  - Doppler is essentially a measurement of the rate of change of the pseudorange
  - A satellite moving away from the receiver (neglecting clock errors) will have a *positive* Doppler measurement value
  - More common sign convention for GPS receivers (Ashtech, Trimble, and others)
- Carrier-phase measurement follows same convention as Doppler measurement (normally)

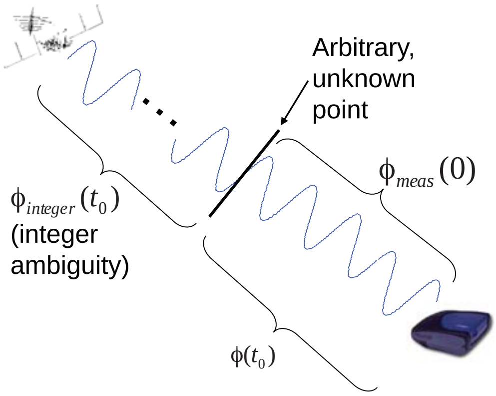
Carrier-Phase (Integrated Doppler) Measurement

- The carrier-phase measurement $\phi_{meas}(t)$ is calculated by integrating the Doppler measurements

$$\text{range}(t) = \underbrace{\int_{t_o}^{t} \Delta f_{meas}(t)dt + \phi(t_0)}_{\substack{\phi_{meas}(t) \\ \text{(can be measured by receiver)}}} + \phi_{integer}(t_0) + \text{clock error} + \text{other errors}$$

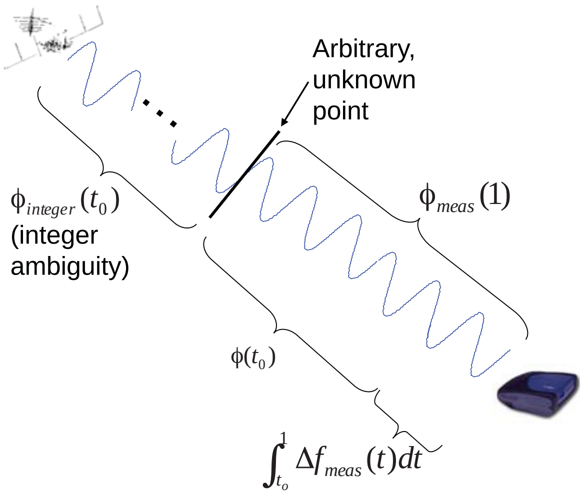
- The integer portion of the initial carrier-phase at the start of the integration ($\phi_{integer}(t_0)$) is known as the "carrier-phase integer ambiguity"
  - Because of this ambiguity, the carrier-phase measurement is not an absolute measurement of position
  - Advanced processing techniques can be used to resolve these carrier-phase ambiguites (carrier-phase ambiguity resolution)
- Alternative way of thinking: carrier-phase measurement is the "beat frequency" between the incoming carrier signal and receiver generated carrier.

Phase Tracking Example
At Start of Phase Lock (Time = 0 seconds)



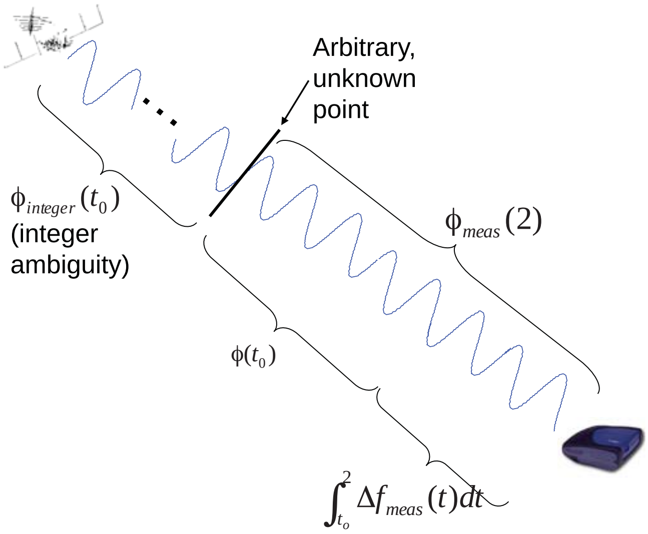
Ignoring clock and other errors

Phase Tracking Example
After Movement (for 1 Second)



Ignoring clock and other errors

Phase Tracking Example After Movement (for 2 Seconds)



Ignoring clock and other errors

Comparison Between Pseudorange and Carrier-Phase Measurements

	Pseudorange	Carrier-Phase
Type of measurement	Range (absolute)	Range (ambiguous)
Measurement precision	~1 m	~0.01 m
Robustness	More robust	Less robust (cycle slips possible)

Necessary for high precision GPS

Carrier-to-Noise Density (C/N_0)

- The carrier-to-noise density is a measure of signal strength
  - The higher the C/N_0, the stronger the signal (and the better the measurements)
  - Units are dB-Hz
  - General rules-of-thumb:
    - $C/N_0 > 40$: Very strong signal
    - $32 < C/N_0 < 40$: Marginal signal
    - $C/N_0 < 32$: Probably losing lock
- C/N_0 tends to be receiver-dependent
  - Can be calculated many different ways
  - Absolute comparisons between receivers not very meaningful
  - Relative comparisons between measurements in a single receiver are very meaningful

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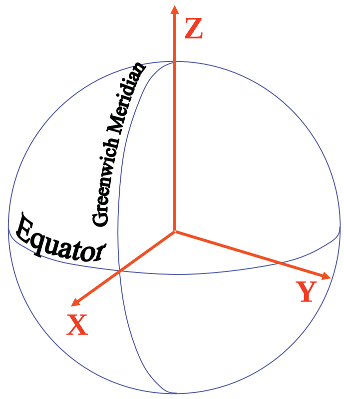
Coordinate Frames

- Giving a set of three coordinates is not sufficient for specifying a position
- Examples:
  - [-1485881.48699, -5152018.35300, 3444641.84728]
  - [-1.85158430, 0.57408361, 1255.323]
  - [-106.08796571, 32.89256771, 1255.323]
- The coordinate frame must also be specified
  - Choice of a coordinate frame is dependent upon the application
  - Most applications can use any defined coordinate frame, but usually one will be more straightforward than others
- Primary coordinate frames used for GPS
  - Earth-Centered Earth-Fixed (ECEF)
  - Geodetic (Longitude - Latitude - Altitude)

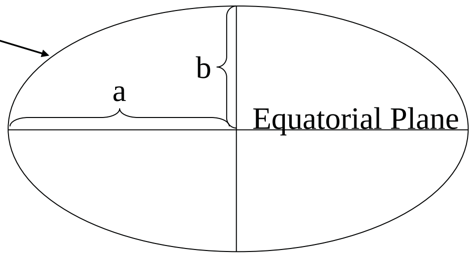
Earth-Centered Earth-Fixed (ECEF) Coordinate Frame

- ECEF frame is
  - Cartesian (orthogonal) reference frame
  - It is a rotating reference frame (w.r.t. inertial space), rotating at earth rate
  - Advantages
    - Easy to calculate distances and vectors between two points
    - Usually computationally efficient
  - Disadvantages
    - Not geographically intuitive



Geodetic Coordinate Frame (WGS-84 Ellipsoid)

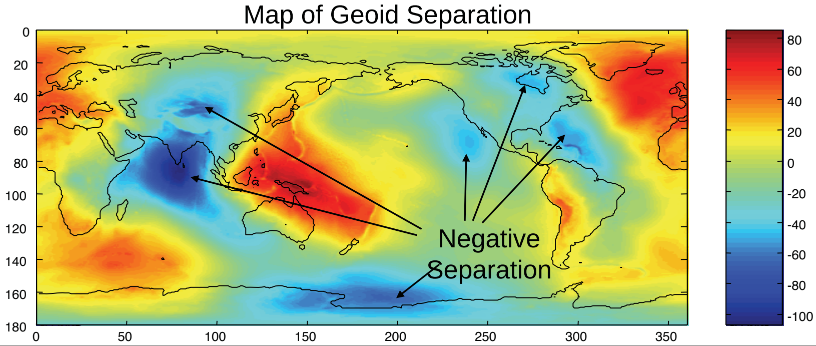
- There are different ways to describe height (or altitude)
  - Distance above the surface of the earth
  - Definition based on gravity
    - Geoid: surface of constant gravitational potential
    - Geoid is a function of topography, earth density variations, and earth rotation rate
    - Geodesy: study of the geoid
  - Definition based upon geometry
    - The geoid can be fit to an ellipsoid (a rotated ellipse)
    - One particular ellipsoidal fit of the geoid is called the WGS-84 ellipsoid



Geoid Separation

- There is a separation between the Geoid (based on gravity) and the WGS-84 ellipsoid (based on a mathematical model)
  - Varies with user position
  - Only critical if interfacing with geoid-based reference systems (such as MSL altitude)

Map of Geoid Separation

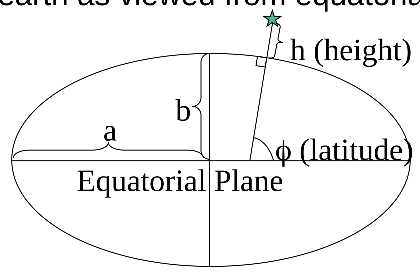


Key WGS-84 Parameters

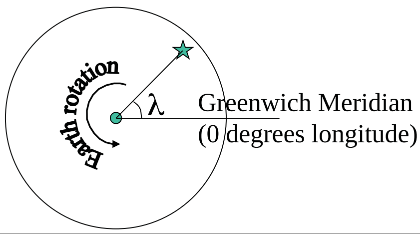
Name	Symbol	Quantity
Semi-major axis	a	6378.137 km
Semi-minor axis	b	6356.7523142 km
Eccentricity	e	0.0818191908426
Earth rotation rate	Ω_ε	7.2921151467E-5 rad/s
Speed of light	c	299792458 m/s
Gravitational parameter	μ	3.986005E14 m^3/s^2
Flattening	f	0.00335281066475

