The Abdus Salam
International Centre
for Theoretical Physics

2458-14

Workshop on GNSS Data Application to Low Latitude Ionospheric Research

6 - 17 May 2013

Inertial Navigation Systems

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Inertial Navigation Systems

Applications of GNSS

Workshop on GNSS Data Application to Low Latitude Ionospheric Research

Trieste – Italy, 06-17 May 2013

Prof. Frank van Graas

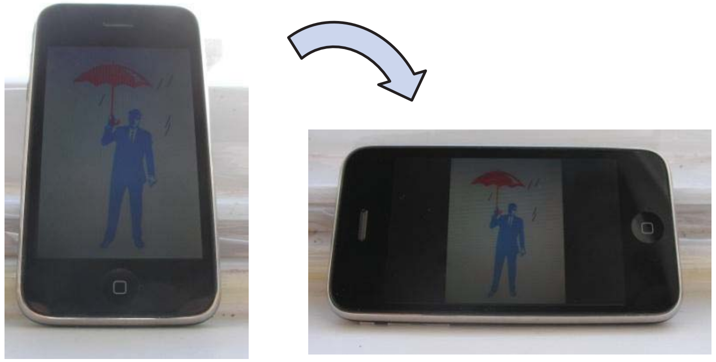
Ohio University

Course Overview

- Introduction to inertial navigation
- Accelerometer operation and error sources
- Gyroscope operation and error sources
- Coordinate frames
- Inertial navigation mechanization
- Dynamics
- Initialization
- Strapdown terminology
- Movement over ellipsoid
- Attitude, velocity, and position updating
- Future trends

Two Inertial Measurement Systems

- Aviation Ring Laser Gyro Standalone Navigation Grade Inertial Reference Unit
- 1 nautical mile per hour performance
- Outputs:
  - 50Hz body rates and accelerations, pitch, roll
  - 25Hz heading
  - 20Hz velocities



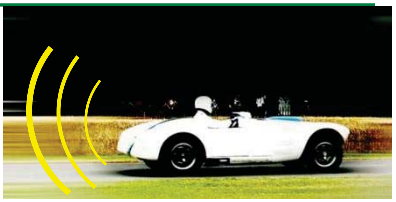
Both systems use inertial sensors and know about the orientation of the device, but only the system on the left can be used to navigate using inertial measurements only.

An inertial navigation system has two components:

1) Sensor package
2) Navigation computer

Autonomous navigation system

What Makes Inertial Navigation Work?



- In order to move, we first need to accelerate
- If we keep track of our acceleration over time, then we can determine how much we move
  - » Use accelerometers to measure acceleration

- If we could also keep track of the direction in which we accelerate, then we can determine how much we move in a particular direction
  - » Use gyroscopes to measure changes in direction over time
- If we knew where we started and in which direction we were pointing, then we can determine our position, velocity and orientation as time goes on
  - » Use initial conditions

Inertial Application Areas

- Navigation Grade
  - » Spacecraft, sub-marine, military aircraft, commercial aircraft, ship
- Tactical Grade

  - » Attitude and heading reference system (artificial horizon), short-term tactical guidance, land vehicles, aircraft autopilot, guidance stabilization, GPS integration
- Short-term (Commercial Grade)
  - » Small unmanned aerial systems (UAS), camera stabilization, airbag sensors, car electronic stability control, video games, smart phones, athletic shoes, motion sensors (e.g. shipping crates, electronics), medical

What About the Ionosphere?

- Primary application:
  - » Mitigate the effects of the ionosphere by aiding the receiver tracking loop with inertial measurements
  - » First need to understand the ionosphere …
- Secondary applications:
  - » Earthquakes and nuclear detonations cause disturbances in the ionosphere
    - Earthquake detection networks are starting to use inertial measurements to obtain the high-frequency content of the quake along with GNSS measurements for the lower frequencies
  - » Tsunami prediction networks are starting to use accelerometers along with GNSS receivers

Short-Term, Tactical, Navigation Grade

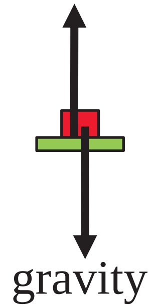
Inertial Grade	Short-Term	Tactical	Navigation Grade
Example Applications	Flight Control Consumer Electronics	Missiles AHRS	Aircraft Navigation System
Gyroscopes	> 10 deg/hr Silicon MEMS	0.1 – 10 deg/hr Fiber Optic, Silicon MEMS	0.001 – 0.1 deg/hr Fiber Optic, Ring Laser
Accelerometers	> 1 mg Silicon MEMS	100 µg – 1 mg Quartz Resonant, Silicon MEMS	10 – 100 µg Quartz Resonant, Silicon MEMS

AHRS = Attitude and Heading Reference System (Artificial Horizon)
MEMS = Micro-Electro-Mechanical Systems
All numbers are root-mean-square (rms) values
Note: Navigation grade INS is integrated with airdata computer for vertical channel stabilization

Accelerometer

- Consider a proof-mass glued to a scale that can measure both positive (push) and negative (pull) weights

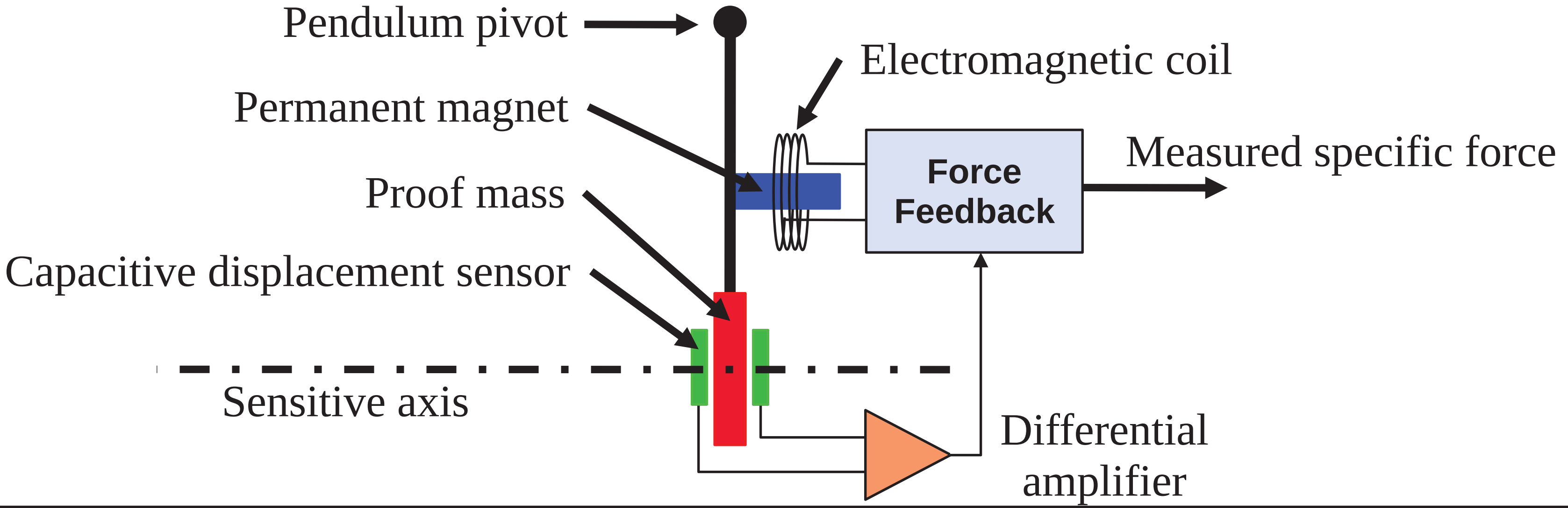
- Most accelerometers (like the one above) measure "specific force," or force per unit mass (in m/s^2), which consists of:
  - » Acceleration specific force = acceleration = a (m/s^2)
  - » Gravitation specific force ≈ 9.8 (m/s^2)
- If the scale is at rest on the floor, then the output of the accelerometer will be +g ≈ 9.8 (m/s^2)



- If the accelerometer is in "free fall," then there is no force on the scale, such that the output = 0 (m/s^2)

Accelerometer Technologies

- Several sensing technologies are in common use:
  - » Force-feedback pendulum
  - » Vibrating beam (or dual vibrating beam)
  - » Micro-machined with electrostatic control of proof mass
- In general, the force is measured indirectly by measuring how much signal is required to keep the proof mass from moving



Delta Velocity instead of Acceleration

- Most accelerometers output ΔV instead of acceleration

$$\Delta V_{t_2,t_1} = \int_{t_1}^{t_2} a(t)dt \quad (\text{m/s})$$

- In words: ΔV is the change in velocity over the time interval Δt measured from t_1 to t_2
  - » For most applications: Δt = 0.01 (s) is sufficient
  - » High-dynamic applications use Δt as small as 0.00025 (s)
- If an accelerometer measures gravity, then $a \approx 9.8$ (m/s^2), such that for Δt = 0.01 (s), ΔV = 0.098 (m/s)

- Convenient format as it simplifies subsequent processing; some sensors already directly measure change; reduces data rate and dynamic range compared to implementation of the single integration outside the sensor

Example Inertial Measurement Unit

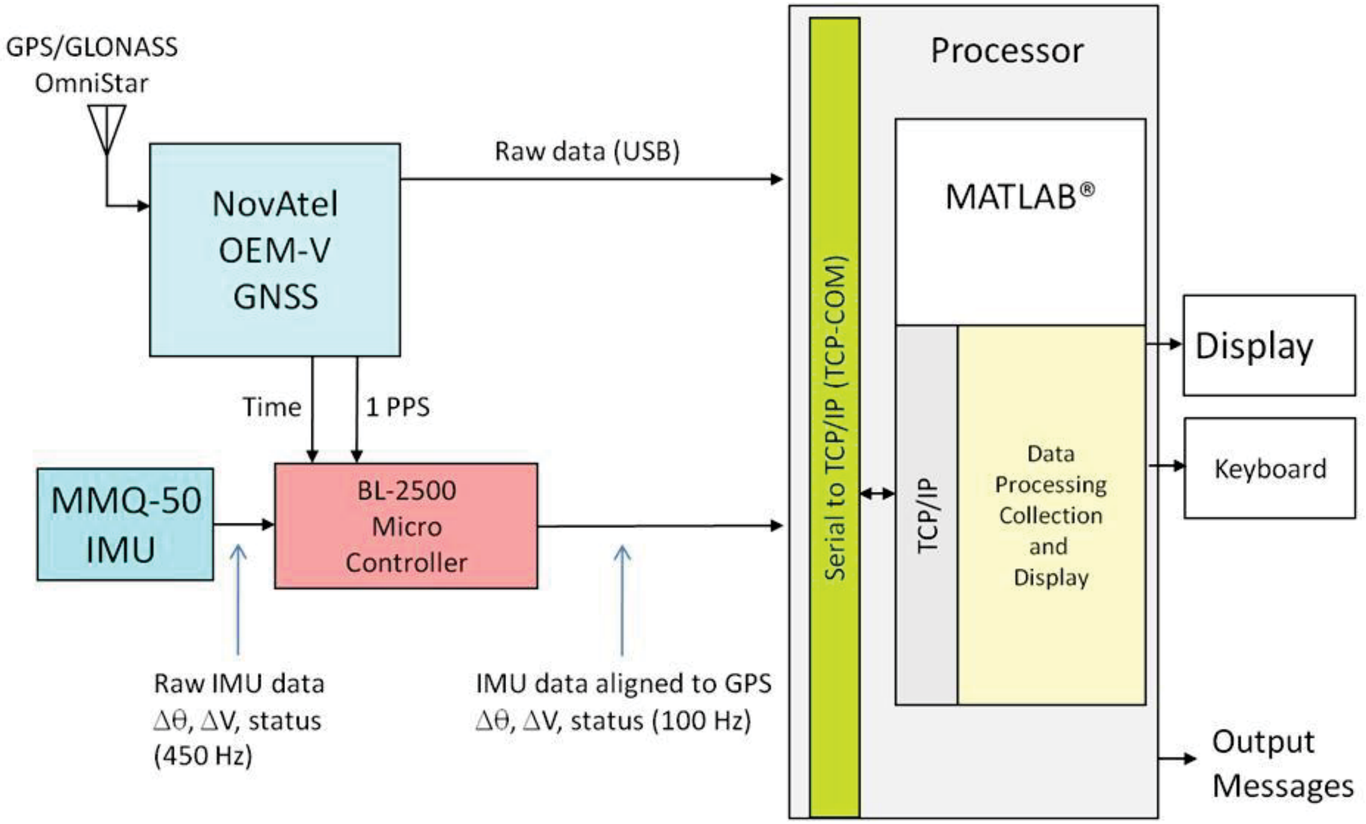
Miniature MEMS Quartz IMU
Systron Donner MMQ™ 50

