2458-10

Workshop on GNSS Data Application to Low Latitude Ionospheric Research

6 - 17 May 2013

Calculation of GPS PNT Solution

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Calculation of GPS PNT Solution

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Overview

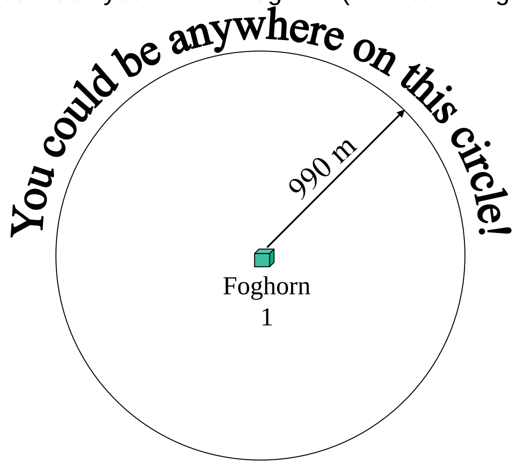
1. Positioning Using Time-of-Arrival
2. GPS Receiver Measurements
3. Coordinate Frames
4. Calculation of Satellite Position
5. GPS Navigation Solution
6. Dilution of Precision

Ranging Using Time-Of-Arrival

- Time-of-arrival (TOA) is one method that can be used to perform positioning
- Basic concept
  - You must know
    - When a signal was transmitted
    - How fast the signal travels
    - Time that the signal was received
  - Then you can determine how far away you are from the signal emitter
- Foghorn example
  - Assume there is a foghorn that goes off at exactly 12:00:00 noon every day
  - You know that the velocity of sound around the foghorn is 330 m/sec
  - You have a device that measures the time when the foghorn blast is received, and it says it heard a foghorn blast at 12:00:03
  - What is the distance between the foghorn and the foghorn "receiver"?
  - Now that you know how far you are from the foghorn, the question is, "Where are you?"

Two-Dimensional Positioning Using Single Range Measurement

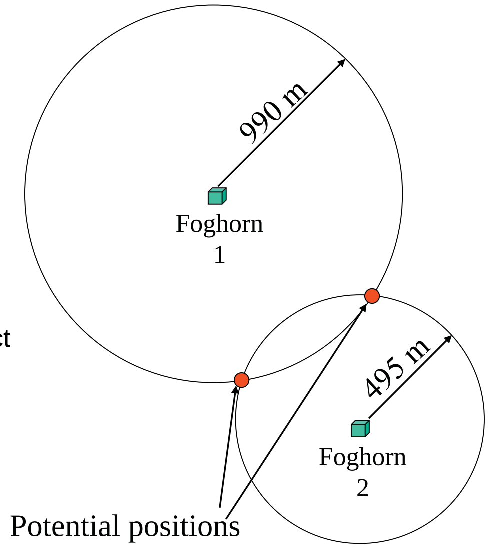
- Range between you and the foghorn (we'll call it foghorn #1) is 990m



- Unable to determine exact position in this case

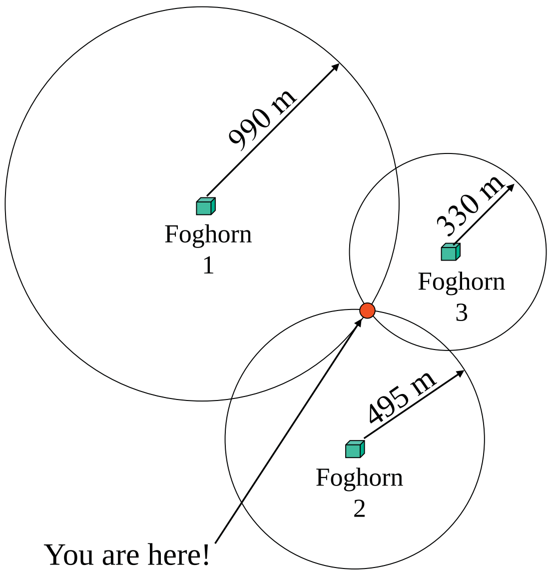
Two-Dimensional Ranging Using Two Measurements

- Now, you take a measurement from foghorn #2 at 12:00:01.5 (for a range of 495 m)
- Yields two potential solutions
  - How would you determine the correct solution?



Resolving Position Ambiguity Using Three Measurements

- You get a third measurement from foghorn #3 at 12:00:01 (Range = 330 m)
  - Now there's a unique solution



Receiver Clock Errors

- The foghorn example assumed that the foghorn “receiver” had a perfectly synchronized clock, so the measurements were perfect
- What happens if there is an unknown receiver clock error?
- Effect on range measurement
  - Without clock error

$$R = v_{sound}\Delta t$$

R = range

v_{sound} = velocity of sound

Δt = transmit/receive time difference

  - With clock error δt

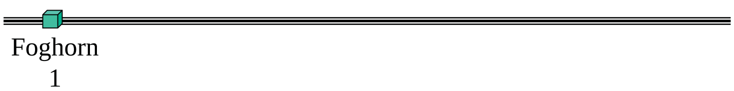
$$R' = v_{sound}\left(\Delta t + \delta t\right)$$

where

R' = range with error (pseudo-range)

Receiver Clock Errors
One-Dimensional Example (1/3)

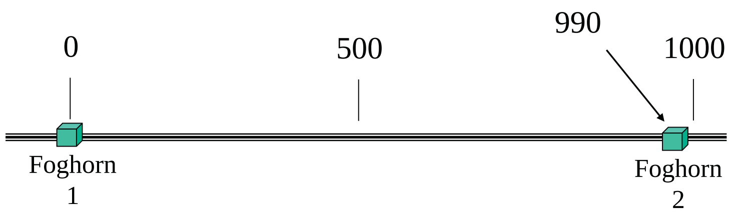
- Now, we’ll look at the foghorn example, except in only one dimension
  - The foghorn(s) and receiver are constrained to be along a line
  - We want to determine the position of the receiver on that line



- If the receiver measured a signal at 12:00:10, where is it on the line?
- Now, assume an unknown clock bias δt in the clock used by the foghorn receiver
- Your foghorn receiver measures a foghorn blast at 12:00:10
- What can you say about where you are?

Receiver Clock Errors
One-Dimensional Example (2/3)

- Clearly, more information is needed
- Assume that there is a second foghorn located 990 m away from the first



- You receive a signal from the second foghorn at 12:00:09
- What can you tell about where you are at this point?

Receiver Clock Errors
One-Dimensional Example (3/3)

- Here are the measurements we have:

$$\text{Pseudorange 1} = 330 \times 10 = 3300 = R_1'$$
$$\text{Pseudorange 2} = 330 \times 9 = 2970 = R_2'$$

- From the pseudorange equation:

$$R_1' = v_{sound}\left(\Delta t_1 + \delta t\right) = \quad x \quad + v_{sound}\delta t = 3300$$
$$R_2' = v_{sound}\left(\Delta t_2 + \delta t\right) = 990 - x + v_{sound}\delta t = 2970$$

- Rearranging terms we get

$$x + v_{sound}\delta t = 3300$$
$$x - v_{sound}\delta t = -1980$$

- We can then solve for the two unknowns

$$\delta t = 8 \text{ seconds}$$
$$x = 660 \text{ m}$$

Does this work?

Receiver Clock Errors
Extending to Three Dimensions

- In the single-dimensional case
  - We needed two measurements to solve for the two unknowns, x and δt.
  - The quantities x and (990 - x) were the "distances" between the position of the receiver and the two foghorns.
- In three-dimensional case
  - We need four measurements to solve for the four unknowns, x, y, z, and δt.
  - The distances between receiver and satellite are not linear equations (as was case in single-dimensional case).
  - The four equations need to be solved simultaneously, for pseudorange measurements $R_1' \ldots R_4'$ and transmitter positions $(x_1,y_1,z_1)\ldots(x_4,y_4,z_4)$:

$$R_1' = \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2} + c\delta t$$
$$R_2' = \sqrt{(x-x_2)^2 + (y-y_2)^2 + (z-z_2)^2} + c\delta t$$
$$R_3' = \sqrt{(x-x_3)^2 + (y-y_3)^2 + (z-z_3)^2} + c\delta t$$
$$R_4' = \sqrt{(x-x_4)^2 + (y-y_4)^2 + (z-z_4)^2} + c\delta t$$

Overview

GPS Measurements (Overview)

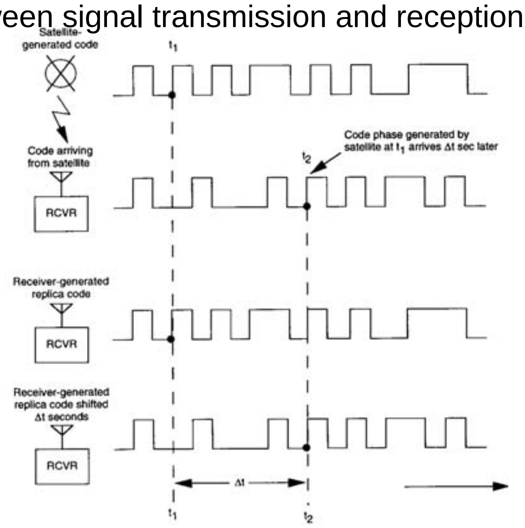
- Each separate tracking loop typically can give 4 different measurement outputs
  - Pseudorange measurement
  - Carrier-phase measurement (sometimes called integrated Doppler)
  - Doppler measurement
  - Carrier-to-noise density C/N_0
- Actual output varies depending upon receiver
  - Ashtech Z-surveyor (or Z-12) gives them all
  - RCVR-3A gives just C/N_0
- Note: We're talking here about *raw measurements*
  - Almost all receivers generate navigation processor outputs (position, velocity, heading, etc.)

Measurement Rates and Timing

- Most receivers take measurements on all channels/ tracking loops simultaneously
  - Measurements time-tagged with the receiver clock (receiver time)
  - The time at which a set of measurements is made is called a data epoch.
- The data rate varies depending upon receiver/ application. Typical data rates:
  - Static surveying: One measurement every 30 seconds (120 measurements per hour)
  - Typical air, land, and marine navigation: 0.5-2 measurement per second (most common)
  - Specialized high-dynamic applications: Up to 50 measurements per second (recent development)

GPS Pseudorange Measurement

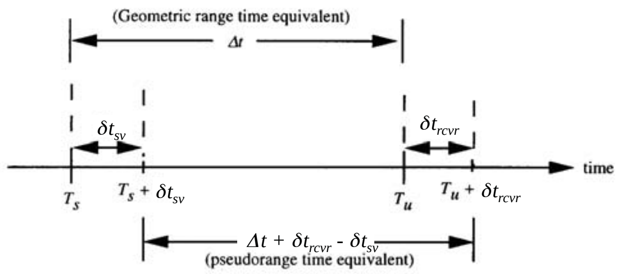
- Pseudorange is a measure of the difference in time between signal transmission and reception



Kaplan (ed.), *Understanding GPS: Principles and Applications*, Artech House, 1996

Effect of Clock Errors on Pseudorange

- Since pseudorange is based on time difference, any clock errors will fold directly into pseudorange



- Small clock errors can result in large pseudorange errors (since clock errors are multiplied by speed of light)
- Satellite clock errors (δt_{sv}) are very small
  - Satellites have atomic time standards
  - Satellite clock corrections transmitted in navigation message
- Receiver clock (δt_{rcvr}) is dominant error

Kaplan (ed.), *Understanding GPS: Principles and Applications*, Artech House, 1996

