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Introduction to Kalman Filters

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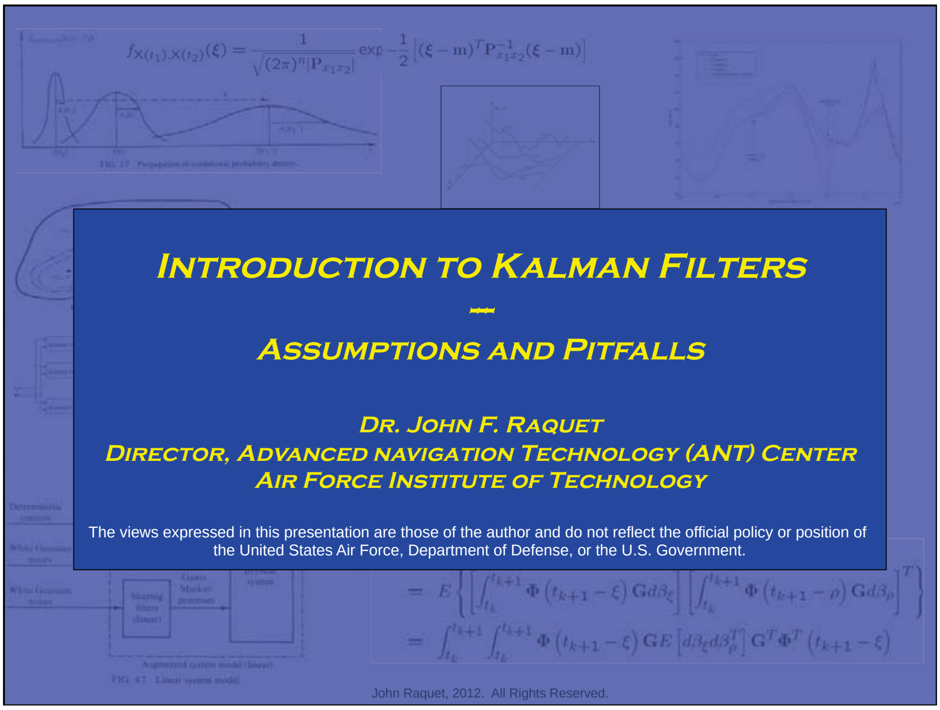

INTRODUCTION TO KALMAN FILTERS

ASSUMPTIONS AND PITFALLS

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Kalman Filtering Overview

- Kalman filtering is an estimation approach that can be applied to navigation
  - Many other application areas
- Concepts to be covered
  - Information describing the system
    - State vector
    - Covariance matrix
  - Propagating state and covariance forward in time
  - Using measurements to update the state and covariance
- Assumptions/Limitations

Kalman Filtering: Information Describing the System (1/2)

- State vector
  - Set of variables that
    - Describe everything you want to know about the system
    - Include all of the information needed to determine how the system changes over time
    - Describe systematic errors in the measurements (anything that's not "noise")
  - Example: Hot air baloon

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

$x, y, z =$ ENU position of balloon

$\dot{x}, \dot{y}, \dot{z} =$ ENU velocity of balloon

  - Does this describe what we want to know?
  - Does this describe how the system changes over time?
  - Would this be a good state vector for a fighter aircraft?
  - Altitude estimation example

Kalman Filtering: Information Describing the System (2/2)

- Covariance matrix
  - The covariance matrix basically describes how well the state is known
    - If the system only gives a state output, it's not that useful.
    - If it outputs the state and tells how accurate it is, then you have information that you can confidently act upon.
    - Hot air balloon example: the system state tells me that I'm 300 m above the ground descending at a rate of 10 m/sec.
      - Need to know covariance matrix as well.
        - » Case 1: Position accuracy = 10 m 1- σ, velocity accuracy = 1 m/sec 1-σ → probably not in danger until ~30 seconds
        - » Case 2: Position accuracy = 400 m 1- σ, velocity accuracy = 15 m/sec 1- σ → you could hit the ground any second!
  - How to interpret covariance matrix $P = E[(\underbrace{\hat{\mathbf{x}}}_{\text{Estimated state}} - \underbrace{\mathbf{x}}_{\text{True state}})(\hat{\mathbf{x}} - \mathbf{x})^T]$
    - Diagonal terms are the error variances of the estimated states
    - Off-diagonal terms are cross-covariances, describing the correlations of the errors between the states