Definition of Geodetic Coordinates (Longitude, Latitude, and Altitude)

- Definition of ellipsoidal height h and latitude ϕ (cross section of earth as viewed from equatorial plane)



- Definition of longitude λ (view from above the north pole)



Geodetic Coordinate Units

- Normally, ellipsoidal altitude (h) is expressed in meters (m).
- Latitude (ϕ) and Longitude (λ) can be expressed in different units
  - Radians
    - Least ambiguous, useful for programming
    - Not as easily recognized geographically
  - Decimal degrees
    - To convert from radians to decimal degrees, multiply by $180/\pi$
    - Not very common
  - Degrees and decimal minutes
    - Integer number of degrees
    - Decimal number of minutes (1 minute = 1/60 degree)
    - Example: 46.596 decimal degrees is 46° 35.76' (reads 46 degrees, 35.76 minutes)
  - Degrees, minutes, and seconds
    - Integer number of degrees
    - Integer number of minutes
    - Decimal number of seconds (1 second = 1/60 minute)
    - Example: 46.596 decimal degrees is 46° 35' 45.6''

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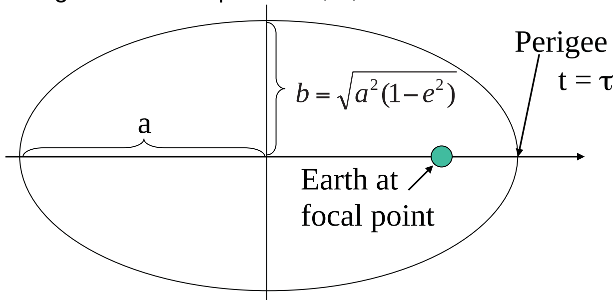
Determining Satellite Position

- In order to determine user position, one must calculate satellite position
- Satellites orbits are primarily based on the Earth's gravity field
- Other forces acting on satellite
  - Gravity from sun, moon, and other planets
  - Atmospheric drag
  - Solar pressure
  - Torques due to Earth's magnetic field
- Orbits are highly predictable
  - Prediction accuracy degrades with time
- Orbits can be described by using a set of "orbital parameters"
  - Six classic orbital parameters
  - Additional parameters to handle orbit variations over time

Classical Orbital Elements (1/2)

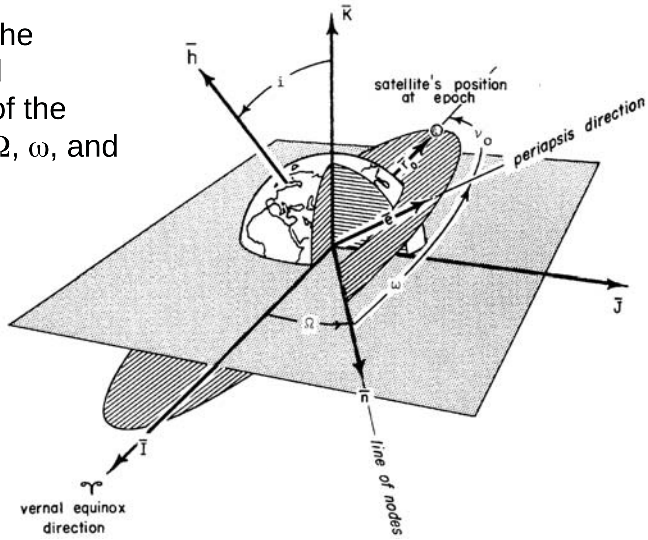
- Classical orbital elements describe
  - Shape of the satellite's orbit (and where the satellite would be on that shape)
  - Position of the orbit relative to inertial (or Earth-fixed) space
- Describing the orbit shape with a, e, and τ



- If given a specific time, you can calculate the position of the satellite on this ellipse

Classical Orbital Elements (2/2)

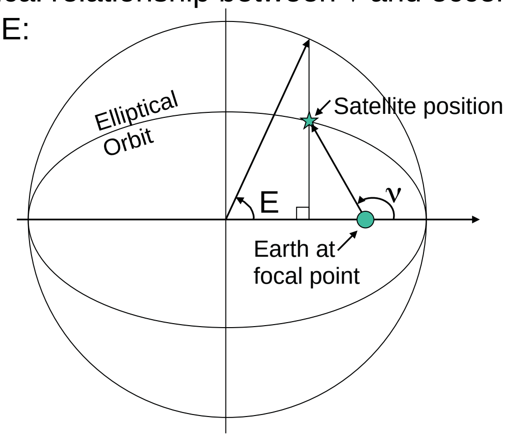
- Describing the position and orientation of the orbit using Ω, ω, and i



Bate, Mueller, and White, "Fundamentals of Astrodynamics," Dover Publications, 1971

Relationship Between True, Eccentric, and Mean Anomaly (1/2)

- True anomaly ν used to directly calculate satellite position on ellipse
- Geometrical relationship between ν and eccentric anomaly E:



Relationship Between True, Eccentric, and Mean Anomaly (2/2)

- Mean anomaly M varies linearly with time (unlike E or ν), so it can be easily calculated

$$M(t) = M_0 + n(t - t_0)$$
$$M_0 = M(t_0)$$
$$n = \sqrt{\frac{\mu}{a^3}} = \text{mean motion}$$

- Eccentric anomaly and mean anomaly related through Kepler's equation

$$M = E - e\sin E$$

- Finally, true anomaly calculated from arctangent* function, using

$$\sin\nu = \frac{\sqrt{1-e^2}\sin E}{1 - e\cos E} \qquad \cos\nu = \frac{\cos E - e}{1 - e\cos E}$$

*Be sure to use the 4-quadrant arctangent function (atan2 in MATLAB).

Where Do We Get the Ephemeris Data? Legacy L1 and L2 Signal Breakdown

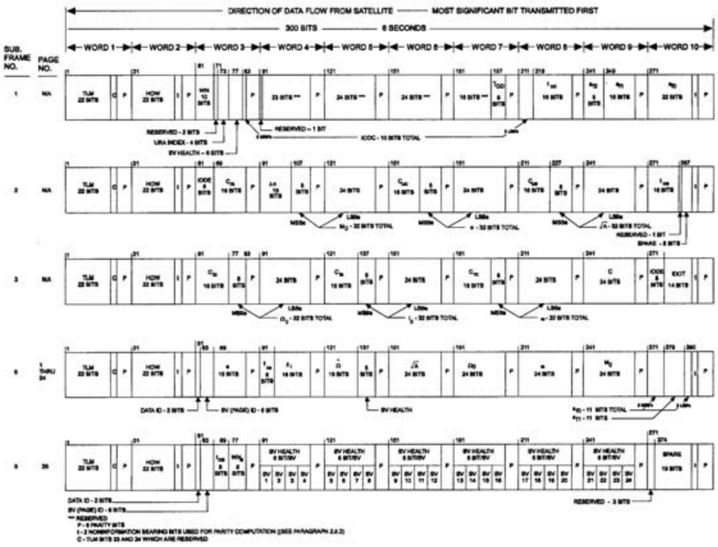
- Note: 50 bps navigation message modulated on all of the codes
- L1 signal
  - P(Y)-code
  - C/A-code modulated on carrier that is 90° out of phase from P-code carrier

$$s_{L1}(t) = \overbrace{A_{P_{L1}} Y(t)N(t)\cos(\omega_1 t)}^{\text{P(Y)-Code}} + \overbrace{A_{C/A} CA(t)N(t)\sin(\omega_1 t)}^{\text{C/A-Code}}$$
$$N(t) = \text{50 bps navigation message}$$
$$A_{P_{L1}} = \text{Amplitude of L1 P-code signal} \approx \text{-163 dBW}$$
$$A_{C/A} = \text{Amplitude of C/A-code signal} \approx \text{-160 dBW}$$
$$\omega_1 = 2\pi f_{L1}$$

- L2 signal
  - P-code only

$$s_{L2}(t) = \overbrace{A_{P_{L2}} Y(t)N(t)\cos(\omega_2 t)}^{\text{P(Y)-Code}}$$
$$A_{P_{L2}} = \text{Amplitude of L2 P-code signal} \approx \text{-166 dBW}$$
$$\omega_2 = 2\pi f_{L2}$$

Data Format of Subframes 1, 2, 3, and 5



Obtained from SPS Signal Spec (http://www.spacecom.af.mil/usspace/gps_support/documents/SPSMAIN.pdf)

GPS Ephemeris Data (From Navigation Message)

- For defining orbit shape and timing

t_{0_e} = Reference time of ephemeris (sec)

$\sqrt{a}$ = Square root of semi-major axis ($m^{1/2}$)

e = Eccentricity

M_0 = Mean anomaly at time t_{0_e} (rad)

- For defining orientation/position of orbit

i_0 = inclination at time t_{0_e} (rad)

Ω_0 = Longitude of ascending node at t_{0_e} (rad)

ω = Argument of perigee at t_{0_e} (rad)

- Correction Terms

$\dot{i}$ = Rate of change of inclination (rad/sec)

$\dot{\Omega}$ = Rate of change of Ω (rad/sec)

Δn = Mean motion correction (rad/sec)

C_{uc}, C_{us} = Argument of latitude correction coefficients

C_{rc}, C_{rs} = Orbital radius correction coefficients

C_{ic}, C_{is} = Inclination correction coefficients

Sample Ephemeris Values
(PRN 10 - 20 Jan 1999)

```
       prn: 10
      week: 993
       t0e: 266400
    sqrt_a: 5.1537e+003
         e: 0.0032
        M0: -0.1952
        i0: 0.9694
    Omega0: -0.7958
     omega: -0.2041
      idot: -3.0894e-010
 Omegadot: -8.4571e-009
   delta_n: 4.6345e-009
```

```
      Cuc: -3.9022e-006
      Cus: 2.3618e-006
      Crc: 339.4063
      Crs: -73.9375
      Cic: 1.8626e-009
      Cis: -3.9116e-008
      toc: 266400
      af0: 3.1394e-005
      af1: 5.6843e-013
      af2: 0
      tgd: -1.8626e-009
    valid: 1
```