Doppler Shift

- For electromagnetic waves (which travel at the speed of light), the received frequency f_R is approximated using the standard Doppler equation

$$f_R = f_T\left(1 - \frac{(\boldsymbol{v}_r \cdot \boldsymbol{a})}{c}\right)$$

f_R = received frequency (Hz)

f_T = transmitted frequency (Hz)

$\boldsymbol{v}_r$ = satellite-to-user relative velocity vector (m/s)

$\boldsymbol{a}$ = unit vector pointing along line-of-sight from user to SV

c = speed of light (m/s)

  - Note that $\boldsymbol{v}_r$ is the (vector) velocity difference

$$\boldsymbol{v}_r = \boldsymbol{v} - \dot{\boldsymbol{u}}$$

$\boldsymbol{v}$ = velocity vector for satellite (m/s)

$\dot{\boldsymbol{u}}$ = velocity vector for user (m/s)

- The Doppler shift Δf is then

$$\Delta f = f_R - f_T \quad \text{(Hz)}$$

Doppler Measurement

- The GPS receiver locks onto the carrier of the GPS signal and measures the received signal frequency
  - Relationship between true and measured received signal frequency: $f_{R_{meas}}$

$$f_R = f_{R_{meas}}(1 + \delta\dot{t}_{rcvr})$$

f_R = true received signal frequency (Hz)

$f_{R_{meas}}$ = measured received signal frequency (Hz)

$\delta\dot{t}_{rcvr}$ = receiver clock drift rate (sec/sec)

  - Doppler measurement formed by differencing the measured received frequency and the transmit frequency:

$$\Delta f_{meas} = f_{R_{meas}} - f_T$$

  - Note: transmit frequency is calculated using information about SV clock drift rate given in navigation message

Doppler Measurement Sign Convention

- Sign convention based on Doppler definition
  - A satellite moving away from the receiver (neglecting clock errors) will have a *negative* Doppler shift

$$f_{R_{meas}} < f_T$$
$$\Delta f_{meas} = f_{R_{meas}} - f_T < 0$$

  - Sign convention used for NovAtel (and possibly other) receivers
- Sign convention based on relationship between Doppler and pseudorange
  - Doppler is essentially a measurement of the rate of change of the pseudorange
  - A satellite moving away from the receiver (neglecting clock errors) will have a *positive* Doppler measurement value
  - More common sign convention for GPS receivers (Ashtech, Trimble, and others)
- Carrier-phase measurement follows same convention as Doppler measurement (normally)

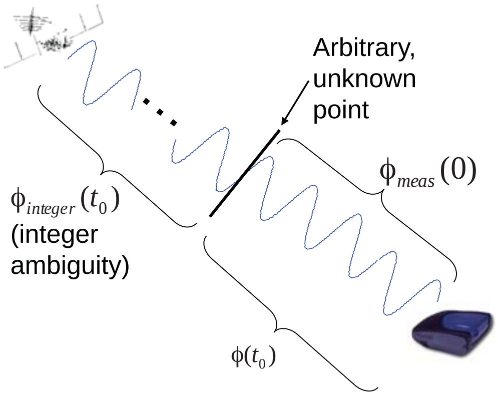
Carrier-Phase (Integrated Doppler) Measurement

- The carrier-phase measurement $\phi_{meas}(t)$ is calculated by integrating the Doppler measurements

$$\text{range}(t) = \underbrace{\int_{t_0}^{t} \Delta f_{meas}(t)dt + \phi(t_0)}_{\substack{\phi_{meas}(t) \\ \text{(can be measured by receiver)}}} + \phi_{integer}(t_0) + \text{clock error} + \text{other errors}$$

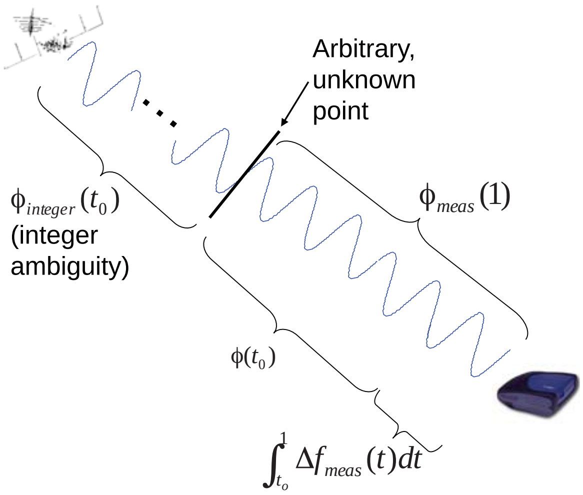
- The integer portion of the initial carrier-phase at the start of the integration ($\phi_{integer}(t_0)$) is known as the “carrier-phase integer ambiguity”
  - Because of this ambiguity, the carrier-phase measurement is not an absolute measurement of position
  - Advanced processing techniques can be used to resolve these carrier-phase ambiguites (carrier-phase ambiguity resolution)
- Alternative way of thinking: carrier-phase measurement is the “beat frequency” between the incoming carrier signal and receiver generated carrier.

Phase Tracking Example
At Start of Phase Lock (Time = 0 seconds)



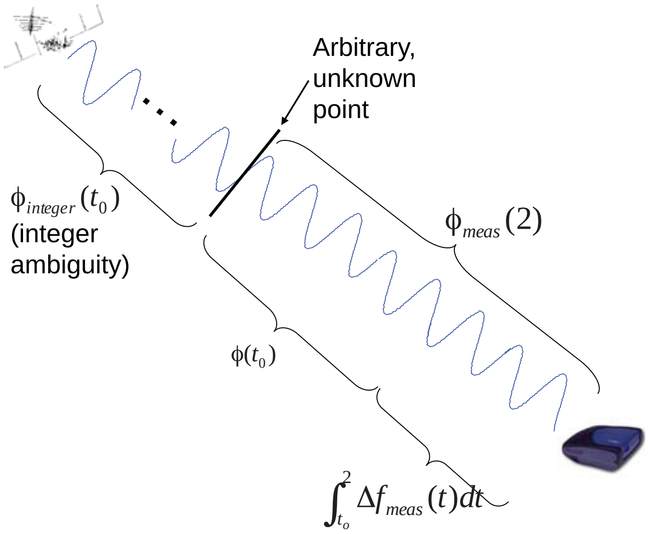
Ignoring clock and other errors

Phase Tracking Example
After Movement (for 1 Second)



Ignoring clock and other errors

Phase Tracking Example After Movement (for 2 Seconds)



Ignoring clock and other errors

Comparison Between Pseudorange and Carrier-Phase Measurements

	Pseudorange	Carrier-Phase
Type of measurement	Range (absolute)	Range (ambiguous)
Measurement precision	~1 m	~0.01 m
Robustness	More robust	Less robust (cycle slips possible)

Necessary for high precision GPS

Carrier-to-Noise Density (C/N_0)

- The carrier-to-noise density is a measure of signal strength
  - The higher the C/N_0, the stronger the signal (and the better the measurements)
  - Units are dB-Hz
  - General rules-of-thumb:
    - $C/N_0 > 40$: Very strong signal
    - $32 < C/N_0 < 40$: Marginal signal
    - $C/N_0 < 32$: Probably losing lock
- C/N_0 tends to be receiver-dependent
  - Can be calculated many different ways
  - Absolute comparisons between receivers not very meaningful
  - Relative comparisons between measurements in a single receiver are very meaningful

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3. **Coordinate Frames**
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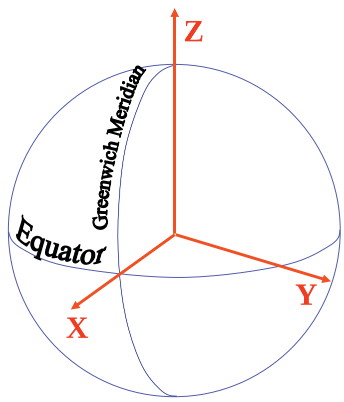
Coordinate Frames

- Giving a set of three coordinates is not sufficient for specifying a position
- Examples:
  - [-1485881.48699, -5152018.35300, 3444641.84728]
  - [-1.85158430, 0.57408361, 1255.323]
  - [-106.08796571, 32.89256771, 1255.323]
- The coordinate frame must also be specified
  - Choice of a coordinate frame is dependent upon the application
  - Most applications can use any defined coordinate frame, but usually one will be more straightforward than others
- Primary coordinate frames used for GPS
  - Earth-Centered Earth-Fixed (ECEF)
  - Geodetic (Longitude - Latitude - Altitude)

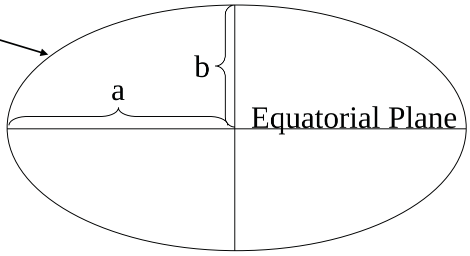
Earth-Centered Earth-Fixed (ECEF) Coordinate Frame

- ECEF frame is
  - Cartesian (orthogonal) reference frame
  - It is a rotating reference frame (w.r.t. inertial space), rotating at earth rate
  - Advantages
    - Easy to calculate distances and vectors between two points
    - Usually computationally efficient
  - Disadvantages
    - Not geographically intuitive



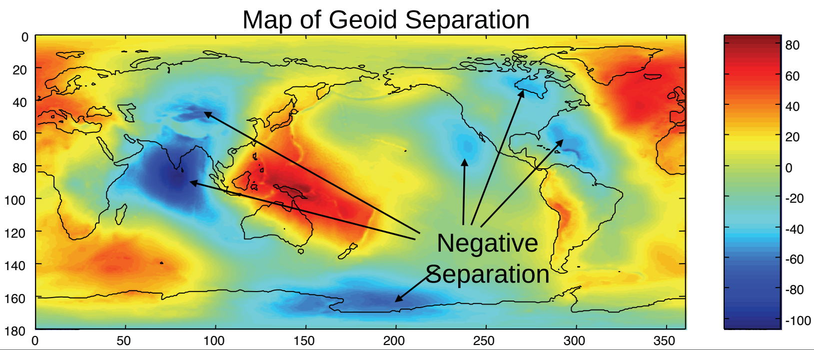
Geodetic Coordinate Frame (WGS-84 Ellipsoid)

- There are different ways to describe height (or altitude)
  - Distance above the surface of the earth
  - Definition based on gravity
    - Geoid: surface of constant gravitational potential
    - Geoid is a function of topography, earth density variations, and earth rotation rate
    - Geodesy: study of the geoid
  - Definition based upon geometry
    - The geoid can be fit to an ellipsoid (a rotated ellipse)
    - One particular ellipsoidal fit of the geoid is called the WGS-84 ellipsoid



Geoid Separation

- There is a separation between the Geoid (based on gravity) and the WGS-84 ellipsoid (based on a mathematical model)
  - Varies with user position
  - Only critical if interfacing with geoid-based reference systems (such as MSL altitude)

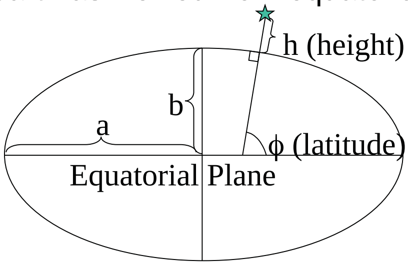


Key WGS-84 Parameters

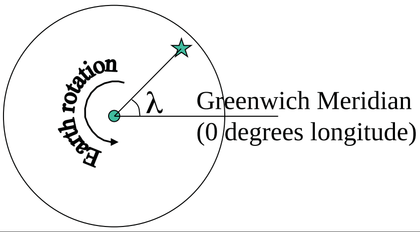
Name	Symbol	Quantity
Semi-major axis	a	6378.137 km
Semi-minor axis	b	6356.7523142 km
Eccentricity	e	0.0818191908426
Earth rotation rate	Ω_ε	7.2921151467E-5 rad/s
Speed of light	c	299792458 m/s
Gravitational parameter	μ	3.986005E14 m^3/s^2
Flattening	f	0.00335281066475