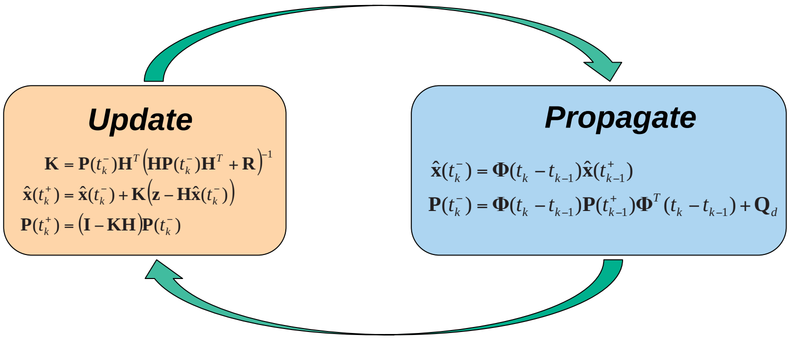
Kalman Filtering: Propagating Covariance and State Forward in Time

- State vector and covariance matrix can be propagated forward in time
  - If you know the current state estimate, you can determine the state estimate at a point in the future
  - If you know the current covariance matrix, you can determine the covariance matrix at a point in the future
  - Information about how the state and covariance changes over time is given in
    - Dynamics matrix $\mathbf{F}$: $\dot{\mathbf{x}} = \mathbf{F}\mathbf{x}$
    - State transition matrix $\mathbf{\Phi}$: $\mathbf{x}(t_1) = \mathbf{\Phi}(t_1 - t_0)\mathbf{x}(t_0)$
  - When propagating covariance forward in time, *process noise* is added to account for
    - Unmodeled dynamics
    - Unmodeled system inputs
    - Anything else that decreases the ability to predict the future state using the current state
  - Process noise increases uncertainty (i.e., larger covariance values)

Kalman Filtering: Measurement Updates

- A measurement gives information about the state values
  - Examples: GPS pseudorange (for position or clock bias) or Doppler (for velocity or clock drift)
- Effects of a measurement update
  - State values are adjusted to reflect the measurement
  - Covariance matrix is adjusted to reflect how well the state is known, now that the measurement is available
    - Measurements always decrease uncertainty (i.e., smaller covariance values)
- Measurement noise
  - Description of how precise the measurement is
  - The effect of measurement on state and covariance determined by tradeoff between
    - Measurement noise (how good the measurement is)
    - Covariance matrix (how well the state is known at this point)
- Relationship between measurement and states given by $\mathbf{H}$ matrix (same as least-squares)

The Kalman Filter Iteration



Update

$$\mathbf{K} = \mathbf{P}(t_k^-)\mathbf{H}^T\left(\mathbf{H}\mathbf{P}(t_k^-)\mathbf{H}^T + \mathbf{R}\right)^{-1}$$

$$\hat{\mathbf{x}}(t_k^+) = \hat{\mathbf{x}}(t_k^-) + \mathbf{K}\left(\mathbf{z} - \mathbf{H}\hat{\mathbf{x}}(t_k^-)\right)$$

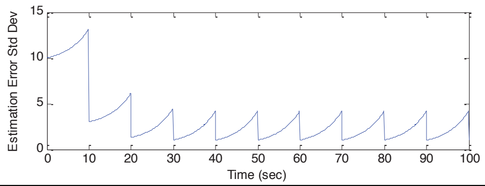
$$\mathbf{P}(t_k^+) = \left(\mathbf{I} - \mathbf{K}\mathbf{H}\right)\mathbf{P}(t_k^-)$$

Propagate

$$\hat{\mathbf{x}}(t_k^-) = \mathbf{\Phi}(t_k - t_{k-1})\hat{\mathbf{x}}(t_{k-1}^+)$$

$$\mathbf{P}(t_k^-) = \mathbf{\Phi}(t_k - t_{k-1})\mathbf{P}(t_{k-1}^+)\mathbf{\Phi}^T(t_k - t_{k-1}) + \mathbf{Q}_d$$