SV Clock Correction Terms (toc, af0, af1, af2)

Calculating Satellite Position

- Set of equations for calculating SV position from ephemeris is given in ICD-GPS-200C (Table 20-IV)
  - IS-GPS-200D can be found at
    www.navcen.uscg.gov/pdf/IS-GPS-200D.pdf
  - Comments
    - Make sure that the correct quadrant is determined when calculating true anomaly (use “atan2” function or equivalent)
    - Output x_k, y_k, z_k are the ECEF coordinates of the SV antenna phase center at time t (in the ECEF coordinate frame at time t)

IS-GPS-200D: Solving for Satellite Position (1 of 4)

Variables obtained from navigation message highlighted with box: □

Table 20-IV. Elements of Coordinate Systems (sheet 1 of 2)	
$\mu = 3.986005 \times 10^{14}$ meters^3/sec^2	WGS 84 value of the earth's gravitational constant for GPS user
$\dot{\Omega}_e = 7.2921151467 \times 10^{-5}$ rad/sec	WGS 84 value of the earth's rotation rate
$A = \left(\boxed{\sqrt{A}}\right)^2$	Semi-major axis
$n_0 = \sqrt{\dfrac{\mu}{A^3}}$	Computed mean motion (rad/sec)
$t_k = t - \boxed{t_{oe}}$*	Time from ephemeris reference epoch

* t is GPS system time at time of transmission, i.e., GPS time corrected for transit time (range/speed of light). Furthermore, t_k shall be the actual total time difference between the time t and the epoch time t_{oe}, and must account for beginning or end of week crossovers. That is, if t_k is greater than 302,400 seconds, subtract 604,800 seconds from t_k. If t_k is less than -302,400 seconds, add 604,800 seconds to t_k.

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IS-GPS-200D: Solving for Satellite Position (2 of 4)

Variables obtained from navigation message highlighted with box: □

Table 20-IV. Elements of Coordinate Systems (sheet 1 of 2) (continued)		
$n = n_0 + \boxed{\Delta n}$		Corrected mean motion
$M_k = \boxed{M_0} + n t_k$		Mean anomaly
$M_k = E_k - \boxed{e} \sin E_k$	← See upcoming slide for how to solve for E_k	Kepler's Equation for Eccentric Anomaly (may be solved by iteration) (radians)
$\nu_k = \tan^{-1}\left\{\dfrac{\sin \nu_k}{\cos \nu_k}\right\}$	← For informational purposes only—not needed in calculations	True Anomaly
$= \tan^{-1}\left\{\dfrac{\sqrt{1-\boxed{e}^2}\,\sin E_k / (1-\boxed{e}\cos E_k)}{(\cos E_k - \boxed{e}) / (1-\boxed{e}\cos E_k)}\right\}$	← Use four-quadrant arctan function ("atan2" in MATLAB)	

IS-GPS-200D, 7 Dec 2004, p 97

IS-GPS-200D: Solving for Satellite Position (3 of 4)

Variables obtained from navigation message highlighted with box: □

Table 20-IV. Elements of Coordinate Systems (sheet 2 of 2)

$E_k = \cos^{-1}\left\{\frac{e + \cos \nu_k}{1 + e \cos \nu_k}\right\}$	← For informational purposes only—not needed in calculations	Eccentric Anomaly
$\Phi_k = \nu_k + \boxed{\omega}$		Argument of Latitude
$\delta u_k = \boxed{c_{us}} \sin 2\Phi_k + \boxed{c_{uc}} \cos 2\Phi_k$	Argument of Latitude Correction	Second Harmonic Perturbations
$\delta r_k = \boxed{c_{rs}} \sin 2\Phi_k + \boxed{c_{rc}} \cos 2\Phi_k$	Radius Correction	
$\delta i_k = \boxed{c_{is}} \sin 2\Phi_k + \boxed{c_{ic}} \cos 2\Phi_k$	Inclination Correction	
$u_k = \Phi_k + \boxed{\delta u_k}$		Corrected Argument of Latitude
$r_k = A(1 - \boxed{e} \cos E_k) + \delta r_k$		Corrected Radius

IS-GPS-200D, 7 Dec 2004, p 98

IS-GPS-200D: Solving for Satellite Position (4 of 4)

Variables obtained from navigation message highlighted with box: □

Table 20-IV. Elements of Coordinate Systems (sheet 2 of 2) (continued)

$i_k = \boxed{i_0} + \delta i_k + (\boxed{IDOT})\, t_k$	Corrected Inclination
$x_k' = r_k \cos u_k$ $y_k' = r_k \sin u_k$	Positions in orbital plane.
$\Omega_k = \boxed{\Omega_0} + (\boxed{\dot{\Omega}} - \dot{\Omega}_e)\, t_k - \dot{\Omega}_e \boxed{t_{oe}}$	Corrected longitude of ascending node.
$x_k = x_k' \cos \Omega_k - y_k' \cos i_k \sin \Omega_k$ $y_k = x_k' \sin \Omega_k + y_k' \cos i_k \cos \Omega_k$ $z_k = y_k' \sin i_k$	Earth-fixed coordinates.

Solution to Kepler's Equation

- Kepler's equation, though simple in form, has no known closed-form solution
  - All solutions are iterative (or approximate)
- Newton's method

$$E_0 = M$$

$$E_{j+1} = E_j + \frac{M - (E_j - e\sin E_j)}{1 - e\cos E_j}$$

- Method used in RCVR-3A software specification

$$E_0 = M + e\sin M$$

$$E_{j+1} = \frac{e(\sin E_j - E_j \cos E_j) + M}{1 - e\cos E_j}$$

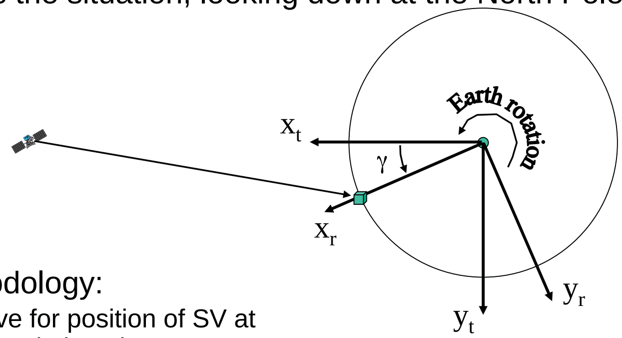
  - RCVR-3A performs two iterations (i.e., stops calculating at E_2)
  - Don't confuse these subscripts with subscripts in ephemeris equations!

Accounting for Signal Travel Time (1 of 3)

- Signal arrives at receiver ***after*** it is transmitted (due to signal travel time)
  - Transmit time: Time the signal was transmitted
  - Receive time: Time the signal was received
- Satellite position should be calculated based upon transmit time
  - When measuring a signal, we don't really care what happened after that signal was transmitted
  - Transmit time should be GPS system time (or as close to it as possible)
  - Very good approximate value of transmit time obtained by subtracting pseudorange (expressed in seconds) from the receive time as indicated by the receiver clock
    - Why?
- What other considerations do we need to make for signal travel time?

Accounting for Signal Travel Time (2 of 3)

- Here's the situation, looking down at the North Pole



- Methodology:
  - Solve for position of SV at transmit time, in ECEF coordinates at transmit time (x_t, y_t, and z_t) using ICD-GPS-200 equations
  - Rotate into ECEF reference frame at the time of reception:

$$\begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix}$$

$$\gamma = \dot{\Omega}_e t_{prop}$$

t_{prop} = Signal propagation time

Accounting for Signal Travel Time (3 of 3)

- Neglecting atmospheric delay, the signal propagation time is calculated by

$$t_{prop} = \frac{\text{geometric range to satellite}}{\text{speed of light}} = \frac{|\boldsymbol{p}_{sv} - \boldsymbol{p}_{rcvr}|}{c}$$

$\boldsymbol{p}_{sv}$ = satellite ECEF position vector

$\boldsymbol{p}_{rcvr}$ = receiver ECEF position vector

- Note that the satellite position is needed to calculate t_{prop} (and vice-versa)
  - Satellite position in ECEF coordinates at transmit time is sufficiently accurate (x_t, y_t, and z_t)
  - Note that receiver position must be known
    - Can be approximate

Overview

Pseudorange Equation

- The pseudorange is the sum of the true range plus the receiver clock error
  - We're assuming (for now) that the receiver clock error is the only remaining error
    - SV clock error has been corrected for
    - All other errors are deemed negligible (or have been corrected)

$$\begin{aligned}\rho_j &= \sqrt{(x_j - x_u)^2 + (y_j - y_u)^2 + (z_j - z_u)^2} + c\delta t_u \\ &= f(x_u, y_u, z_u, \delta t_u)\end{aligned}$$

$$\begin{aligned}\rho_j &= \text{pseudorange measurement from satellite } j \text{ (m)} \\ x_j, y_j, z_j &= \text{ECEF position of satellite } j \text{ (m)} \\ x_u, y_u, z_u &= \text{ECEF position of user (m)} \\ \delta t_u &= \text{receiver clock error (sec)}\end{aligned}$$

- For now, only use one type of pseudorange (L1 C/A, L1 P, or L2 P)

Statement of the Problem

- At a given measurement epoch, the GPS receiver generates *n* pseudorange measurements (from *n* different satellites)

$$\begin{aligned}
\rho_1 &= \sqrt{(x_1 - x_u)^2 + (y_1 - y_u)^2 + (z_1 - z_u)^2} + c\delta t_u \\
\rho_2 &= \sqrt{(x_2 - x_u)^2 + (y_2 - y_u)^2 + (z_2 - z_u)^2} + c\delta t_u \\
\rho_3 &= \sqrt{(x_3 - x_u)^2 + (y_3 - y_u)^2 + (z_3 - z_u)^2} + c\delta t_u \\
&\vdots \\
\rho_n &= \sqrt{(x_n - x_u)^2 + (y_n - y_u)^2 + (z_n - z_u)^2} + c\delta t_u
\end{aligned}$$