Definition of Geodetic Coordinates (Longitude, Latitude, and Altitude)

- Definition of ellipsoidal height h and latitude ϕ (cross section of earth as viewed from equatorial plane)



- Definition of longitude λ (view from above the north pole)



Geodetic Coordinate Units

- Normally, ellipsoidal altitude (h) is expressed in meters (m).
- Latitude (ϕ) and Longitude (λ) can be expressed in different units
  - Radians
    - Least ambiguous, useful for programming
    - Not as easily recognized geographically
  - Decimal degrees
    - To convert from radians to decimal degrees, multiply by $180/\pi$
    - Not very common
  - Degrees and decimal minutes
    - Integer number of degrees
    - Decimal number of minutes (1 minute = 1/60 degree)
    - Example: 46.596 decimal degrees is 46° 35.76' (reads 46 degrees, 35.76 minutes)
  - Degrees, minutes, and seconds
    - Integer number of degrees
    - Integer number of minutes
    - Decimal number of seconds (1 second = 1/60 minute)
    - Example: 46.596 decimal degrees is 46° 35' 45.6''

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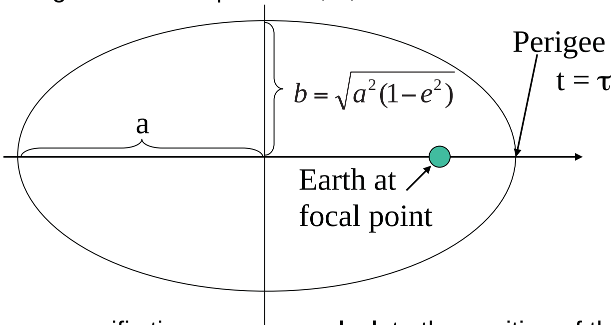
Determining Satellite Position

- In order to determine user position, one must calculate satellite position
- Satellites orbits are primarily based on the Earth's gravity field
- Other forces acting on satellite
  - Gravity from sun, moon, and other planets
  - Atmospheric drag
  - Solar pressure
  - Torques due to Earth's magnetic field
- Orbits are highly predictable
  - Prediction accuracy degrades with time
- Orbits can be described by using a set of "orbital parameters"
  - Six classic orbital parameters
  - Additional parameters to handle orbit variations over time

Classical Orbital Elements (1/2)

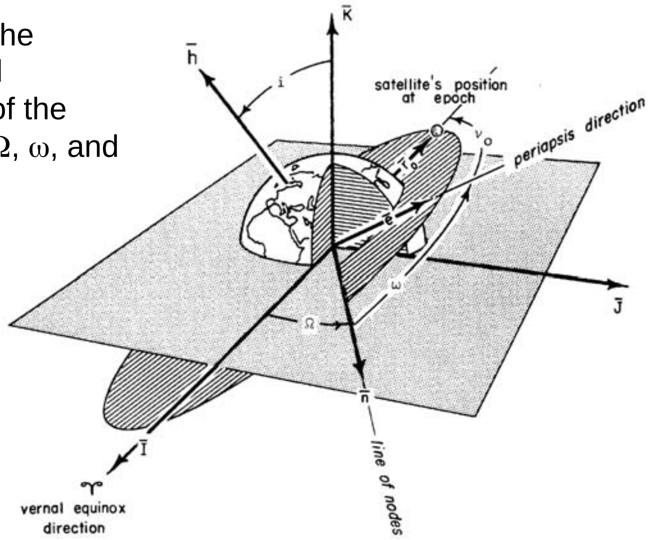
- Classical orbital elements describe
  - Shape of the satellite's orbit (and where the satellite would be on that shape)
  - Position of the orbit relative to inertial (or Earth-fixed) space
- Describing the orbit shape with a, e, and τ



- If given a specific time, you can calculate the position of the satellite on this ellipse

Classical Orbital Elements (2/2)

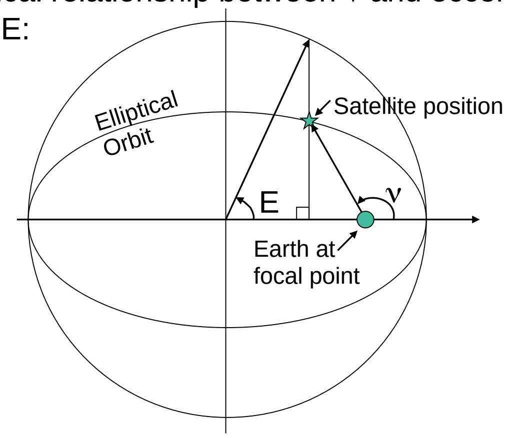
- Describing the position and orientation of the orbit using Ω, ω, and i



Bate, Mueller, and White, "Fundamentals of Astrodynamics," Dover Publications, 1971

Relationship Between True, Eccentric, and Mean Anomaly (1/2)

- True anomaly ν used to directly calculate satellite position on ellipse
- Geometrical relationship between ν and eccentric anomaly E:



Relationship Between True, Eccentric, and Mean Anomaly (2/2)

- Mean anomaly M varies linearly with time (unlike E or ν), so it can be easily calculated

$$M(t) = M_0 + n(t - t_0)$$
$$M_0 = M(t_0)$$
$$n = \sqrt{\frac{\mu}{a^3}} = \text{mean motion}$$

- Eccentric anomaly and mean anomaly related through Kepler's equation

$$M = E - e\sin E$$

- Finally, true anomaly calculated from arctangent* function, using

$$\sin\nu = \frac{\sqrt{1-e^2}\sin E}{1 - e\cos E} \qquad \cos\nu = \frac{\cos E - e}{1 - e\cos E}$$

*Be sure to use the 4-quadrant arctangent function (atan2 in MATLAB).

Where Do We Get the Ephemeris Data? Legacy L1 and L2 Signal Breakdown

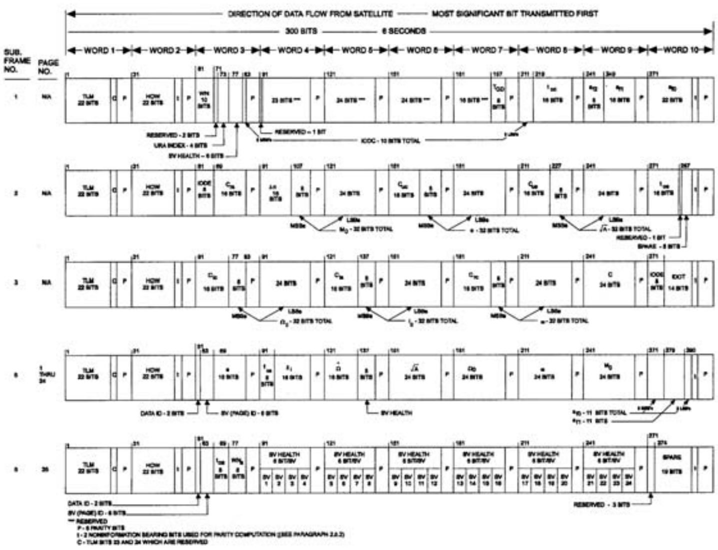
- Note: 50 bps navigation message modulated on all of the codes
- L1 signal
  - P(Y)-code
  - C/A-code modulated on carrier that is 90° out of phase from P-code carrier

$$s_{L1}(t) = \underbrace{A_{P_{L1}} Y(t) N(t)\cos(\omega_1 t)}_{\text{P(Y)-Code}} + \underbrace{A_{C/A} CA(t) N(t)\sin(\omega_1 t)}_{\text{C/A-Code}}$$
$$N(t) = \text{50 bps navigation message}$$
$$A_{P_{L1}} = \text{Amplitude of L1 P-code signal} \approx \text{-163 dBW}$$
$$A_{C/A} = \text{Amplitude of C/A-code signal} \approx \text{-160 dBW}$$
$$\omega_1 = 2\pi f_{L1}$$

- L2 signal
  - P-code only $s_{L2}(t) = \underbrace{A_{P_{L2}} Y(t) N(t)\cos(\omega_2 t)}_{\text{P(Y)-Code}}$

$$A_{P_{L2}} = \text{Amplitude of L2 P-code signal} \approx \text{-166 dBW}$$
$$\omega_2 = 2\pi f_{L2}$$

Data Format of Subframes 1, 2, 3, and 5



Obtained from SPS Signal Spec (http://www.spacecom.af.mil/usspace/gps_support/documents/SPSMAIN.pdf)

GPS Ephemeris Data (From Navigation Message)

- For defining orbit shape and timing

t_{0_e} = Reference time of ephemeris (sec)

$\sqrt{a}$ = Square root of semi-major axis (m$^{1/2}$)

e = Eccentricity

M_0 = Mean anomaly at time t_{0_e} (rad)

- For defining orientation/position of orbit

i_0 = inclination at time t_{0_e} (rad)

Ω_0 = Longitude of ascending node at t_{0_e} (rad)

ω = Argument of perigee at t_{0_e} (rad)

- Correction Terms

$\dot{i}$ = Rate of change of inclination (rad/sec)

$\dot{\Omega}$ = Rate of change of Ω (rad/sec)

Δn = Mean motion correction (rad/sec)

C_{uc}, C_{us} = Argument of latitude correction coefficients

C_{rc}, C_{rs} = Orbital radius correction coefficients

C_{ic}, C_{is} = Inclination correction coefficients

Sample Ephemeris Values
(PRN 10 - 20 Jan 1999)

```
      prn: 10
     week: 993
      t0e: 266400
   sqrt_a: 5.1537e+003
        e: 0.0032
       M0: -0.1952
       i0: 0.9694
   Omega0: -0.7958
    omega: -0.2041
     idot: -3.0894e-010
Omegadot: -8.4571e-009
  delta_n: 4.6345e-009
```

```
  Cuc: -3.9022e-006
  Cus: 2.3618e-006
  Crc: 339.4063
  Crs: -73.9375
  Cic: 1.8626e-009
  Cis: -3.9116e-008
  toc: 266400
  af0: 3.1394e-005
  af1: 5.6843e-013
  af2: 0
  tgd: -1.8626e-009
valid: 1
```

SV Clock Correction Terms (toc, af0, af1, af2)

Calculating Satellite Position

- Set of equations for calculating SV position from ephemeris is given in ICD-GPS-200C (Table 20-IV)
  - IS-GPS-200D can be found at
    www.navcen.uscg.gov/pdf/IS-GPS-200D.pdf
  - Comments
    - Make sure that the correct quadrant is determined when calculating true anomaly (use “atan2” function or equivalent)
    - Output x_k, y_k, z_k are the ECEF coordinates of the SV antenna phase center at time t (in the ECEF coordinate frame at time t)

IS-GPS-200D: Solving for Satellite Position (1 of 4)

Variables obtained from navigation message highlighted with box: $\square$

Table 20-IV. Elements of Coordinate Systems (sheet 1 of 2)	
$\mu = 3.986005 \times 10^{14}$ meters^3/sec^2	WGS 84 value of the earth's gravitational constant for GPS user
$\dot{\Omega}_e = 7.2921151467 \times 10^{-5}$ rad/sec	WGS 84 value of the earth's rotation rate
$A = \left(\boxed{\sqrt{A}}\right)^2$	Semi-major axis
$n_0 = \sqrt{\frac{\mu}{A^3}}$	Computed mean motion (rad/sec)
$t_k = t - \boxed{t_{oe}}$*	Time from ephemeris reference epoch

* t is GPS system time at time of transmission, i.e., GPS time corrected for transit time (range/speed of light). Furthermore, t_k shall be the actual total time difference between the time t and the epoch time t_{oe}, and must account for beginning or end of week crossovers. That is, if t_k is greater than 302,400 seconds, subtract 604,800 seconds from t_k. If t_k is less than -302,400 seconds, add 604,800 seconds to t_k.