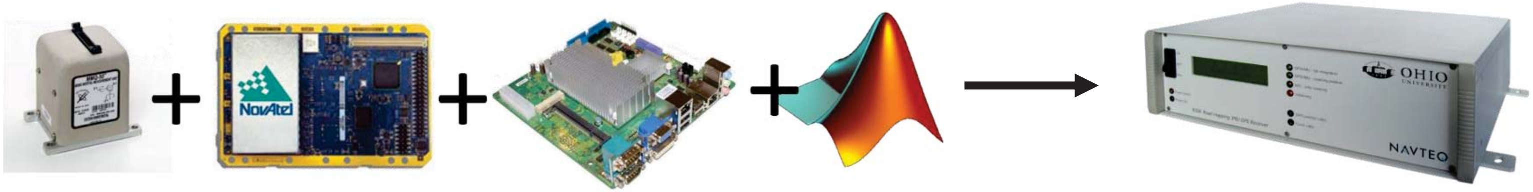
From: www.systron.com

Accelerometers	
Range	±10 g
Bias turn-on to turn-on stability	≤ 2.5 mg, 1σ
Bias in-run stability	≤ 3 mg, 1σ
Velocity random walk	0.5 mg/√Hz
Scale factor error	≤ 0.5%
Alignment	≤ 5 mrad
Bandwidth	50 Hz

Gyroscopes	
Range	200°/sec
Bias turn-on to turn-on stability	≤ 100°/hr, 1σ
Bias instability	≤ 4-15°/hr
Angle random walk	0.3°/√hr
Scale factor error	≤ 0.5%
Alignment	≤ 5 mrad
Bandwidth (-90° phase shift)	50 Hz

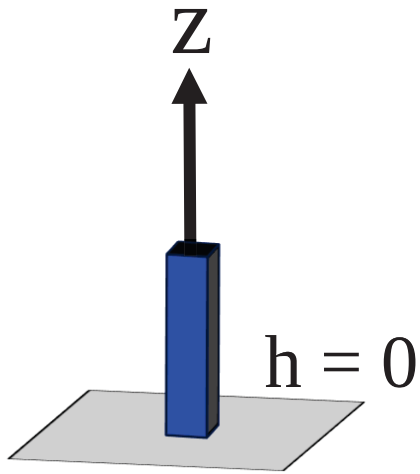
Hardware Integration Research Platform

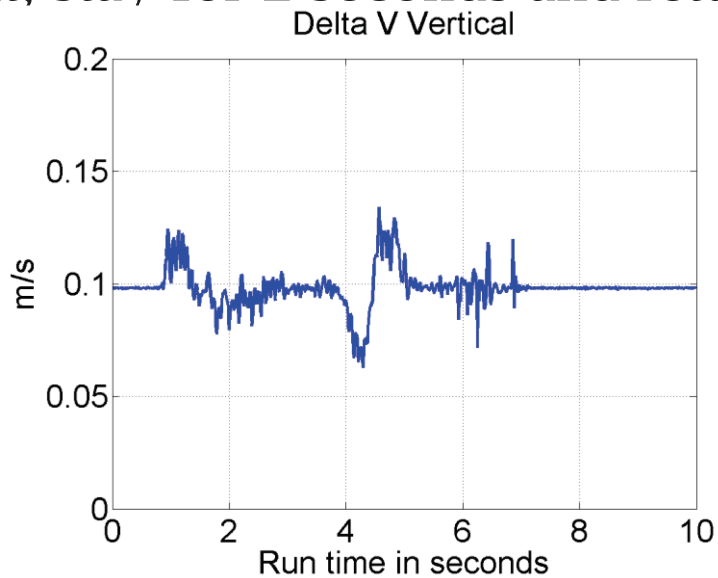




Inertial Navigation in the Vertical Direction

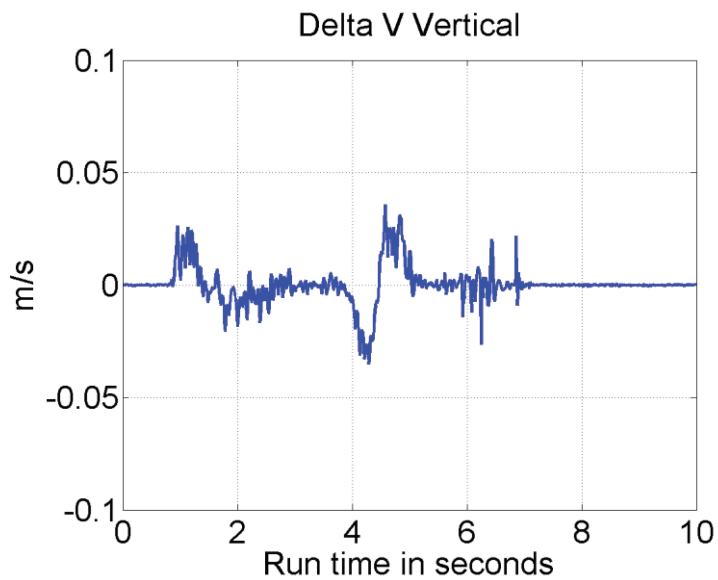
- Accelerometer sensitive axis in the Z direction (perpendicular to local level defined by gravity)
- Scenario:
  - » t = 0: Start at height = 0 (h = 0)
  - » t = 1 s to 2 s: accelerate upwards, reach highest point, stay for 2 seconds and return to h = 0





MEMS Accelerometer: Vertical Acceleration

- Assume
  - » Accelerometer does not have any errors except for noise
  - » Gravity does not vary as a function of height
- Use first second of data to estimate gravity and subtract it from the measured ΔV ("zero-velocity update")

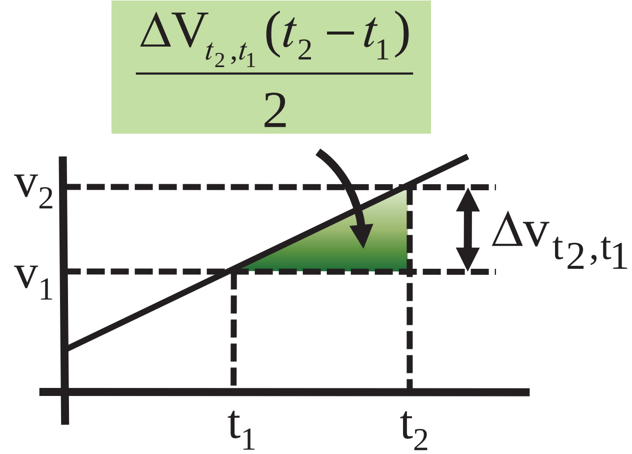


Vertical Velocity and Displacement

- To obtain velocity and displacement from ΔV measurements:

$$v_{t_2} = v_{t_1} + \Delta V_{t_2,t_1}$$

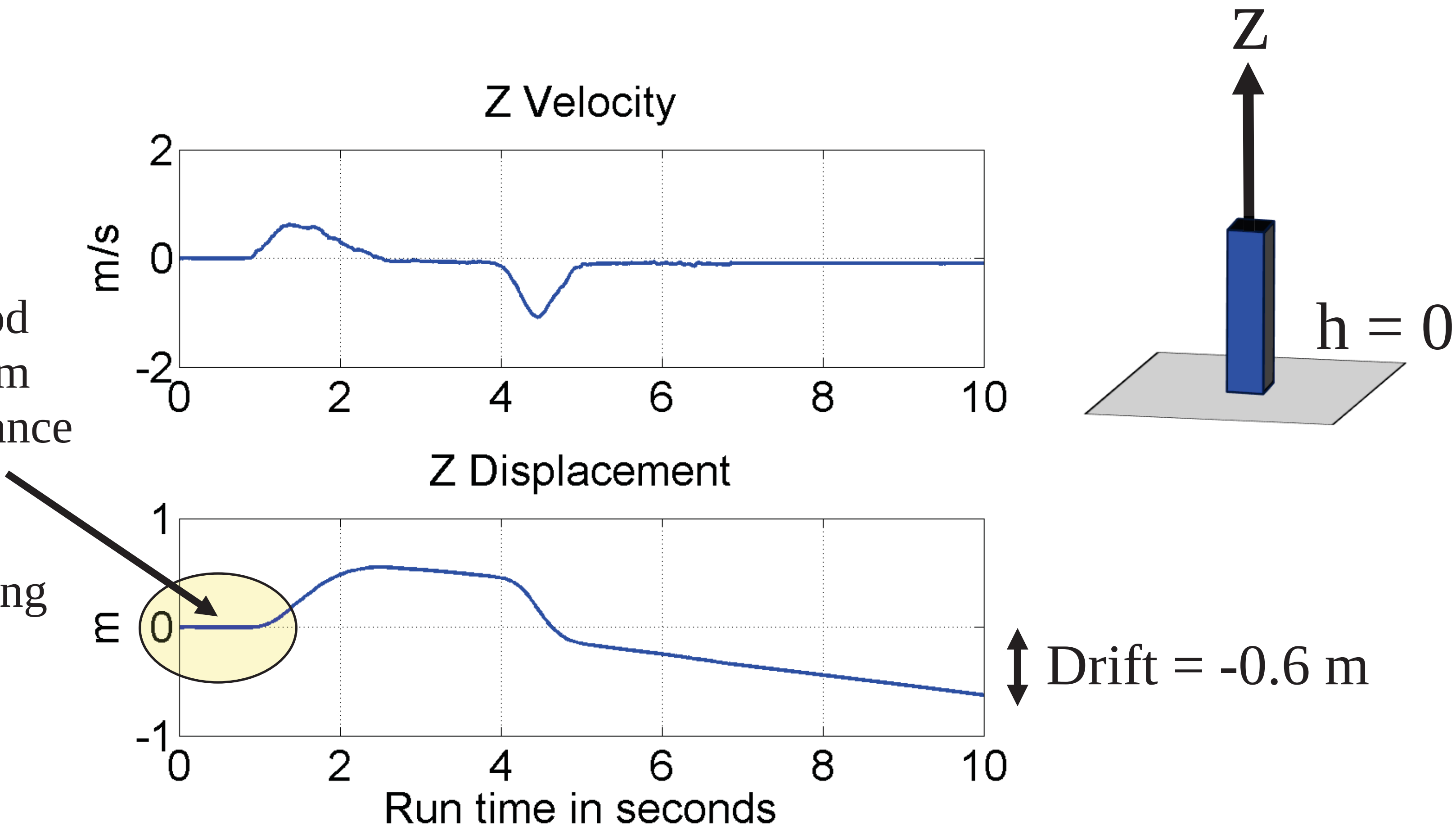
$$r_{t_2} = r_{t_1} + v_{t_1}(t_2 - t_1) + \frac{\Delta V_{t_2,t_1}(t_2 - t_1)}{2}$$