- Goal: Determine user position and clock error, expressed in state-vector form as

$$\mathbf{x} = \begin{bmatrix} x_u \\ y_u \\ z_u \\ c\delta t_u \end{bmatrix}$$

Solving the Pseudorange Equations

- The *n* pseudorange equations are non-linear (so no easy solution)
- Ways to solve
  - Closed form solutions
    - Complicated
    - May not give as much insight
  - Iterative techniques based on linearization
    - Often using least-squares estimation
    - Arguably the simplest approach
    - Approach covered in this course
  - Kalman filtering
    - Similar to least-squares approach, except with additional ability to handle measurements over a period of time
    - Will discuss briefly
- What is linearization?
  - Pick a nominal (or approximate) solution
  - Linearize about that point, resulting in a set of linear equations
  - Solve the linear equations
  - Will use Taylor series expansion for linearization

Taylor Series Expansion (1/2)

- Taylor series expansion (1 variable)

$$f(a+\Delta a) = f(a) + \Delta a \frac{df}{da} + \frac{(\Delta a)^2}{2!}\frac{d^2 f}{da^2} + \frac{(\Delta a)^3}{3!}\frac{d^3 f}{da^3} + \ldots$$

- This can be used to linearize about a certain value of the independent variable a.
  - Example: the function $f(t) = 2 + 3t - 6t^2$ is a non-linear function in t
  - Suppose we want to linearize about the point $\hat{t} = 2$
  - The complete Taylor series expression is

$$f(\hat{t}+\Delta t) = f(\hat{t}) + \Delta t \frac{df}{dt} + \frac{(\Delta t)^2}{2!}\frac{d^2 f}{dt^2}$$

$$= 2 + 3\hat{t} - 6\hat{t}^2 + \Delta t(3 - 12\hat{t}) + \frac{(\Delta t)^2}{2}(-12)$$

  - To linearize, we set $\hat{t} = 2$ and neglect higher order (non-linear) terms of Δt
    - Valid for perturbations (i.e., small values of Δt)

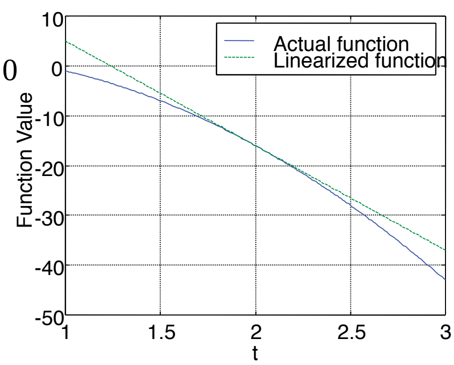
Taylor Series Expansion (2/2)

- (Continued example) Linearized form

$$f(\hat{t}+\Delta t)\Big|_{\hat{t}=2} = f(\hat{t}=2) + \Delta t \frac{df}{dt}\Big|_{\hat{t}=2} + \frac{(\Delta t)^2}{2!}\frac{d^2 f}{dt^2}\Big|_{\hat{t}=2} \rightarrow 0$$

$$= 2 + 3(2) - 6(2^2) + \Delta t(3 - 12(2))$$

$$= -16 - 21\Delta t$$



- First order Taylor series for function in two variables:

$$f(\hat{a}+\Delta a, \hat{b}+\Delta b) = f(\hat{a},\hat{b}) + \Delta a \frac{\partial f}{\partial a}\Big|_{\hat{a},\hat{b}} + \Delta b \frac{\partial f}{\partial b}\Big|_{\hat{a},\hat{b}} + \text{h.o.t.}$$

Linearization of Pseudorange Equations (1/5)

- First, define a nominal state (position and clock error) as

$$\hat{\mathbf{x}} = \begin{bmatrix} \hat{x}_u \\ \hat{y}_u \\ \hat{z}_u \\ c\delta\hat{t}_u \end{bmatrix} = \text{nominal (approximate) state}$$

- An approximate (or expected) pseudorange can then be calculated for satellite *j*

$$\begin{aligned} \hat{\rho}_j &= \sqrt{(x_j - \hat{x}_u)^2 + (y_j - \hat{y}_u)^2 + (z_j - \hat{z}_u)^2} + c\delta\hat{t}_u \\ &= f(\hat{x}_u, \hat{y}_u, \hat{z}_u, c\delta\hat{t}_u) \end{aligned}$$

  – This approximate (expected) pseudorange is the pseudorange that we would expect to have if our position and clock error were actually $\hat{x}_u, \hat{y}_u, \hat{z}_u$, and $c\delta\hat{t}_u$.

Linearization of Pseudorange Equations (2/5)

- Relationship between true and approximate position and time

$$\begin{aligned} x_u &= \hat{x}_u + \Delta x_u \\ y_u &= \hat{y}_u + \Delta y_u \\ z_u &= \hat{z}_u + \Delta z_u \\ c\delta t_u &= c\delta\hat{t}_u + \Delta c\delta t_u \end{aligned}$$

  – Vector form:

$$\mathbf{x}_u = \hat{\mathbf{x}}_u + \Delta\mathbf{x}_u$$

- Based on these relations, we can write

$$f(x_u, y_u, z_u, c\delta t_u) = f(\hat{x}_u + \Delta x_u, \hat{y}_u + \Delta y_u, \hat{z}_u + \Delta z_u, c\delta\hat{t}_u + \Delta c\delta t_u)$$

- To linearize, right-hand side of equation can be evaluated using a first order Taylor series expansion

Linearization of Pseudorange Equations (3/5)

- First order Taylor series expansion of pseudorange function:

$$f(\hat{x}_u+\Delta x_u,\hat{y}_u+\Delta y_u,\hat{z}_u+\Delta z_u,c\delta\hat{t}_u+\Delta c\delta t_u)=f(\hat{x}_u,\hat{y}_u,\hat{z}_u,c\delta\hat{t}_u)$$
$$+\frac{\partial f(\hat{x}_u,\hat{y}_u,\hat{z}_u,c\delta\hat{t}_u)}{\partial\hat{x}_u}\Delta x_u+\frac{\partial f(\hat{x}_u,\hat{y}_u,\hat{z}_u,c\delta\hat{t}_u)}{\partial\hat{y}_u}\Delta y_u$$
$$+\frac{\partial f(\hat{x}_u,\hat{y}_u,\hat{z}_u,c\delta\hat{t}_u)}{\partial\hat{z}_u}\Delta z_u+\frac{\partial f(\hat{x}_u,\hat{y}_u,\hat{z}_u,c\delta\hat{t}_u)}{\partial c\delta\hat{t}_u}\Delta c\delta t_u$$
$$+\text{h.o.t.}$$

- The partial derivatives are

$$\frac{\partial f(\hat{x}_u,\hat{y}_u,\hat{z}_u,\delta\hat{t}_u)}{\partial\hat{x}_u}=-\frac{x_j-\hat{x}_u}{\hat{r}_j}\qquad\frac{\partial f(\hat{x}_u,\hat{y}_u,\hat{z}_u,\delta\hat{t}_u)}{\partial\hat{y}_u}=-\frac{y_j-\hat{y}_u}{\hat{r}_j}$$
$$\frac{\partial f(\hat{x}_u,\hat{y}_u,\hat{z}_u,\delta\hat{t}_u)}{\partial\hat{z}_u}=-\frac{z_j-\hat{z}_u}{\hat{r}_j}\qquad\frac{\partial f(\hat{x}_u,\hat{y}_u,\hat{z}_u,\delta\hat{t}_u)}{\partial c\delta\hat{t}_u}=1$$

$$\hat{r}_j=\sqrt{(x_j-\hat{x}_u)^2+(y_j-\hat{y}_u)^2+(z_j-\hat{z}_u)^2}$$

Linearization of Pseudorange Equations (4/5)

- Using above results, linearized pseudorange equation is

$$\rho_j=\hat{\rho}_j-\frac{x_j-\hat{x}_u}{\hat{r}_j}\Delta x_u-\frac{y_j-\hat{y}_u}{\hat{r}_j}\Delta y_u-\frac{z_j-\hat{z}_u}{\hat{r}_j}\Delta z_u+\Delta c\delta t_u$$

- This can be simplified to $\Delta\rho_j=a_{xj}\Delta x_u+a_{yj}\Delta y_u+a_{zj}\Delta z_u-\Delta c\delta t_u$ where

$$\Delta\rho_j=\hat{\rho}_j-\rho_j$$
$$a_{xj}=\frac{x_j-\hat{x}_u}{\hat{r}_j},\qquad a_{yj}=\frac{y_j-\hat{y}_u}{\hat{r}_j},\qquad a_{zj}=\frac{z_j-\hat{z}_u}{\hat{r}_j}$$

Linearization of Pseudorange Equations (5/5)

- Original (nonlinear) equations for n measurements

$$
\begin{aligned}
\rho_1 &= \sqrt{(x_1-x_u)^2+(y_1-y_u)^2+(z_1-z_u)^2}+c\delta\, t_u \\
\rho_2 &= \sqrt{(x_2-x_u)^2+(y_1-y_u)^2+(z_1-z_u)^2}+c\delta\, t_u \\
\rho_3 &= \sqrt{(x_3-x_u)^2+(y_3-y_u)^2+(z_3-z_u)^2}+c\delta\, t_u \\
&\vdots \\
\rho_n &= \sqrt{(x_n-x_u)^2+(y_n-y_u)^2+(z_n-z_u)^2}+c\delta\, t_u
\end{aligned}
$$

- Linearized (error) equations for the same n measurements

$$
\begin{aligned}
\Delta\rho_1 &= a_{x1}\Delta x_u + a_{y1}\Delta y_u + a_{z1}\Delta z_u - \Delta c\delta t_u \\
\Delta\rho_2 &= a_{x2}\Delta x_u + a_{y2}\Delta y_u + a_{z2}\Delta z_u - \Delta c\delta t_u \\
\Delta\rho_3 &= a_{x3}\Delta x_u + a_{y3}\Delta y_u + a_{z3}\Delta z_u - \Delta c\delta t_u \\
&\vdots \\
\Delta\rho_n &= a_{xn}\Delta x_u + a_{yn}\Delta y_u + a_{zn}\Delta z_u - \Delta c\delta t_u
\end{aligned}
$$