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IS-GPS-200D: Solving for Satellite Position (2 of 4)

Variables obtained from navigation message highlighted with box: $\square$

Table 20-IV. Elements of Coordinate Systems (sheet 1 of 2) (continued)		
$n = n_0 + \boxed{\Delta n}$		Corrected mean motion
$M_k = \boxed{M_0} + nt_k$		Mean anomaly
$M_k = E_k - \boxed{e}\sin E_k$	← See upcoming slide for how to solve for E_k	Kepler's Equation for Eccentric Anomaly (may be solved by iteration) (radians)
$\nu_k = \tan^{-1}\left\{\frac{\sin \nu_k}{\cos \nu_k}\right\}$	← For informational purposes only—not needed in calculations	True Anomaly
$= \tan^{-1}\left\{\frac{\sqrt{1-\boxed{e}^2}\sin E_k / (1-\boxed{e}\cos E_k)}{(\cos E_k - \boxed{e}) / (1-\boxed{e}\cos E_k)}\right\}$	← Use four-quadrant arctan function ("atan2" in MATLAB)	

IS-GPS-200D, 7 Dec 2004, p 97

IS-GPS-200D: Solving for Satellite Position (3 of 4)

Variables obtained from navigation message highlighted with box: $\boxed{}$

Table 20-IV. Elements of Coordinate Systems (sheet 2 of 2)

$E_k = \cos^{-1}\left\{\dfrac{e+\cos v_k}{1+e\cos v_k}\right\}$	← For informational purposes only—not needed in calculations	Eccentric Anomaly
$\Phi_k = v_k + \boxed{\omega}$		Argument of Latitude
$\delta u_k = \boxed{c_{us}}\sin 2\Phi_k + \boxed{c_{uc}}\cos 2\Phi_k$	Argument of Latitude Correction	Second Harmonic Perturbations
$\delta r_k = \boxed{c_{rs}}\sin 2\Phi_k + \boxed{c_{rc}}\cos 2\Phi_k$	Radius Correction	
$\delta i_k = \boxed{c_{is}}\sin 2\Phi_k + \boxed{c_{ic}}\cos 2\Phi_k$	Inclination Correction	
$u_k = \Phi_k + \boxed{\delta u_k}$		Corrected Argument of Latitude
$r_k = A(1 - \boxed{e}\cos E_k) + \delta r_k$		Corrected Radius

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IS-GPS-200D: Solving for Satellite Position (4 of 4)

Variables obtained from navigation message highlighted with box: $\boxed{}$

Table 20-IV. Elements of Coordinate Systems (sheet 2 of 2) (continued)

$i_k = \boxed{i_0} + \delta i_k + (\boxed{IDOT})\, t_k$	Corrected Inclination
$x_k' = r_k\cos u_k$ $y_k' = r_k\sin u_k$	Positions in orbital plane.
$\Omega_k = \boxed{\Omega_0} + (\boxed{\dot{\Omega}} - \dot{\Omega}_e)\, t_k - \dot{\Omega}_e \boxed{t_{oe}}$	Corrected longitude of ascending node.
$x_k = x_k'\cos\Omega_k - y_k'\cos i_k\sin\Omega_k$ $y_k = x_k'\sin\Omega_k + y_k'\cos i_k\cos\Omega_k$ $z_k = y_k'\sin i_k$	Earth-fixed coordinates.

Solution to Kepler's Equation

- Kepler's equation, though simple in form, has no known closed-form solution
  - All solutions are iterative (or approximate)
- Newton's method

$$E_0 = M$$
$$E_{j+1} = E_j + \frac{M - (E_j - e\sin E_j)}{1 - e\cos E_j}$$

- Method used in RCVR-3A software specification

$$E_0 = M + e\sin M$$
$$E_{j+1} = \frac{e(\sin E_j - E_j \cos E_j) + M}{1 - e\cos E_j}$$

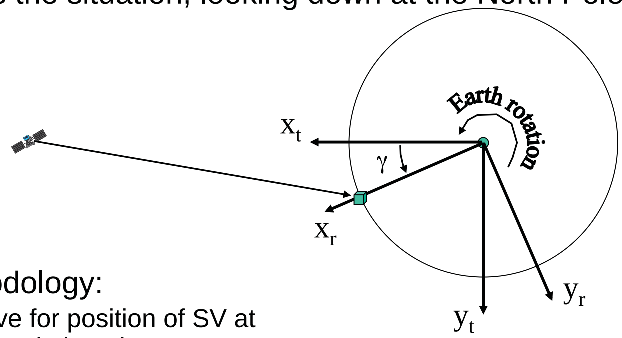
  - RCVR-3A performs two iterations (i.e., stops calculating at E_2)
  - Don't confuse these subscripts with subscripts in ephemeris equations!

Accounting for Signal Travel Time (1 of 3)

- Signal arrives at receiver ***after*** it is transmitted (due to signal travel time)
  - Transmit time: Time the signal was transmitted
  - Receive time: Time the signal was received
- Satellite position should be calculated based upon transmit time
  - When measuring a signal, we don't really care what happened after that signal was transmitted
  - Transmit time should be GPS system time (or as close to it as possible)
  - Very good approximate value of transmit time obtained by subtracting pseudorange (expressed in seconds) from the receive time as indicated by the receiver clock
    - Why?
- What other considerations do we need to make for signal travel time?

Accounting for Signal Travel Time (2 of 3)

- Here's the situation, looking down at the North Pole



- Methodology:
  - Solve for position of SV at transmit time, in ECEF coordinates at transmit time (x_t, y_t, and z_t) using ICD-GPS-200 equations
  - Rotate into ECEF reference frame at the time of reception:

$$\begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix}$$

$$\gamma = \dot{\Omega}_e t_{prop}$$

t_{prop} = Signal propagation time

Accounting for Signal Travel Time (3 of 3)

- Neglecting atmospheric delay, the signal propagation time is calculated by

$$t_{prop} = \frac{\text{geometric range to satellite}}{\text{speed of light}} = \frac{\left|\boldsymbol{p}_{sv} - \boldsymbol{p}_{rcvr}\right|}{c}$$

$\boldsymbol{p}_{sv}$ = satellite ECEF position vector

$\boldsymbol{p}_{rcvr}$ = receiver ECEF position vector

- Note that the satellite position is needed to calculate t_{prop} (and vice-versa)
  - Satellite position in ECEF coordinates at transmit time is sufficiently accurate (x_t, y_t, and z_t)
  - Note that receiver position must be known
    - Can be approximate

Overview

Pseudorange Equation

- The pseudorange is the sum of the true range plus the receiver clock error
  - We're assuming (for now) that the receiver clock error is the only remaining error
    - SV clock error has been corrected for
    - All other errors are deemed negligible (or have been corrected)

$$\begin{aligned}\rho_j &= \sqrt{(x_j - x_u)^2 + (y_j - y_u)^2 + (z_j - z_u)^2} + c\delta t_u \\ &= f(x_u, y_u, z_u, \delta t_u)\end{aligned}$$

$$\begin{aligned}\rho_j &= \text{pseudorange measurement from satellite } j \text{ (m)} \\ x_j, y_j, z_j &= \text{ECEF position of satellite } j \text{ (m)} \\ x_u, y_u, z_u &= \text{ECEF position of user (m)} \\ \delta t_u &= \text{receiver clock error (sec)}\end{aligned}$$

- For now, only use one type of pseudorange (L1 C/A, L1 P, or L2 P)

Statement of the Problem

- At a given measurement epoch, the GPS receiver generates n pseudorange measurements (from n different satellites)

$$\begin{aligned}
\rho_1 &= \sqrt{(x_1 - x_u)^2 + (y_1 - y_u)^2 + (z_1 - z_u)^2} + c\delta\, t_u \\
\rho_2 &= \sqrt{(x_2 - x_u)^2 + (y_2 - y_u)^2 + (z_2 - z_u)^2} + c\delta\, t_u \\
\rho_3 &= \sqrt{(x_3 - x_u)^2 + (y_3 - y_u)^2 + (z_3 - z_u)^2} + c\delta\, t_u \\
&\vdots \\
\rho_n &= \sqrt{(x_n - x_u)^2 + (y_n - y_u)^2 + (z_n - z_u)^2} + c\delta\, t_u
\end{aligned}$$

- Goal: Determine user position and clock error, expressed in state-vector form as

$$\mathbf{x} = \begin{bmatrix} x_u \\ y_u \\ z_u \\ c\delta\, t_u \end{bmatrix}$$

Solving the Pseudorange Equations

- The n pseudorange equations are non-linear (so no easy solution)
- Ways to solve
  - Closed form solutions
    - Complicated
    - May not give as much insight
  - Iterative techniques based on linearization
    - Often using least-squares estimation
    - Arguably the simplest approach
    - Approach covered in this course
  - Kalman filtering
    - Similar to least-squares approach, except with additional ability to handle measurements over a period of time
    - Will discuss briefly
- What is linearization?
  - Pick a nominal (or approximate) solution
  - Linearize about that point, resulting in a set of linear equations
  - Solve the linear equations
  - Will use Taylor series expansion for linearization

Taylor Series Expansion (1/2)

- Taylor series expansion (1 variable)

$$f(a+\Delta a) = f(a) + \Delta a \frac{df}{da} + \frac{(\Delta a)^2}{2!}\frac{d^2 f}{da^2} + \frac{(\Delta a)^3}{3!}\frac{d^3 f}{da^3} + \ldots$$

- This can be used to linearize about a certain value of the independent variable a.
  - Example: the function $f(t) = 2 + 3t - 6t^2$ is a non-linear function in t
  - Suppose we want to linearize about the point $\hat{t} = 2$
  - The complete Taylor series expression is

$$f(\hat{t} + \Delta t) = f(\hat{t}) + \Delta t \frac{df}{dt} + \frac{(\Delta t)^2}{2!}\frac{d^2 f}{dt^2}$$

$$= 2 + 3\hat{t} - 6\hat{t}^2 + \Delta t(3 - 12\hat{t}) + \frac{(\Delta t)^2}{2}(-12)$$

  - To linearize, we set $\hat{t} = 2$ and neglect higher order (non-linear) terms of Δt
    - Valid for perturbations (i.e., small values of Δt)

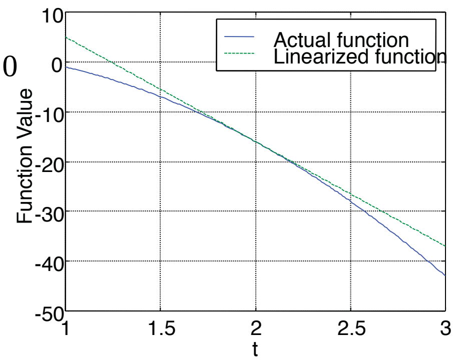
Taylor Series Expansion (2/2)