Example of Estimation Error Over Time



Measurement Model

Linear Measurement Model

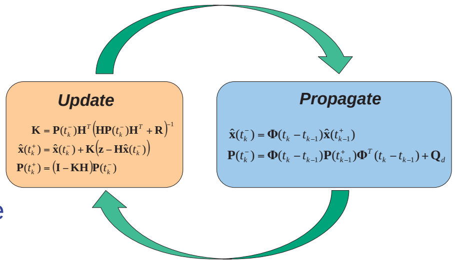
$$\mathbf{z} = \mathbf{H}\mathbf{x} + \mathbf{v}$$

measurement

sensitivity matrix

state

meas noise



Update

$$\mathbf{K} = \mathbf{P}(t_k^-)\mathbf{H}^T\left(\mathbf{H}\mathbf{P}(t_k^-)\mathbf{H}^T + \mathbf{R}\right)^{-1}$$

$$\hat{\mathbf{x}}(t_k^+) = \hat{\mathbf{x}}(t_k^-) + \mathbf{K}\left(\mathbf{z} - \mathbf{H}\hat{\mathbf{x}}(t_k^-)\right)$$

$$\mathbf{P}(t_k^+) = \left(\mathbf{I} - \mathbf{K}\mathbf{H}\right)\mathbf{P}(t_k^-)$$

Propagate

$$\hat{\mathbf{x}}(t_k^-) = \mathbf{\Phi}(t_k - t_{k-1})\hat{\mathbf{x}}(t_{k-1}^+)$$

$$\mathbf{P}(t_k^-) = \mathbf{\Phi}(t_k - t_{k-1})\mathbf{P}(t_{k-1}^+)\mathbf{\Phi}^T(t_k - t_{k-1}) + \mathbf{Q}_d$$

Measurement noise is described by the measurement noise covariance matrix:

$$\mathbf{R} = E[\mathbf{v}\mathbf{v}^T] = \begin{bmatrix} E[v_1^2] & E[v_1 v_2] & \cdots & E[v_1 v_n] \\ E[v_1 v_2] & E[v_2^2] & & \\ \vdots & & \ddots & \\ E[v_1 v_n] & & & E[v_n^2] \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & & \\ \vdots & & \ddots & \\ \sigma_{1n} & & & \sigma_n^2 \end{bmatrix}$$

- Key assumptions (in semi-nontechnical language)
  - Measurement errors $\mathbf{v}$ are Gaussian (follow a “bell curve”)
  - Measurement errors $\mathbf{v}$ are “white” (completely random from measurement to measurement)
  - Measurement model is linear

What if Measurements are Non-Linear

- Example of non-linear measurements: a range (distance) measurement (such as with GPS)
- Can use non-linear measurement model

Nonlinear

$$\mathbf{z} = \mathbf{h}(\mathbf{x}) + \mathbf{v}$$

Linear

$$\mathbf{z} = \mathbf{H}\mathbf{x} + \mathbf{v}$$

- Kalman filter is then modified to become an “Extended Kalman Filter” (EKF)
  - Requires linearization about the estimated solution
  - Because of this, an EKF is not, technically speaking, truly optimal like the KF
  - In many cases it would be “nearly optimal”—depends on the nature of the linearization

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Dynamics Model

Discrete-Time Dynamics Model

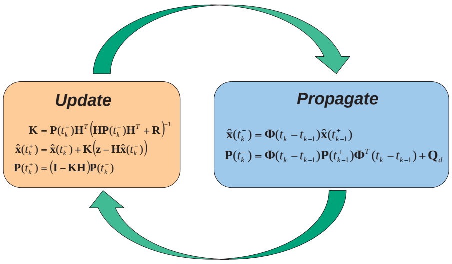
$$\mathbf{x}(t_k^-) = \mathbf{\Phi}\mathbf{x}(t_{k-1}^+) + \mathbf{w}_d$$

state after propagation

dynamics matrix

state before propagation

process noise



Update

$$\mathbf{K} = \mathbf{P}(t_k^-)\mathbf{H}^T\left(\mathbf{H}\mathbf{P}(t_k^-)\mathbf{H}^T + \mathbf{R}\right)^{-1}$$
$$\hat{\mathbf{x}}(t_k^+) = \hat{\mathbf{x}}(t_k^-) + \mathbf{K}\left(\mathbf{z} - \mathbf{H}\hat{\mathbf{x}}(t_k^-)\right)$$
$$\mathbf{P}(t_k^+) = \left(\mathbf{I} - \mathbf{K}\mathbf{H}\right)\mathbf{P}(t_k^-)$$