Solving the Linearized Pseudorange Equations Using Least-Squares (1/2)

- We can express the set of pseudorange equations in matrix form

$$\Delta\boldsymbol{\rho} = \mathbf{H}\Delta\mathbf{x}$$

$$
\Delta\boldsymbol{\rho} = \begin{bmatrix} \Delta\rho_1 \\ \Delta\rho_2 \\ \Delta\rho_3 \\ \vdots \\ \Delta\rho_n \end{bmatrix}
\qquad
\mathbf{H} = \begin{bmatrix} a_{x1} & a_{y1} & a_{z1} & -1 \\ a_{x2} & a_{y2} & a_{z2} & -1 \\ a_{x3} & a_{y3} & a_{z3} & -1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{xn} & a_{yn} & a_{zn} & -1 \end{bmatrix}
\qquad
\Delta\mathbf{x} = \begin{bmatrix} \Delta x_u \\ \Delta y_u \\ \Delta z_u \\ \Delta c\delta t_u \end{bmatrix}
$$

- Three possible cases
  - $n < 4$: Underdetermined case
    - Cannot solve for $\Delta\mathbf{x}$
    - Is there still useable information?
  - $n = 4$: Uniquely determined case
    - One valid solution for $\Delta\mathbf{x}$ (generally)
    - Solved by calculating $\mathbf{H}^{-1}$ ($\Delta\mathbf{x} = \mathbf{H}^{-1}\Delta\boldsymbol{\rho}$)
  - $n > 4$: Overdetermined case
    - No solution that perfectly solves equation (generally)
    - Can use least-squares techniques (which pick solution that minimizes the square of the error)

Solving the Linearized Pseudorange Equations Using Least-Squares (2/2)

- Basic least-squares solution (no measurement weighting)

$$\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\boldsymbol{\rho}$$

  - Reasonable approach for single-point positioning in presence of SA
- Solution with measurement weighting (weighted least-squares)
  - Useful when
    - Measurements have different error statistics
    - Measurement errors are correlated
  - Measurement error covariance matrix $\mathbf{C}_\rho$
    - Diagonal terms are measurement error variances
    - Off-diagonal terms show cross-correlation between measurement errors

$$\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{C}_\rho^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{C}_\rho^{-1}\Delta\boldsymbol{\rho}$$

  - Note that this is identical to unweighted case if $\mathbf{C}_\rho = \mathbf{I}$ (identity matrix)

Measurement Residuals

- For overdetermined system, generally no valid solution for $\Delta\mathbf{x}$ that solves measurement equation, so

$$\Delta\boldsymbol{\rho} \neq \mathbf{H}\Delta\mathbf{x}$$

- Measurement residuals ($\mathbf{v}$)
  - Corrections that, when applied to measurements, would result in solution of above equation
  - Least-squares minimizes the sum of squares of these residuals

$$\mathbf{v} = \Delta\boldsymbol{\rho} - \mathbf{H}\Delta\mathbf{x}$$

$$\Delta\boldsymbol{\rho} = \mathbf{H}\Delta\mathbf{x} + \mathbf{v}$$

Iterating the Nominal State

- Linearized equations (and resulting $\mathbf{H}$ matrix) calculated using nominal state $\hat{\mathbf{x}}_u$
- Linearization valid when
  - Nominal state is close to true state
  - $\Delta\mathbf{x}$ is "small"
- If $\hat{\mathbf{x}}_u$ is not very accurate (i.e., $\Delta\mathbf{x}$ is large), iteration is required
  - For each iteration, a new value of $\hat{\mathbf{x}}_u$ is calculated based upon the old value and the corrections $\Delta\mathbf{x}$

$$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$$

  - This new value of $\hat{\mathbf{x}}_u$ is then used to recalculate the corrections $\Delta\mathbf{x}$ (which should be smaller this time)
- Solution must converge
  - For standard GPS positioning, not much of a problem (will generally converge with an initial guess at the center of the Earth)
  - For more non-linear situations (e.g., using pseudolites), this can be more of a problem

Correcting for Satellite Clock Error

- Single point positioning only estimates receiver clock error
  - Assumes all other errors are negligible
  - Requires correction of satellite clock error
- Clock correction (from IS-GPS-200D)

$$\rho_{corr} = \rho + c\Delta t_{sv}$$

$$\Delta t_{sv} = a_{f_0} + a_{f_1}(t - t_{0_c}) + a_{f_2}(t - t_{0_c})^2 + \Delta t_r$$

$$\Delta t_r = Fe\sqrt{a}\sin E_k$$

ρ_{corr} = pseudorange corrected for SV clock error

ρ = original (raw) pseudorange measurement

Δt_{sv} = SV clock correction

$a_{f_0}, a_{f_1}, a_{f_2}, t_{0_c}$ = SV clock correction parameters from nav message

Δt_r = relativity correction (since not circular orbit)

F = constant = $-4.442807633 \times 10^{-10}$ sec/(meter)$^{1/2}$

e = eccentricity from nav message

$\sqrt{a}$ = square root of semi-major axis from nav message

E_k = Eccentric anomaly (from SV position calculation)

Determining Signal Transmit Time (1/2)

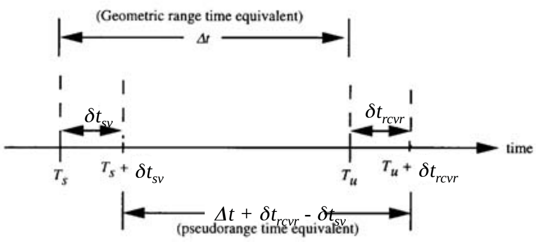
- For satellite position calculation, need true GPS transmit time of the signal (T_s)
  - Receiver provides time of reception according to the receiver clock ($T_u + \delta t_{rcvr}$)
  - From diagram below, if the pseudorange time equivalent is subtracted from the receive time, then the result is the true transmit time plus the satellite clock error

$$\underbrace{T_u + \delta t_{rcvr}}_{\text{receive time}} - \frac{PR}{c} = T_s + \delta t_{sv}$$

PR = pseudorange measurement (m)

$$\underbrace{T_u + \delta t_{rcvr}}_{\text{receive time}} - \frac{PR}{c} - \underbrace{\delta t_{sv}}_{} = T_s$$

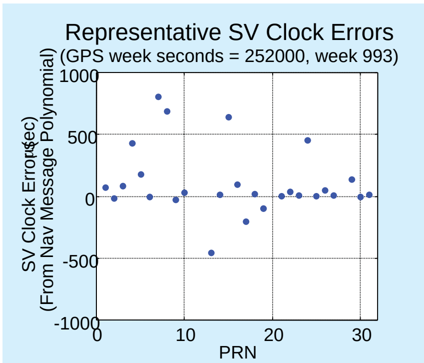
same as Δt_{sv} from the previous slide



Determining Signal Transmit Time (2/2)

- Effect of neglecting δt_{sv} for SV positioning[1]
  - Satellite clock error can grow to up to ~1 msec:
  - Typical satellite velocity is 3900 m/s
  - Worst-case position error from neglecting δt_{sv}

    3900 m/s x 0.001 s = **3.9 m**
  - Effect of neglecting δt_{sv}
    - Single point positioning: Can be significant (but not with SA)
    - Differential positioning: effectively cancelled out (acts like 3.9 m satellite position error)



[1]The SV clock error δt_{sv} will have a significant effect on the actual pseudorange measurement. This page only describes the impact of δt_{sv} on determining the position of the satellite.

Use of Dual Frequency Measurements to Calculate Ionospheric Delay

- L1 ionospheric delay calculated by

$$\Delta S_{iono,corr_{L1}} = \left(\frac{f_2^2}{f_2^2 - f_1^2}\right)(\rho_{L1} - \rho_{L2})$$

$\Delta S_{iono,corr_{L1}}$ = L1 ionospheric delay (m)

f_1, f_2 = L1 and L2 carrier frequencies

ρ_{L1}, ρ_{L2} = L1 and L2 pseudorange measurements

- L2 ionospheric delay can be calculated by

$$\Delta S_{iono,corr_{L2}} = \left(\frac{f_1}{f_2}\right)^2 \Delta S_{iono,corr_{L1}}$$

- Ionospheric-free pseudorange:

$$\rho_{IF} = \frac{\rho_{L2} - \gamma\rho_{L1}}{1-\gamma}, \qquad \gamma = \left(\frac{f_{L1}}{f_{L2}}\right)^2 = \left(\frac{77}{60}\right)^2$$