- (Continued example) Linearized form

$$f(\hat{t} + \Delta t)\Big|_{\hat{t}=2} = f(\hat{t}=2) + \Delta t \frac{df}{dt}\Big|_{\hat{t}=2} + \cancelto{0}{\frac{(\Delta t)^2}{2!}\frac{d^2 f}{dt^2}}\Big|_{\hat{t}=2}$$

$$= 2 + 3(2) - 6(2^2) + \Delta t(3 - 12(2))$$

$$= -16 - 21\Delta t$$



- First order Taylor series for function in two variables:

$$f(\hat{a} + \Delta a, \hat{b} + \Delta b) = f(\hat{a}, \hat{b}) + \Delta a \frac{\partial f}{\partial a}\Big|_{\hat{a},\hat{b}} + \Delta b \frac{\partial f}{\partial b}\Big|_{\hat{a},\hat{b}} + \text{h.o.t.}$$

Linearization of Pseudorange Equations (1/5)

- First, define a nominal state (position and clock error) as

$$\hat{\mathbf{x}} = \begin{bmatrix} \hat{x}_u \\ \hat{y}_u \\ \hat{z}_u \\ c\delta\hat{t}_u \end{bmatrix} = \text{nominal (approximate) state}$$

- An approximate (or expected) pseudorange can then be calculated for satellite *j*

$$\begin{aligned} \hat{\rho}_j &= \sqrt{(x_j - \hat{x}_u)^2 + (y_j - \hat{y}_u)^2 + (z_j - \hat{z}_u)^2} + c\delta\hat{t}_u \\ &= f(\hat{x}_u, \hat{y}_u, \hat{z}_u, c\delta\hat{t}_u) \end{aligned}$$

  - This approximate (expected) pseudorange is the pseudorange that we would expect to have if our position and clock error were actually $\hat{x}_u, \hat{y}_u, \hat{z}_u$, and $c\delta\hat{t}_u$.

Linearization of Pseudorange Equations (2/5)

- Relationship between true and approximate position and time

$$\begin{aligned} x_u &= \hat{x}_u + \Delta x_u \\ y_u &= \hat{y}_u + \Delta y_u \\ z_u &= \hat{z}_u + \Delta z_u \\ c\delta t_u &= c\delta\hat{t}_u + \Delta c\delta t_u \end{aligned}$$

  - Vector form:

$$\mathbf{x}_u = \hat{\mathbf{x}}_u + \Delta\mathbf{x}_u$$

- Based on these relations, we can write

$$f(x_u, y_u, z_u, c\delta t_u) = f(\hat{x}_u + \Delta x_u, \hat{y}_u + \Delta y_u, \hat{z}_u + \Delta z_u, c\delta\hat{t}_u + \Delta c\delta t_u)$$

- To linearize, right-hand side of equation can be evaluated using a first order Taylor series expansion

Linearization of Pseudorange Equations (3/5)

- First order Taylor series expansion of pseudorange function:

$$\begin{aligned} f(\hat{x}_u+\Delta x_u, \hat{y}_u+\Delta y_u, \hat{z}_u+\Delta z_u, c\delta\hat{t}_u+\Delta c\delta t_u) &= f(\hat{x}_u, \hat{y}_u, \hat{z}_u, c\delta\hat{t}_u) \\ &+ \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, c\delta\hat{t}_u)}{\partial \hat{x}_u}\Delta x_u + \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, c\delta\hat{t}_u)}{\partial \hat{y}_u}\Delta y_u \\ &+ \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, c\delta\hat{t}_u)}{\partial \hat{z}_u}\Delta z_u + \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, c\delta\hat{t}_u)}{\partial c\delta\hat{t}_u}\Delta c\delta t_u \\ &+ \text{h.o.t.} \end{aligned}$$

- The partial derivatives are

$$\frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \delta\hat{t}_u)}{\partial \hat{x}_u} = -\frac{x_j-\hat{x}_u}{\hat{r}_j} \qquad \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \delta\hat{t}_u)}{\partial \hat{y}_u} = -\frac{y_j-\hat{y}_u}{\hat{r}_j}$$

$$\frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \delta\hat{t}_u)}{\partial \hat{z}_u} = -\frac{z_j-\hat{z}_u}{\hat{r}_j} \qquad \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \delta\hat{t}_u)}{\partial c\delta\hat{t}_u} = 1$$

$$\hat{r}_j = \sqrt{(x_j-\hat{x}_u)^2+(y_j-\hat{y}_u)^2+(z_j-\hat{z}_u)^2}$$

Linearization of Pseudorange Equations (4/5)

- Using above results, linearized pseudorange equation is

$$\rho_j = \hat{\rho}_j - \frac{x_j-\hat{x}_u}{\hat{r}_j}\Delta x_u - \frac{y_j-\hat{y}_u}{\hat{r}_j}\Delta y_u - \frac{z_j-\hat{z}_u}{\hat{r}_j}\Delta z_u + \Delta c\delta t_u$$

- This can be simplified to $\Delta\rho_j = a_{xj}\Delta x_u + a_{yj}\Delta y_u + a_{zj}\Delta z_u - \Delta c\delta t_u$ where

$$\Delta\rho_j = \hat{\rho}_j - \rho_j$$

$$a_{xj} = \frac{x_j-\hat{x}_u}{\hat{r}_j}, \qquad a_{yj} = \frac{y_j-\hat{y}_u}{\hat{r}_j}, \qquad a_{zj} = \frac{z_j-\hat{z}_u}{\hat{r}_j}$$

Linearization of Pseudorange Equations (5/5)

- Original (nonlinear) equations for n measurements

$$\begin{aligned}
\rho_1 &= \sqrt{(x_1 - x_u)^2 + (y_1 - y_u)^2 + (z_1 - z_u)^2} + c\delta\, t_u \\
\rho_2 &= \sqrt{(x_2 - x_u)^2 + (y_1 - y_u)^2 + (z_1 - z_u)^2} + c\delta\, t_u \\
\rho_3 &= \sqrt{(x_3 - x_u)^2 + (y_3 - y_u)^2 + (z_3 - z_u)^2} + c\delta\, t_u \\
&\vdots \\
\rho_n &= \sqrt{(x_n - x_u)^2 + (y_n - y_u)^2 + (z_n - z_u)^2} + c\delta\, t_u
\end{aligned}$$

- Linearized (error) equations for the same n measurements

$$\begin{aligned}
\Delta\rho_1 &= a_{x1}\Delta x_u + a_{y1}\Delta y_u + a_{z1}\Delta z_u - \Delta c\delta t_u \\
\Delta\rho_2 &= a_{x2}\Delta x_u + a_{y2}\Delta y_u + a_{z2}\Delta z_u - \Delta c\delta t_u \\
\Delta\rho_3 &= a_{x3}\Delta x_u + a_{y3}\Delta y_u + a_{z3}\Delta z_u - \Delta c\delta t_u \\
&\vdots \\
\Delta\rho_n &= a_{xn}\Delta x_u + a_{yn}\Delta y_u + a_{zn}\Delta z_u - \Delta c\delta t_u
\end{aligned}$$

Solving the Linearized Pseudorange Equations Using Least-Squares (1/2)

- We can express the set of pseudorange equations in matrix form

$$\Delta\boldsymbol{\rho} = \mathbf{H}\Delta\mathbf{x}$$

$$\Delta\boldsymbol{\rho} = \begin{bmatrix} \Delta\rho_1 \\ \Delta\rho_2 \\ \Delta\rho_3 \\ \vdots \\ \Delta\rho_n \end{bmatrix} \qquad \mathbf{H} = \begin{bmatrix} a_{x1} & a_{y1} & a_{z1} & -1 \\ a_{x2} & a_{y2} & a_{z2} & -1 \\ a_{x3} & a_{y3} & a_{z3} & -1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{xn} & a_{yn} & a_{zn} & -1 \end{bmatrix} \qquad \Delta\mathbf{x} = \begin{bmatrix} \Delta x_u \\ \Delta y_u \\ \Delta z_u \\ \Delta c\delta t_u \end{bmatrix}$$

- Three possible cases
  - $n < 4$: Underdetermined case
    - Cannot solve for $\Delta\mathbf{x}$
    - Is there still useable information?
  - $n = 4$: Uniquely determined case
    - One valid solution for $\Delta\mathbf{x}$ (generally)
    - Solved by calculating $\mathbf{H}^{-1}$ ($\Delta\mathbf{x} = \mathbf{H}^{-1}\Delta\boldsymbol{\rho}$)
  - $n > 4$: Overdetermined case
    - No solution that perfectly solves equation (generally)
    - Can use least-squares techniques (which pick solution that minimizes the square of the error)

Solving the Linearized Pseudorange Equations Using Least-Squares (2/2)

- Basic least-squares solution (no measurement weighting)

$$\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\boldsymbol{\rho}$$

  - Reasonable approach for single-point positioning in presence of SA
- Solution with measurement weighting (weighted least-squares)
  - Useful when
    - Measurements have different error statistics
    - Measurement errors are correlated
  - Measurement error covariance matrix $\mathbf{C}_\rho$
    - Diagonal terms are measurement error variances
    - Off-diagonal terms show cross-correlation between measurement errors

$$\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{C}_\rho^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{C}_\rho^{-1}\Delta\boldsymbol{\rho}$$

  - Note that this is identical to unweighted case if $\mathbf{C}_\rho = \mathbf{I}$ (identity matrix)

Measurement Residuals

- For overdetermined system, generally no valid solution for $\Delta\mathbf{x}$ that solves measurement equation, so

$$\Delta\boldsymbol{\rho} \neq \mathbf{H}\Delta\mathbf{x}$$

- Measurement residuals ($\mathbf{v}$)
  - Corrections that, when applied to measurements, would result in solution of above equation
  - Least-squares minimizes the sum of squares of these residuals

$$\mathbf{v} = \Delta\boldsymbol{\rho} - \mathbf{H}\Delta\mathbf{x}$$

$$\Delta\boldsymbol{\rho} = \mathbf{H}\Delta\mathbf{x} + \mathbf{v}$$

Iterating the Nominal State

- Linearized equations (and resulting $\mathbf{H}$ matrix) calculated using nominal state $\hat{\mathbf{x}}_u$
- Linearization valid when
  - Nominal state is close to true state
  - $\Delta\mathbf{x}$ is "small"
- If $\hat{\mathbf{x}}_u$ is not very accurate (i.e., $\Delta\mathbf{x}$ is large), iteration is required
  - For each iteration, a new value of $\hat{\mathbf{x}}_u$ is calculated based upon the old value and the corrections $\Delta\mathbf{x}$

$$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$$

  - This new value of $\hat{\mathbf{x}}_u$ is then used to recalculate the corrections $\Delta\mathbf{x}$ (which should be smaller this time)
- Solution must converge
  - For standard GPS positioning, not much of a problem (will generally converge with an initial guess at the center of the Earth)
  - For more non-linear situations (e.g., using pseudolites), this can be more of a problem

Correcting for Satellite Clock Error

- Single point positioning only estimates receiver clock error
  - Assumes all other errors are negligible
  - Requires correction of satellite clock error
- Clock correction (from IS-GPS-200D)

$$\rho_{corr} = \rho + c\Delta t_{sv}$$

$$\Delta t_{sv} = a_{f_0} + a_{f_1}(t - t_{0_c}) + a_{f_2}(t - t_{0_c})^2 + \Delta t_r$$

$$\Delta t_r = Fe\sqrt{a}\sin E_k$$

ρ_{corr} = pseudorange corrected for SV clock error
ρ = original (raw) pseudorange measurement
Δt_{sv} = SV clock correction
$a_{f_0}, a_{f_1}, a_{f_2}, t_{0_c}$ = SV clock correction parameters from nav message
Δt_r = relativity correction (since not circular orbit)
F = constant = $-4.442807633 \times 10^{-10}$ sec/(meter)$^{1/2}$
e = eccentricity from nav message
$\sqrt{a}$ = square root of semi-major axis from nav message
E_k = Eccentric anomaly (from SV position calculation)

