HOLOGRAPHIC ENTANGLEMENT ENTROPY & CAUSAL HOLOGRAPHIC INFORMATION

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Workshop on Ultracold Atoms and Gauge Theories

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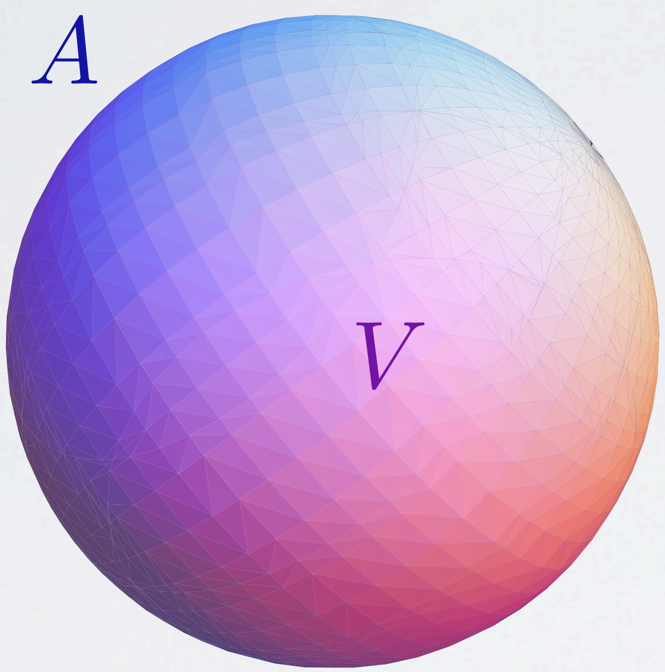
OUTLINE

- Motivation & Background
- Holographic Entanglement Entropy
- Causal Holographic Information
- Summary & Outlook

Holography

- in theory of gravity, # of qubits describing a region ≦ its surface area

[‘t Hooft, Susskind, Bousso]



entropy S is not extensive:

$$S \not\sim V$$

instead, $S \sim A$

- Motivated by considerations of black holes

Black holes

- Black hole = region of spacetime which cannot communicate with the external Universe
- In Nature, results as endpoint of gravitational collapse
- In general relativity, specific solution of Einstein's equations:

$$R_{ab} - \frac{1}{2} R\, g_{ab} + \Lambda\, g_{ab} = 8\pi\, \mathcal{T}_{ab}$$

- Boundary of a black hole = event horizon
- Important properties: horizon area A and surface gravity κ

Black hole thermodynamics

Laws of BH mechanics mimic laws of thermodynamics:

0. κ is constant over horizon for stationary BH
1. $dM = (1/8\pi)\,\kappa\,dA + \Omega_H\,dJ$
2. $\delta A \geq 0$ in any process
3. Impossible to achieve $\kappa = 0$ by a physical process

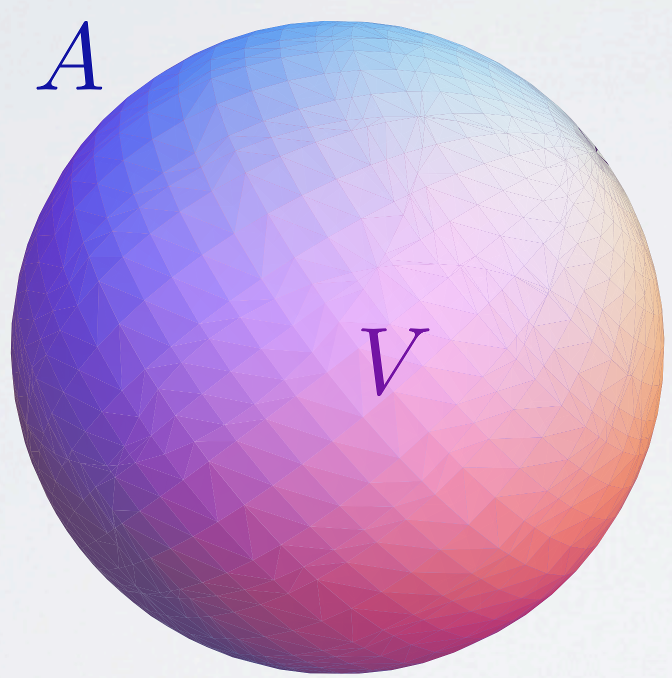
0. T is constant over system in thermal equilibrium
1. $dE = T\,dS +$ work terms
2. $\delta S \geq 0$ in any process
3. Impossible to achieve $T = 0$ by a physical process

- hence natural to identify $S_{BH} = \dfrac{A}{4\,\hbar}$ and $T_{BH} = \dfrac{\hbar\,\kappa}{2\,\pi}$
- T substantiated by semi-classical calculations [Hawking]: black holes radiate
- entropy bound [Bekenstein] motivated holographic principle ['t Hooft, Susskind, Bousso]
- Natural question: statistical mechanics origin of BH?

Holography

~ in theory of gravity, # of qubits describing a region ≦ its surface area

['t Hooft, Susskind, Bousso]



entropy S is not extensive:

$$S \not\sim V$$

instead, $S \sim A$

~ More than just counting of # qubits: physical equivalence between two theories formulated in different # of spacetime dimensions

Concrete realization: AdS/CFT correspondence:

AdS/CFT correspondence

String theory (∋ gravity) $\Longleftrightarrow$ gauge theory (CFT)

"in bulk" asymp. AdS × S

"on boundary"

[Maldacena]

'soup can' diagram of AdS:

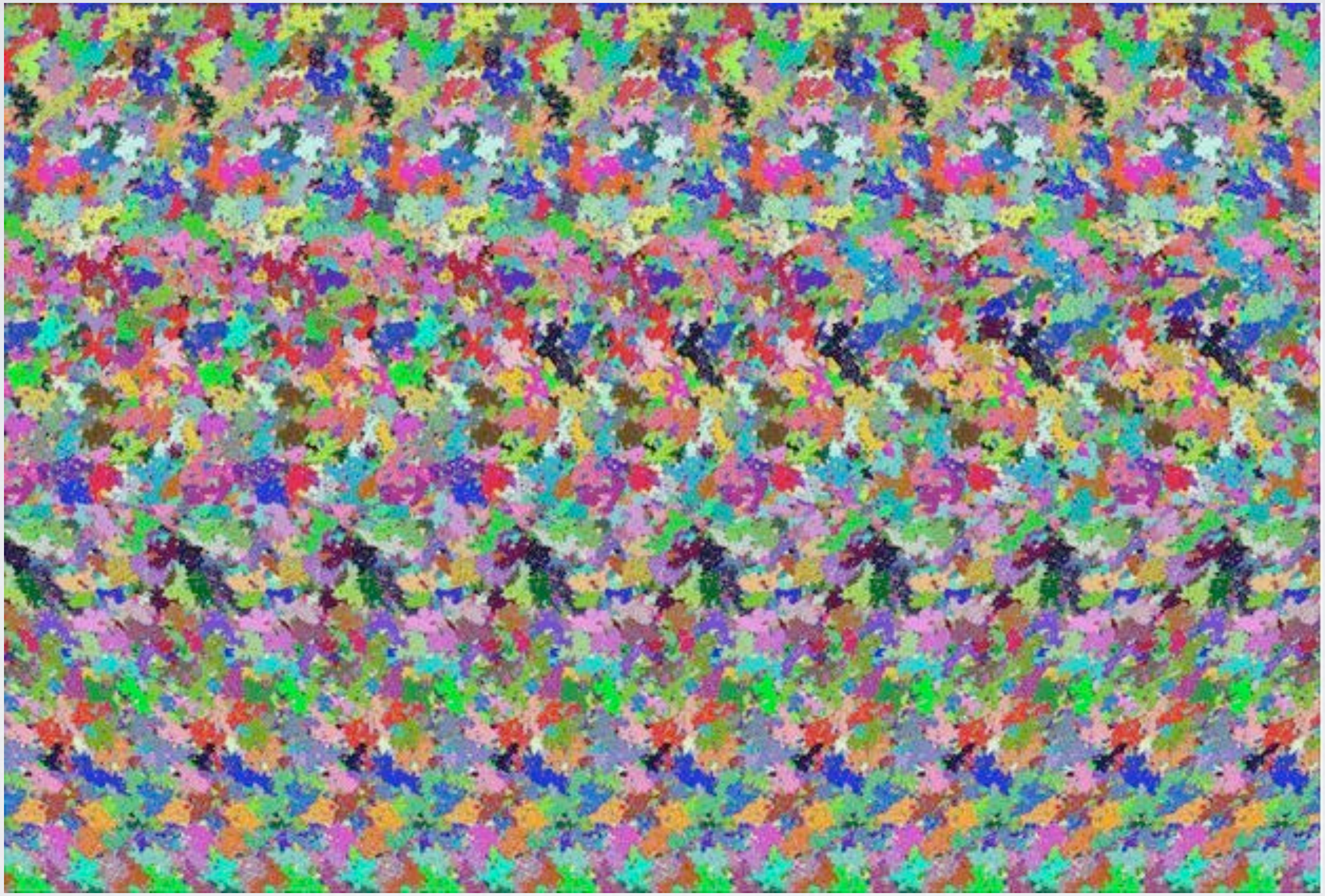
here label is everything...

Key aspects:

* Gravitational theory maps to non-gravitational one!
* *Holographic*: gauge theory lives in fewer dimensions.

AdS/CFT correspondence

- better analogy: stereogram...



...but infinitely more complicated

AdS/CFT correspondence

String theory ($\ni$ gravity) $\Longleftrightarrow$ gauge theory (CFT)

"in bulk" asymp. AdS × K *"on boundary"*

Key aspects:

- ✳ Gravitational theory maps to non-gravitational one!
- ✳ *Holographic*: gauge theory lives in fewer dimensions.
- ✳ Strong/weak coupling duality.