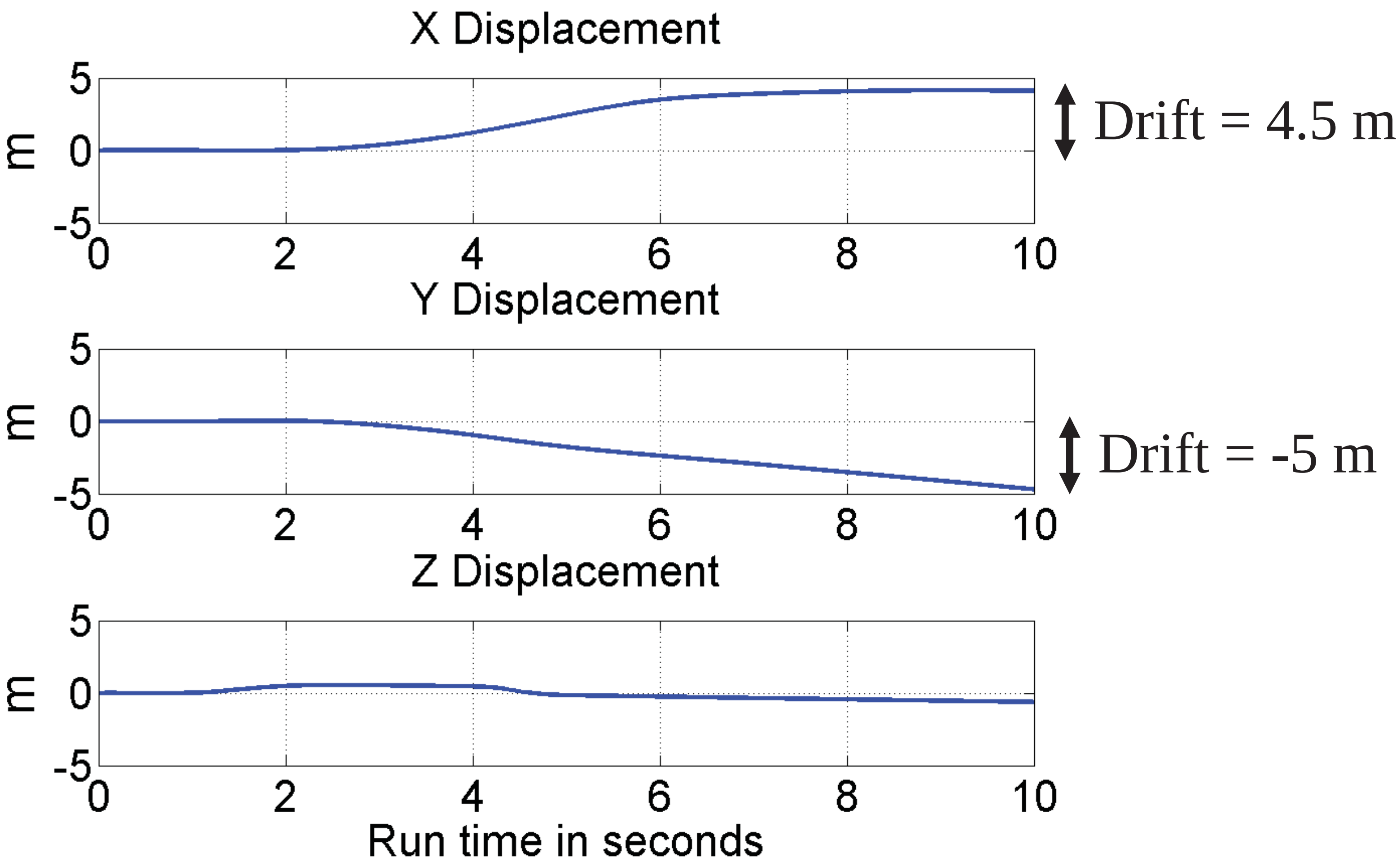
- Note that acceleration is time-varying, not constant
  - » In general, cannot directly use: $r(t) = r_0 + v_0 t + \frac{1}{2} a t^2$ (because it assumes constant acceleration)
  - » However, can use this by integrating in small steps and assuming that acceleration is constant during each step

Measured Vertical Velocity and Displacement

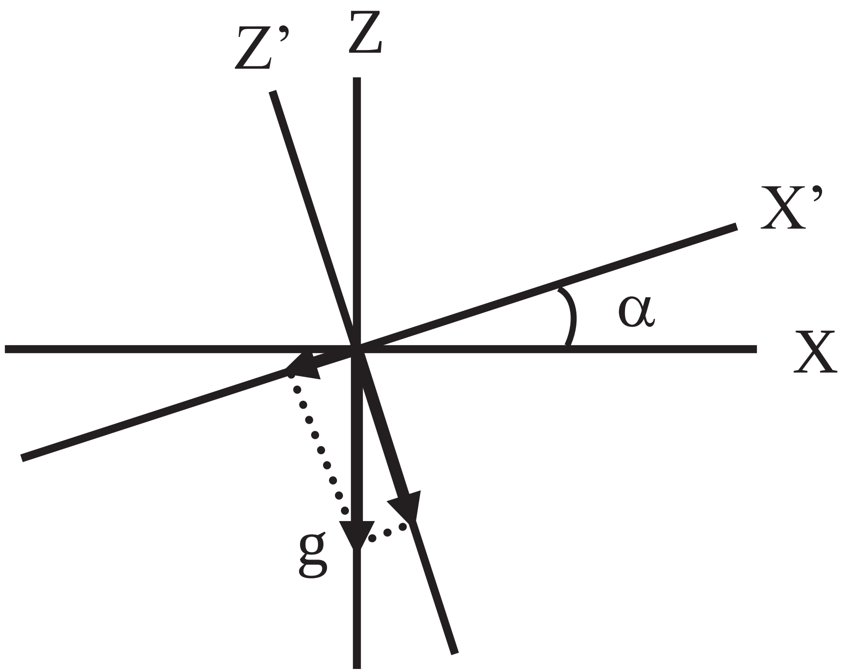
Very good short-term performance for tracking loop aiding



What Happened in the X and Y Directions?



Leveling Error



Z': g cos(α) ≈ g

X': g sin(α) ≈ αg

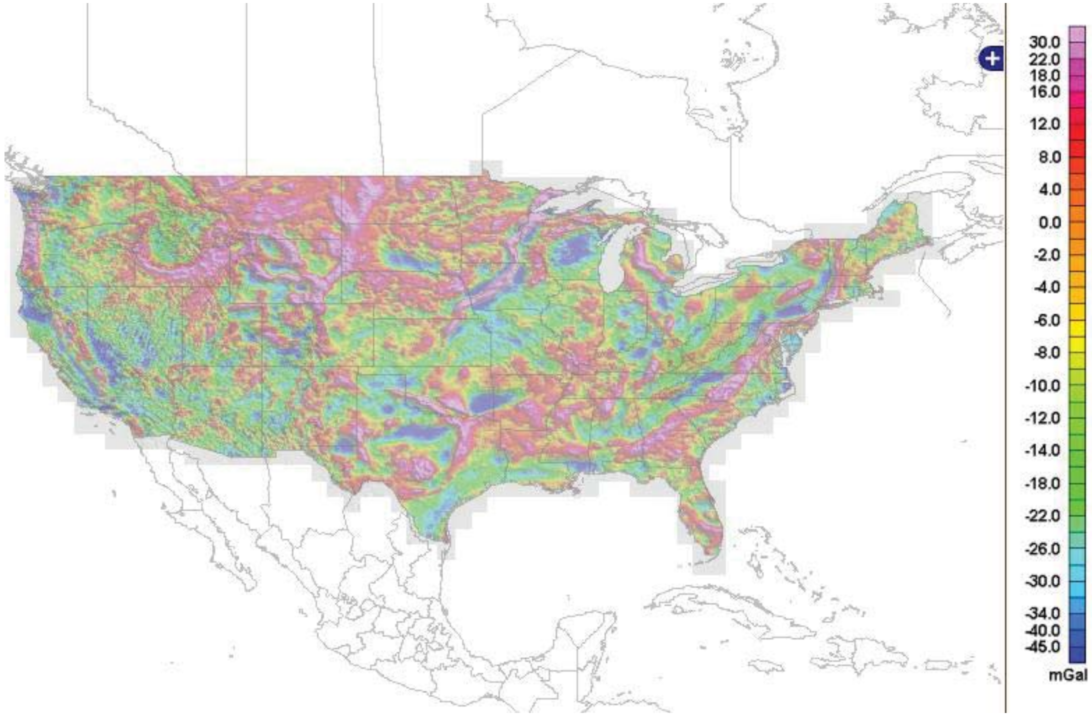
If α = 2 degrees, then

X' acceleration: 0.34 m/s²

After 5 s: ΔX=4.28 m

Other Complications

- Accelerometers have many errors in addition to noise
- Gravity is not constant (anomalies), doesn't always point to the center of the Earth (deflections), and varies as a function of height.



Gravity anomaly map

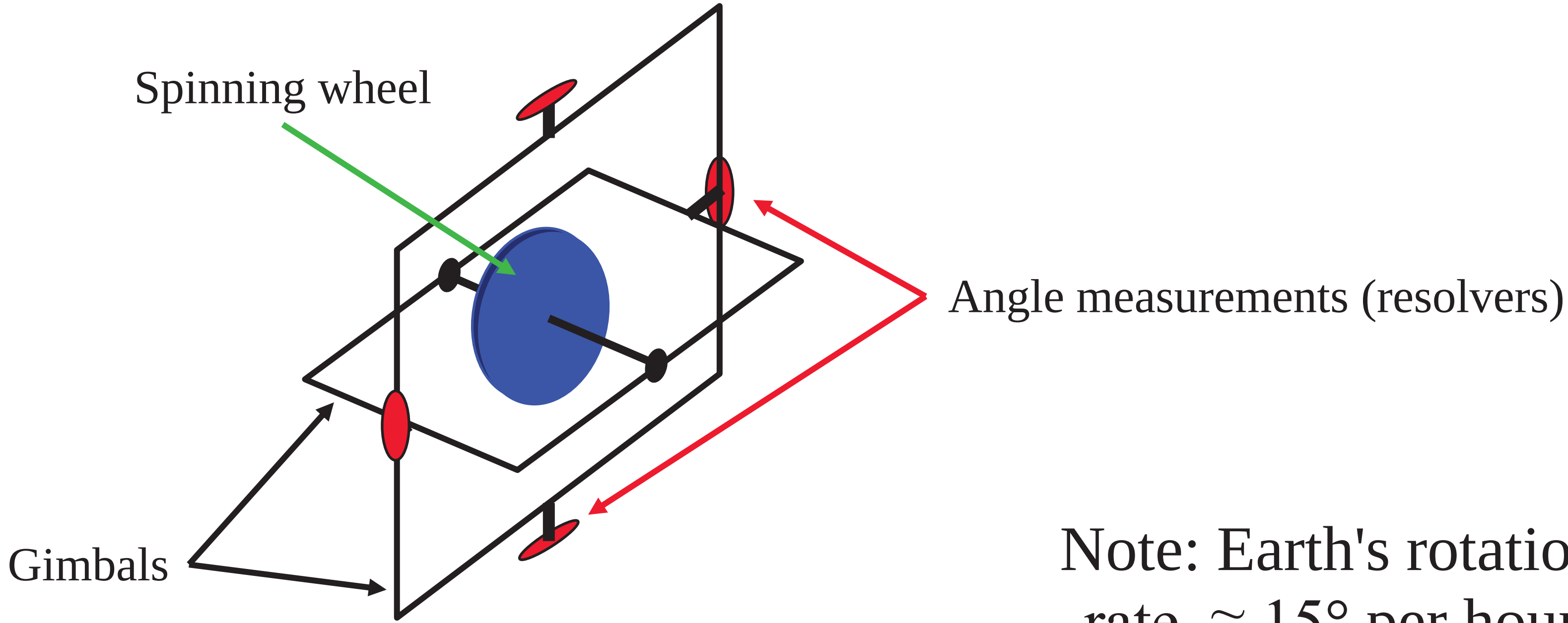
1 mGal = 10^{-5} m/s^2

From: http://mrdata.usgs.gov/geophysics/gravity.html

For the WGS 84 Earth Gravitational Model: http://earth-info.nga.mil/GandG/wgs84/gravitymod/ and http://www.ngs.noaa.gov/GEOID/GEOID96/ngdc-indx.html

Gyroscope

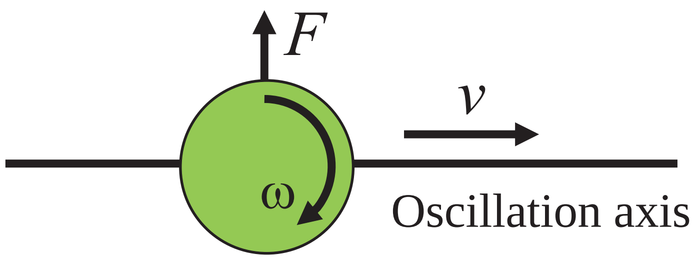
- Gyroscopes are used to measure orientation or changes in orientation and are often based on the principle of conservation of angular momentum.