- Multipath and measurement noise will corrupt this measurement of ionosphere

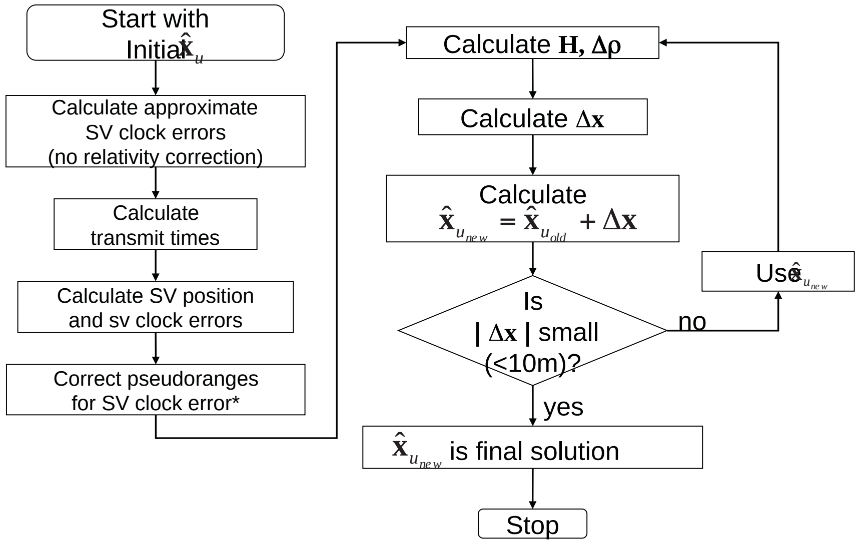
Correcting for Satellite Group Delay

- Each satellite has a slight time bias between the L1 and the L2 signals
  - Not desired, but it's there nonetheless
  - Will affect dual-frequency users, unless it's accounted for
  - Can be measured and/or calibrated out
  - This calibration is accounted for when the control segment generates the satellite clock correction terms from broadcast nav message: $a_{f_0}, a_{f_1}, a_{f_2}$, and t_{0_c}
  - However, this is all designed for the dual-frequency user! Single frequency users need to remove the effect of this dual-frequency correction on their Δt_{sv} value
- Single frequency users must apply the group delay term (TGD) from the nav message to their SV clock correction term (from p. 90 of ICD-GPS-200C)

$$(\Delta t_{sv})_{L1} = \Delta t_{sv} - T_{GD}$$

$$(\Delta t_{sv})_{L2} = \Delta t_{sv} - \left(\frac{77}{60}\right)^2 T_{GD}$$

Single Point Positioning Algorithm



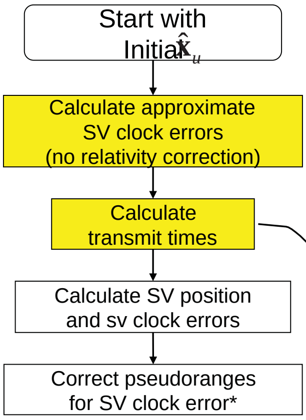
*include group delay correction, if a single-frequency user

GPS Positioning Example

- We'll look at a single case to give an example
- Situation
  - Receiver measurement time (GPS week seconds): 220937
  - Initial $\hat{\mathbf{x}}_u$: [506071.529 -4882278.667 4109624.557 15.807]
    - Initial guess of position (in error by ~50 km)
    - Initial clock error expressed in m
  - Measurements:

PRN	Pseudorange
12	25022759.323
2	22075351.532
26	21929350.580
15	22677087.545
29	21039894.608
21	24757444.127
30	24032696.422

Example: Calculation of Transmit Time



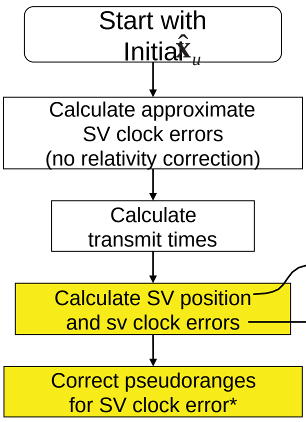
$$\Delta t_{sv_{approx}} = a_{f_0} + a_{f_1}(t - t_{0_c}) + a_{f_2}(t - t_{0_c})^2$$

(Receive time = 220937)

PRN	PR/c (s)	Approx SV Clock Error (s)	Transmit Time (s)
12	0.083466941	-0.000359172	220936.916892231
2	0.073635447	0.000199964	220936.926164589
26	0.073148440	0.000247391	220936.926604169
15	0.075642622	-0.000107130	220936.924464508
29	0.070181534	-0.000043329	220936.929861795
21	0.082581944	0.000065081	220936.917352974
30	0.080164446	0.000075192	220936.919760362

$$\underbrace{T_u + \delta t_{rcvr}}_{\text{receive time}} - \frac{PR}{c} - \delta t_{sv} = T_s$$

Example: SV Position and Clock Error and Pseudorange Correction



PRN	SV ECEF X (m)	SV ECEF Y (m)	SV ECEF Z (m)	SV Clock Err (s)
12	-9924909.896	-22418412.217	-10238600.462	-0.000359164
2	19519446.654	-12864169.870	12106498.214	0.000199964
26	9518973.908	-24465469.002	-347289.566	0.000247426
15	6420995.137	-25601178.700	-2907089.329	-0.000107131
29	-8265550.815	-16497554.935	19234406.500	-0.000043326
21	-22299549.612	-6458590.524	13615554.839	0.000065090
30	-18044425.181	-19566072.431	-289953.964	0.000075168

Orbital calculations + time-of-transit rotation correction

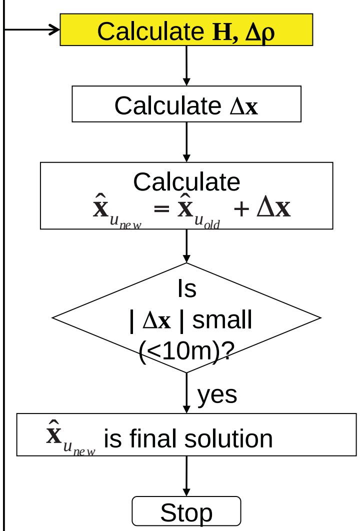
$$\Delta t_{sv} = a_{f_0} + a_{f_1}(t - t_{0_c}) + a_{f_2}(t - t_{0_c})^2 + \Delta t_r$$

PRN	Original PR (m)	Corrected PR (m)	Group Delay Tgd (ns)
12	25022759.323	24915088.264	-12.107193470
2	22075351.532	22135304.471	-17.229467630
26	21929350.580	22003529.125	-6.519258022
15	22677087.545	22644973.532	-10.244548320
29	21039894.608	21026908.513	-8.847564459
21	24757444.127	24776961.101	-11.641532180
30	24032696.422	24055233.876	-8.381903172

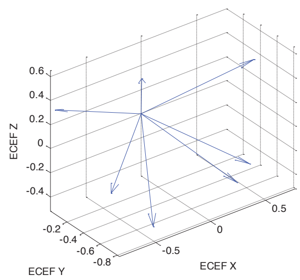
$$\rho_{corr} = \rho + c\left(\Delta t_{sv} - T_{GD}\right)$$

*include group delay correction, if a single-frequency user

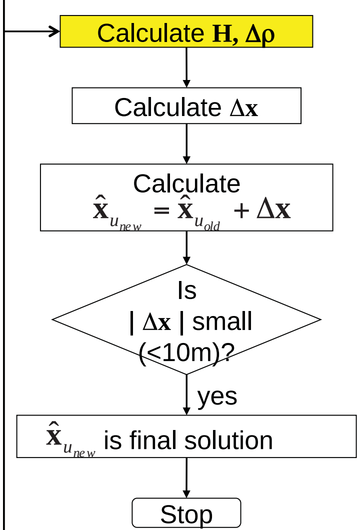
Example: H Matrix (Iteration 1)



$$\mathbf{H} = \begin{bmatrix} -0.4182 & -0.7030 & -0.5752 & -1 \\ 0.8597 & -0.3609 & 0.3616 & -1 \\ 0.4094 & -0.8896 & -0.2025 & -1 \\ 0.2610 & -0.9143 & -0.3096 & -1 \\ -0.4179 & -0.5533 & 0.7205 & -1 \\ -0.9212 & -0.0637 & 0.3840 & -1 \\ -0.7709 & -0.6102 & -0.1828 & -1 \end{bmatrix}$$



Example: Δρ (Iteration 1)



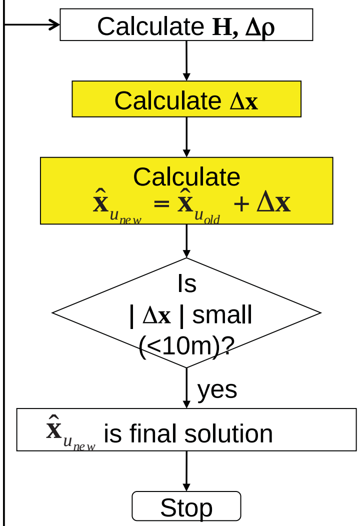
Calculated

Measured (corrected)

$$\Delta\rho = \hat{\rho} - \rho_{corr}$$

PRN	Calculated PR	Measured PR	Delta-Rho
12	24943810.919	24915088.264	28722.655
2	22117181.292	22135304.471	-18123.179
26	22013598.807	22003529.125	10069.682
15	22660408.867	22644973.532	15435.335
29	20990847.857	21026908.513	-36060.657
21	24757718.148	24776961.101	-19242.953
30	24064325.866	24055233.876	9091.990

Example: Solution and Residuals (Iteration 1)



$$\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\rho$$

$$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$$

$\hat{\mathbf{x}}_{u_{new}}$	$\hat{\mathbf{x}}_{u_{old}}$	$\Delta\mathbf{x}$
506068.143	506071.529	-3.386
-4882283.665	-4882278.667	-4.998
4059632.252	4109624.557	-49992.305
63.927	15.807	48.120

Residuals: $\mathbf{v} = \Delta\rho - \mathbf{H}\Delta\mathbf{x}$

PRN	$\mathbf{v}$	$\Delta\rho$	$\mathbf{H}\Delta\mathbf{x}$
12	9.162	28722.655	28713.493
2	1.699	-18123.179	-18124.878
26	-6.800	10069.682	10076.482
15	-0.178	15435.335	15435.513
29	4.853	-36060.657	-36065.510
21	-3.299	-19242.953	-19239.654
30	-5.436	9091.990	9097.426

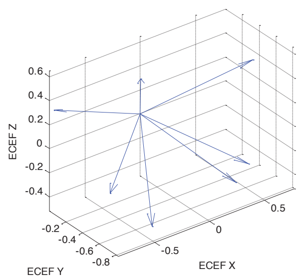
Example: H Matrix (Iterations 1 and 2)

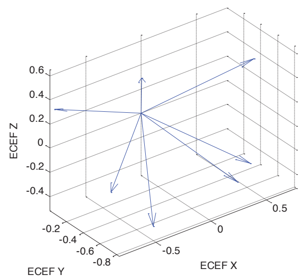
Iteration 1

$$\mathbf{H} = \begin{bmatrix} -0.4182 & -0.7030 & -0.5752 & -1 \\ 0.8597 & -0.3609 & 0.3616 & -1 \\ 0.4094 & -0.8896 & -0.2025 & -1 \\ 0.2610 & -0.9143 & -0.3096 & -1 \\ -0.4179 & -0.5533 & 0.7205 & -1 \\ -0.9212 & -0.0637 & 0.3840 & -1 \\ -0.7709 & -0.6102 & -0.1828 & -1 \end{bmatrix}$$