Invaluable tool to:

~ Use gravity on AdS to learn about strongly coupled field theory
(as successfully implemented in e.g. AdS/QCD & AdS/CMT programs)

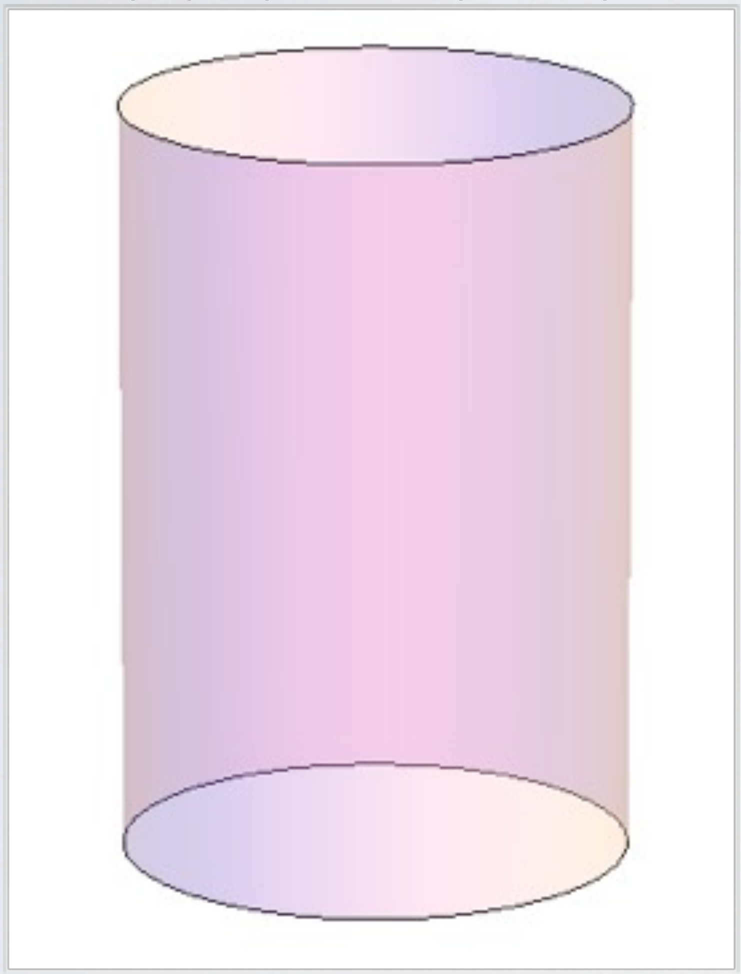
~ Use the gauge theory to define & study quantum gravity in AdS

Pre-requisite: Understand the AdS/CFT 'dictionary'...

Bulk geometries and CFT states

different bulk geometries $\longleftrightarrow$ different states in CFT

(asymptotically AdS)

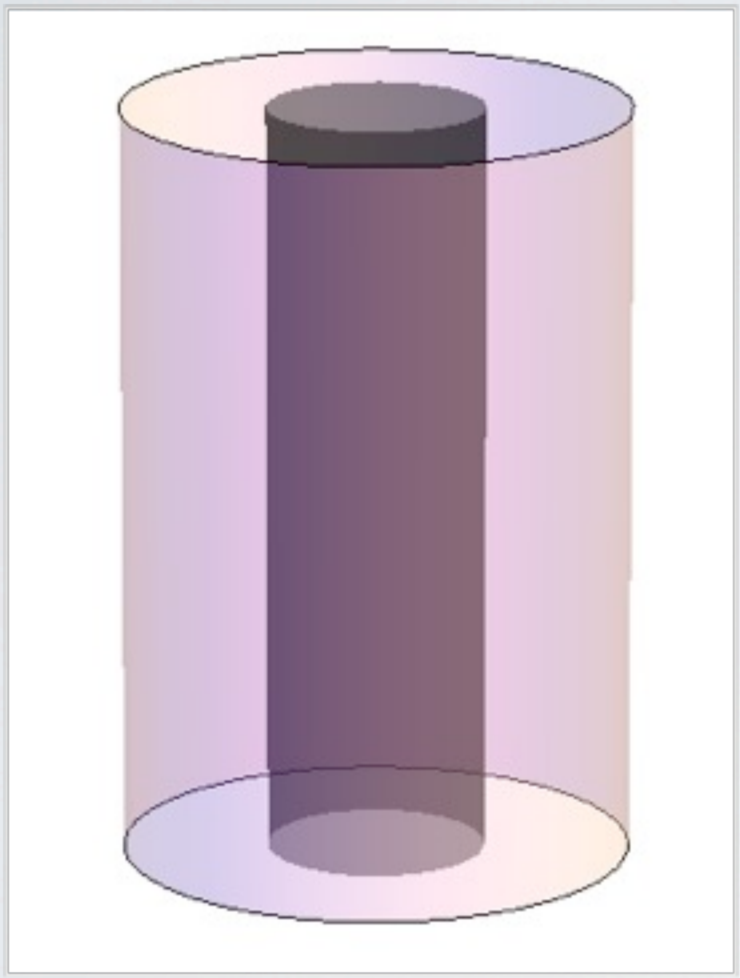


- Pure AdS $\longleftrightarrow$ vacuum state in CFT

Finite-mass deformations of the bulk geometry result in non-zero boundary stress tensor

Black holes in equilibrium

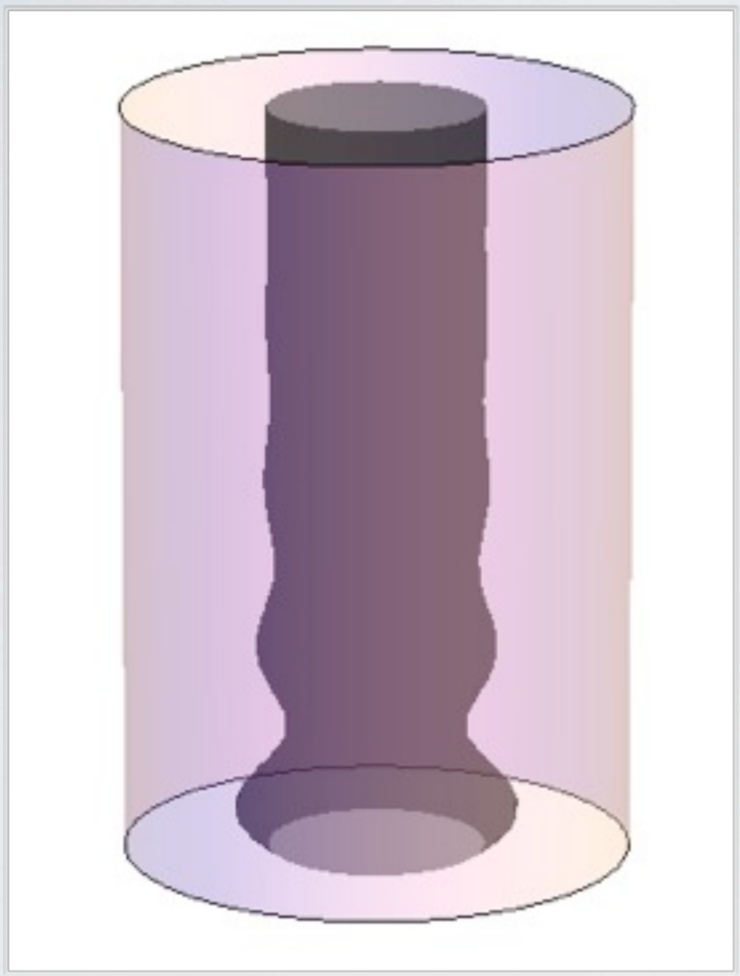
different bulk geometries $\longleftrightarrow$ different states in CFT



- Pure AdS $\longleftrightarrow$ vacuum state in CFT
- Black hole $\longleftrightarrow$ thermal state in CFT

Small deviations from equilibrium

evolving bulk geometries $\longleftrightarrow$ corresponding dynamics



- Pure AdS $\longleftrightarrow$ vacuum state in CFT
- Black hole $\longleftrightarrow$ thermal state in CFT
- Quasinormal modes of perturbed black hole $\longleftrightarrow$ approach to thermal equilibrium [Horowitz, VH]
- Horizon response properties $\longleftrightarrow$ transport coefficients in CFT [Kovtun, Son, Starinets]

- Generic long-wavelength dynamics of a black hole $\longleftrightarrow$ relativistic conformal fluid dynamics [Bhattacharyya, VH, Minwalla, Rangamani]

Diagnostics of bulk geometry

The bulk metric can be extracted using various CFT probes (which are described by geometrical quantities in the bulk):

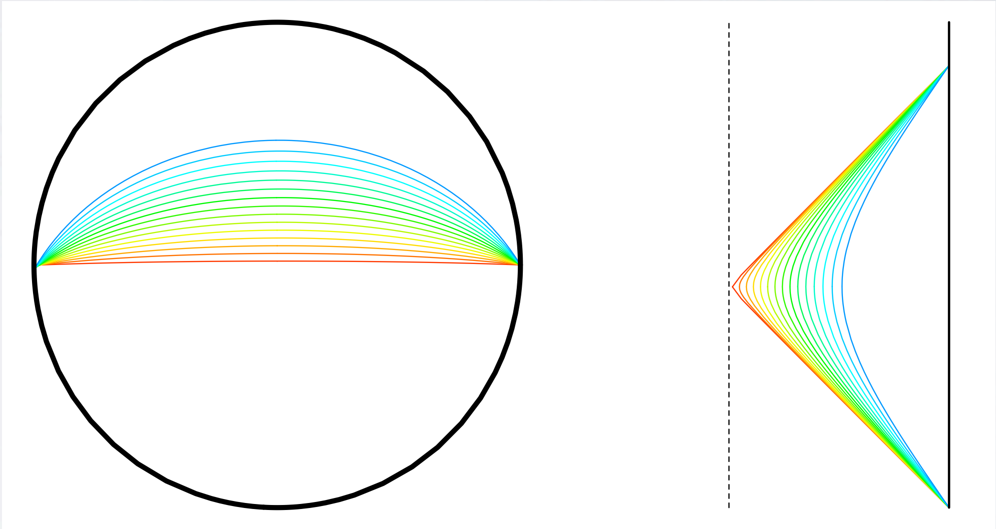
Examples:

CFT probe	bulk quantity
✳ expectation values of local gauge-invariant operators	asymptotic fall-off of corresponding conjugate field
✳ correlation functions of local gauge-invariant operators	in WKB approx., proper length of corresponding geodesic
✳ Wilson loop exp. vals.	area of string worldsheet
✳ entanglement entropy	vol of extremal co-dim.2 surface

E.g.: bulk-cone singularities

- Singularities in boundary correlation functions are sensitive to null geodesics through the bulk. [VH, Liu, Rangamani]

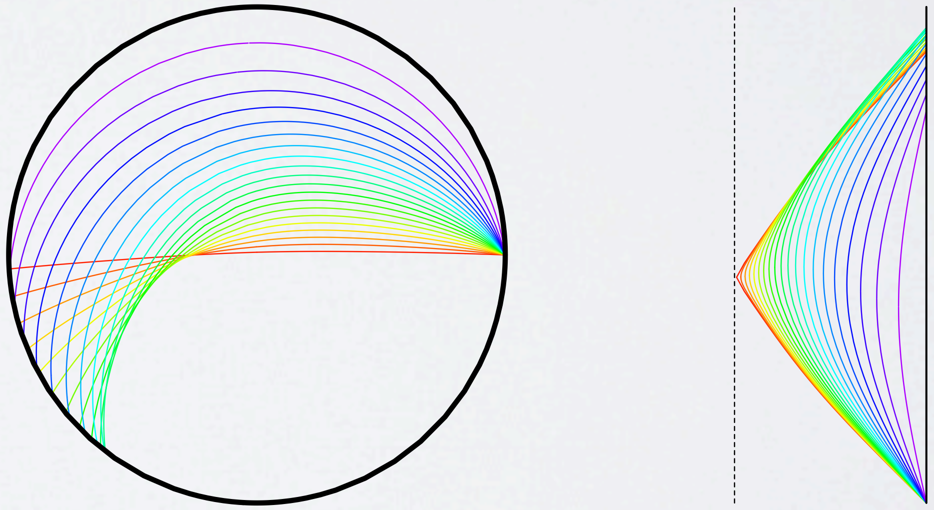
Null geodesics in AdS:



const. t

(r , t) plane

cf. Null geodesics in AdS `star' geometry:



- Can be used to extract bulk metric from singularity locus

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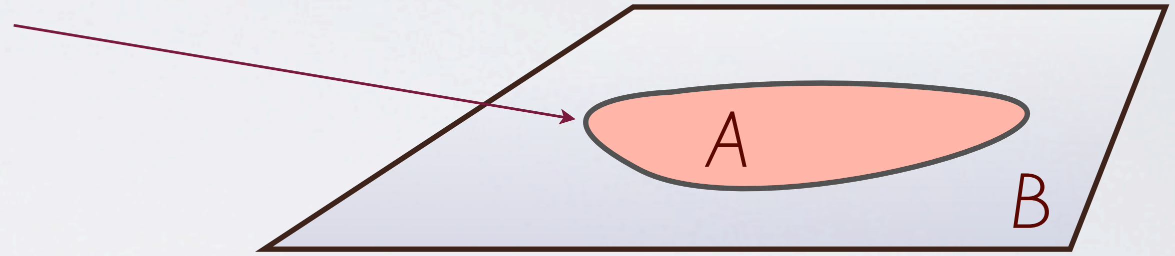
Entanglement Entropy (EE)

- Divide a quantum system into two parts and use EE to characterize the amount of correlations between them
- For QFT these parts can be spatial regions, separated by a smooth entangling surface



- Construct the reduced density matrix $\rho_{\mathcal{A}}$ for region A, by integrating out degrees of freedom in outside region B.
- This characterizes information available in A
- Entanglement entropy of A is the von Neumann entropy of $\rho_{\mathcal{A}}$

$$\mathcal{S}_{\mathcal{A}} = -\mathrm{Tr}\,\rho_{\mathcal{A}}\,\log \rho_{\mathcal{A}}$$

Entanglement Entropy (EE)

Applications:

- Quantum Information theory: computational resource
- Condensed Matter theory: diagnostic to characterize topological phases, quantum critical points, ...
- Quantum Gravity: suggested as origin of black hole entropy [Bombelli,Koul,Lee&Sorkin, Srednicki, Frolov&Novikov, Callan&Wilczek, Susskind ...] and in fact as origin of macroscopic spacetime [van Raamsdonk et.al.]

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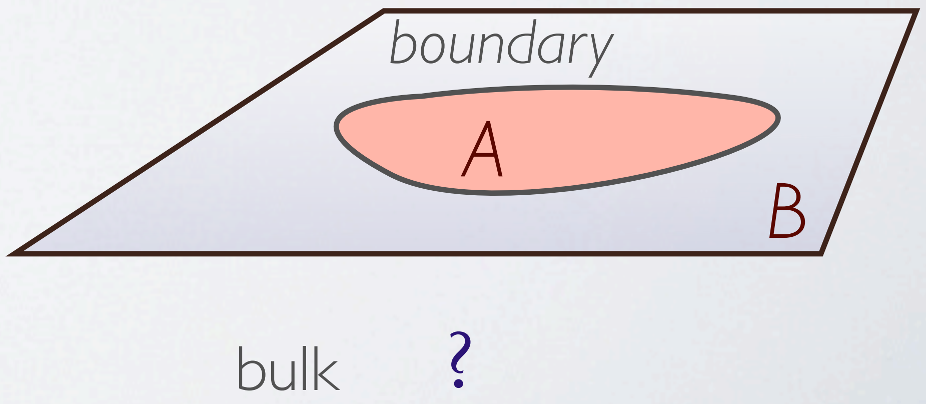
But: EE = non-local quantity, difficult to measure & to calculate

AdS/CFT to the rescue?

⤳ Is there a natural bulk dual of EE?

(= "Holographic EE")

Yes!

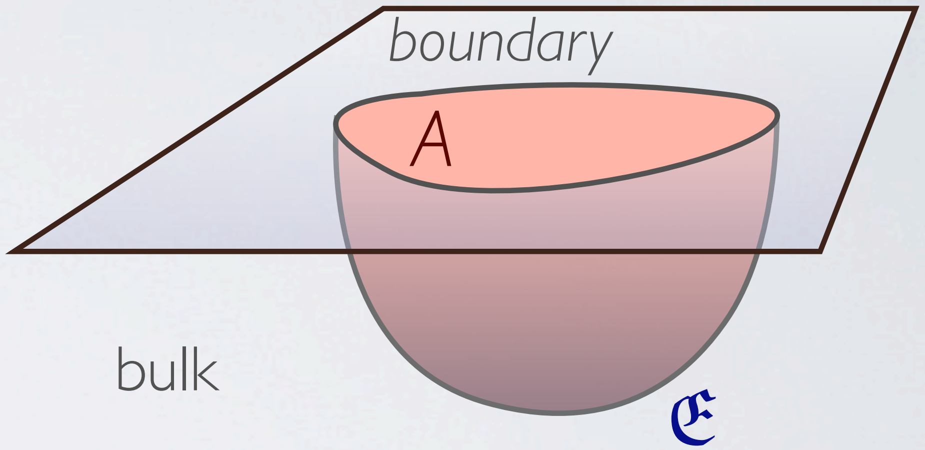


Holographic Entanglement Entropy

Proposal [Ryu & Takayanagi] for static configurations:

- In the bulk this is captured by area of minimal co-dimension 2 bulk surface $\mathfrak{E}$ anchored on $\partial\mathcal{A}$.

$$EE \equiv S_{\mathcal{A}} = \min_{\partial\mathfrak{E}=\partial\mathcal{A}} \frac{\text{Area}(\mathfrak{E})}{4\,G_N}$$



In time-dependent situations, prescription must be covariantized...

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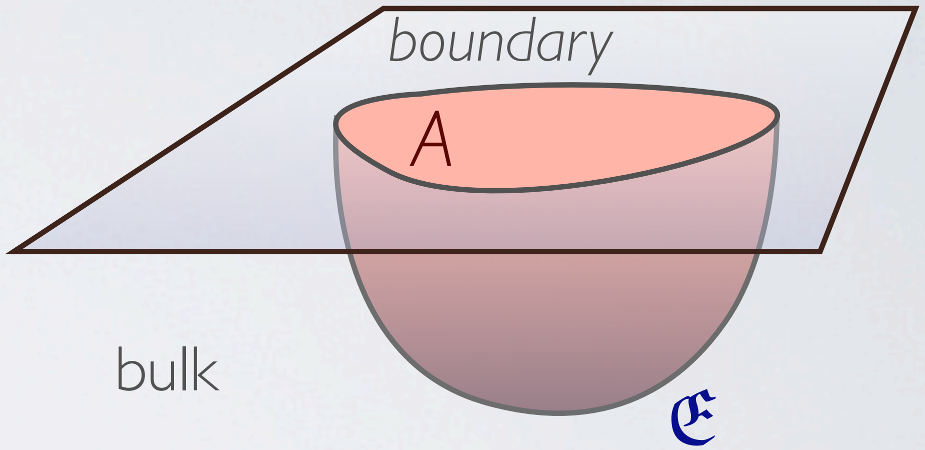

$$EE \equiv S_{\mathcal{A}} = \min_{\partial\mathfrak{E}=\partial\mathcal{A}} \frac{\text{Area}(\mathfrak{E})}{4\,G_N}$$

In time-dependent situations, prescription must be covariantized:

[VH, Rangamani, Takayanagi]

- minimal surface → extremal surface
- equivalently, $\mathfrak{E}$ is the surface with zero null expansions; cf. light sheet construction [Bousso]

In case of multiple surfaces, $S_{\mathcal{A}}$ is given by the minimal area extremal surface homologous to $\mathcal{A}$. [Headrick, Takayanagi, et.al.]