Note: Earth's rotation rate $\approx 15°$ per hour (7.292115×10^{-5} rad/sec)

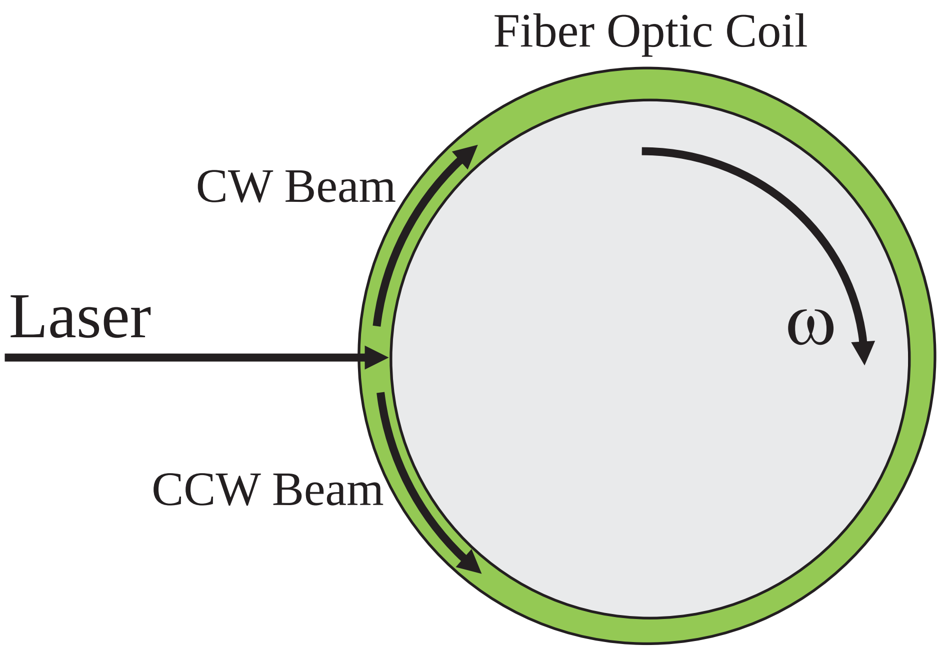
Some Gyroscope Technologies

- Mechanical (see previous slide) – measures actual angles relative to some orientation
- Micro Electro-Mechanical System (MEMS) Gyroscopes use the Coriolis effect: $F = -2m(\omega \times v)$ on a vibrating proof mass
  » Single beam oscillator, balanced or tuning fork oscillators, shell or wine glass or cylindrical oscillators



- Ring Laser Gyroscopes (RLG) and Fiber Optic Gyroscopes (FOG) use the Sagnac effect to measure angular rate
  » RLG: laser beam around a closed path with mirrors
  » FOG: lasers through optical fiber

Fiber Optic Gyroscope



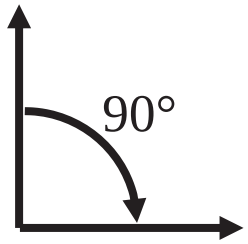
- Path for the CW beam is longer due to the rotation
- Path for the CCW beam is shorter due to the rotation

Measure the interference pattern between the CW and CCW beam to determine $\Delta\theta$

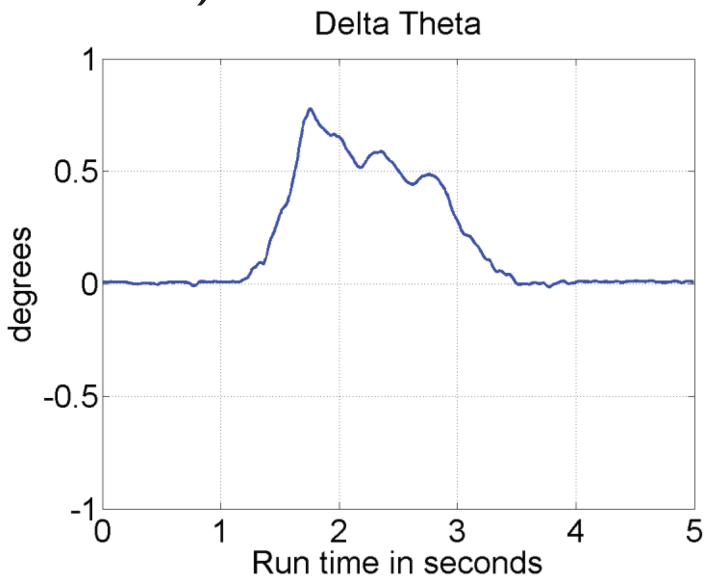
- Benefits include reliability and high bandwidth (small mass enables strapdown without vibration isolation)

MEMS Gyroscope Example

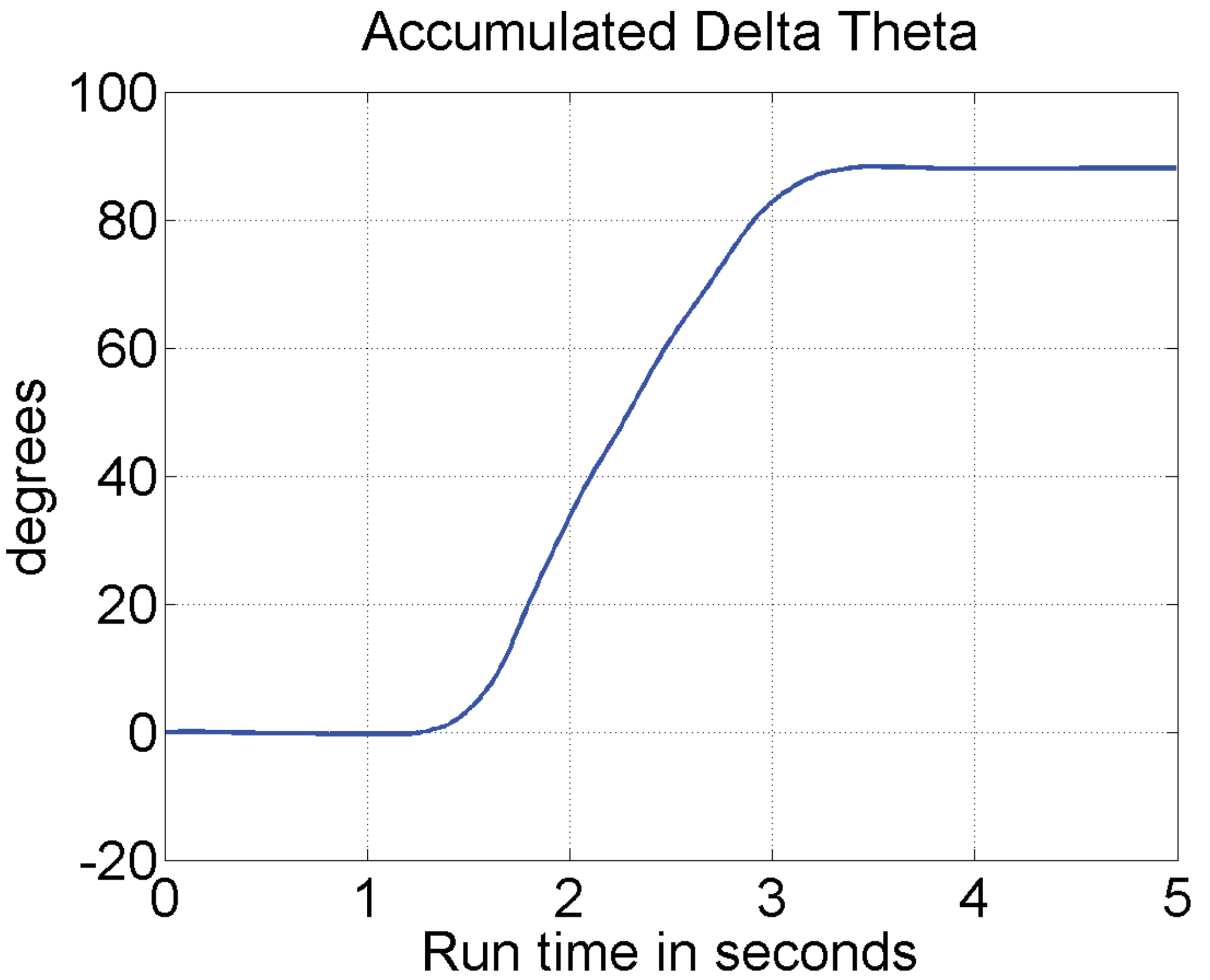
- Stationary, followed by a clock-wise (CW) rotation of 90°
  - » t = 0: Start at angle = 0°
  - » t = 1.5 s to 3.5 s: rotate CW up to 90°



- Use stationary portion to estimate the gyro bias
- Integrate (accumulate) Δθ to determine the change in angle



Measured Angle (around Z Axis)

