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Real one-dimensional dynamics: real and complex methods

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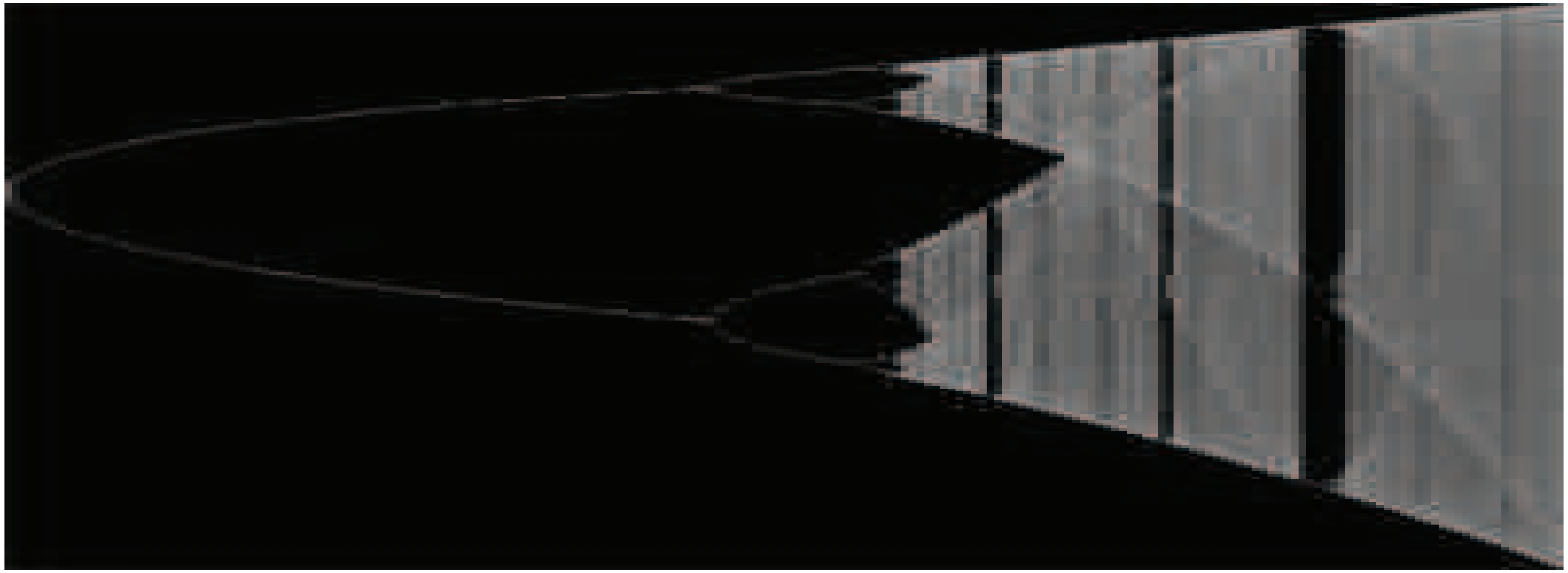
Throughout these talks we assume that N is an *interval or a circle* and that $f\colon N \to N$ is *real analytic*. For example:

$$f(x) = ax(1-x) \text{ or } f(x) = x^2 + c$$



One of the reasons real one-dimensional dynamics has been such an exciting field is because

- the theory is *far from trivial, yet almost complete*;
- the theory can be considered as a *model for what can happen in higher dimensions*.

- My first talk will be about theorems that can be obtained by real tools.
- The later talks will then discuss why one introduces complex tools

Real methods: (for example) results about attractors

Notation: $\omega(x)$ is the set of *accumulation points* of the sequence $x, f(x), f^2(x), \ldots$.

It would be great to describe *all* orbits of f, but it turns out to be much more fruitful to describe attractors and ergodic properties.

We say that a compact forward invariant set is a *topological* resp. *metric attractor* if

$$B(X) = \{x; \omega(x) \subset X\}$$

is of *second Baire category* (i.e. countable intersection of open and dense) resp. *has positive Lebesgue measure*, and if for any $X' \subsetneq X$, $B(X')$ does not have this property.

