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Density of Hyperbolicity and Complex Methods (Second Lecture)

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Second Lecture: Density of Hyperbolicity and Complex Methods

Density of Hyperbolicity

The *simplest situation* is when the *attractors of f are all hyperbolic periodic orbits*: such f called **hyperbolic** (also called *Axiom A*).
It would be nice if *every map can be approximated by a hyperbolic map*. This problem goes back in some form to

- *Fatou*, who stated this as a conjecture in the 1920's.
- *Smale* gave this problem 'naively' as a thesis problem in the 1960's, see [Sma80].
- *Jakobson* proved that the set of hyperbolic maps is dense in the C^1 topology, see [Jak71];
- The quadratic case $x \mapsto ax(1-x)$ was proved in a major breakthrough in the the mid 90's by *Lyubich [Lyu97]* and also *Graczyk and Swiatek [GŚ97]*.
- *Blokh and Misiurewicz* [BM00] proved a partial result towards the density of hyperbolic maps in the C^2 topology.
- *Shen* [She04] then proved the C^2 density of hyperbolic maps.

The general result is:

Theorem (Density of hyperbolicity for real polynomials, [KSvS07a])

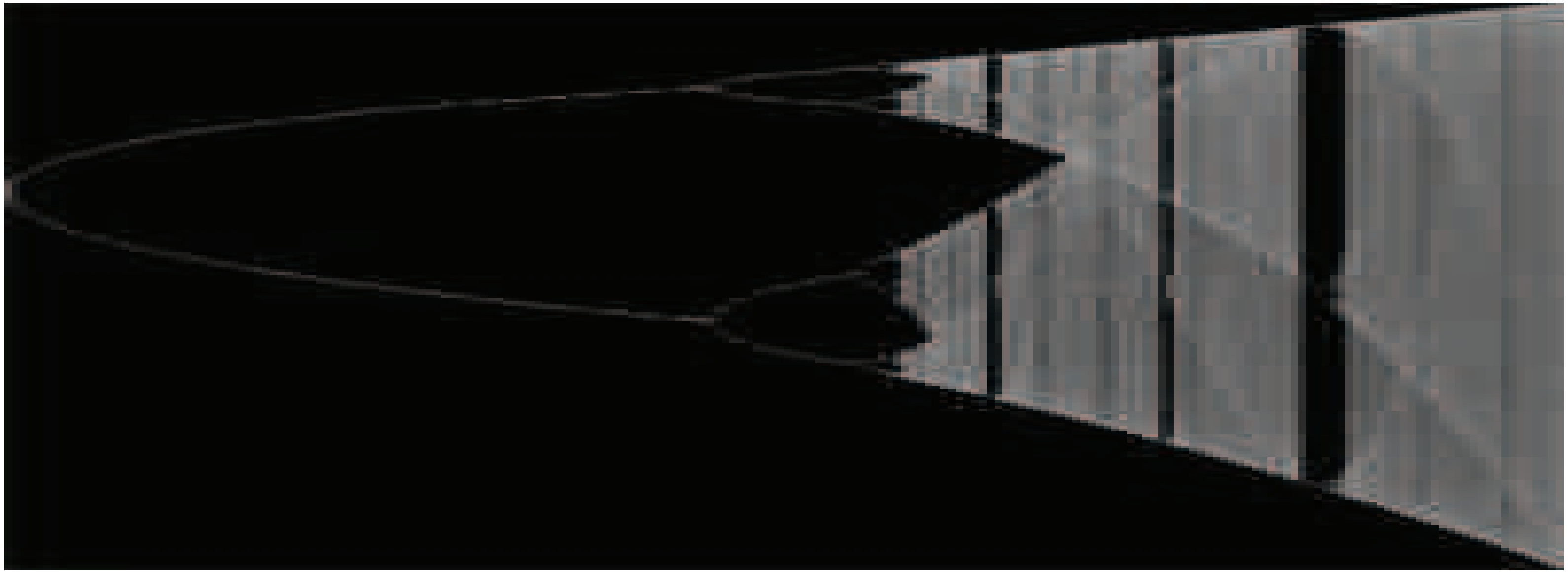
Any real polynomial can be approximated by hyperbolic real polynomials of the same degree.

The above theorem allows us to solve the 2nd part of Smale's eleventh problem for the 21st century. [Sma00]:

Theorem (Density of hyperbolicity for smooth one-dimensional maps, [KSvS07b])

Hyperbolic maps are dense in the space of C^k maps of the compact interval or the circle, $k = 1, 2, \dots, \infty, \omega$.