Evidence

* Leading contribution correctly reproduces the area law
* Recover known results of EE for intervals in 2-d CFT [Calabrese&Cardy] both in vacuum and in thermal state
* Derivation of holographic EE for spherical entangling surfaces [Cassini,Huerta,&Myers]
* Attempted proof by [Fursaev] elaborated & refined by [Headrick, Faulkner, Maldacena]

Further suggestive evidence:

* Automatically satisfies $S_{\mathcal{A}} = S_{\mathcal{B}}$ for pure states
* Automatically satisfies strong subadditivity [Lieb&Ruskai] & Araki-Lieb inequality -- easy to prove on the gravity side, far harder within field theory

Application 1: proof of SSA

- strong subadditivity:

$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \cup \mathcal{A}_2} + S_{\mathcal{A}_1 \cap \mathcal{A}_2}$$

$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \setminus \mathcal{A}_2} + S_{\mathcal{A}_2 \setminus \mathcal{A}_1}$$

Application 1: proof of SSA

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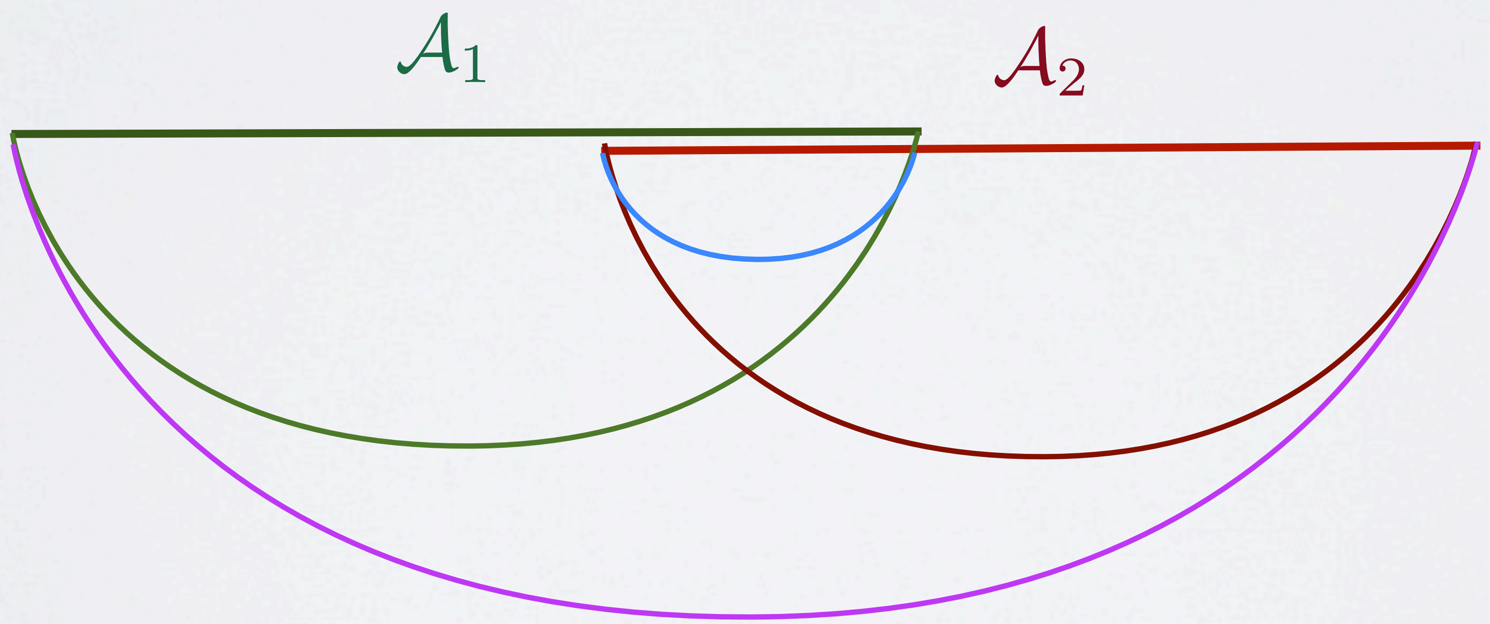
- proof in static configurations [Headrick&Takayanagi]



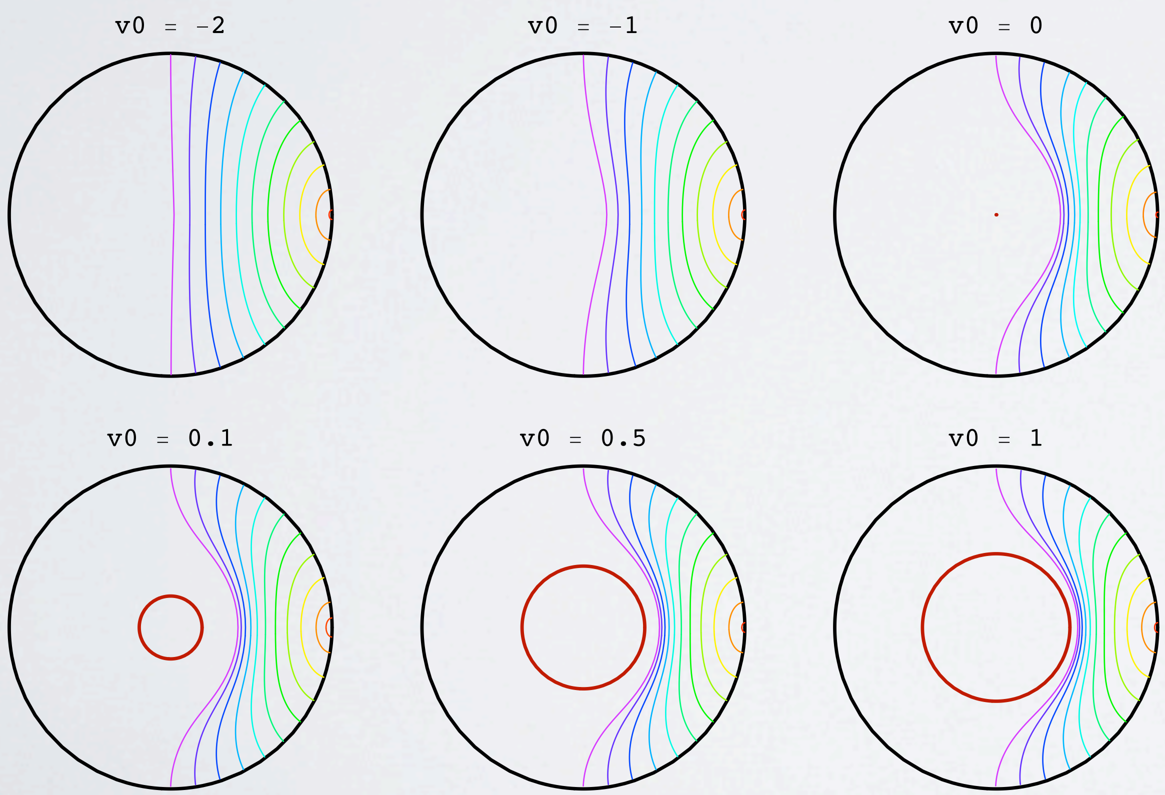
In time-dependent configurations more involved but true [Headrick et.al., Wall]

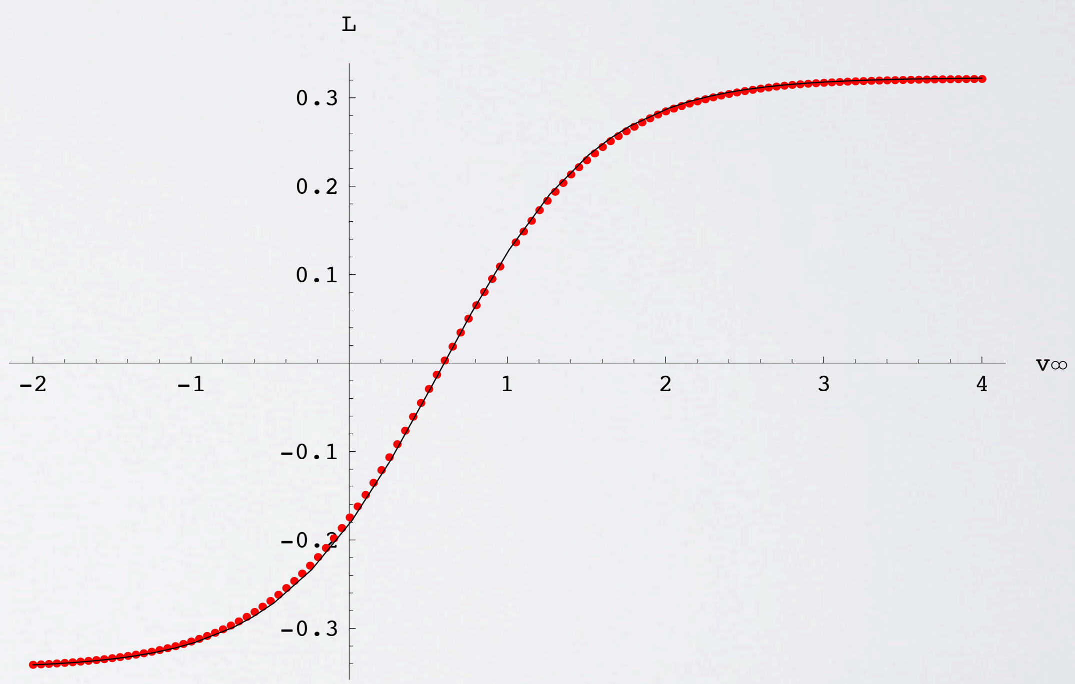
Application 2: Thermalization

Entanglement entropy growth during thermalisation:
Bulk geometry = collapsing black hole (in 3-d):

behaviour of extremal surfaces
at times *v0* during collapse

corresponding entanglement entropy:





[VH, Rangamani, Takayanagi]

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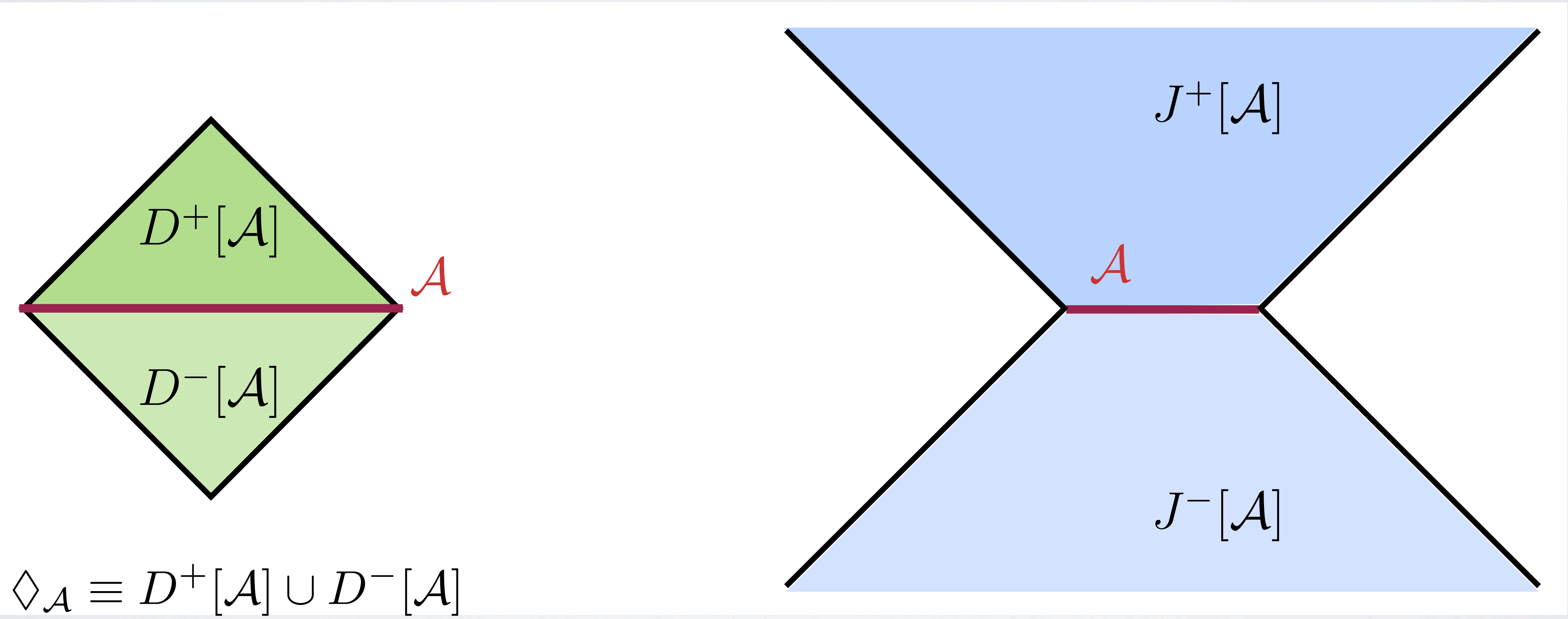
New & Simpler Construct

What is the *most natural* bulk region associated to a given region $\mathcal{A}$ on the bdy?

- 'natural': try to be minimalistic, use only bulk causality
- Take $\mathcal{A}$ to be *d-1* dimensional spatial region on bdy of asymp. AdS_{d+1} bulk spacetime.
- The unique minimal construction gives a bulk causal wedge associated with $\mathcal{A}$, and a corresponding *d-1* dimensional bulk surface $\Xi_{\mathcal{A}}$
- Using geometrical information, we can associate a number $\chi_{\mathcal{A}}$ to $\mathcal{A}$, corresponding to area of $\Xi_{\mathcal{A}}$

Causal construction

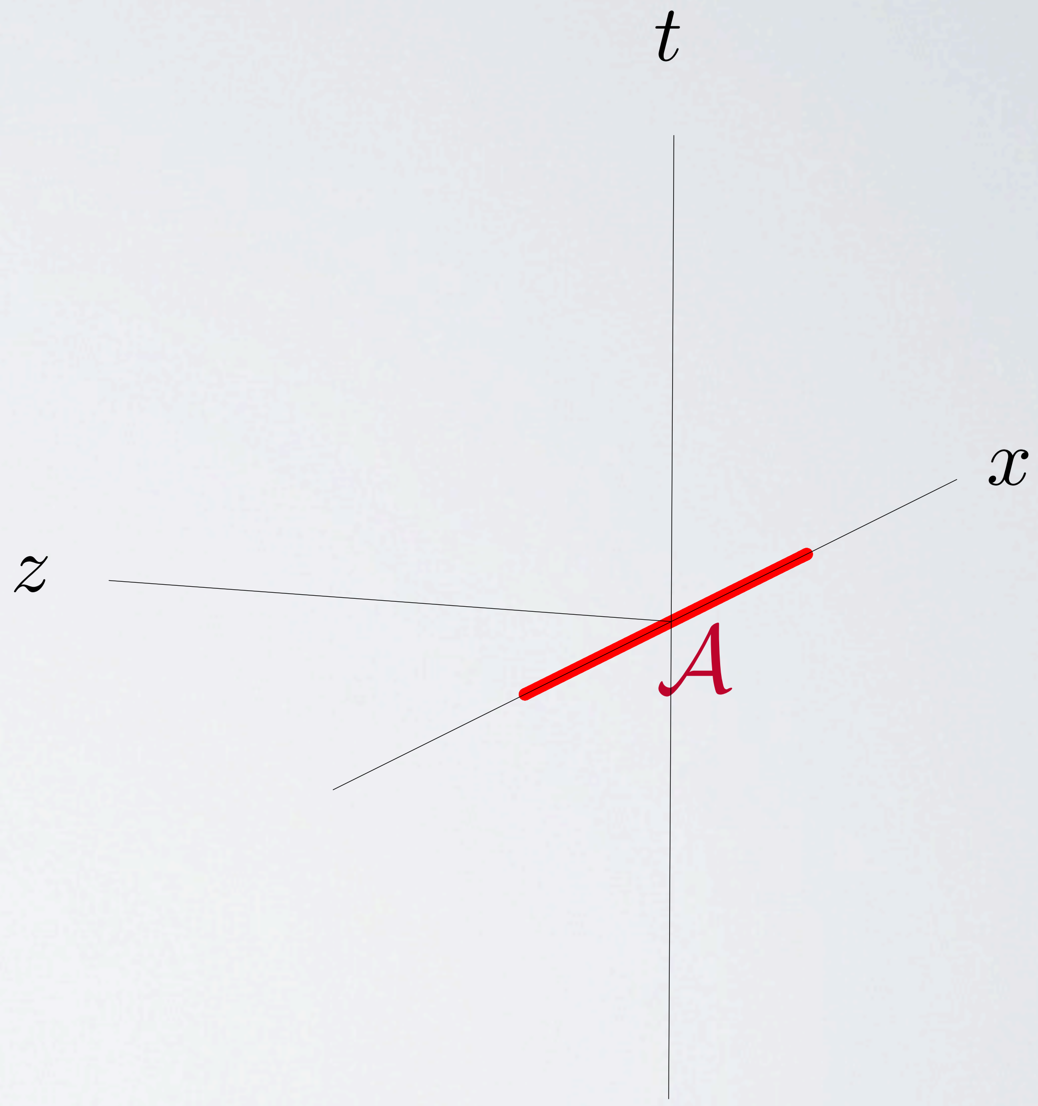
- domain of dependence $D^{\pm}[\mathcal{A}]$ = region which must influence or be influenced by events in $\mathcal{A}$
- domain of influence $I^{\pm}[\mathcal{A}]$ = region which can influence or be influenced by events in $\mathcal{A}$



- Given $\rho_{\mathcal{A}}$, we can determine observables in the entire $\Diamond_{\mathcal{A}}$