Determining Signal Transmit Time (1/2)

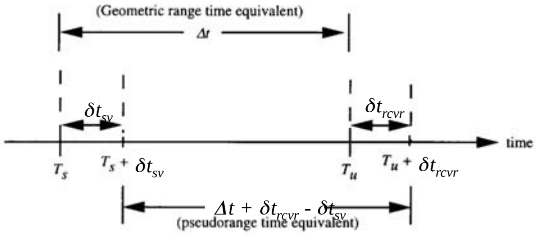
- For satellite position calculation, need true GPS transmit time of the signal (T_s)
  - Receiver provides time of reception according to the receiver clock ($T_u + \delta t_{rcvr}$)
  - From diagram below, if the pseudorange time equivalent is subtracted from the receive time, then the result is the true transmit time plus the satellite clock error

$$\underbrace{T_u + \delta t_{rcvr}}_{\text{receive time}} - \frac{PR}{c} = T_s + \delta t_{sv}$$

PR = pseudorange measurement (m)

$$\underbrace{T_u + \delta t_{rcvr}}_{\text{receive time}} - \frac{PR}{c} - \underbrace{\delta t_{sv}}_{} = T_s$$

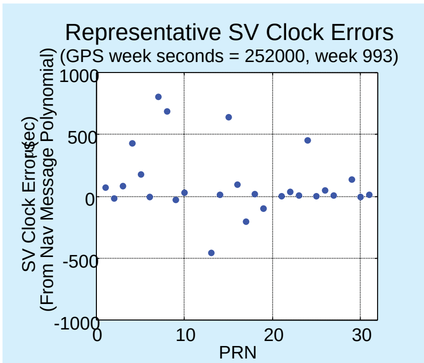
same as Δt_{sv} from the previous slide



Determining Signal Transmit Time (2/2)

- Effect of neglecting δt_{sv} for SV positioning[1]
  - Satellite clock error can grow to up to ~1 msec:
  - Typical satellite velocity is 3900 m/s
  - Worst-case position error from neglecting δt_{sv}

    3900 m/s x 0.001 s = **3.9 m**
  - Effect of neglecting δt_{sv}
    - Single point positioning: Can be significant (but not with SA)
    - Differential positioning: effectively cancelled out (acts like 3.9 m satellite position error)



[1]The SV clock error δt_{sv} will have a significant effect on the actual pseudorange measurement. This page only describes the impact of δt_{sv} on determining the position of the satellite.

Use of Dual Frequency Measurements to Calculate Ionospheric Delay

- L1 ionospheric delay calculated by

$$\Delta S_{iono,corr_{L1}} = \left(\frac{f_2^2}{f_2^2 - f_1^2}\right)(\rho_{L1} - \rho_{L2})$$

$$\Delta S_{iono,corr_{L1}} = \text{L1 ionospheric delay (m)}$$

$$f_1, f_2 = \text{L1 and L2 carrier frequencies}$$

$$\rho_{L1}, \rho_{L2} = \text{L1 and L2 pseudorange measurements}$$

- L2 ionospheric delay can be calculated by

$$\Delta S_{iono,corr_{L2}} = \left(\frac{f_1}{f_2}\right)^2 \Delta S_{iono,corr_{L1}}$$

- Ionospheric-free pseudorange:

$$\rho_{IF} = \frac{\rho_{L2} - \gamma\rho_{L1}}{1-\gamma}, \qquad \gamma = \left(\frac{f_{L1}}{f_{L2}}\right)^2 = \left(\frac{77}{60}\right)^2$$

- Multipath and measurement noise will corrupt this measurement of ionosphere

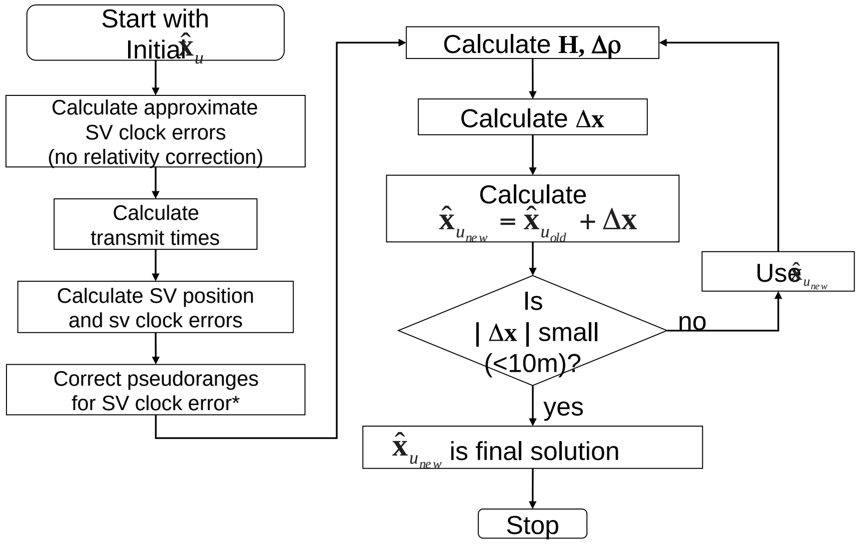
Correcting for Satellite Group Delay

- Each satellite has a slight time bias between the L1 and the L2 signals
  - Not desired, but it's there nonetheless
  - Will affect dual-frequency users, unless it's accounted for
  - Can be measured and/or calibrated out
  - This calibration is accounted for when the control segment generates the satellite clock correction terms from broadcast nav message: $a_{f_0}, a_{f_1}, a_{f_2}$, and t_{0_c}
  - However, this is all designed for the dual-frequency user! Single frequency users need to remove the effect of this dual-frequency correction on their Δt_{sv} value
- Single frequency users must apply the group delay term (TGD) from the nav message to their SV clock correction term (from p. 90 of ICD-GPS-200C)

$$(\Delta t_{sv})_{L1} = \Delta t_{sv} - T_{GD}$$

$$(\Delta t_{sv})_{L2} = \Delta t_{sv} - \left(\frac{77}{60}\right)^2 T_{GD}$$

Single Point Positioning Algorithm



*include group delay correction, if a single-frequency user

GPS Positioning Example

- We'll look at a single case to give an example
- Situation
  - Receiver measurement time (GPS week seconds): 220937
  - Initial $\hat{\mathbf{x}}_u$: [506071.529 -4882278.667 4109624.557 15.807]

    Initial guess of position (in error by ~50 km)

    Initial clock error expressed in m
  - Measurements:

PRN	Pseudorange
12	25022759.323
2	22075351.532
26	21929350.580
15	22677087.545
29	21039894.608
21	24757444.127
30	24032696.422

Example: Calculation of Transmit Time

Start with Initial $\hat{\mathbf{x}}_u$

Calculate approximate SV clock errors (no relativity correction)

Calculate transmit times

Calculate SV position and sv clock errors

Correct pseudoranges for SV clock error*

$$\Delta t_{sv_{approx}} = a_{f_0} + a_{f_1}(t - t_{0_c}) + a_{f_2}(t - t_{0_c})^2$$

(Receive time = 220937)

PRN	PR/c (s)	Approx SV Clock Error (s)	Transmit Time (s)
12	0.083466941	-0.000359172	220936.916892231
2	0.073635447	0.000199964	220936.926164589
26	0.073148440	0.000247391	220936.926604169
15	0.075642622	-0.000107130	220936.924464508
29	0.070181534	-0.000043329	220936.929861795
21	0.082581944	0.000065081	220936.917352974
30	0.080164446	0.000075192	220936.919760362

$$\underbrace{T_u + \delta t_{rcvr}}_{\text{receive time}} - \frac{PR}{c} - \underbrace{\delta t_{sv}} = T_s$$

Example: SV Position and Clock Error and Pseudorange Correction

Start with Initial $\hat{\mathbf{x}}_u$

Calculate approximate SV clock errors (no relativity correction)

Calculate transmit times

Calculate SV position and sv clock errors

Correct pseudoranges for SV clock error*

PRN	SV ECEF X (m)	SV ECEF Y (m)	SV ECEF Z (m)	SV Clock Err (s)
12	-9924909.896	-22418412.217	-10238600.462	-0.000359164
2	19519446.654	-12864169.870	12106498.214	0.000199964
26	9518973.908	-24465469.002	-347289.566	0.000247426
15	6420995.137	-25601178.700	-2907089.329	-0.000107131
29	-8265550.815	-16497554.935	19234406.500	-0.000043326
21	-22299549.612	-6458590.524	13615554.839	0.000065090
30	-18044425.181	-19566072.431	-289953.964	0.000075168

Orbital calculations + time-of-transit rotation correction

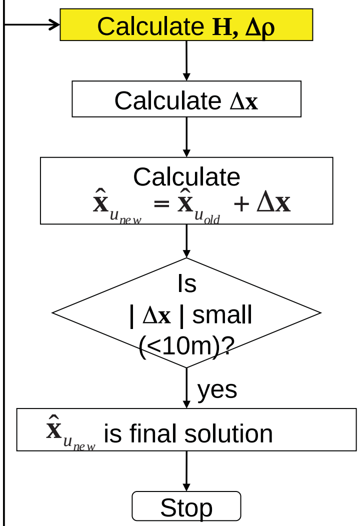
$$\Delta t_{sv} = a_{f_0} + a_{f_1}(t - t_{0_c}) + a_{f_2}(t - t_{0_c})^2 + \Delta t_r$$

PRN	Original PR (m)	Corrected PR (m)	Group Delay Tgd (ns)
12	25022759.323	24915088.264	-12.107193470
2	22075351.532	22135304.471	-17.229467630
26	21929350.580	22003529.125	-6.519258022
15	22677087.545	22644973.532	-10.244548320
29	21039894.608	21026908.513	-8.847564459
21	24757444.127	24776961.101	-11.641532180
30	24032696.422	24055233.876	-8.381903172

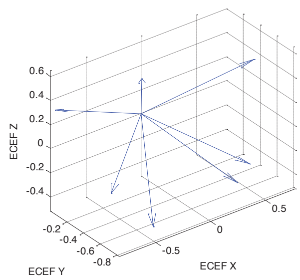
$$\rho_{corr} = \rho + c(\Delta t_{sv} - T_{GD})$$

*include group delay correction, if a single-frequency user

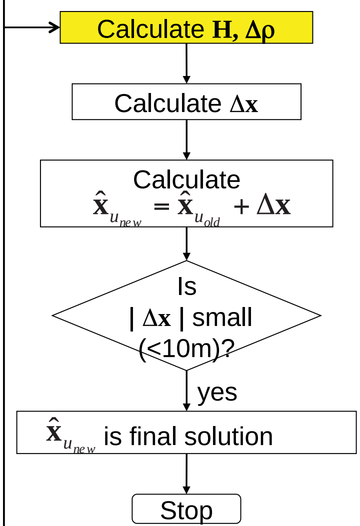
Example: H Matrix (Iteration 1)



$$\mathbf{H} = \begin{bmatrix} -0.4182 & -0.7030 & -0.5752 & -1 \\ 0.8597 & -0.3609 & 0.3616 & -1 \\ 0.4094 & -0.8896 & -0.2025 & -1 \\ 0.2610 & -0.9143 & -0.3096 & -1 \\ -0.4179 & -0.5533 & 0.7205 & -1 \\ -0.9212 & -0.0637 & 0.3840 & -1 \\ -0.7709 & -0.6102 & -0.1828 & -1 \end{bmatrix}$$