For quadratic maps $f_a = ax(1-x)$, the above theorems assert that the periodic windows are dense in the bifurcation diagram.



The quadratic case turns out to be special, because in this case certain return maps become almost linear. This special behaviour does not even hold for maps of the form $x \mapsto x^4 + c$.

Comments on the strategy of proof: local versus global perturbations

It turns out that it is often *not possible to perturb a map to a hyperbolic map by local methods* (in the C^k topology, $k \geq 2$).

Instead one shows that a non-hyperbolic map is essentially uniquely determined by its conjugacy class: *if f and g are conjugate then show they are quasi-symmetrically conjugate*. This approach goes back to Sullivan.

In [KSvS07a] we showed that this rigidity holds for polynomials with certain additional restrictions (e.g. all critical points real).

In fact, it holds in general:

Quasi-symmetric rigidity

Theorem (SvS)

Assume that $f, g\colon [0,1] \to [0,1]$ *are real analytic, topologically conjugate and that the topologically conjugacy is a bijection between*

- *the set of critical points and the order of corresponding critical points is the same;*
- *the set of parabolic periodic points.*

Then the conjugacy *between* f *and* g *is* quasi-symmetric*.*

A homeomorphism $h\colon [0,1] \to [0,1]$ is called *quasi-symmetric* if there exists $K < \infty$ so that

$$\frac{1}{K} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq K$$

for all $x - t, x, x + t \in [0,1]$.

- Note that f and g can only have finitely many parabolic periodic orbits (see [MdMvS92]
- All conditions are necessary
- Previous results:
  - *Khanin and Teplinsky* show this for critical circle maps (building on earlier work of de *Faria, de Melo and Yampolsky*).
  - *Levin + vS* show that for covering maps with *one inflection point c*, one can obtain a qs conjugacy restricted to $\omega(c)$, provided $\omega(c)$ is either minimal or every periodic orbit in $\omega(c)$ is repelling.
  - *Kozlovski-Shen-vS* for real polynomials with only real critical points.
- In our proof complex methods are essential.
- Interestingly, the proof even goes through to the C^3 *category* (*Trevor Clark*). Trevor and I decided to merge our results and publish this in a joint paper.

Quasi-symmetric rigidity $\implies$ density of hyperbolicity?

Let's explain this for the family $z^2 + c$.

- Assume, *by contradiction*, that there exists a non-trivial interval of parameters $[c_l, c_r]$ so that the corresponding map are all non-hyperbolic.
- Hence, $\forall c \in [c_l, c_r]$ and for all $n \geq 0$ one has $f_c^n(0) \neq 0$.
- So *all maps f_c with $c \in [c_l, c_r]$ are topologically conjugate*.
- Assume that $[c_l, c_r]$ is a *maximal* interval with this property (that this interval is *closed when f is non-hyperbolic* follows from kneading theory).
- By *qs-rigidity* Thm, $f_c, f_{c'}$ are *qs-conjugate* $\forall c, c' \in [c_l, c_r]$.

Now assume that $c_l \neq c_r$. Then we will use *quasiconformal maps* to obtain an *open* neighbourhood $O \supset [c_l, c_r]$ so that for all $c, c' \in O$ the maps $f_c, f_{c'}$ are also topologically conjugate.

This *contradicts maximality of $[c_l, c_r]$*. Hence $c_l = c_r$, and density follows.

So let's go to the complex plane!

Quasiconformal homeomorphisms

An orientation preserving homeomorphism h is called *K-quasiconformal* if

- there exists a constant $K < \infty$ such that for *Lebesgue almost all* $x \in \mathbb{C}$

$$\limsup_{r \to 0} \frac{\sup_{|y-x|=r} |h(y) - h(x)|}{\inf_{|y-x|=r} |h(y) - h(x)|} \leq K.$$

- If $K = 1$ then h is conformal.
- Such maps are, for example, Hölder and Lebesgue almost everywhere differentiable (as maps from $\mathbb{C} = \mathbb{R}^2$ to $\mathbb{C} = \mathbb{R}^2$).
- (In general, a conjugacy **cannot** be C^1, because then multipliers at periodic points would be the same.)

In this case *qs*-rigidity $\implies$ qc-rigidity.

Assume first that $f(z) = z^2 + c_l$ and $\tilde{f}(z) = z^2 + c_r$ are *qs-conjugate*.