Causal Wedge construction

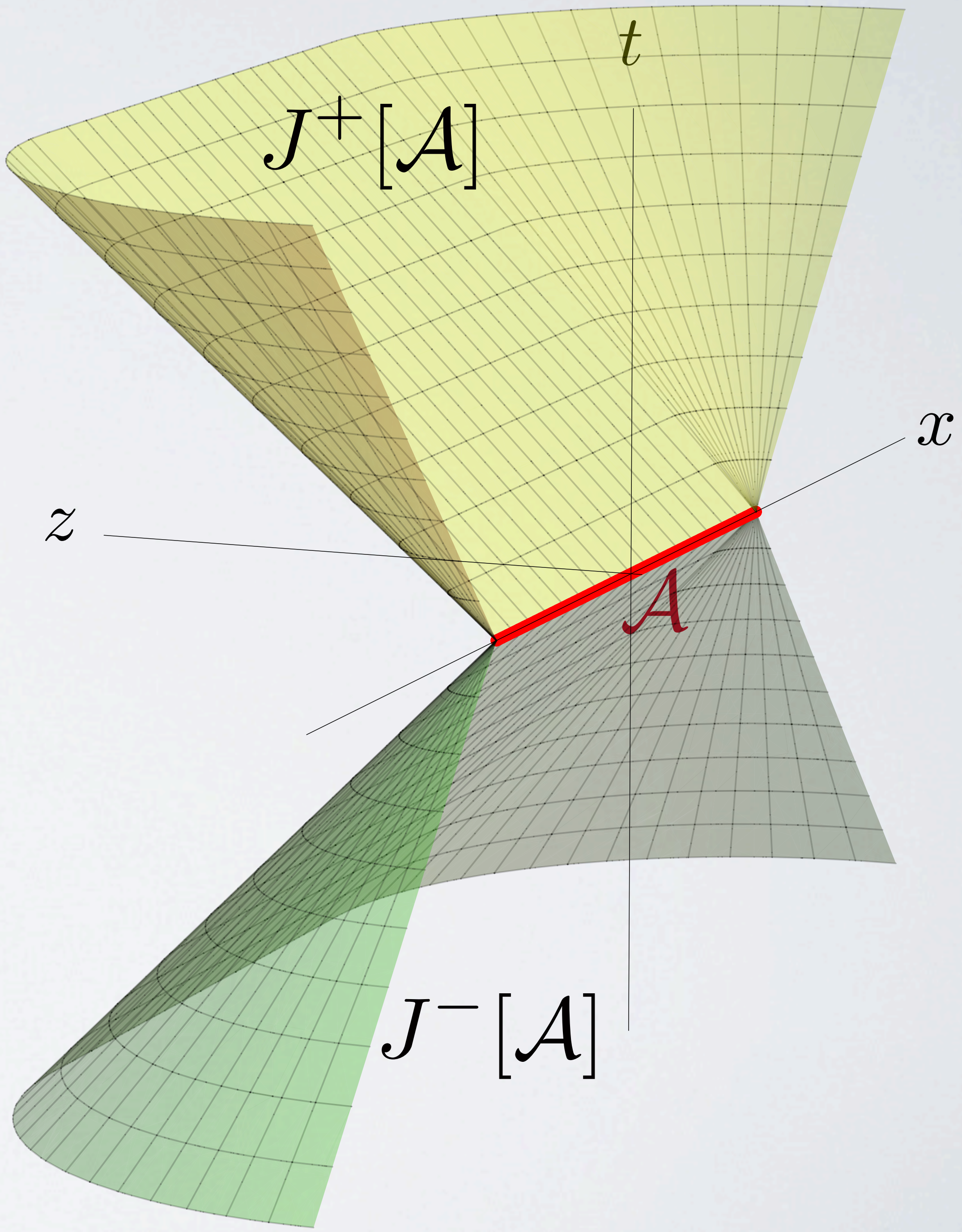
- Consider a bdy region $\mathcal{A}$



Causal Wedge construction

- Consider a bdy region $\mathcal{A}$
- Its bulk domains of influence extend arb. deep into the bulk and have trivial intersection

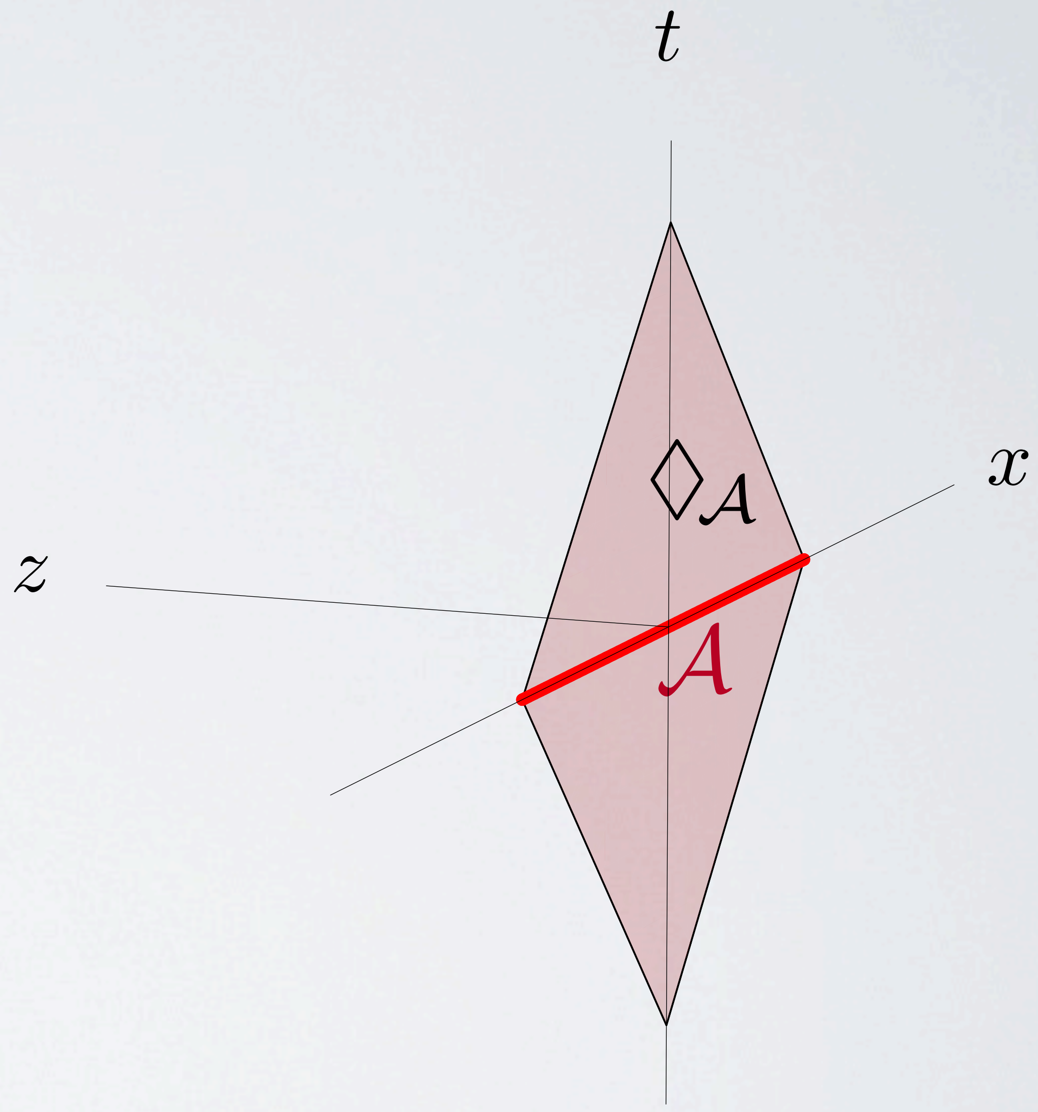
(and bulk domain of dependence of $\mathcal{A}$ is just the region $\mathcal{A}$ itself).



Causal Wedge construction

- Consider a bdy region $\mathcal{A}$
- Its bulk domains of influence extend arb. deep into the bulk and have trivial intersection
- Consider a bdy domain of dependence of $\mathcal{A}$, denoted $\lozenge_{\mathcal{A}}$

(observables in the entire region $\lozenge_{\mathcal{A}}$ can be determined solely from the initial conditions specified on $\mathcal{A}$)



Causal Wedge construction

- Consider a bdy region $\mathcal{A}$
- Its bulk domains of influence extend arb. deep into the bulk and have trivial intersection
- Consider a bdy domain of dependence of $\mathcal{A}$, denoted $\Diamond_{\mathcal{A}}$
- Its bulk domains of influence extend arb. deep, but their intersection doesn't