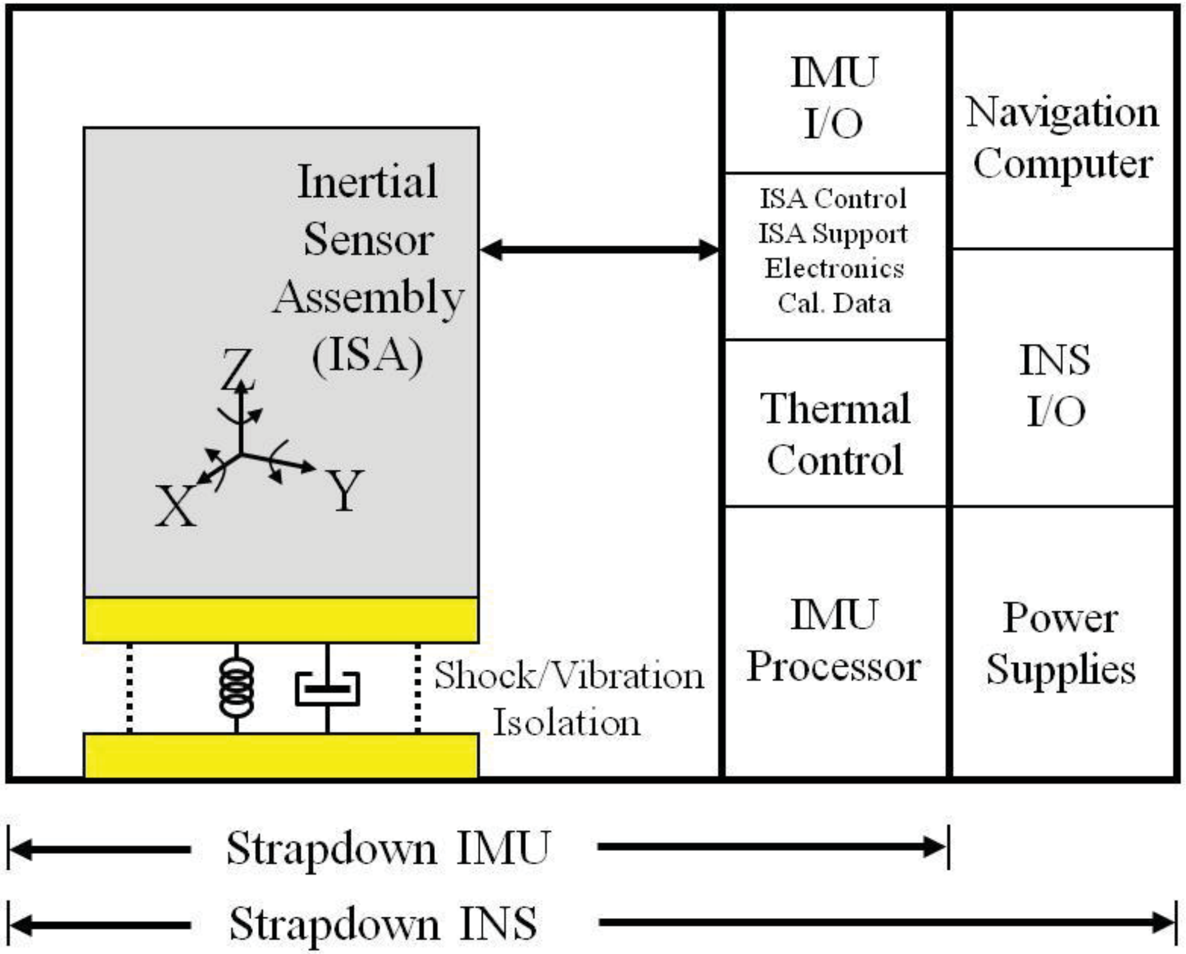


Gimballed and Strapdown

- Early days: gimballed
  - » Platform with accelerometers is held level and north-pointing such that position is calculated "directly" by integrating the ΔV's

- Today: mostly strapdown
  - » Sensors are "rigidly" mounted to the vehicle/device and leveling and north-pointing is implemented in software.

Strapdown INS Terminology



Ref: Proposed IEEE Inertial Systems Terminology Standard and Other Inertial Sensor Standards, Randall K. Curey, Michael E. Ash, Leroy O. Thielman, and Cleon H. Barker, IEEE PLANS 2004 (IEEE Std 1559).

IMU Accelerometer Parameters of Interest

- Input dynamic range (g)
- Bias stability during operation (mg)
- Bias error at turn-on at room temperature (mg)
- Bias error variation over temperature (mg)
- Noise (mg, 1-σ in a x-Hz bandwidth)
- Velocity random walk (m/s/$\sqrt{hr}$)
- Bias vibration sensitivity (mg/g^2)
- Scale factor error (% of full scale)
- Scale factor linearity (%)
- Frequency response (-3-dB bandwidth in Hz)
- Cross-axis sensitivity (%)

IMU Gyroscope Parameters of Interest

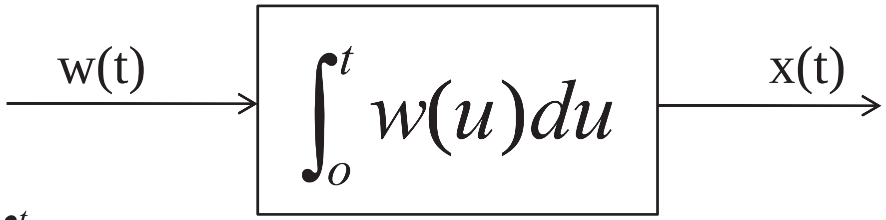
- Input dynamic range (deg/s)
- Bias stability during operation (deg/s)
- Bias error at turn-on at room temperature (deg/s)
- Bias error variation over temperature (deg/s)
- Noise (deg/s, 1-σ in a x-Hz bandwidth)
- Angular random walk (deg/s/$\sqrt{\text{hr}}$)
- Bias vibration sensitivity (deg/s/g^2)
- g-sensitivity (deg/s/g)
- Scale factor error (% of full scale)
- Scale factor linearity (%)
- Frequency response (-3-dB bandwidth in Hz)
- Cross-axis sensitivity (%)

Gyroscope Drift & Angle Random Walk

- Gyroscope drift is expressed in terms of degrees per hour and represents the long-term (average) angular drift of the gyro
- Gyroscope noise is the short-term variation in the output of the gyro
  - » Can be measured in deg/sec
  - » Can be expressed as a Power Spectral Density (PSD) with units of deg/sec/√Hz or $(\text{deg/sec})^2/\text{Hz}$ to express the output noise as a function of bandwidth
  - » To track angular changes, output is integrated over time to find the angle as a function of time. Angular changes exhibit Angle Random Walk (ARW) in units of deg/√hr
    - If ARW = 1 deg/√s, then
      - » After 1 s, σ of the angular change will be 1 deg
      - » After 100 s, σ of the angular change will be 10 deg
      - » After 1000 s, σ of the angular change will be 31.6 deg

Random Walk Process

- "Take successive steps in random directions"
- In continuous time, this process can be generated by feeding an integrator with white noise.



$$x(t) = \int_0^t w(u)du$$

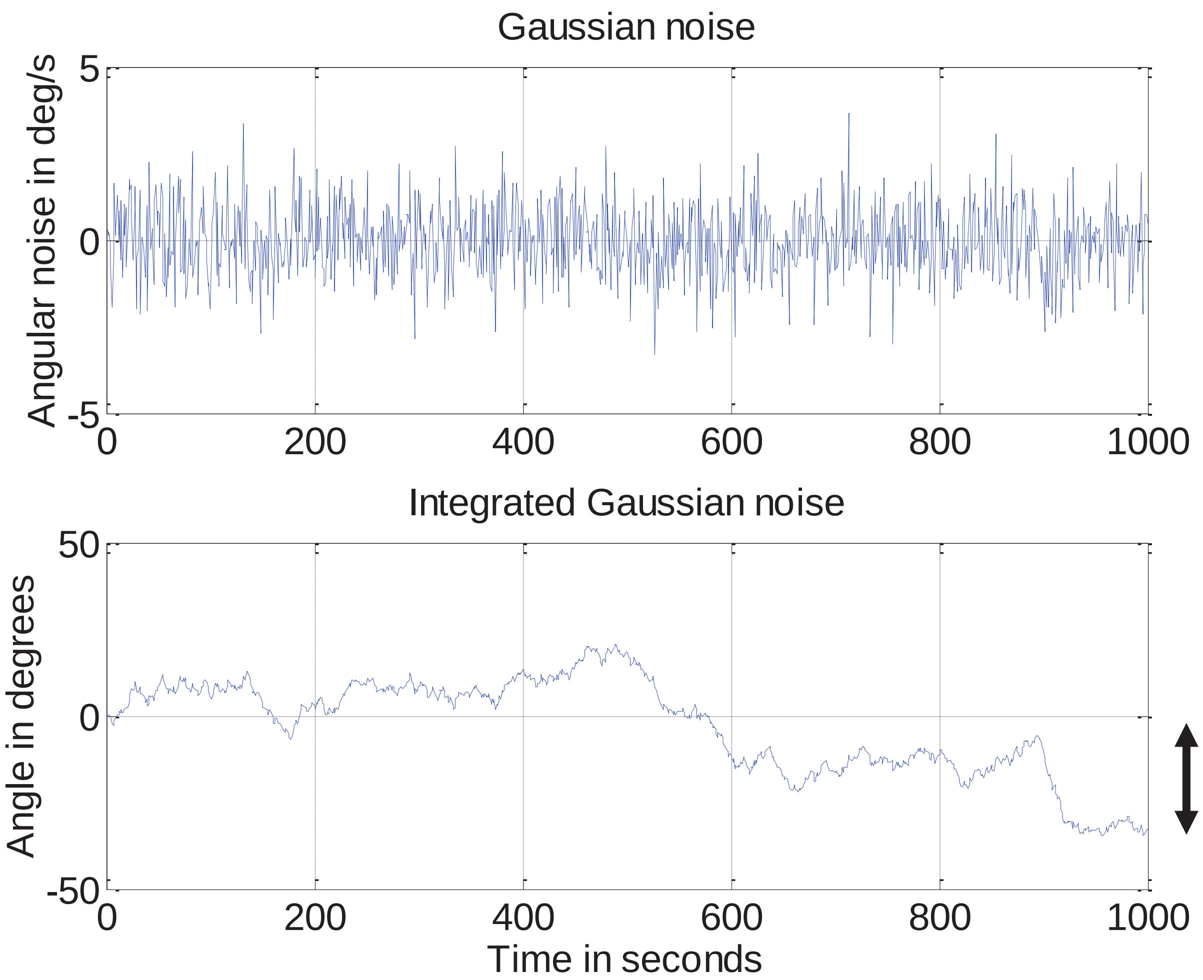
$$E\{x(t)\} = E\left\{\int_0^t w(u)du\right\} = 0$$

$$E\{x^2(t)\} = E\left\{\int_0^t w(u)du \int_0^t w(v)dv\right\} = \int_0^t \int_0^t E\{w(u)w(v)\}dudv$$

$$E\{x^2(t)\} = \int_0^t \int_0^t \delta(u-v)dudv = \int_0^t dv = t$$

- Mean is zero, but variance increases linearly with time.