Propagate

$$\hat{\mathbf{x}}(t_k^-) = \mathbf{\Phi}(t_k - t_{k-1})\hat{\mathbf{x}}(t_{k-1}^+)$$
$$\mathbf{P}(t_k^-) = \mathbf{\Phi}(t_k - t_{k-1})\mathbf{P}(t_{k-1}^+)\mathbf{\Phi}^T(t_k - t_{k-1}) + \mathbf{Q}_d$$

Process noise is described by the measurement noise covariance matrix:

$$\mathbf{Q}_d = E[\mathbf{w}_d\mathbf{w}_d^T] = \begin{bmatrix} E[w_1^2] & E[w_1w_2] & \cdots & E[w_1w_n] \\ E[w_1w_2] & E[w_2^2] & & \\ \vdots & & \ddots & \\ E[w_1w_n] & & & E[w_n^2] \end{bmatrix}$$

- Key assumptions (in semi-nontechnical language)
  - Process noise $\mathbf{w}_d$ is Gaussian (follow a “bell curve”)
  - Process noise $\mathbf{w}_d$ is “white” (completely random from epoch to epoch)
  - Dynamics model is perfectly known

Initialization and “Time Constant” of a KF

- Things needed in order to initialize a filter
  - Initial state estimate
  - Initial covariance matrix
  - Measurement model(s)
  - Propagation model(s)
- Time constant (not meant in a precise, technical way)
  - Defines how long a measurement will affect the filter
  - In theory, every measurement will affect the filter for the rest of time
  - In practice, this may not be the case so much
    - Example: Case in which there is high propagation noise—old measurements are significantly “de-weighted” relative to new measurements
  - Warning: Even in a case where a filter has a “short” time constant (i.e., measurements lose impact fairly quickly), a large measurement error (blunder) can have a devastating impact

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Kalman Filter Example: Hot Air Balloon

- Scenario: Want to estimate height of a hot air balloon on a windy day, starting at 800 m

- What I have
  - Radar altimeter to measure height above ground (assume ground height is known)
    - Meas error modeled as Gaussian with 2m standard deviation
  - Stochastic process model for how the wind affects the height of the balloon
  - Initial uncertainty modeled as Gaussian with standard deviation of 10m (height) and 1m/s (vertical velocity)
- What I want to know
  - Height estimate and standard deviation
  - Vertical velocity estimate and standard deviation

Stochastic Process Model

- State vector: $\mathbf{x} = \begin{bmatrix} h \\ \dot{h} \end{bmatrix}$
  - h: balloon height (m)
  - $\dot{h}$: baloon vertical velocity (m/s)
- Continuous time process model:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \mathbf{w}(t) \qquad E[\mathbf{w}(t)\mathbf{w}^T(t+\tau)] = Q\delta(\tau) \qquad Q = \begin{bmatrix} 0 & 0 \\ 0 & 0.01 \end{bmatrix}$$

Dirac delta

Process noise

Process noise matrix

Dynamics matrix F

- Discrete time process model: $\mathbf{x}(t_k) = \Phi(\Delta t)\mathbf{x}(t_{k-1}) + \mathbf{w}_d(t_k)$

$$\Phi(\Delta t) = e^{F\Delta t} = \begin{bmatrix} 1 & \Delta t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 0 & 0 \end{bmatrix} \quad \text{for} \quad \Delta t = t_k - t_{k-1} = 0.5\,\text{sec}$$

$$E[\mathbf{w}_d(t_k)\mathbf{w}_d^T(t_k)] = Q_d = \begin{bmatrix} 0.0004 & 0.0013 \\ 0.0013 & 0.005 \end{bmatrix}$$

Kalman Filter Propagation Equations

- Propagate state: $\hat{\mathbf{x}}(t_k) = \Phi\hat{\mathbf{x}}(t_{k-1})$
- Propagate covariance: $P(t_k) = \Phi P(t_{k-1})\Phi^T + Q_d$
- Example
  - Initial conditions:

$$\hat{\mathbf{x}}(0) = \begin{bmatrix} 800 \\ 0 \end{bmatrix} \qquad P(0) = \begin{bmatrix} 10^2 & 0 \\ 0 & 1^2 \end{bmatrix}$$