t

$J^+[\Diamond_{\mathcal{A}}]$

x

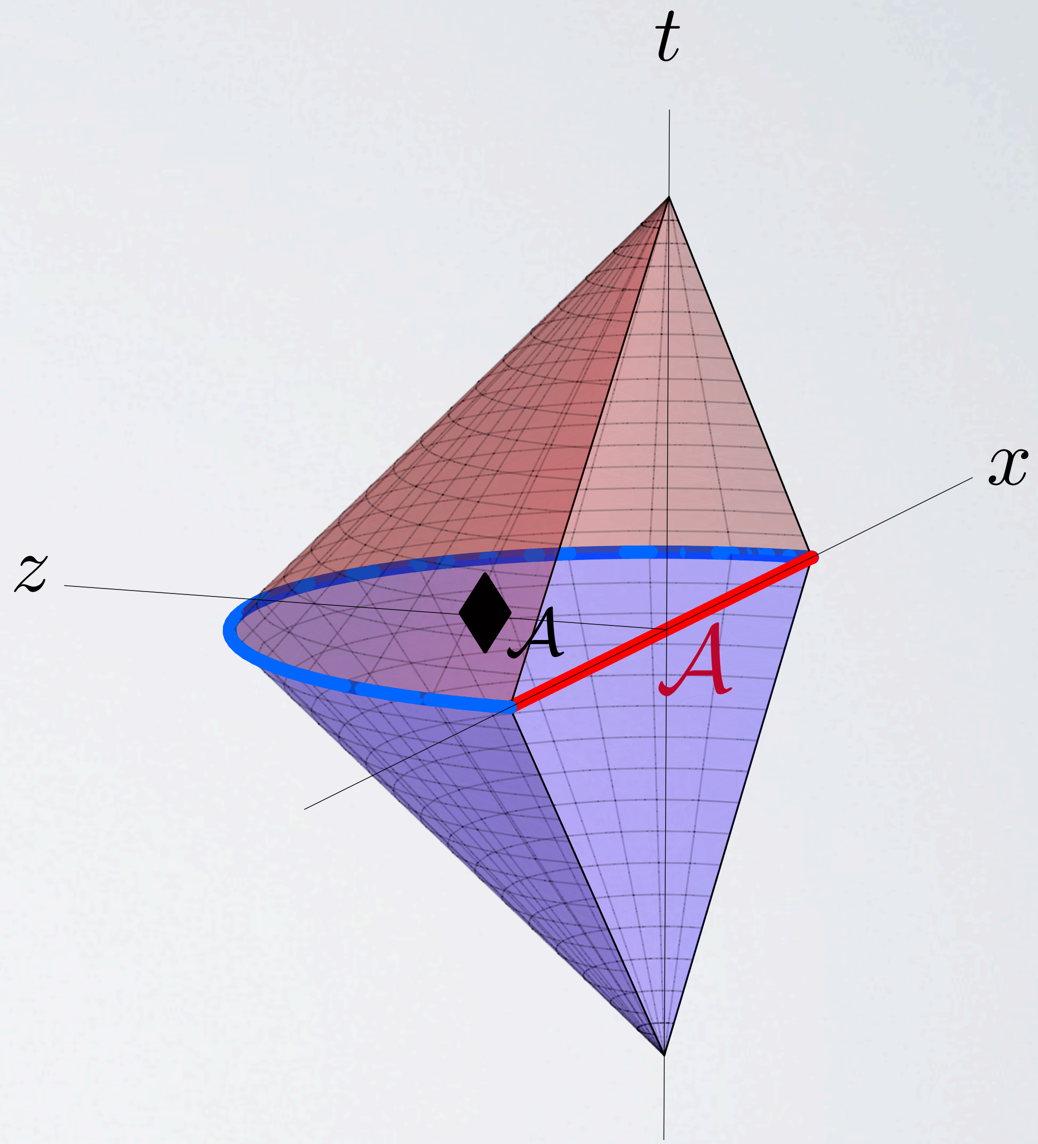
z

$\mathcal{A}$

$J^-[\Diamond_{\mathcal{A}}]$

Causal Wedge construction

- Consider a bdy region $\mathcal{A}$
- Its bulk domains of influence extend arb. deep into the bulk and have trivial intersection
- Consider a bdy domain of dependence of $\mathcal{A}$, denoted $\lozenge_{\mathcal{A}}$
- Its bulk domains of influence extend arb. deep, but their intersection doesn't
- This defines for us the bulk causal wedge of $\mathcal{A}$, denoted $\blacklozenge_{\mathcal{A}}$



Causal Wedge

- Bulk causal wedge $\blacklozenge_{\mathcal{A}}$

$$\blacklozenge_{\mathcal{A}} \equiv J^-[\lozenge_{\mathcal{A}}] \cap J^+[\lozenge_{\mathcal{A}}]$$

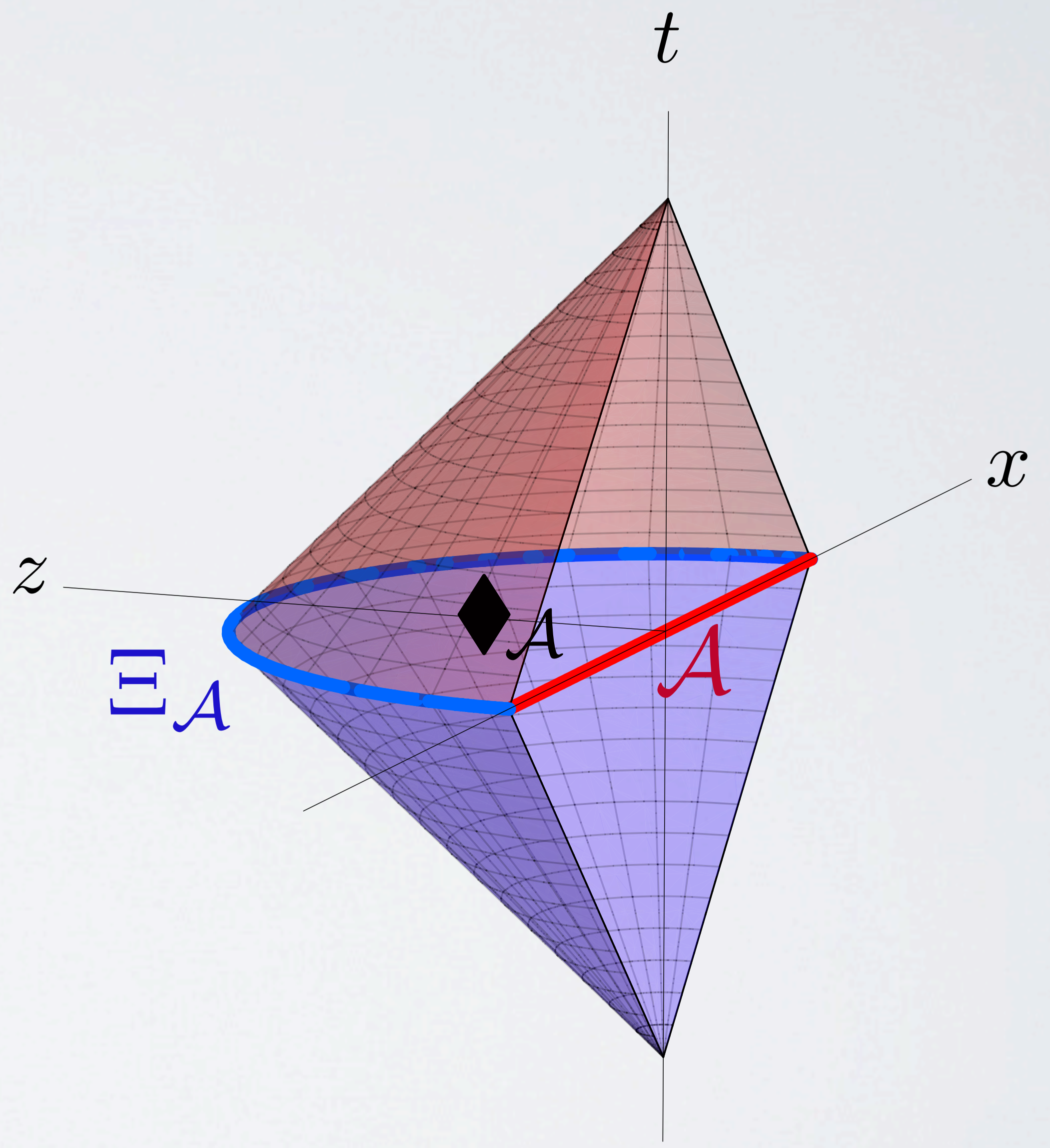
= { bulk causal curves which begin and end on $\lozenge_{\mathcal{A}}$ }

- Causal information surface $\Xi_{\mathcal{A}}$

$$\Xi_{\mathcal{A}} \equiv \partial_+(\blacklozenge_{\mathcal{A}}) \cap \partial_-(\blacklozenge_{\mathcal{A}})$$

- Causal holographic information $\chi_{\mathcal{A}}$

$$\chi_{\mathcal{A}} \equiv \frac{\text{Area}(\Xi_{\mathcal{A}})}{4\, G_N}$$



[VH & Rangamani]

Conjectured meaning of $\chi_{\mathcal{A}}$:

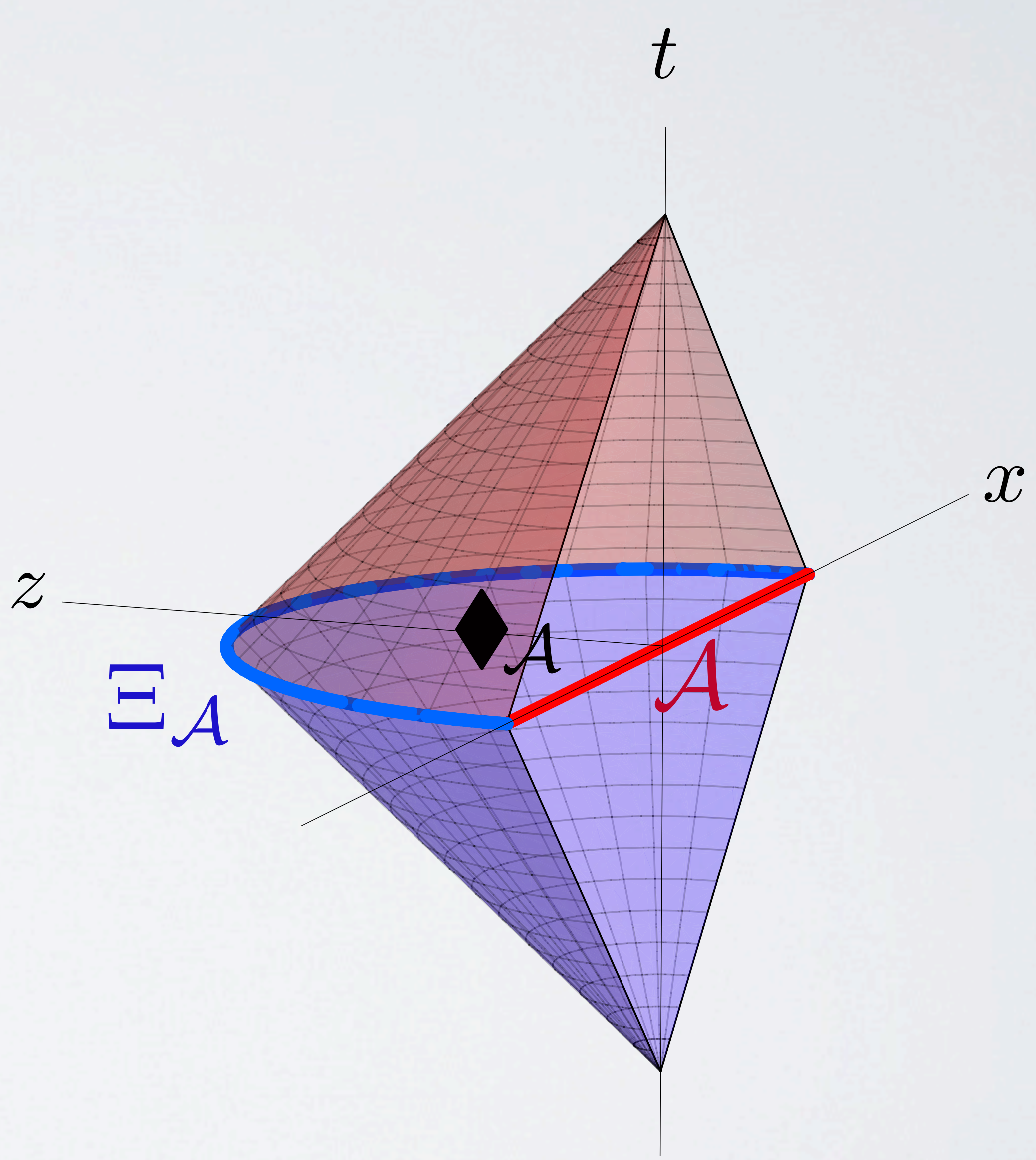
- We conjecture that $\chi_{\mathcal{A}}$ characterizes the amount of information contained in $\mathcal{A}$ which can be used to reconstruct the bulk geometry (entirely in $\blacklozenge_{\mathcal{A}}$ but possibly further)...
  - cons. set of local bulk `observers' starting & ending on bdy inside $\lozenge_{\mathcal{A}}$
  - these have access to full $\blacklozenge_{\mathcal{A}}$, but the info contained can be reduced:
    - bulk evolution: suffices to consider just Cauchy slice for $\blacklozenge_{\mathcal{A}}$
    - holography: suffices to consider just screen: natural region associated to $\mathcal{A} = \Xi_{\mathcal{A}}$
  - hence natural to identify $\chi_{\mathcal{A}}$ with amount of info contained in $\mathcal{A}$
- This has entropy-like behavior, however, it does *not* correspond to a Von Neumann entropy:
  - e.g. it violates strong subadditivity.
- However, it provides a bound on Entanglement entropy;
  - and coincides in special, maximally-entangled, cases.