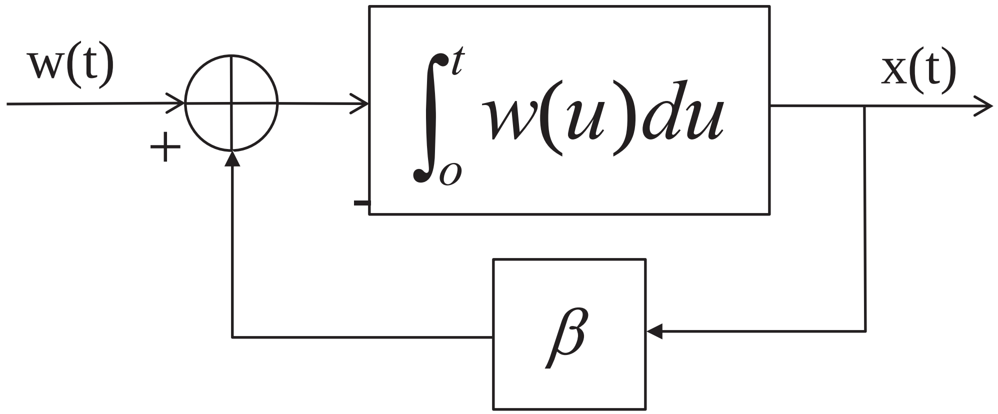
Gyro Random Walk



σ of angular change after 1000 s

Gyroscope and Accelerometer Time-Varying Biases

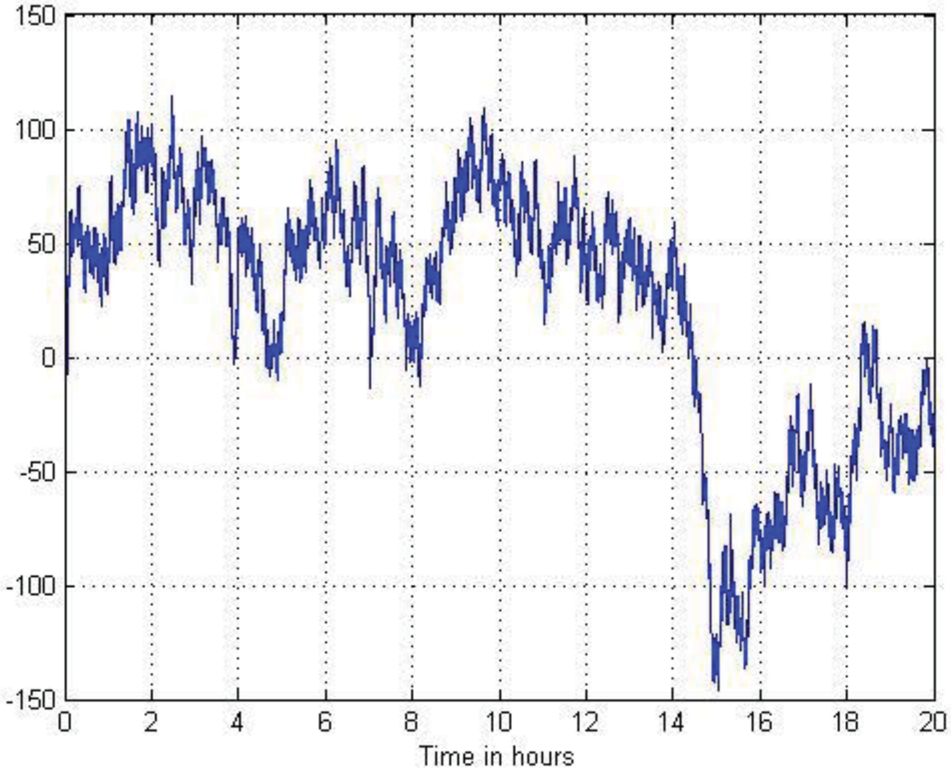
- Gyroscope and accelerometer biases are often modeled using a first-order Gauss-Markov process: a stationary Gaussian process with an exponential autocorrelation function.



- Continuous time: $\dot{x} = -\beta x + w$ time constant = 1/β
- Discrete time: $x_{k+1} = e^{-\beta \Delta t} x_k + w_k$ Var(x) = $\sigma^2/(2\beta)$

where: w_k is an uncorrelated zero-mean Gaussian sequence

1^{st} Order Gauss-Markov Example



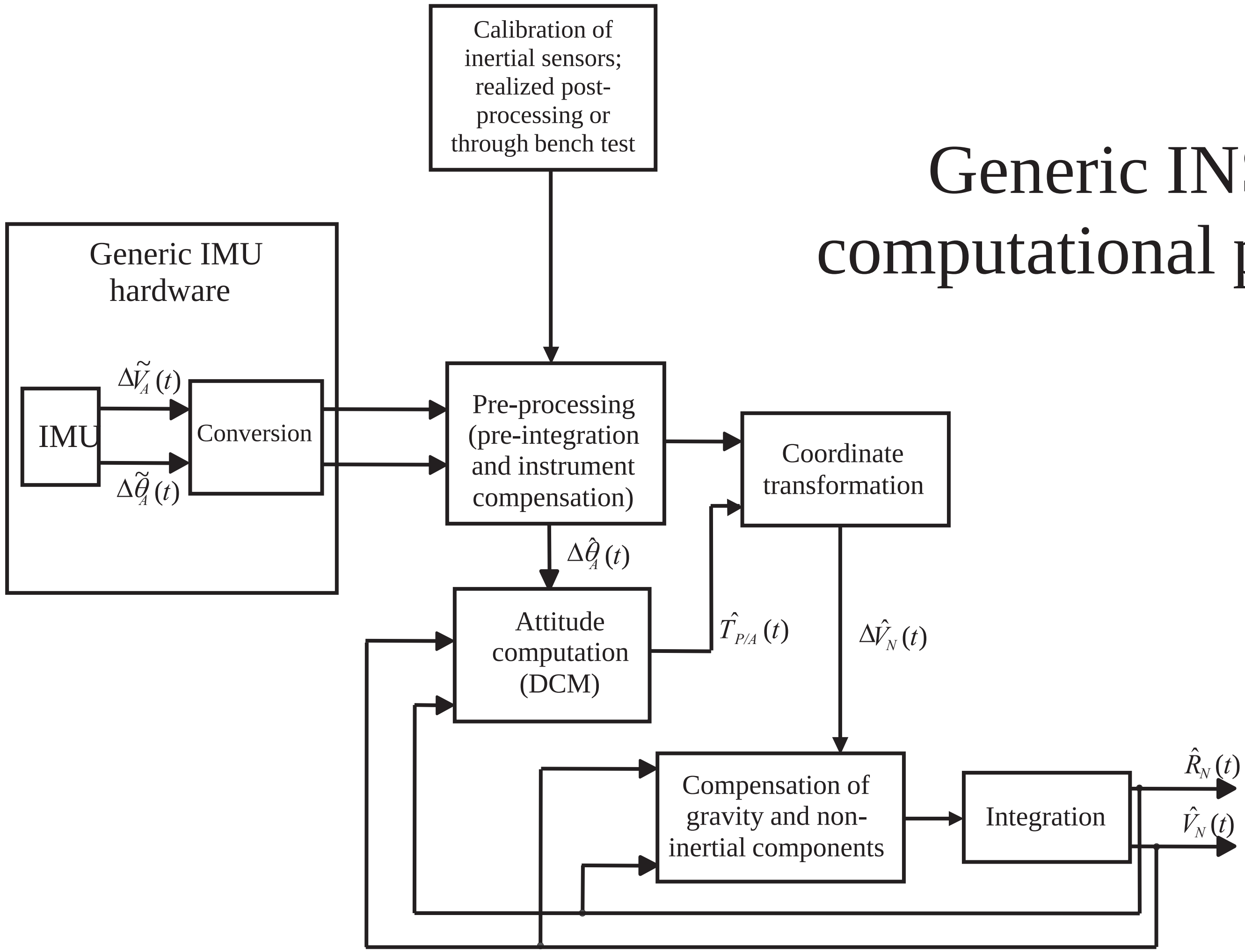

```
N = 7200*10;      % 20 hours
beta = 1/7200;    % time constant is 2 hours
dt = 1;           % time interval is 1 second
w = randn(N,1);   % generate unity Gaussian
t= zeros(N,1);    % pre-allocate memory for speed
x = zeros(N,1);   % pre-allocate memory for speed
x(1,1) = 0;       % initial condition
t(1,1) = 0;       % initial condition
for i = 2 : 1 : N
   t(i,1) = i;
   x(i,1) = exp(-beta*dt) * x(i-1,1) + w(i,1);
end
```

$Var(x) = \sigma^2/(2\beta)=3600$

$Std(x) = 60$

1^{st} Order Gauss-Markov processes are often used in engineering problems
Mean and variance statistics are "stable", but as can be seen from the above example, the process is not always "well-behaved."

INS Functional Components



Generic INS computational process

Conversion of Measured Data to State Estimates

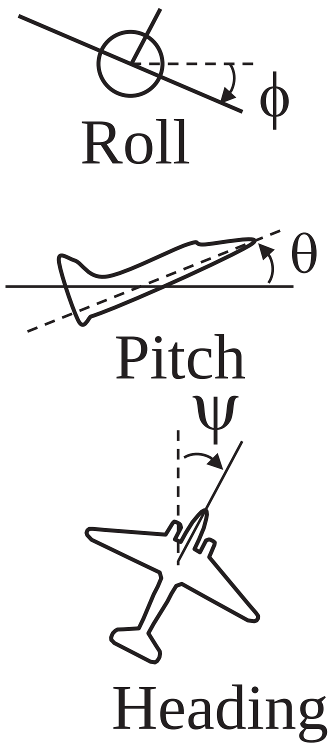
- Gyro increments are converted into angular orientation changes.
- Accelerometer outputs are combined with gravity effect and transformed into velocity changes in navigation axes.
- Result is combined with known kinematical adjustments and propagated into position increments.
- Current position estimate at each measurement time is used to predict observations.
- Residuals (innovations) are formed by comparing predictions versus measurements.
- Residuals (innovations) are weighted to produce updated states.
- Position, velocity, and misorientation state updates are assimilated into current estimates, resetting their *a posteriori* estimates to zero.

Aircraft Euler Angles

$$[\phi]_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \quad [\theta]_y = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$[\psi]_z = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{cases} \phi & \text{Roll} \\ \theta & \text{Pitch} \\ \psi & \text{Heading} \end{cases}$$

$$T_{A/P} = C_n^b = [\phi]_x [\theta]_y [\psi]_z$$



Transformation from Platform to Aircraft $T_{A/P} = C_n^b$
(Platform = Navigation Frame)

Small Angles

$$T_{P/A} = \begin{bmatrix} \cos\psi\cos\theta & -\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi & \sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi \\ \sin\psi\cos\theta & \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi & -\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

- For small angles:

$$\sin\psi \approx \psi, \quad \sin\theta \approx \theta, \quad \sin\phi \approx \phi, \quad \cos\psi \approx 1, \quad \cos\theta \approx 1, \quad \cos\phi \approx 1$$

$$T_{P/A} = \begin{bmatrix} 1 & -\psi & \theta \\ \psi & 1 & -\phi \\ -\theta & \phi & 1 \end{bmatrix} = I + \begin{bmatrix} 0 & -\psi & \theta \\ \psi & 0 & -\phi \\ -\theta & \phi & 0 \end{bmatrix}$$

$$T_{A/P} = T_{P/A}^{'} = \begin{bmatrix} 1 & \psi & -\theta \\ -\psi & 1 & \phi \\ \theta & -\phi & 1 \end{bmatrix} = I + \begin{bmatrix} 0 & \psi & -\theta \\ -\psi & 0 & \phi \\ \theta & -\phi & 0 \end{bmatrix}$$

Gimbal Lock

$$T_{P/A} = \begin{bmatrix} \cos\psi\cos\theta & -\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi & \sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi \\ \sin\psi\cos\theta & \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi & -\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

When $\theta = \frac{\pi}{2}$, then $T_{P/A} = \begin{bmatrix} 0 & \sin(\phi-\psi) & \cos(\phi-\psi) \\ 0 & \cos(\phi-\psi) & -\sin(\phi-\psi) \\ -1 & 0 & 0 \end{bmatrix}$

- Now only the difference between the roll and heading angle determines the transformation from Aircraft to Platform.
- The roll and heading angles are not unique.
- Same occurs when the pitch angle = -90°, then the sum of the roll and heading angles determines the transformation.

- For aircraft, both roll and heading undergo a discontinuity when the pitch angle passes through ±90° → gimbal lock.