  - First time step:

$$\hat{\mathbf{x}}(0.5) = \Phi\hat{\mathbf{x}}(0) = \begin{bmatrix} 1 & 0.5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 800 \\ 0 \end{bmatrix} = \begin{bmatrix} 800 \\ 0 \end{bmatrix}$$

$$P(0.5) = \Phi P(0)\Phi^T + Q_d$$

$$= \begin{bmatrix} 1 & 0.5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0.5 & 0 \end{bmatrix} + \begin{bmatrix} 0.0004 & 0.0013 \\ 0.0013 & 0.005 \end{bmatrix}$$

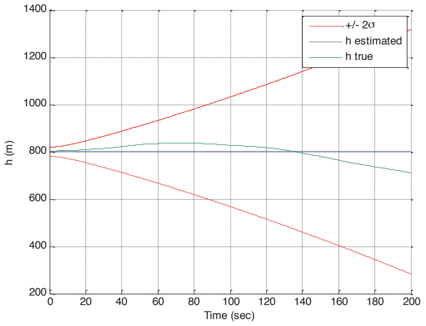
$$= \begin{bmatrix} 100.25 & 0.5 \\ 0.5 & 1 \end{bmatrix} + \begin{bmatrix} 0.0004 & 0.0013 \\ 0.0013 & 0.005 \end{bmatrix}$$

$$= \begin{bmatrix} 100.2504 & 0.5013 \\ 0.5013 & 1.005 \end{bmatrix} \longleftarrow$$

Notice that this is higher (more uncertainty) than when we started!

Propagation Example—No Measurements (Single Run)

h



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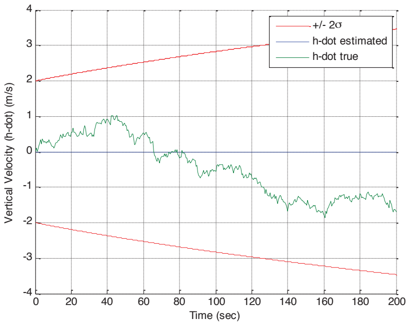

Propagation Example—No Measurements (25 Runs)

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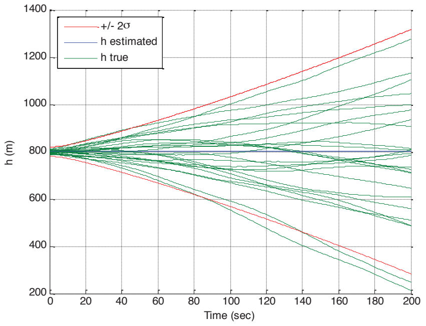

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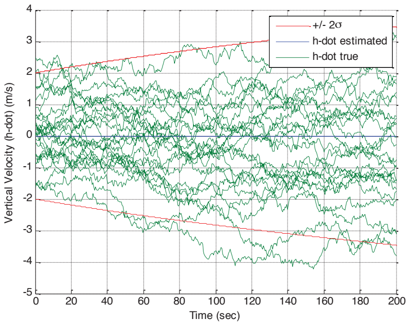

Measurement Update at t=40 Sec

- At t=40 sec, a measurement of 820.97 m is taken (remember, meas error standard deviation is 2 m)
- Measurement model: $\mathbf{z} = \mathbf{Hx} + \mathbf{v}$ $\quad \mathbf{R} = E[\mathbf{vv}^T]$
  - In this case: $z = h_{meas} + v = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} h \\ \dot{h} \end{bmatrix}}_{\mathbf{x}} + v \qquad R = E\left[v^2\right] = 4$

(Note that this is a scalar measurement)

- Step 1: Propagate up to measurement time: $t_k = 80\,\mathrm{sec}$

$$\hat{\mathbf{x}}(t_k) = \begin{bmatrix} 800 \\ 0 \end{bmatrix} \qquad P(t_k) = \begin{bmatrix} 1913.3 & 48 \\ 48 & 1.40 \end{bmatrix}$$