Iteration 2

$$\mathbf{H} = \begin{bmatrix} -0.4187 & -0.7038 & -0.5739 & -1 \\ 0.8590 & -0.3606 & 0.3635 & -1 \\ 0.4096 & -0.8900 & -0.2003 & -1 \\ 0.2612 & -0.9149 & -0.3076 & -1 \\ -0.4172 & -0.5524 & 0.7217 & -1 \\ -0.9204 & -0.0636 & 0.3857 & -1 \\ -0.7712 & -0.6104 & -0.1808 & -1 \end{bmatrix}$$





Example: $\Delta\rho$ (Iterations 1 and 2)

Iteration 1

Calculated → $\hat{\rho}$; Measured (corrected) → ρ_{corr}

$$\Delta\rho = \hat{\rho} - \rho_{corr}$$

PRN	Calculated PR	Measured PR	Delta-Rho
12	24943810.919	24915088.264	28722.655
2	22117181.292	22135304.471	-18123.179
26	22013598.807	22003529.125	10069.682
15	22660408.867	22644973.532	15435.335
29	20990847.857	21026908.513	-36060.657
21	24757718.148	24776961.101	-19242.953
30	24064325.866	24055233.876	9091.990

Iteration 2

Calculated → $\hat{\rho}$; Measured (corrected) → ρ_{corr}

$$\Delta\rho = \hat{\rho} - \rho_{corr}$$

PRN	Calculated PR	Measured PR	Delta-Rho
12	24915130.980	24915088.264	42.716
2	22135355.242	22135304.471	50.771
26	22003576.788	22003529.125	47.662
15	22645023.243	22644973.532	49.711
29	21026941.948	21026908.513	33.435
21	24777000.804	24776961.101	39.703
30	24055278.650	24055233.876	44.773

Example: Solution and Residuals (Iterations 1 and 2)

Iteration 1

$$\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\boldsymbol{\rho}$$

$$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$$

$\hat{\mathbf{x}}_{u_{new}}$	$\hat{\mathbf{x}}_{u_{old}}$	$\Delta\mathbf{x}$
506068.143	506071.529	-3.386
-4882283.665	-4882278.667	-4.998
4059632.252	4109624.557	-49992.305
63.927	15.807	48.120

Residuals: $\mathbf{v} = \Delta\boldsymbol{\rho} - \mathbf{H}\Delta\mathbf{x}$

PRN	$\mathbf{v}$	$\Delta\boldsymbol{\rho}$	$\mathbf{H}\Delta\mathbf{x}$
12	9.162	28722.655	28713.493
2	1.699	-18123.179	-18124.878
26	-6.800	10069.682	10076.482
15	-0.178	15435.335	15435.513
29	4.853	-36060.657	-36065.510
21	-3.299	-19242.953	-19239.654
30	-5.436	9091.990	9097.426

Iteration 2

$$\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\boldsymbol{\rho}$$

$$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$$

$\hat{\mathbf{x}}_{u_{new}}$	$\hat{\mathbf{x}}_{u_{old}}$	$\Delta\mathbf{x}$
506075.869	506068.143	7.726
-4882274.608	-4882283.665	9.057
4059622.275	4059632.252	-9.977
13.120	63.927	-50.807

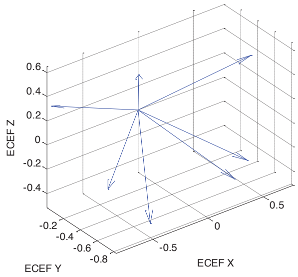
Residuals: $\mathbf{v} = \Delta\boldsymbol{\rho} - \mathbf{H}\Delta\mathbf{x}$

PRN	$\mathbf{v}$	$\Delta\boldsymbol{\rho}$	$\mathbf{H}\Delta\mathbf{x}$
12	-4.208	42.716	46.924
2	0.220	50.771	50.551
26	-0.248	47.662	47.910
15	2.103	49.711	47.609
29	-1.946	33.435	35.381
21	0.431	39.703	39.272
30	3.648	44.773	41.125

Example: H Matrix (Iterations 2 and 3)

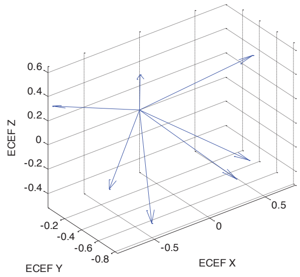
Iteration 2

$$\mathbf{H} = \begin{bmatrix} -0.4187 & -0.7038 & -0.5739 & -1 \\ 0.8590 & -0.3606 & 0.3635 & -1 \\ 0.4096 & -0.8900 & -0.2003 & -1 \\ 0.2612 & -0.9149 & -0.3076 & -1 \\ -0.4172 & -0.5524 & 0.7217 & -1 \\ -0.9204 & -0.0636 & 0.3857 & -1 \\ -0.7712 & -0.6104 & -0.1808 & -1 \end{bmatrix}$$



Iteration 3

$$\mathbf{H} = \begin{bmatrix} -0.4187 & -0.7038 & -0.5739 & -1 \\ 0.8590 & -0.3606 & 0.3635 & -1 \\ 0.4096 & -0.8900 & -0.2003 & -1 \\ 0.2612 & -0.9149 & -0.3076 & -1 \\ -0.4172 & -0.5524 & 0.7217 & -1 \\ -0.9204 & -0.0636 & 0.3857 & -1 \\ -0.7712 & -0.6104 & -0.1808 & -1 \end{bmatrix}$$



Example: Δρ (Iterations 2 and 3)

Iteration 2

Calculated → $\hat{\rho}$; Measured (corrected) → ρ_{corr}

$$\Delta\rho = \hat{\rho} - \rho_{corr}$$

PRN	Calculated PR	Measured PR	Delta-Rho
12	24915130.980	24915088.264	42.716
2	22135355.242	22135304.471	50.771
26	22003576.788	22003529.125	47.662
15	22645023.243	22644973.532	49.711
29	21026941.948	21026908.513	33.435
21	24777000.804	24776961.101	39.703
30	24055278.650	24055233.876	44.773

Iteration 3

Calculated → $\hat{\rho}$; Measured (corrected) → ρ_{corr}

$$\Delta\rho = \hat{\rho} - \rho_{corr}$$

PRN	Calculated PR	Measured PR	Delta-Rho
12	24915084.055	24915088.264	-4.208
2	22135304.691	22135304.471	0.220
26	22003528.878	22003529.125	-0.248
15	22644975.634	22644973.532	2.103
29	21026906.567	21026908.513	-1.946
21	24776961.532	24776961.101	0.431
30	24055237.525	24055233.876	3.648

Example: Solution and Residuals (Iterations 2 and 3)

Iteration 2 $\Delta\mathbf{x} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\Delta\boldsymbol{\rho}$

$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$

$\hat{\mathbf{x}}_{u_{new}}$	$\hat{\mathbf{x}}_{u_{old}}$	$\Delta\mathbf{x}$
506075.869	506068.143	7.726
-4882274.608	-4882283.665	9.057
4059622.275	4059632.252	-9.977
13.120	63.927	-50.807

Residuals: $\mathbf{v} = \Delta\boldsymbol{\rho} - \mathbf{H}\Delta\mathbf{x}$

PRN	$\mathbf{v}$	$\Delta\boldsymbol{\rho}$	$\mathbf{H}\Delta\mathbf{x}$
12	-4.208	42.716	46.924
2	0.220	50.771	50.551
26	-0.248	47.662	47.910
15	2.103	49.711	47.609
29	-1.946	33.435	35.381
21	0.431	39.703	39.272
30	3.648	44.773	41.125

Iteration 3 $\Delta\mathbf{x} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\Delta\boldsymbol{\rho}$

$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$

$\hat{\mathbf{x}}_{u_{new}}$	$\hat{\mathbf{x}}_{u_{old}}$	$\Delta\mathbf{x}$
506075.869	506075.869	0.000
-4882274.608	-4882274.608	0.000
4059622.275	4059622.275	0.000
13.120	13.120	0.000

↑ On order of 10^{-6}

Residuals: $\mathbf{v} = \Delta\boldsymbol{\rho} - \mathbf{H}\Delta\mathbf{x}$

PRN	$\mathbf{v}$	$\Delta\boldsymbol{\rho}$	$\mathbf{H}\Delta\mathbf{x}$
12	-4.208	42.716	46.924
2	0.220	50.771	50.551
26	-0.248	47.662	47.910
15	2.103	49.711	47.609
29	-1.946	33.435	35.381
21	0.431	39.703	39.272
30	3.648	44.773	41.125

Convergence

- Practically speaking, getting the system to converge with GNSS is easy
  - Example showed case where initial guess was 50 km in error
  - Can start with the center of the Earth as a guess, and it would only add an iteration or two
  - Normally, a receiver will use its last solution as a starting point, so only a single iteration is necessary
- Nonlinearities (which drive the need for iteration) are more severe when dealing with pseudolites
  - Much closer to receiver than satellite
  - H matrix varies more quickly as a function of position

Overview

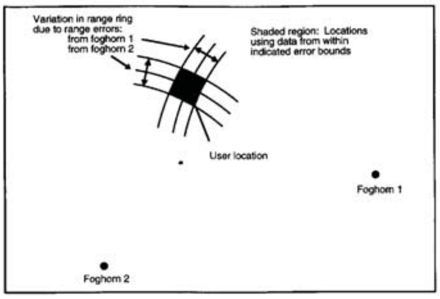
Measurement Domain vs. Position Domain

- Pseudorange errors are errors in "measurement domain"
  - Errors in the measurements themselves
  - UERE is one example
- Ultimately, we'd like to know errors in "position domain"
  - The position errors that result when using the measurements
  - Errors in position domain are different than measurement errors!
    - Can be larger
    - Can be smaller
  - Dependent on measurement geometry
- Mathematical representation
  - We have covariance matrix of measurements ($\mathbf{C}_\rho$).
  - We want covariance matrix of calculated position and clock error ($\mathbf{C}_x$)
- In GPS applications, this problem is approached using concept called Dilution of Precision (DOP)