- **Fact:** Any qs-homeomorphism h on $\mathbb{R}$ can be extended to a K-quasiconformal-homeomorphism H on $\mathbb{C}$.
- Hence $\exists$ a qc map H so that $H \circ f = \tilde{f} \circ H$ on $\mathbb{R}$ and near ∞.
- Now define a sequence of lifts H_n inductively by $H_0 = H$ and $\tilde{f} \circ H_{n+1} = H_n \circ f$. This can be done, see blackboard.
- *H_n is again K-qc for any n with the same K.*
- $H_{n+1} = H_n$ on ever larger sets.
- **Fact:** The space of K-quasiconformal maps is compact.
- Hence H_n converges to some K-qc homeomorphism H.
- Therefore $\tilde{f} \circ H = H \circ f$.

$\tilde{f} = H \circ f \circ H^{-1}$ for some qc-homeo H. So what?

Now we use the *Measurable Riemann Mapping Theorem:*

- $DH(z)$ exists for a.e. z.
- So $DH(z)$ sends ellipse based at z to circle based at $H(z)$.
- One can associate to this ellipse some number $\mu(z) \in \mathbb{D} = \{w; |w| < 1\}$ where $|\mu(z)|$ is the eccentricity of the ellipse.
- By this theorem, associated to $t\mu(z)$ there is another qc map H_t with the same long and short axis and eccentricity $t|\mu(z)|$.
- Normalize so that $H_t(0) = 0$ and $H_t(x)/x \to 1$ as $x \to \infty$.
- *Since $\tilde{f} = H \circ f \circ H^{-1}$ is holomorphic, the map $f_t = H_t \circ f \circ H_t^{-1}$ is again conformal*, see blackboard.
- $\tilde{f}_t$ has a unique critical point, is holomorphic and the normalisation implies that $f_t(z) = z^2 + c(t)$.

What's useful about $f_t = H_t \circ f \circ H_t^{-1}$?

Reminder: $f_t(z) = H_t \circ f \circ H_t^{-1}(z) = z^2 + c(t)$

- $H_0 = id \implies f_0 = H_0 \circ f \circ H_0^{-1} = f = z^2 + c_l \implies$ $c(0) = c_l$;
- $f_1 = H \circ f \circ H^{-1} = \tilde{f} = z^2 + c_r \implies c(1) = c_r$.
- By the Measurable Riemann Mapping Theorem, *$t \mapsto f_t(0)$ is holomorphic*. Hence *$t \mapsto c(t)$ is holomorphic.*
- By construction, $t \mapsto c(t)$ is real and has no critical points.
- Hence for $t > 1$, $t \approx 1$ one has $c(t) > c_r$ and the map f_t is still conjugate to f.
- $\implies$ *open neighbourhood of $[c_l, c_r]$ of conjugate maps*.

Together this shows *qs-rigidity $\implies$ density of hyperbolicity.*

qs-rigidity $\Longrightarrow$ density of hyperbolicity for real polynomials with real critical points

If the two qs-conjugate polynomials **only** *have real critical points* then one can generalise this argument:

- use an inductive dimension reduction:
- restrict to algebraic varieties of the form $\{f; f^n(c_1) = c_2\}$ of lower and lower dimension.

qs-rigidity $\implies$ density of hyperbolicity for real polynomials

- If the two qs-conjugate polynomials $f, \tilde{f}$ have **non-real** *critical points* then $f, \tilde{f}$ qs-conjugate $\not\Rightarrow$ $f, \tilde{f}$ are qc-conjugate.
- Lifting $\tilde{f} \circ H_{n+1} = H_n \circ f$ not possible: one has no information about the orbits of the complex critical points.
- Want all critical points to be *captured* (in hyperbolic basin).
- *Step 1:* Consider one-parameter families f_t, $t \in [-1, 1]$ of *regular maps*: each *neutral* periodic orbit of f_t has a *critical point in its basin*.
- *Step 2:* Moreover, assume that f_0 and f_1 are **not** conjugate, and that captured critical points for f_0 remain captured for f_t. *Thm:* $\exists t \approx 0$ so that f_t has new captured critical points. Here use *holomorphic motion* and *geometric control* for certain *complex box mappings*. (Not soft...)

How to construct regular families?