Example: Δρ (Iteration 1)



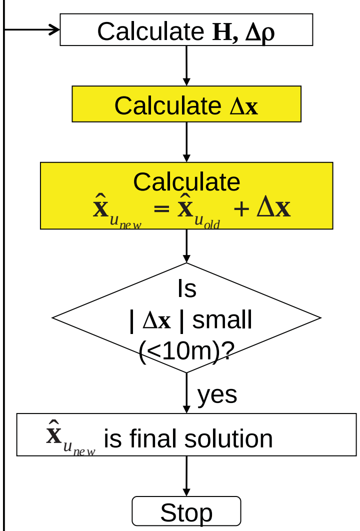
Calculated

Measured (corrected)

$$\Delta\rho = \hat{\rho} - \rho_{corr}$$

PRN	Calculated PR	Measured PR	Delta-Rho
12	24943810.919	24915088.264	28722.655
2	22117181.292	22135304.471	-18123.179
26	22013598.807	22003529.125	10069.682
15	22660408.867	22644973.532	15435.335
29	20990847.857	21026908.513	-36060.657
21	24757718.148	24776961.101	-19242.953
30	24064325.866	24055233.876	9091.990

Example: Solution and Residuals (Iteration 1)



$$\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\rho$$

$$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$$

$\hat{\mathbf{x}}_{u_{new}}$	$\hat{\mathbf{x}}_{u_{old}}$	$\Delta\mathbf{x}$
506068.143	506071.529	-3.386
-4882283.665	-4882278.667	-4.998
4059632.252	4109624.557	-49992.305
63.927	15.807	48.120

Residuals: $\mathbf{v} = \Delta\rho - \mathbf{H}\Delta\mathbf{x}$

PRN	$\mathbf{v}$	$\Delta\rho$	$\mathbf{H}\Delta\mathbf{x}$
12	9.162	28722.655	28713.493
2	1.699	-18123.179	-18124.878
26	-6.800	10069.682	10076.482
15	-0.178	15435.335	15435.513
29	4.853	-36060.657	-36065.510
21	-3.299	-19242.953	-19239.654
30	-5.436	9091.990	9097.426

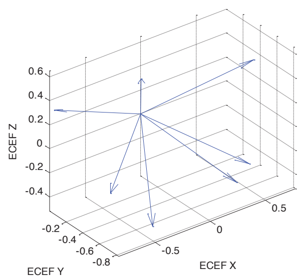
Example: H Matrix (Iterations 1 and 2)

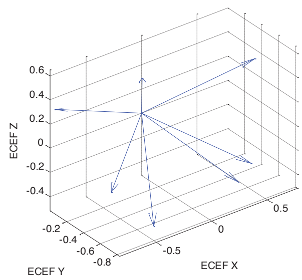
Iteration 1

$$\mathbf{H} = \begin{bmatrix} -0.4182 & -0.7030 & -0.5752 & -1 \\ 0.8597 & -0.3609 & 0.3616 & -1 \\ 0.4094 & -0.8896 & -0.2025 & -1 \\ 0.2610 & -0.9143 & -0.3096 & -1 \\ -0.4179 & -0.5533 & 0.7205 & -1 \\ -0.9212 & -0.0637 & 0.3840 & -1 \\ -0.7709 & -0.6102 & -0.1828 & -1 \end{bmatrix}$$

Iteration 2

$$\mathbf{H} = \begin{bmatrix} -0.4187 & -0.7038 & -0.5739 & -1 \\ 0.8590 & -0.3606 & 0.3635 & -1 \\ 0.4096 & -0.8900 & -0.2003 & -1 \\ 0.2612 & -0.9149 & -0.3076 & -1 \\ -0.4172 & -0.5524 & 0.7217 & -1 \\ -0.9204 & -0.0636 & 0.3857 & -1 \\ -0.7712 & -0.6104 & -0.1808 & -1 \end{bmatrix}$$





Example: $\Delta\rho$ (Iterations 1 and 2)

Iteration 1

Calculated

Measured (corrected)

$$\Delta\rho = \hat{\rho} - \rho_{corr}$$

PRN	Calculated PR	Measured PR	Delta-Rho
12	24943810.919	24915088.264	28722.655
2	22117181.292	22135304.471	-18123.179
26	22013598.807	22003529.125	10069.682
15	22660408.867	22644973.532	15435.335
29	20990847.857	21026908.513	-36060.657
21	24757718.148	24776961.101	-19242.953
30	24064325.866	24055233.876	9091.990

Iteration 2

Calculated

Measured (corrected)

$$\Delta\rho = \hat{\rho} - \rho_{corr}$$

PRN	Calculated PR	Measured PR	Delta-Rho
12	24915130.980	24915088.264	42.716
2	22135355.242	22135304.471	50.771
26	22003576.788	22003529.125	47.662
15	22645023.243	22644973.532	49.711
29	21026941.948	21026908.513	33.435
21	24777000.804	24776961.101	39.703
30	24055278.650	24055233.876	44.773

Example: Solution and Residuals (Iterations 1 and 2)

Iteration 1

$$\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\boldsymbol{\rho}$$

$$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$$

$\hat{\mathbf{x}}_{u_{new}}$	$\hat{\mathbf{x}}_{u_{old}}$	$\Delta\mathbf{x}$
506068.143	506071.529	-3.386
-4882283.665	-4882278.667	-4.998
4059632.252	4109624.557	-49992.305
63.927	15.807	48.120

Residuals: $\mathbf{v} = \Delta\boldsymbol{\rho} - \mathbf{H}\Delta\mathbf{x}$

PRN	$\mathbf{v}$	$\Delta\boldsymbol{\rho}$	$\mathbf{H}\Delta\mathbf{x}$
12	9.162	28722.655	28713.493
2	1.699	-18123.179	-18124.878
26	-6.800	10069.682	10076.482
15	-0.178	15435.335	15435.513
29	4.853	-36060.657	-36065.510
21	-3.299	-19242.953	-19239.654
30	-5.436	9091.990	9097.426

Iteration 2

$$\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\boldsymbol{\rho}$$

$$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$$

$\hat{\mathbf{x}}_{u_{new}}$	$\hat{\mathbf{x}}_{u_{old}}$	$\Delta\mathbf{x}$
506075.869	506068.143	7.726
-4882274.608	-4882283.665	9.057
4059622.275	4059632.252	-9.977
13.120	63.927	-50.807

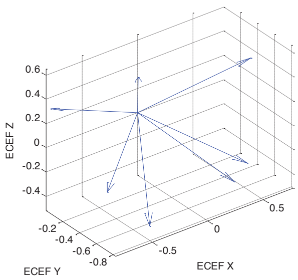
Residuals: $\mathbf{v} = \Delta\boldsymbol{\rho} - \mathbf{H}\Delta\mathbf{x}$

PRN	$\mathbf{v}$	$\Delta\boldsymbol{\rho}$	$\mathbf{H}\Delta\mathbf{x}$
12	-4.208	42.716	46.924
2	0.220	50.771	50.551
26	-0.248	47.662	47.910
15	2.103	49.711	47.609
29	-1.946	33.435	35.381
21	0.431	39.703	39.272
30	3.648	44.773	41.125

Example: H Matrix (Iterations 2 and 3)

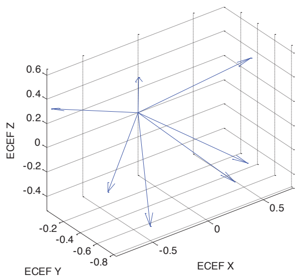
Iteration 2

$$\mathbf{H} = \begin{bmatrix} -0.4187 & -0.7038 & -0.5739 & -1 \\ 0.8590 & -0.3606 & 0.3635 & -1 \\ 0.4096 & -0.8900 & -0.2003 & -1 \\ 0.2612 & -0.9149 & -0.3076 & -1 \\ -0.4172 & -0.5524 & 0.7217 & -1 \\ -0.9204 & -0.0636 & 0.3857 & -1 \\ -0.7712 & -0.6104 & -0.1808 & -1 \end{bmatrix}$$



Iteration 3

$$\mathbf{H} = \begin{bmatrix} -0.4187 & -0.7038 & -0.5739 & -1 \\ 0.8590 & -0.3606 & 0.3635 & -1 \\ 0.4096 & -0.8900 & -0.2003 & -1 \\ 0.2612 & -0.9149 & -0.3076 & -1 \\ -0.4172 & -0.5524 & 0.7217 & -1 \\ -0.9204 & -0.0636 & 0.3857 & -1 \\ -0.7712 & -0.6104 & -0.1808 & -1 \end{bmatrix}$$



Example: Δρ (Iterations 2 and 3)

Iteration 2

Calculated

Measured (corrected)

$$\Delta\rho = \hat{\rho} - \rho_{corr}$$

PRN	Calculated PR	Measured PR	Delta-Rho
12	24915130.980	24915088.264	42.716
2	22135355.242	22135304.471	50.771
26	22003576.788	22003529.125	47.662
15	22645023.243	22644973.532	49.711
29	21026941.948	21026908.513	33.435
21	24777000.804	24776961.101	39.703
30	24055278.650	24055233.876	44.773

Iteration 3

Calculated

Measured (corrected)

$$\Delta\rho = \hat{\rho} - \rho_{corr}$$

PRN	Calculated PR	Measured PR	Delta-Rho
12	24915084.055	24915088.264	-4.208
2	22135304.691	22135304.471	0.220
26	22003528.878	22003529.125	-0.248
15	22644975.634	22644973.532	2.103
29	21026906.567	21026908.513	-1.946
21	24776961.532	24776961.101	0.431
30	24055237.525	24055233.876	3.648

Example: Solution and Residuals (Iterations 2 and 3)

Iteration 2 $\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\boldsymbol{\rho}$

$$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$$

$\hat{\mathbf{x}}_{u_{new}}$	$\hat{\mathbf{x}}_{u_{old}}$	$\Delta\mathbf{x}$
506075.869	506068.143	7.726
-4882274.608	-4882283.665	9.057
4059622.275	4059632.252	-9.977
13.120	63.927	-50.807

Residuals: $\mathbf{v} = \Delta\boldsymbol{\rho} - \mathbf{H}\Delta\mathbf{x}$

PRN	$\mathbf{v}$	$\Delta\boldsymbol{\rho}$	$\mathbf{H}\Delta\mathbf{x}$
12	-4.208	42.716	46.924
2	0.220	50.771	50.551
26	-0.248	47.662	47.910
15	2.103	49.711	47.609
29	-1.946	33.435	35.381
21	0.431	39.703	39.272
30	3.648	44.773	41.125

Iteration 3 $\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\boldsymbol{\rho}$

$$\hat{\mathbf{x}}_{u_{new}} = \hat{\mathbf{x}}_{u_{old}} + \Delta\mathbf{x}$$

$\hat{\mathbf{x}}_{u_{new}}$	$\hat{\mathbf{x}}_{u_{old}}$	$\Delta\mathbf{x}$
506075.869	506075.869	0.000
-4882274.608	-4882274.608	0.000
4059622.275	4059622.275	0.000
13.120	13.120	0.000