Four-Parameter Set (Quaternions)

- A quaternion is described by (nonsingular, unambiguous):

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \mathrm{E}_1 \sin\left(\frac{\theta}{2}\right) \\ \mathrm{E}_2 \sin\left(\frac{\theta}{2}\right) \\ \mathrm{E}_3 \sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$|\mathbf{q}| = 1$$

$$T_{A/P}\mathbf{E} = \mathbf{E}$$

$$E_1^2 + E_2^2 + E_3^2 = 1$$

$$\theta = \arccos\left(\frac{T_{11} + T_{22} + T_{33} - 1}{2}\right)$$

- Example:

$$T_{A/P} = \begin{bmatrix} q_4^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 q_2 + q_4 q_3) & 2(q_1 q_3 - q_4 q_2) \\ 2(q_1 q_2 - q_4 q_3) & q_4^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2 q_3 + q_4 q_1) \\ 2(q_1 q_3 + q_4 q_2) & 2(q_2 q_3 - q_4 q_1) & q_4^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

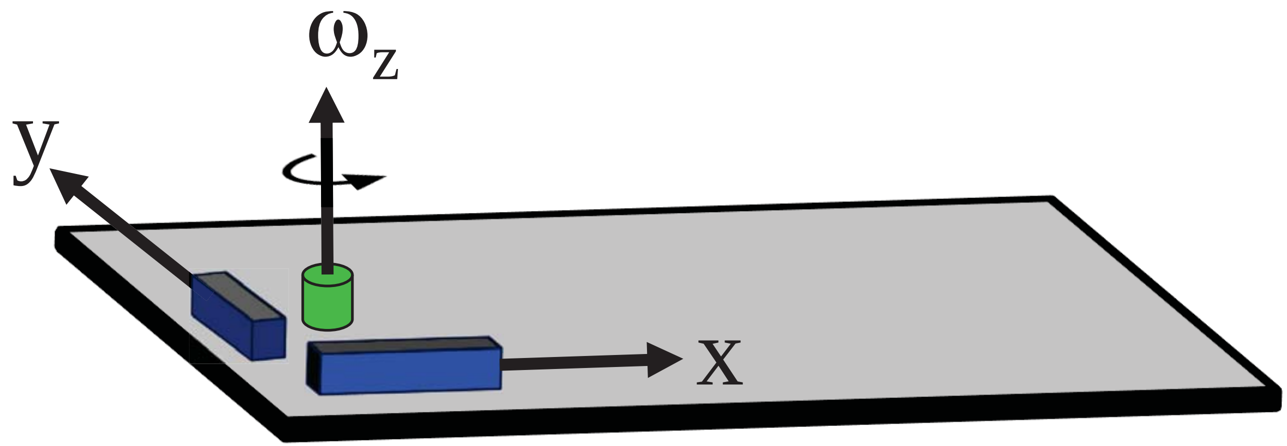
$$E_1 = 1, \quad E_2 = E_3 = 0, \quad \text{then } q_1 = \sin(\theta/2), \quad q_2 = q_3 = 0$$

$$2q_1 q_4 = 2\sin(\theta/2)\cos(\theta/2) = \sin(\theta)$$

$$q_4^2 - q_1^2 = \cos^2(\theta/2) - \sin^2(\theta/2) = \cos(\theta)$$

$$T_{A/P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

Inertial Navigation on a Flat Non-Rotating Earth



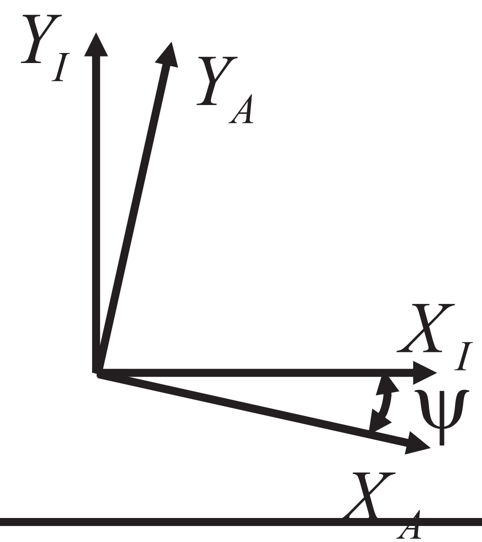
$$\dot{\psi} = \omega_Z$$

$$F_{X,I} = F_{X,A}\cos(\psi) + F_{Y,A}\sin(\psi)$$

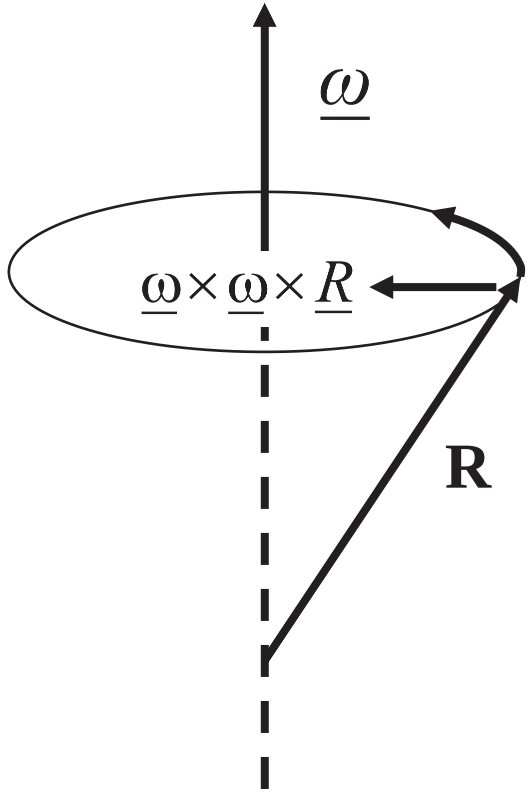
$$F_{Y,I} = -F_{X,A}\sin(\psi) + F_{Y,A}\cos(\psi)$$

$$\dot{v}_{X,I} = F_{X,I} \qquad \dot{x}_{X,I} = v_{X,I}$$

$$\dot{v}_{Y,I} = F_{Y,I} \qquad \dot{x}_{Y,I} = v_{Y,I}$$



Rotation around an Axis in Inertial Space



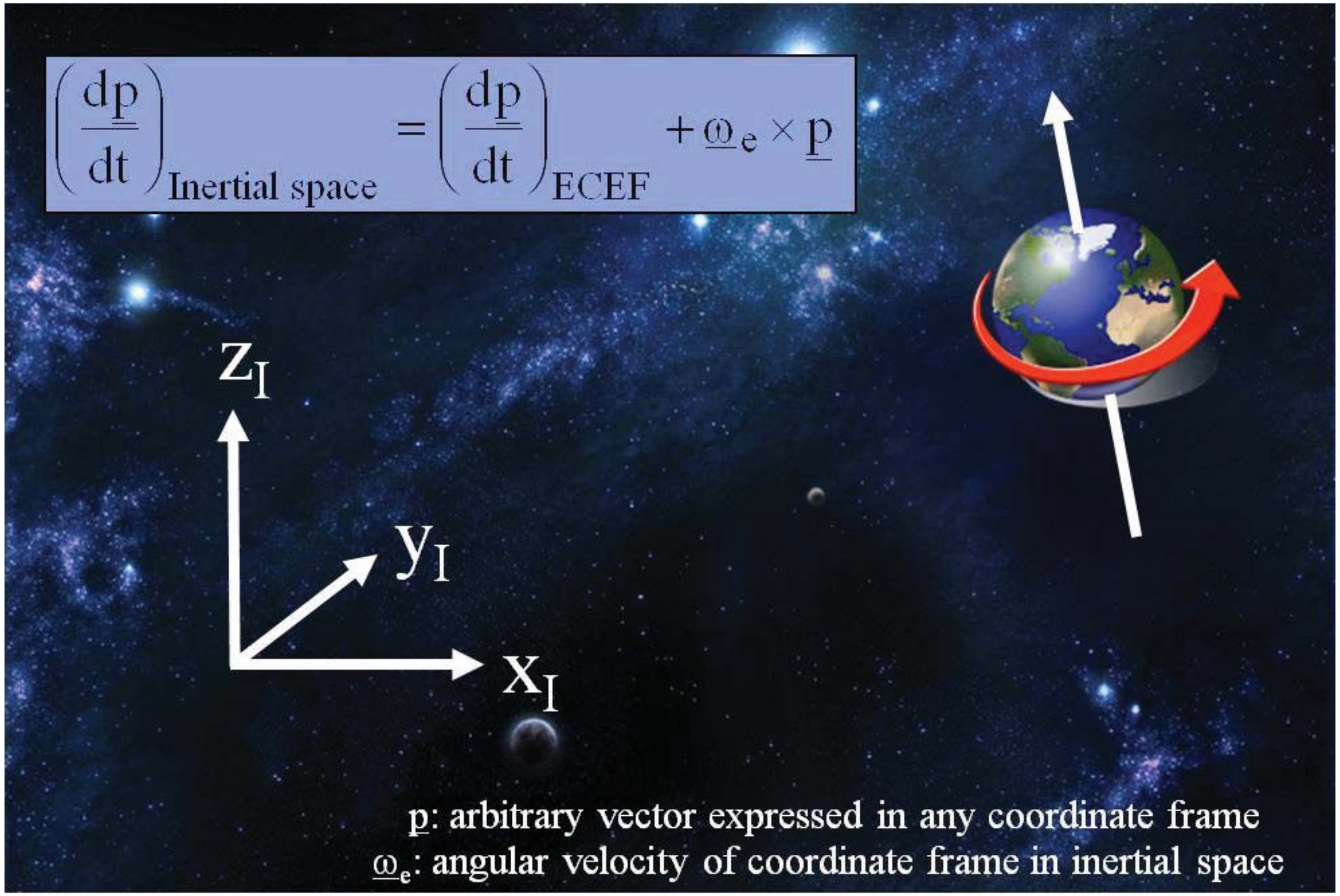
$\underline{\omega}$ *angular velocity in rad/s*

$$\underline{v} = \underline{\omega} \times \underline{R}$$

$$\underline{a} = \underline{\omega} \times \underline{v}$$

$$\underline{a} = \underline{\omega} \times \left(\underline{\omega} \times \underline{R}\right)$$

Inertial Space & Earth-Centered-Earth-Fixed



As derived in: Goldstein, H., *Classical Mechanics*, 2nd Ed., 1980, pp. 174-176.

Apply to radius and inertial space velocity vectors

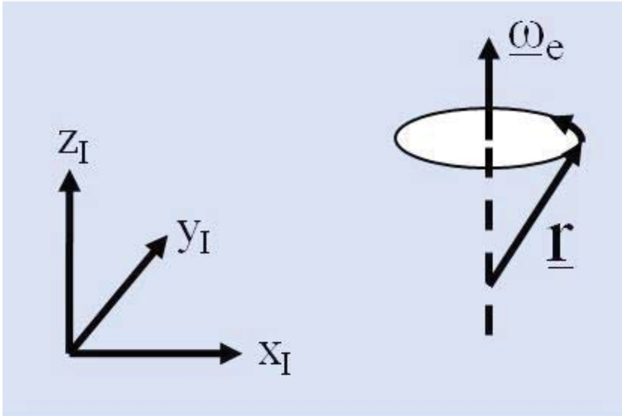
$$\left(\frac{d\underline{p}}{dt}\right)_I = \left(\frac{d\underline{p}}{dt}\right)_E + \underline{\omega}_e \times \underline{p}$$

Arbitrary vector

$$\begin{cases} \left(\dfrac{d\underline{r}}{dt}\right)_I = \left(\dfrac{d\underline{r}}{dt}\right)_E + \underline{\omega}_e \times \underline{r} \\ \left(\dfrac{d\underline{v}_I}{dt}\right)_I = \left(\dfrac{d\underline{v}_I}{dt}\right)_E + \underline{\omega}_e \times \underline{v}_I \end{cases}$$

Radius vector

Velocity relative to inertial space



$$\begin{cases} \underline{v}_I = \underline{v}_E + \underline{\omega}_e \times \underline{r} \\ \left(\dfrac{d\underline{v}_I}{dt}\right)_I = \left(\dfrac{d(\underline{v}_E + \underline{\omega}_e \times \underline{r})}{dt}\right)_E + \underline{\omega}_e \times (\underline{v}_E + \underline{\omega}_e \times \underline{r}) \end{cases}$$

Radius from center of ECEF

Earth angular velocity (constant) relative to inertial space

Velocity relative to ECEF

$$\begin{cases} \underline{v}_I = \underline{v}_E + \underline{\omega}_e \times \underline{r} \\ \underline{a}_I = \underline{a}_E + \left(\dfrac{d(\underline{\omega}_e \times \underline{r})}{dt}\right)_E + \underline{\omega}_e \times \underline{v}_E + \underline{\omega}_e \times (\underline{\omega}_e \times \underline{r}) \end{cases}$$

$$\begin{cases} \underline{v}_I = \underline{v}_E + \underline{\omega}_e \times \underline{r} \\ \underline{a}_I = \underline{a}_E + 2(\underline{\omega}_e \times \underline{v}_E) + \underline{\omega}_e \times (\underline{\omega}_e \times \underline{r}) \end{cases}$$

Based on: Goldstein, H., *Classical Mechanics*, 2nd Ed., 1980, p. 177.