- *Step 3:* Approximate f by a polynomial $\tilde{f}$ of the same degree without neutral periodic orbits and *same captured critical pts*.
- *Step 4:* All maps C^3 near $\tilde{f}$ are *regular*.
- *Step 5:* Locally perturb $\tilde{f}$ to a *C^3 hyperbolic map g* (here use 'complex bounds')!!! Note *$\tilde{f}$ and g are not C^∞ close at all*.
- *Step 6:* Approximate the smooth map g by a *polynomial map G of much higher degree*.
- *Step 7:* Consider the family $f_t = \tilde{f} + tG$. By Step 4 this a *regular family*.
- *Step 8:* Using Step 2: $\exists t \approx 0$ so that f_t is hyperbolic. However, f_t has *much higher degree*.
- *Step 9:* f_t and $\tilde{f}$ are C^0 close on a large disc $\mathbb{D}_R$. Hence, using the so-called *Straightening Theorem*, $\exists$ a real polynomial $\hat{f}$
  - of the same degree as f
  - conjugate to f_t on $\mathbb{D}_{R/2}$
  - still close to $\tilde{f}$.

Density of families of entire families

Theorem (Hyperbolicity for entire maps (with Lasse Rempe))

Let f be an entire function with a finite number of critical values and either

- *f is bounded is on the real axis.*
- *some sector condition is satisfied.*

Then there exist orientation preserving homeomorphisms ϕ, ψ arbitrarily close to the identity such that $g := \psi \circ f \circ \phi^{-1}$ is entire and hyperbolic.

Application: trigonometric polynomials

Consider generalized trigonometric polynomial $F_\mu : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$:

$$F_\mu(t) = D\cdot t + \mu_1 + \mu_{2m}\sin(2\pi mt) + \sum_{j=1}^{m-1}(\mu_{2j}\sin(2\pi jt) + \mu_{2j+1}\cos(2\pi jt)).$$

Note that if $\mu, \mu' \in \mathbb{R}^{2m}$ with $\mu_1 - \mu_1' \in \mathbb{Z}$, then $f_\mu = f_{\mu'}$. So choose $\mu = (\mu_1, \ldots, \mu_{2m}) \in \Delta$, where

$$\Delta := \{\mu \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}^{2m-1} \ : \ \mu_{2m} > 0 \text{ and } f_\mu \text{ is } 2m-\text{multimodal }\}.$$

For example: the *Arnol'd family* $x \mapsto x + \alpha + \beta\sin(2\pi x)$. In this

case we have the following theorem:

Theorem (Density of hyperbolicity and rigidity in the trigonometric family, joint with Lasse Rempe)

Hyperbolic parameters in Δ for which f_μ are dense. *Furthermore,*

1. *Consider the set* $[\mu_0]$ *of parameters* μ *for which* f_μ *is topologically conjugate to* f_{μ_0} *by an order-preserving homeomorphism of the circle. Then* $[\mu_0]$ *has at most m components.*
2. *If* f_{μ_0} *has no periodic attractors on the circle, then each component of* $[\mu_0]$ *is equal to a point.*

This answers the conjectures posed by de Melo, Salomão and Vargas.

Here

- we need to pay attention to points that go repeatedly to infinity and back again and show absence of line fields on this set.
- We also need to show that $f, \tilde{f}$ are qs conjugate on the real line.
- In the polynomial case this was not fully needed, but now we do not have a straightening theorem.

Summary:

$$\text{real method} \implies \text{real bounds} \implies \begin{cases} \text{Koebe} \\ \text{complex bounds} \end{cases}$$

$$\text{complex method} \implies \begin{cases} \text{quasiconformal maps} \\ \text{Measurable Riemann Mapping Theorem} \\ \text{Holomorphic Motion} \end{cases}$$

How to construct qs-symmetries?

One approach is to use Carleson box construction.

We shall use complex methods, namely a complex analogue of the nice interval (puzzle pieces) and then to use our

QC-Criterion: For any $\epsilon > 0$ there exists a constant K with the following property.
Let $\phi\colon \Omega \to \tilde{\Omega}$ be a qc homeomorphism between two Jordan domains. Let $X \subset \Omega$ consist of pairwise disjoint topological disks (possibly infinitely many).
Assume that the following hold

- the components of X are topological discs with *ϵ-bounded geometry* each of which *ϵ-well-inside Ω* (and the same holds for $\phi(X)$).
- *ϕ is 1-qc on $\Omega - X$*.

Then there exists a *new K-qc homeo $\tilde{\psi}\colon \Omega \to \tilde{\Omega}$* which *agrees with ϕ on $\partial\Omega$*.