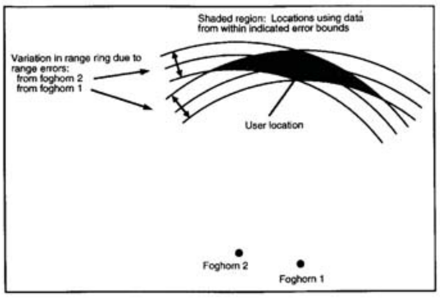
Effect of Geometry on Positioning Accuracy (Foghorn Example)

Consider the foghorn example, except allow for a measurement error

Good Geometry Example



Poor Geometry Example



Obtaining $\mathbf{C_x}$ from Least-Squares Analysis (1/2)

- Definition of $\mathbf{C_x}$

$$C_x = \begin{bmatrix} \sigma_{x_u}^2 & \sigma_{x_u y_u} & \sigma_{x_u z_u} & \sigma_{x_u \delta t_u} \\ \sigma_{x_u y_u} & \sigma_{y_u}^2 & \sigma_{y_u z_u} & \sigma_{y_u \delta t_u} \\ \sigma_{x_u z_u} & \sigma_{y_u z_u} & \sigma_{z_u}^2 & \sigma_{z_u \delta t_u} \\ \sigma_{x_u \delta t_u} & \sigma_{y_u \delta t_u} & \sigma_{z_u \delta t_u} & \sigma_{\delta t_u}^2 \end{bmatrix}$$

where, for example,

$$\sigma_{x_u}^2 = E\left[(x_u - E[x_u])^2\right]$$

$= \text{variance of } x_u$

$$\sigma_{x_u y_u} = E\left[(x_u - E[x_u])(y_u - E[y_u])\right]$$

$= \text{covariance of } x_u \text{ and } y_u$

- Definition of $\mathbf{C}_\rho$

$$\mathbf{C}_\rho = \begin{bmatrix} \sigma_{\rho_1}^2 & \sigma_{\rho_1\rho_2} & \cdots & \sigma_{\rho_1\rho_n} \\ \sigma_{\rho_1\rho_2} & \sigma_{\rho_2}^2 & \cdots & \sigma_{\rho_2\rho_n} \\ \vdots & \vdots & \ddots & \sigma_{\rho_3\rho_n} \\ \sigma_{\rho_1\rho_n} & \sigma_{\rho_2\rho_n} & \sigma_{\rho_3\rho_n} & \sigma_{\rho_n}^2 \end{bmatrix}$$

Obtaining $\mathbf{C_x}$ from Least-Squares Analysis (2/2)

- According to least-squares theory:

$$\mathbf{C_x} = \left(\mathbf{H}^T \mathbf{C}_\rho^{-1} \mathbf{H}\right)^{-1}$$

  – Basic assumptions
    - Measurement errors are zero-mean
    - Measurement errors have a Gaussian distribution
- Recall that the least-squares solution with measurement weighting was

$$\Delta\mathbf{x} = \left(\mathbf{H}^T \mathbf{C}_\rho^{-1} \mathbf{H}\right)^{-1} \mathbf{H}^T \mathbf{C}_\rho^{-1} \Delta\boldsymbol{\rho}$$
$$= \mathbf{C_x} \mathbf{H}^T \mathbf{C}_\rho^{-1} \Delta\boldsymbol{\rho}$$

  – Consider case where the nominal position and clock error (used to calculate $\Delta\boldsymbol{\rho}$) are actually the true position and clock error
    - The $\Delta\boldsymbol{\rho}$ represents the measurement *errors*
    - The $\Delta\mathbf{x}$ represents the position and clock *errors*
    - The $\mathbf{C_x}$ matrix is a multiplier for the measurement errors ($\Delta\boldsymbol{\rho}$)
      – "Large" $\mathbf{C_x}$ values → large position errors
      – "Small" $\mathbf{C_x}$ values → small position errors

Dilution of Precision (DOP)

- In GPS, the concept of Dilution of Precision (DOP) is used
  – Based upon covariance matrix of position and clock errors ($\mathbf{C_x}$)
  – Additional assumptions
    - All measurements have the same variance

$$\sigma_{\rho_1}^2 = \sigma_{\rho_2}^2 = \ldots = \sigma_{\rho_n}^2 = \sigma_\rho^2$$

    - Measurement errors are uncorrelated (i.e.,covariance values are zero)

$$\sigma_{\rho_j \rho_k} = 0, \quad j \neq k$$

  – Using these assumptions

$$\mathbf{C}_\rho = \mathbf{I}\sigma_\rho^2$$

and

$$\mathbf{C_x} = \left(\mathbf{H}^T \mathbf{H}\right)^{-1} \sigma_\rho^2$$

  – The matrix $\left(\mathbf{H}^T \mathbf{H}\right)^{-1}$ is called the DOP matrix
    - Directly relates measurement errors to position errors

Use of Local-Level Coordinate Frame (1/2)

- Normally, DOPs describe errors in geodetic (local-level) coordinate frame (east, north, up), rather than the ECEF frame.
  - Need to modify the H matrix so that the errors refer to the local-level frame
  - Original H matrix (used to calculate position)

$$\mathbf{H}^E = \begin{bmatrix} \mathbf{a}_1^{E^T} & 1 \\ \mathbf{a}_2^{E^T} & 1 \\ \vdots & \vdots \\ \mathbf{a}_n^{E^T} & 1 \end{bmatrix}$$

    - "**a**" vectors are unit line-of-sight vectors between user and SV in *ECEF frame*
    - This will give the $\mathbf{C}_\mathbf{x}$ matrix described previously
  - New H matrix for DOP calculations

$$\mathbf{H}^G = \begin{bmatrix} \mathbf{a}_1^{G^T} & 1 \\ \mathbf{a}_2^{G^T} & 1 \\ \vdots & \vdots \\ \mathbf{a}_n^{G^T} & 1 \end{bmatrix}$$

    - "**a**" vectors are now unit line-of-sight vectors between user and SV in *geodetic (ENU) frame*

Use of Local-Level Coordinate Frame (2/2)

- Local-level "**a**" vectors can be calculated using direction cosine matrix (DCM)

$$\mathbf{a}^G = \mathbf{C}_E^G \mathbf{a}^E$$

$\mathbf{C}_E^G$ = DCM that rotates from ECEF to geodetic (E,N,U) frame

$$\mathbf{C}_E^G = \left(\mathbf{C}_G^E\right)^{-1} = \left(\mathbf{C}_G^E\right)^T$$

- When $\mathbf{H}^G$ is used to calculate the covariance $\mathbf{C}_\mathbf{x} = \left(\mathbf{H}^{G^T}\mathbf{H}^G\right)^{-1}\sigma_\rho^2$, then $\mathbf{C}_\mathbf{x}$ is defined as

$$\mathbf{C}_\mathbf{x} = \begin{bmatrix} \sigma_e^2 & \sigma_{en} & \sigma_{eu} & \sigma_{e\delta t_u} \\ \sigma_{en} & \sigma_n^2 & \sigma_{nu} & \sigma_{n\delta t_u} \\ \sigma_{eu} & \sigma_{nu} & \sigma_u^2 & \sigma_{u\delta t_u} \\ \sigma_{e\delta t_u} & \sigma_{n\delta t_u} & \sigma_{u\delta t_u} & \sigma_{\delta t_u}^2 \end{bmatrix}$$

  - This is what we desire to describe using DOPs

DOP Values

- Desirable to characterize the $\mathbf{C_x}$ matrix using a single number
  - For DOPs
    - Cross-correlation terms ignored
    - Root-Sum-Square (RSS) value of variables of interest, normalized by σ_{UERE}
    - Example:

$$GDOP = \frac{\sqrt{\sigma_e^2 + \sigma_n^2 + \sigma_u^2 + \sigma_{\delta t_u}^2}}{\sigma_{UERE}}$$

    - GDOP can be calculated directly from DOP matrix

$$\left(\mathbf{H}^{G^T}\mathbf{H}^G\right)^{-1} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{bmatrix} \qquad GDOP = \sqrt{D_{11} + D_{22} + D_{33} + D_{44}}$$

    - Note that GDOP relates UERE with RSS of errors

Key relationship!

$$\sqrt{\sigma_e^2 + \sigma_n^2 + \sigma_u^2 + \sigma_{\delta t_u}^2} = GDOP \times \sigma_{UERE}$$

Types of DOPs

- The "Big Three"
  - GDOP (Geometric DOP)

$$GDOP = \sqrt{D_{11} + D_{22} + D_{33} + D_{44}}$$

$$\sqrt{\sigma_e^2 + \sigma_n^2 + \sigma_u^2 + \sigma_{\delta t_u}^2} = GDOP \times \sigma_{UERE}$$

  - PDOP (Position DOP)

$$PDOP = \sqrt{D_{11} + D_{22} + D_{33}}$$

$$\sqrt{\sigma_e^2 + \sigma_n^2 + \sigma_u^2} = PDOP \times \sigma_{UERE}$$

  - HDOP (Horizontal DOP)

$$HDOP = \sqrt{D_{11} + D_{22}}$$

$$\sqrt{\sigma_e^2 + \sigma_n^2} = HDOP \times \sigma_{UERE}$$

- Less common (for navigators, at least!)
  - VDOP (Vertical DOP)

$$VDOP = \sqrt{D_{33}}$$

$$\sqrt{\sigma_u^2} = VDOP \times \sigma_{UERE}$$

  - TDOP (Time DOP)

$$TDOP = \sqrt{D_{44}}$$

$$\sqrt{\sigma_{\delta t_u}^2} = TDOP \times \sigma_{UERE}$$

    - Note: time is in units of meters

Typical DOP Plot

Dayton Ohio – 24 Apr 2003 – All Visible SVs (above 10° elevation)