$\nwarrow \qquad \nearrow$

These are consistent with previous slides

Measurement Update at t=40 Sec

(continued)

- Step 2: Measurement update

State: $\hat{\mathbf{x}}(t_k) = \hat{\mathbf{x}}(t_k) + K\left[z(t_k) - H\hat{\mathbf{x}}(t_k)\right]$

Covariance: $P(t_k) = \left[I - KH\right]P(t_k)$

Kalman gain: $K = P(t_k)H^T\left[HP(t_k)H^T + R\right]^{-1}$

- In this case, since $H = \begin{bmatrix} 1 & 0 \end{bmatrix}$:

from previous slide: $P(t_k) = \begin{bmatrix} 1913.3 & 48 \\ 48 & 1.40 \end{bmatrix}$

$$K = \begin{bmatrix} \dfrac{P_{1,1}}{P_{1,1} + R} \\ \dfrac{P_{2,1}}{P_{1,1} + R} \end{bmatrix} = \begin{bmatrix} \dfrac{1913.3}{1913.3 + 4} \\ \dfrac{48}{3.9917 + 4} \end{bmatrix} = \begin{bmatrix} 0.998 \\ 0.025 \end{bmatrix}$$

$$\hat{\mathbf{x}}(t_k) = \hat{\mathbf{x}}(t_k) + K\left[z(t_k) - H\hat{\mathbf{x}}(t_k)\right]$$

$$= \begin{bmatrix} 800 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.998 \\ 0.025 \end{bmatrix} \underbrace{\begin{bmatrix} 820.97 - 800 \end{bmatrix}}_{\text{residual}}$$

$$\hat{\mathbf{x}}(t_k) = \begin{bmatrix} 820.926 \\ 0.525 \end{bmatrix}$$

$$P(t_k) = \underbrace{\begin{bmatrix} 0.0021 & 0 \\ -0.0250 & 1 \end{bmatrix}}_{I-KH} \underbrace{\begin{bmatrix} 1913.3 & 48 \\ 48 & 1.40 \end{bmatrix}}_{P(t_k)} = \begin{bmatrix} 3.992 & 0.100 \\ 0.100 & 0.198 \end{bmatrix}$$

(Before measurement)

Filter Propagation/Measurement Incorporation Example

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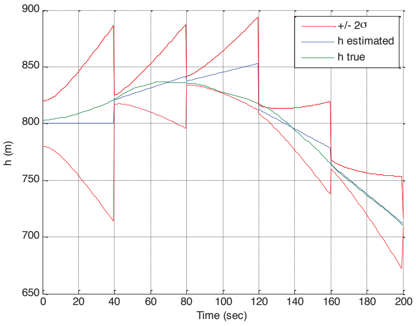

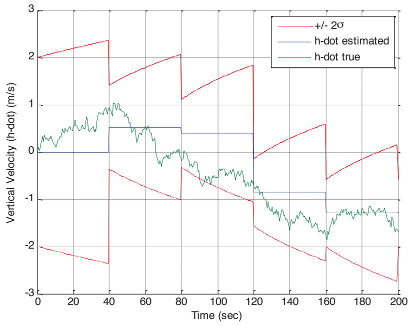

Errors in Filter Estimate for Same Case as Previous Slide

- These plots show the DIFFERENCE between the estimated state (blue on previous slide) and the true state (green on previous slide)

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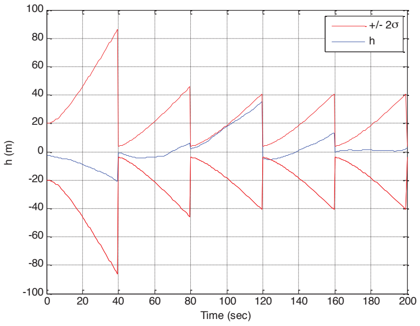