On order of 10^{-6}

Residuals: $\mathbf{v} = \Delta\boldsymbol{\rho} - \mathbf{H}\Delta\mathbf{x}$

PRN	$\mathbf{v}$	$\Delta\boldsymbol{\rho}$	$\mathbf{H}\Delta\mathbf{x}$
12	-4.208	42.716	46.924
2	0.220	50.771	50.551
26	-0.248	47.662	47.910
15	2.103	49.711	47.609
29	-1.946	33.435	35.381
21	0.431	39.703	39.272
30	3.648	44.773	41.125

Convergence

- Practically speaking, getting the system to converge with GNSS is easy
  - Example showed case where initial guess was 50 km in error
  - Can start with the center of the Earth as a guess, and it would only add an iteration or two
  - Normally, a receiver will use its last solution as a starting point, so only a single iteration is necessary
- Nonlinearities (which drive the need for iteration) are more severe when dealing with pseudolites
  - Much closer to receiver than satellite
  - H matrix varies more quickly as a function of position

Overview

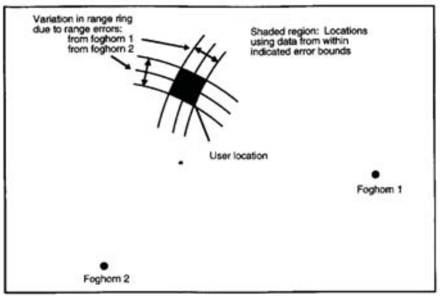
Measurement Domain vs. Position Domain

- Pseudorange errors are errors in "measurement domain"
  - Errors in the measurements themselves
  - UERE is one example
- Ultimately, we'd like to know errors in "position domain"
  - The position errors that result when using the measurements
  - Errors in position domain are different than measurement errors!
    - Can be larger
    - Can be smaller
  - Dependent on measurement geometry
- Mathematical representation
  - We have covariance matrix of measurements ($\mathbf{C}_\rho$).
  - We want covariance matrix of calculated position and clock error ($\mathbf{C}_x$)
- In GPS applications, this problem is approached using concept called Dilution of Precision (DOP)

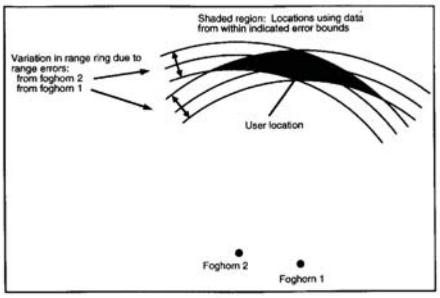
Effect of Geometry on Positioning Accuracy (Foghorn Example)

Consider the foghorn example, except allow for a measurement error

Good Geometry Example



Poor Geometry Example



Obtaining $\mathbf{C_x}$ from Least-Squares Analysis (1/2)

- Definition of $\mathbf{C_x}$

$$C_x = \begin{bmatrix} \sigma_{x_u}^2 & \sigma_{x_u y_u} & \sigma_{x_u z_u} & \sigma_{x_u \delta t_u} \\ \sigma_{x_u y_u} & \sigma_{y_u}^2 & \sigma_{y_u z_u} & \sigma_{y_u \delta t_u} \\ \sigma_{x_u z_u} & \sigma_{y_u z_u} & \sigma_{z_u}^2 & \sigma_{z_u \delta t_u} \\ \sigma_{x_u \delta t_u} & \sigma_{y_u \delta t_u} & \sigma_{z_u \delta t_u} & \sigma_{\delta t_u}^2 \end{bmatrix}$$

where, for example,

$$\sigma_{x_u}^2 = E\left[(x_u - E[x_u])^2\right]$$
$= \text{variance of } x_u$

$$\sigma_{x_u y_u} = E\left[(x_u - E[x_u])(y_u - E[y_u])\right]$$
$= \text{covariance of } x_u \text{ and } y_u$

- Definition of $\mathbf{C_\rho}$

$$\mathbf{C}_\rho = \begin{bmatrix} \sigma_{\rho_1}^2 & \sigma_{\rho_1\rho_2} & \cdots & \sigma_{\rho_1\rho_n} \\ \sigma_{\rho_1\rho_2} & \sigma_{\rho_2}^2 & \cdots & \sigma_{\rho_2\rho_n} \\ \vdots & \vdots & \ddots & \sigma_{\rho_3\rho_n} \\ \sigma_{\rho_1\rho_n} & \sigma_{\rho_2\rho_n} & \sigma_{\rho_3\rho_n} & \sigma_{\rho_n}^2 \end{bmatrix}$$

Obtaining $\mathbf{C_x}$ from Least-Squares Analysis (2/2)

- According to least-squares theory:

$$\mathbf{C_x} = \left(\mathbf{H}^T \mathbf{C}_\rho^{-1} \mathbf{H}\right)^{-1}$$

  - Basic assumptions
    - Measurement errors are zero-mean
    - Measurement errors have a Gaussian distribution
- Recall that the least-squares solution with measurement weighting was

$$\Delta\mathbf{x} = \left(\mathbf{H}^T \mathbf{C}_\rho^{-1} \mathbf{H}\right)^{-1} \mathbf{H}^T \mathbf{C}_\rho^{-1} \Delta\boldsymbol{\rho}$$
$$= \mathbf{C_x} \mathbf{H}^T \mathbf{C}_\rho^{-1} \Delta\boldsymbol{\rho}$$

  - Consider case where the nominal position and clock error (used to calculate $\Delta\boldsymbol{\rho}$) are actually the true position and clock error
    - The $\Delta\boldsymbol{\rho}$ represents the measurement *errors*
    - The $\Delta\mathbf{x}$ represents the position and clock *errors*
    - The $\mathbf{C_x}$ matrix is a multiplier for the measurement errors ($\Delta\boldsymbol{\rho}$)
      - "Large" $\mathbf{C_x}$ values → large position errors
      - "Small" $\mathbf{C_x}$ values → small position errors

Dilution of Precision (DOP)

- In GPS, the concept of Dilution of Precision (DOP) is used
  - Based upon covariance matrix of position and clock errors ($\mathbf{C_x}$)
  - Additional assumptions
    - All measurements have the same variance

$$\sigma_{\rho_1}^2 = \sigma_{\rho_2}^2 = \ldots = \sigma_{\rho_n}^2 = \sigma_\rho^2$$

    - Measurement errors are uncorrelated (i.e.,covariance values are zero)

$$\sigma_{\rho_j \rho_k} = 0, \quad j \neq k$$

  - Using these assumptions

$$\mathbf{C}_\rho = \mathbf{I}\sigma_\rho^2$$

and

$$\mathbf{C_x} = \left(\mathbf{H}^T \mathbf{H}\right)^{-1} \sigma_\rho^2$$

  - The matrix $\left(\mathbf{H}^T \mathbf{H}\right)^{-1}$ is called the DOP matrix
    - Directly relates measurement errors to position errors

Use of Local-Level Coordinate Frame (1/2)

- Normally, DOPs describe errors in geodetic (local-level) coordinate frame (east, north, up), rather than the ECEF frame.
  - Need to modify the H matrix so that the errors refer to the local-level frame
  - Original H matrix (used to calculate position)

$$\mathbf{H}^E = \begin{bmatrix} \mathbf{a}_1^{E^T} & 1 \\ \mathbf{a}_2^{E^T} & 1 \\ \vdots & \vdots \\ \mathbf{a}_n^{E^T} & 1 \end{bmatrix}$$

    - "**a**" vectors are unit line-of-sight vectors between user and SV in *ECEF frame*
    - This will give the $\mathbf{C}_\mathbf{x}$ matrix described previously
  - New H matrix for DOP calculations

$$\mathbf{H}^G = \begin{bmatrix} \mathbf{a}_1^{G^T} & 1 \\ \mathbf{a}_2^{G^T} & 1 \\ \vdots & \vdots \\ \mathbf{a}_n^{G^T} & 1 \end{bmatrix}$$

    - "**a**" vectors are now unit line-of-sight vectors between user and SV in *geodetic (ENU) frame*

Use of Local-Level Coordinate Frame (2/2)

- Local-level "**a**" vectors can be calculated using direction cosine matrix (DCM)

$$\mathbf{a}^G = \mathbf{C}_E^G \mathbf{a}^E$$

$\mathbf{C}_E^G$ = DCM that rotates from ECEF to geodetic (E,N,U) frame

$$\mathbf{C}_E^G = \left(\mathbf{C}_G^E\right)^{-1} = \left(\mathbf{C}_G^E\right)^T$$

- When $\mathbf{H}^G$ is used to calculate the covariance $\mathbf{C}_\mathbf{x} = \left(\mathbf{H}^{G^T}\mathbf{H}^G\right)^{-1}\sigma_\rho^2$, then $\mathbf{C}_\mathbf{x}$ is defined as

$$\mathbf{C}_\mathbf{x} = \begin{bmatrix} \sigma_e^2 & \sigma_{en} & \sigma_{eu} & \sigma_{e\delta t_u} \\ \sigma_{en} & \sigma_n^2 & \sigma_{nu} & \sigma_{n\delta t_u} \\ \sigma_{eu} & \sigma_{nu} & \sigma_u^2 & \sigma_{u\delta t_u} \\ \sigma_{e\delta t_u} & \sigma_{n\delta t_u} & \sigma_{u\delta t_u} & \sigma_{\delta t_u}^2 \end{bmatrix}$$

  - This is what we desire to describe using DOPs

DOP Values

- Desirable to characterize the $\mathbf{C_x}$ matrix using a single number
  - For DOPs
    - Cross-correlation terms ignored
    - Root-Sum-Square (RSS) value of variables of interest, normalized by σ_{UERE}
    - Example:

$$GDOP = \frac{\sqrt{\sigma_e^2 + \sigma_n^2 + \sigma_u^2 + \sigma_{\delta t_u}^2}}{\sigma_{UERE}}$$

    - GDOP can be calculated directly from DOP matrix

$$\left(\mathbf{H}^{G^T}\mathbf{H}^G\right)^{-1} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{bmatrix} \qquad GDOP = \sqrt{D_{11} + D_{22} + D_{33} + D_{44}}$$

    - Note that GDOP relates UERE with RSS of errors

Key relationship!

$$\sqrt{\sigma_e^2 + \sigma_n^2 + \sigma_u^2 + \sigma_{\delta t_u}^2} = GDOP \times \sigma_{UERE}$$

Types of DOPs

- The "Big Three"
  - GDOP (Geometric DOP)

$$GDOP = \sqrt{D_{11} + D_{22} + D_{33} + D_{44}}$$

$$\sqrt{\sigma_e^2 + \sigma_n^2 + \sigma_u^2 + \sigma_{\delta t_u}^2} = GDOP \times \sigma_{UERE}$$

  - PDOP (Position DOP)

$$PDOP = \sqrt{D_{11} + D_{22} + D_{33}}$$

$$\sqrt{\sigma_e^2 + \sigma_n^2 + \sigma_u^2} = PDOP \times \sigma_{UERE}$$

  - HDOP (Horizontal DOP)

$$HDOP = \sqrt{D_{11} + D_{22}}$$

$$\sqrt{\sigma_e^2 + \sigma_n^2} = HDOP \times \sigma_{UERE}$$

- Less common (for navigators, at least!)
  - VDOP (Vertical DOP)

$$VDOP = \sqrt{D_{33}}$$

$$\sqrt{\sigma_u^2} = VDOP \times \sigma_{UERE}$$

  - TDOP (Time DOP)

$$TDOP = \sqrt{D_{44}}$$

$$\sqrt{\sigma_{\delta t_u}^2} = TDOP \times \sigma_{UERE}$$

    - Note: time is in units of meters

Typical DOP Plot

Dayton Ohio – 24 Apr 2003 – All Visible SVs (above 10° elevation)