Why complex methods: density of hyperbolicity and rigidity

The nicest maps are those where each attractor is a hyperbolic periodic orbits. These maps are called the *hyperbolic*.

We will sketch a proof that - in some sense - most maps are hyperbolic.

These latter results rely on constructing an extension of f to the complex plane.

This interplay of real and complex methods in interval dynamics will be one of the main topics of these lectures.

Real bounds and ergodic properties.

Description of attractors

Theorem

Each map has at least one and at most a finite number of attractors. *If X is an attractor then one of the following:*

1. X *is a* periodic attractor*;*
2. $X = \omega(c)$ *where c is a critical point of f so that $\omega(c)$ is a* Cantor set *which is* minimal *and has zero Lebesgue measure;*
3. X *is equal to a* finite union of intervals *which contains a critical point (or equal to the entire space N)*

Definition: An invariant set is called *minimal* if each forward orbit is dense in $\omega(c)$.

Corollary (Denjoy): An attractor of a circle diffeomorphism is either the whole circle or a periodic orbit.

Examples

Examples of maps $f\colon [0,1] \to [0,1]$:

- $f(x) = 2x(1-x)$. Then $x = 1/2$ is an *attracting fixed point.*
- $f(x) = ax(1-x)$ is an *infinitely renormalizable map* where the parameter a is at the accumulation of period doubling: there exists a sequence of intervals J_n and integers $p(n)$ so that

  $$J_n, \dots, f^{p(n)-1}(J_n) \text{ have disjoint interiors}$$

  and

  $$f^{p(n)}(J_n) \subset J_n.$$

  The resulting attractor is a Cantor set. It turns out this NOT the only example of a Cantor attractor.
- $f(x) = 4x(1-x)$. In this case the attractor is $[0,1]$ and the *map is conjugate to a tent map with slope* ± 2. There are infinitely many periodic orbits (or all periods), but a.e. point $x \in [0,1]$ has a dense orbit in $[0,1]$.

Historical Background

- The proof in the non-invertible case has a long history: [Guc79, dMvS89, BL89, Lyu89, MdMvS92, vSV04]

- Note that the *objects in the classification* in the theorem is the same, regardless whether the attractor is a topological or a metric one.
- Milnor posed the question whether a metric attractor is also a topological attractor (and vice versa). The answer turns out to be NO, as we will see.

Nice Intervals and first return maps

For *simple* maps such as $f(x) = 4x(1-x)$ one describe the map through a *Bernoulli or Markov setting*:

$$J_1 \to J_1 \cup J_2, \quad J_2 \to J_1 \cup J_2$$

where $J_1 = [0, 1/2]$ and $J_2 = [1/2, 1]$. However, for most maps this is not possible.

Instead: use first return maps to so-called nice intervals.

- Let I be an interval, and assume that there exists a *(minimal)* $n > 0$ so that $f^n(x) \in I$. Then we denote the component of $f^{-n}(I)$ containing x by by $\mathcal{L}_x(I)$.
- An interval is called **nice** if no iterate of $x \in \partial I$ ever gets mapped into the interior of I.
- If I is nice then $\mathcal{L}_x(I) \subset I$ whenever $x \in I$.
- This makes it useful to work with first return domains.

- If I is nice, then a pullback $\mathcal{L}_x(I)$ is also nice.
- Two *pullbacks* of I *are either disjoint, or one is contained in the other*.
- Nice intervals are easy to find.
- Indeed, let's say f is unimodal. Then take a periodic orbit, choose p in the orbit 'closest to the critical point'. Then $I = [p, p']$ is a nice interval where p' so that $f(p') = p$.
- The *first return map* $R_I : Dom(I) \to I$ to I has (usually infinitely many) diffeomorphic branches and a folding branch.

One of the main challenges is to control the distortion. If all the branches were linear, then one knows essentially everything.