Force in Inertial Space (measured by accelerometer)

$$\underline{F}_I = m\underline{a}_I$$

$$\text{with: } \underline{a}_I = \underline{a}_E + 2(\underline{\omega}_e \times \underline{v}_E) + \underline{\omega}_e \times (\underline{\omega}_e \times \underline{r})$$

$$\text{or: } \underline{F}_I = m\underline{a}_I = m\underline{a}_E + 2m(\underline{\omega}_e \times \underline{v}_E) + m\underline{\omega}_e \times (\underline{\omega}_e \times \underline{r})$$

$$\underline{F}_{specific} = m\underline{a}_E - 2m(\underline{\omega}_e \times \underline{v}_E) - m\underline{\omega}_e \times (\underline{\omega}_e \times \underline{r})$$

Note: $1\,\text{cm/s}^2$

after 5 min

displacement:

$\frac{1}{2}(0.01)(300)^2 = 450\text{ m}$

Coriolis force

$2\omega_e v < 1.5 \times 10^{-4}\,v$

if $v = 300\text{ m/s}$

$2\omega_e v < 4.5\ \text{cm/s}^2$

Centrifugal force (points outwards)

$\omega_e^2 r \approx 3.4\,\text{cm/s}^2$

Sum of earth's gravitational field and centrifugal force = gravity field

Based on: Goldstein, H., *Classical Mechanics*, 2nd Ed., 1980, pp. 177-179.

Movement over Ellipsoid

Radius of curvature in a meridian

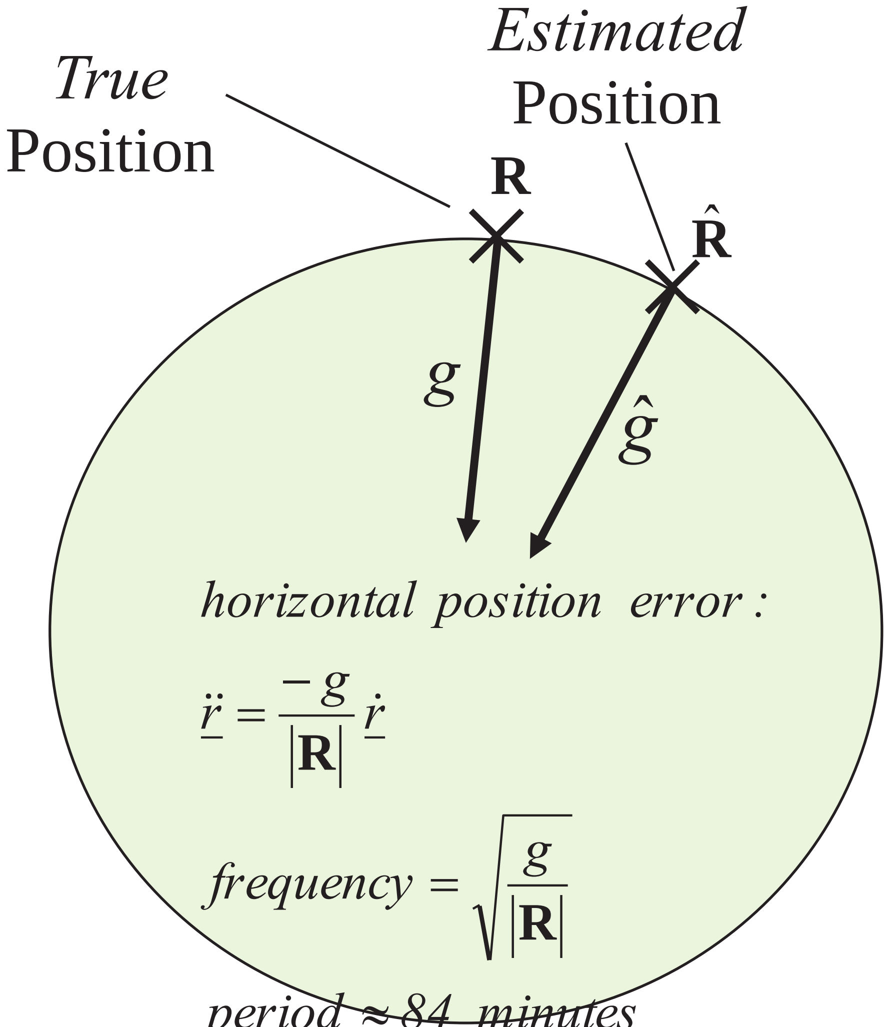
$$R_M = \frac{a_E(1 - e_E^2)}{\left[1 - e_E^2 \sin^2(Lat)\right]^{3/2}}$$

Radius of curvature in planes parallel to the equator

$$R_P = \frac{a_E}{\sqrt{1 - e_E^2 \sin^2(Lat)}}$$

$$a_E = 6378137 \text{ m}; \; e_E^2 = (2 - f) f; \; f = \frac{1}{298.257223563}$$

Basis of Schuler Effect



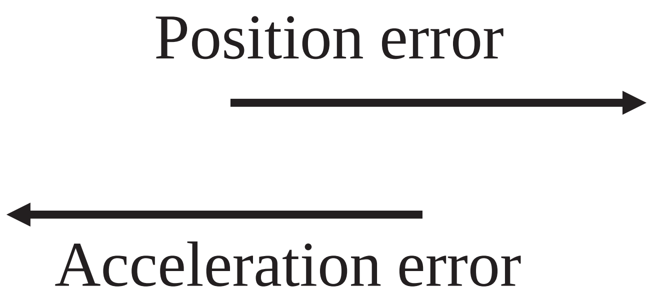
horizontal position error :

$$\ddot{\underline{r}} = \frac{-g}{|\mathbf{R}|}\dot{\underline{r}}$$

$$frequency = \sqrt{\frac{g}{|\mathbf{R}|}}$$

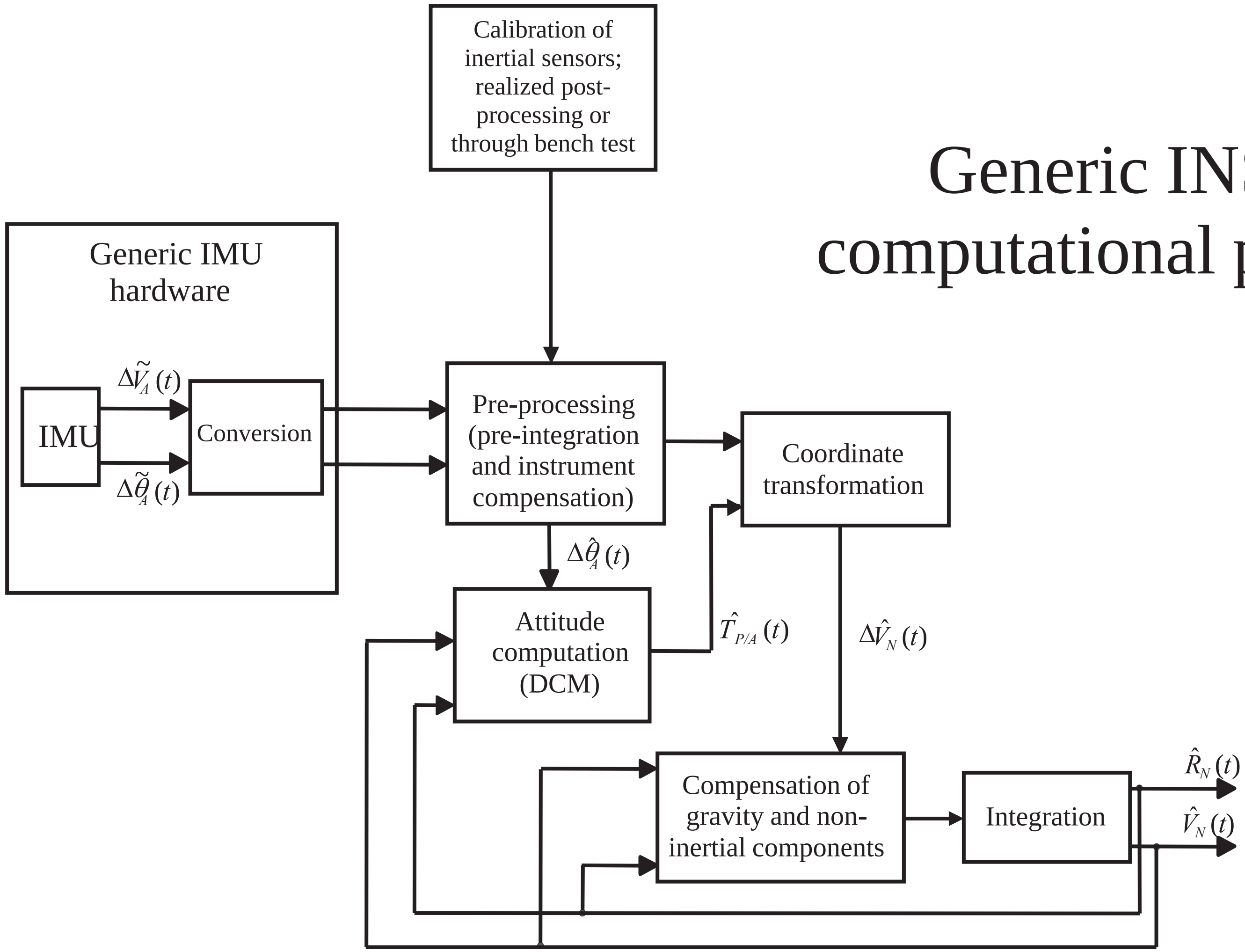
$$period \approx 84 \ \ minutes$$

Computed gravity vector $\hat{\mathbf{g}}$ has a *backward* directional component



INS Functional Components

Generic INS computational process



Initialization

- Position is initialized with
  - » Estimated longitude
  - » Estimated geodetic latitude
  - » Estimated altitude above ellipsoid (baro-altimeter)
  - » Zero wander angle
- Pitch and roll angles are calculated from the accelerometers (gravity defines the apparent vertical)
- Heading is obtained from gyrocompassing
  - » Need high-quality gyroscopes
  - » If not available → integration with other sensors
- Form corresponding matrix $\mathbf{T}_{P/E}$ from Earth to nav-reference (local level, wander-azimuth) coordinates

Short Term INS Position Error

Position Error in Cruise – Short Term (<< 84 min)

Position Error at time t =

Initial Position Error +

Velocity Error $\times\ t$ +

Leveling Error $\times\ g \times \frac{1}{2} t^2$ +

Accelerometer Bias $\times \frac{1}{2} t^2$ +

Drift Rate $\times\ g \times 1/6\ t^3$

Future Trends

- Cost of inertial sensors continues to drop, while performance continues to increase
  - » Non-tactical grade
  - » Automotive, consumer applications
- Low-cost navigation grade IMUs are not on the horizon
- Performance improvements feasible through advanced calibration techniques
- Continued developments in the area of integration with GNSS and other navigation sensors
  - » Ultra-tight integration with carrier phase
    - Long signal integration and tracking loop design
  - » Integration with software-defined radios and batch processing (open-loop tracking)