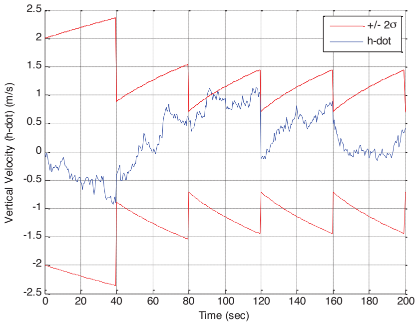

Errors in Filter Estimate
25 Monte Carlo Runs

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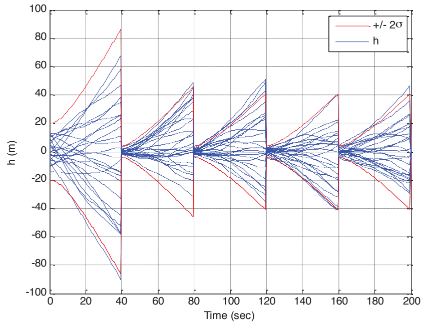
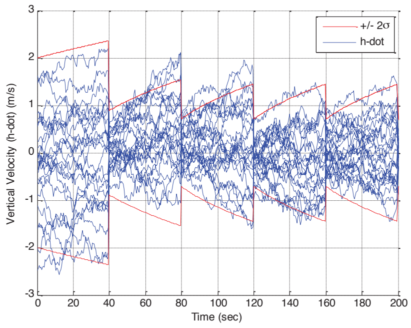




Modeling Errors

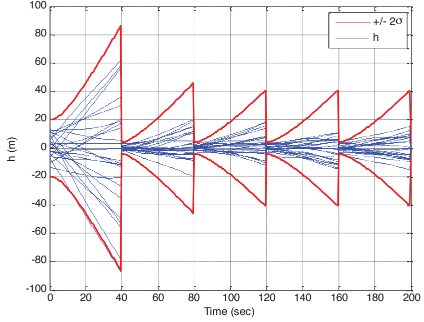
- In the previous example, the filter had a perfect model
- What happens if there is a process noise modeling error?
- Example:

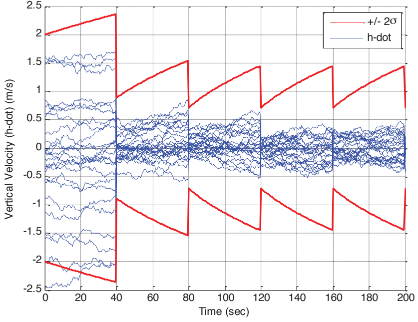
True: $Q = \begin{bmatrix} 0 & 0 \\ 0 & 0.001 \end{bmatrix}$

Modeled: $Q = \begin{bmatrix} 0 & 0 \\ 0 & 0.01 \end{bmatrix}$

h

$\dot{h}$



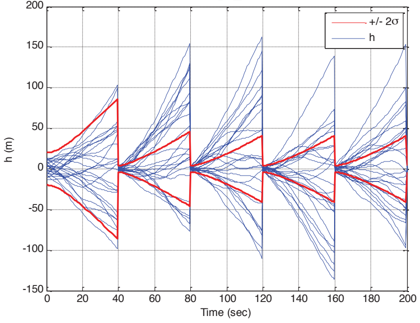


Another Modeling Error Example

True: $Q = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix}$

Modeled: $Q = \begin{bmatrix} 0 & 0 \\ 0 & 0.01 \end{bmatrix}$

h



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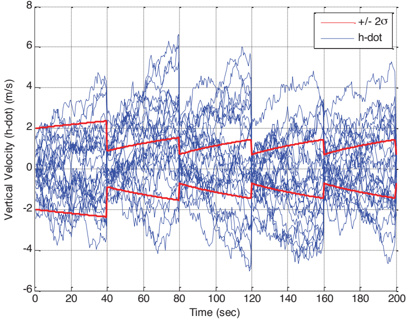

Other Comments on Example

- This was a simple example, but more complex examples (more states, more complicated measurement model) work the same way
- For GNSS systems, the H matrix is the same as the H matrix used for least-squares solutions
  - Measurement model is nonlinear, so Extended Kalman Filter (EKF) is used
- Kalman filter will give optimal results when all of its assumptions are met
  - Measurement errors are zero mean, white, Gaussian noise
  - Process noise (discrete-time) is zero mean, white, Gaussian noise
  - Measurement model and process model are known and correct
  - Measurements and process model are linear functions of the state
- If any of these are not met, it is not technically optical any more
  - However, it still may give “good” results
- Often, the modeling aspects of the problem are a more significant challenge than the filter itself

One Final Thought…

Life is a Kalman Filter

(really, it is!)