Notation and terminology

- If $T = [a - x, a + x]$ is an interval and $\tau > 0$ then we define
$$\tau \cdot T = [a - \tau x, a + \tau x].$$
- reminder: $\mathcal{L}_x(I)$ is the component of $f^{-n}(I)$ containing x where n is minimal so that $f^n(x) \in I$.
- $J_1, \dots, J_k$ have *intersection multiplicity* m if any point x is contained in at most m of the intervals $J_1, \dots, J_k$.
- If $f^n(x) \in T$ where n is minimal, then the *pullback* of T is $\mathcal{L}_x(T)$ and the pullback *chain* is the collection of intervals $\mathcal{L}_x(T), \mathcal{L}_{f(x)}(T), \mathcal{L}_{f^{n-1}(x)}(T), \mathcal{L}_{f^n(x)}(T) = T$.
- Let f have b critical points c_i with critical order l_i. Then we say that f has type $\underline{b} = (l_1, \dots, l_b)$.

Tools: Schwarzian derivative

The theorem require estimates on high iterates of a map. Since f is non-linear, as there are critical points, this is not so easy.

- **Schwarzian derivative:** Define

$$Sf(x) = \frac{f'''(x)f'(x) - (3/2)f''(x)}{[f'(x)]^2}.$$

Then $S(f \circ g) = Sf[g'(x)]^2 + Sg$. Hence

$$Sf < 0 \implies Sf^n < 0 \text{ for all } n \in \mathbb{N}.$$

- **Koebe:** Then for $\delta > 0$ there exists K so that the following holds. If $Sg < 0$ and $g \colon T \to T' := g(T)$ is a diffeomorphism then for each $x, y \in J$ so that $g(x) \in (1-\delta)g(T)$ one has $|Dg(x)|/|Dg(y)| \leq K$. *See blackboard*

Negative Schwarzian appear naturally:

- Let f be a polynomial of degree ≥ 2 with real coefficients and assume that *all zeros of Df are real. Then $Sf < 0$.*

  (Hint: By assumption $Df(x) = A\prod_{j=1}^{n}(x - a_j)$ where a_j are real. Then

  $$Sf(x) = 2\sum_{i<j}\frac{1}{(x - a_i)(x - a_j)} - \frac{3}{2}\left[\sum_{i}\frac{1}{(x - a_i)}\right]^2.$$

  It is not hard to see that this is negative for x real. (There is a more insightful way of showing this, which we will discuss briefly below.)

Many papers assumes that a map has negative Schwarzian. This simplifies because

- each periodic attractor has a critical point in its immediate basin
- one has Koebe control on diffeomorphic branches.

It turns out that the assumption $Sf < 0$ - with extra work - can always be replaced by assuming:

all periodic points of f are hyperbolic and repelling.

Cross-Ratio distortion

- Schwarzian derivative is closely related to cross-ratio (there is a formula...)
- Here $J \subset T$, then $C(T,J) = \dfrac{|T||J|}{|L||R|}$ *where* L, R *are the components of* $T - J$.
- If $f \colon T \to f(T)$ is a continuous bijection then one can consider the expansion of the cross-ratio:

$$\frac{C(fT, fJ)}{C(T,J)}.$$

Cross-Ratio and Poincaré metric

- Expansion of cross-ratio corresponds to $Sf < 0$ and also to contraction of Poincaré metric.
- One can put the *Poincaré metric* on $\mathbb{C}_T = (\mathbb{C} - \mathbb{R}) \cup T$. Then $C(T, J)$ is equal to the Poincaré metric of J.

Another way of showing $Sf < 0$ for certain polynomials:

- Take $f\colon \mathbb{C} \to \mathbb{C}$ a real polynomial with only real critical points and so that $f|T$ is a diffeomorphism.
- Then define $f^{-1}\colon \mathbb{C}_{f(T)} \to \mathbb{C}_T$ by analytic continuation.
- Example $f(z) = z^2$, $T = [1, 2]$, $f(T) = [1, 4]$, see blackboard.
- The map $f^{-1}\colon \mathbb{C}_{f(T)} \to f^{-1}(\mathbb{C}_{f(T)})$ is a conformal bijection and therefore an isomorphism w.r.t. the Poincaré metric on *these* sets.
- Since $f^{-1}(\mathbb{C}_{f(T)}) \subset \mathbb{C}_T$ we get that $f^{-1}\colon \mathbb{C}_{f(T)} \to \mathbb{C}_T$ is contracts w.r.t. the Poincaré metric on these sets. This proves $Sf < 0$.

Distortion control due to disjointness

Many maps do not have negative Schwarzian. It turns out that one can use distortion of cross-ratios instead.

Theorem (Koebe in the case of disjoint intervals)

Assume that $J \subset T$ and $f^n|T$ is a diffeomorphism, and the intersection multiplicity of $T, \ldots, f^{n-1}(T)$ is at most m.

Then Koebe holds (with bounds depending on m).

Real Bounds

One of the most basic tools in real one-dimensional dynamics are real bounds

Theorem

[vSV04] Assume that I is a nice interval with $x \in I$ and assume that $R_I(x) \notin \mathcal{L}_x(I)$. Then $(1+\delta)\mathcal{L}_x^2(I) \subset \mathcal{L}_x(I)$.

Let's explain some of the ideas behind the proof and why this is helpful.

The smallest interval argument

Let $J, \ldots, f^n(J)$ be disjoint intervals. Then one of them, $f^k(J)$, is the smallest. Assume the smallest is **not** the left or right most interval. Then the smallest $f^k(J)$ has two larger intervals $f^l(J)$ and $f^r(J)$ to its right and its left. Now take

$$T' = [f^l(J), f^r(J)] \supset f^k(J)$$

and pullback $T' \supset f^n(J)$ to $T \supset J$.

- the resulting chain has multiplicity ≤ 3;
- $(1+\delta)T \supset J$.

The spiral structure argument

Let's take $x \in I$ and assume that x visits I at least n times. Let $J_0, J_1, J_2, \ldots$ be the return domains x visits consecutively.

Fix $\rho > 0$. For each n there exists $i_0 \leq n$ so that

- P1 $(1+\rho)J_{i_0} \subset I$;
- P2 J_{i_0} has (at least) one ρ-small side and J_{i_0+1} is contained in a ρ-small side of J_{i_0};
- P3 For all $i < i_0$ properties [P1] and [P2] do not hold (which means that one has a spiral structure up to time i_0) and the interval J_{i_0+1} breaks the spiral structure;
- P4 Properties [P1] and [P2] do not hold and the spiral structure holds until time n.

In each situation one obtains space, see blackboard.

Combining these ideas one obtains the proof of the real bounds.

Corollary (Absence of wandering intervals)

If $W, f(W), \ldots$ are disjoint, then $f^n(W)$ converges to an attracting periodic orbit.

Sketch of proof of Corollary (proof by contradiction):

- By 'surgery' one can always assume that f has periodic orbits (and therefore has nice intervals), see blackboard.
- Let x be an accumulation point of the orbit of W.

Sketch of the proof of absence of wandering intervals

Using (a small part of) the proof of real bounds, one can show that there exists a *nested* sequence of nice intervals $I_n \supset I'_n \ni x$ so that

- $(1+\rho)I'_n \subset I_n$
- *the first visit of* W *to* I_n *is contained in* I'_n
- after some further time W enters I_{n+1}.
- *Any pullback of* I_{n+1} *which intersects* I'_n *is contained in* I'_n.

This gives

$$(1+\rho')\mathcal{L}_W(I'_n) \subset \mathcal{L}_W(I_n) \text{ and } \mathcal{L}_W(I_{n+1}) \subset \mathcal{L}_W(I'_n).$$

Combining this gives

$$(1+\rho')\mathcal{L}_W(I_{n+1}) \subset \mathcal{L}_W(I_n)$$

and therefore

$$(1+\rho')^n W \subset (1+\rho')^n \mathcal{L}_W(I_{n+1}) \subset \mathcal{L}_W(I_0).$$

But since the length of $(1+\rho')^n W$ tends to infinity, this gives a contradiction.

Application of real bounds

Theorem [Koebe without disjointness]

Assume all periodic orbits of f are hyperbolic and repelling.

Then Koebe holds on diffeomorphic branches.

Examples of results one can obtain using real methods: Invariant measures

Definition: f has an *absolutely continuous invariant probabiltiy measure* or an *acip*, if there exists a *probability* measure μ, so that

- $\mu(f^{-1}(B)) = \mu(B)$ for each measurable set B and
- $\mu(B)$ *is small when* B *has small Lebesgue measure*.

Theorem

The map $f(x) = 4x(1-x)$ has an acip.

- It is *enough to show* that $\forall \epsilon > 0$ there exists $\delta > 0$ so that for each measurable set $A \subset [0,1]$ of Lebesgue measure $< \delta$ one has $f^{-n}(A)$ has Lebesgue measure $< \epsilon$ for any $n \geq 0$.
- If A is an interval which does not contain 0 or 1, then this immediately follows from Koebe.
- One can reduce the general case to this situation.

Invariant Measures

There is a long history of results on this (going back to the 50's), with well-known results by Misiurewicz, Benedicks-Carleson, Collet-Eckmann, Nowicki-vS and others. The sharpest result is:

Theorem ([BSvS03] and [BRLSvS08])

Assume that f is real analytic and has no periodic attractors. Then there exists constant $C(\underline{b})$ such that if

$$\liminf_n |(f^n)'(f(c))| \geq C$$

for each critical point c then f has an acip.

There is a remarkable sequel to this result, by Rivera-Lettelier and & Shen: one has superpolynomial decay of mixing in this case.

Method of Proof: decomposing pullbacks

- Consider a set A of small size
- Aim: estimate $f^{-n}(A)$.
- Distinguish components of $f^{-n}(A)$.
- A 'good' component J is one for where exists a neighbourhood $T \supset J$ so that
  - $f^n|T$ is a diffeomorphism
  - $f^n(T) \supset (1+\xi)f^n(J)$ where ξ is large
- The Lebesgue measure of all good components of $f^{-n}(A)$ is obtained in this way
- Other components: decompose these branches and use an inductive estimate.

Wild attractors

Theorem (Existence of Wild attractors, [BKNvS96])

There exist maps of the form $f(z) = z^d + c$ with $c \in \mathbb{R}$ and d even (and large) with an invariant Cantor set which is a metric, but not a topological attractor.

Theorem (Non-existence of Wild attractors in the quadratic case, [Lyu94])

Assume that f is unimodal and has a **quadratic** *critical point then the notions of topological and metric attractor coincide.*

Note that wild attractors also exist for certain real polynomials of higher degree with only non-degenerate critical points.

Method of proof: Random Walks

- One can decompose the space into disjoint intervals J_n
- Each interval J_n maps diffeomorphically onto a countable union of such intervals
- One has sufficient control on non-linearity
- Probabilistic proofs to show what happens with points, see blackboard.

Intemingled attractors

A map with several critical points can have several attractors whose basins are **intermingled**:

Theorem ([vS96])

There exists a polynomial $f : [0, 1] \to [0, 1]$ *with two critical points with two disjoint invariant Cantor sets* Λ_i *so that the basin of each of these sets is dense and has positive Lebesgue measure.*

Some questions for Marco Martens

Question (Wild attractors for two-dimensional diffeomorphisms)

Let $M = S^2$. Are there diffeomorphisms $f : M \to M$ which have wild Cantor attractors*? (That is, metric but not topological attractor.) It is well-known that Hénon maps can have a Cantor set as an attractor, see [GvST89], [DCLM05] and Martens' lectures. Do these Cantor sets necessarily have to be of solenoidal type?*

Question (Wandering domains for Hénon maps)

Let H be a Hénon map. Is it possible for H to have wandering domains, i.e. is it possible that there exists an open set U so that $U, f(U), \dots$ are all disjoint and so that U is not contained in the basin of a periodic attractor?

Remark: *Dima Turaev* has proven that cubic diffeomorphism of the plane can have such wandering domains.