Main question:

What is the CFT interpretation of $\Xi_{\mathcal{A}}$ and $\chi_{\mathcal{A}}$?

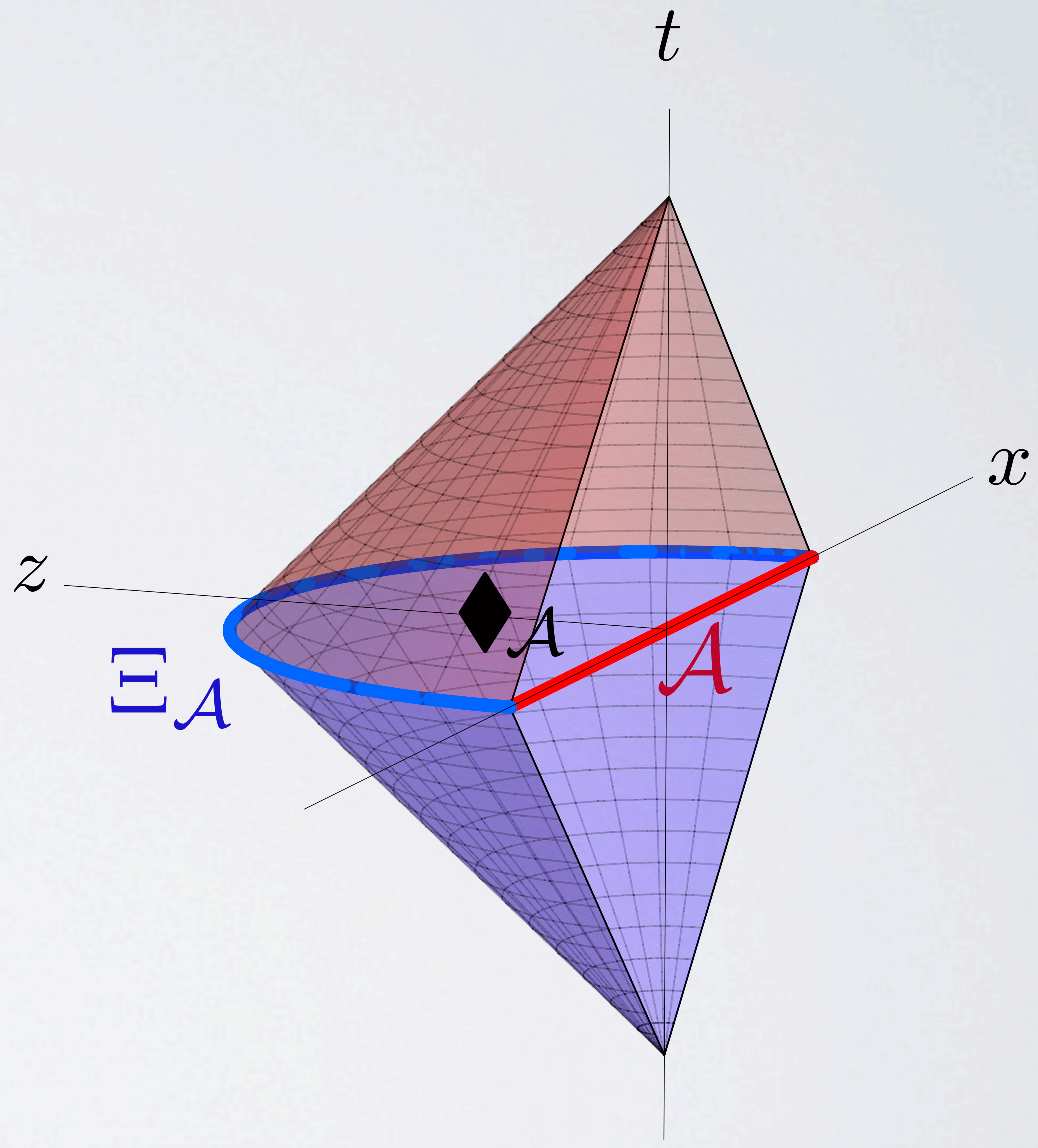
Gather hints

by considering geometrical properties and behavior of $\Xi_{\mathcal{A}}$ ...



General properties of $\Xi_{\mathcal{A}}$:

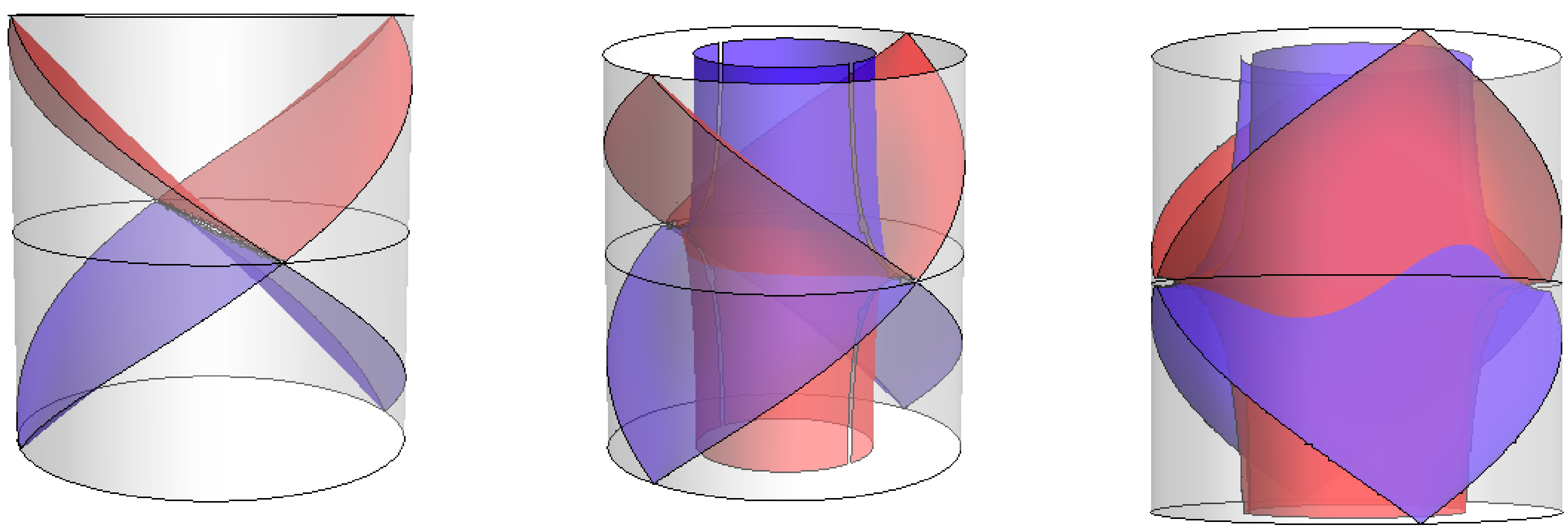
- Causal information surface $\Xi_{\mathcal{A}}$ is a *d-1* dimensional spacelike bulk surface which:
  - is anchored on $\partial\mathcal{A}$
  - lies within (on boundary of) $\blacklozenge_{\mathcal{A}}$
  - reaches deepest into the bulk from among surfaces in $\blacklozenge_{\mathcal{A}}$
  - is a minimal-area surface among surfaces on $\partial(\blacklozenge_{\mathcal{A}})$ anchored on $\partial\mathcal{A}$
- However, $\Xi_{\mathcal{A}}$ is in general not an extremal surface $\mathfrak{E}_{\mathcal{A}}$ in the bulk.



Cases when $\Xi_{\mathcal{A}}$ and $\mathfrak{E}_{\mathcal{A}}$ coincide:

- However, in all cases where one is able to compute entanglement entropy in QFT from first principles, independently of coupling, the surfaces $\mathfrak{E}_{\mathcal{A}}$ and $\Xi_{\mathcal{A}}$ agree!
- = When EE can be related to thermal entropy... cf. [Myers et.al.]

bdy:	CFT vacuum:	thermal density matrix:	grand canonical density matrix:
bulk:	pure AdS:	static BTZ:	rotating BTZ:



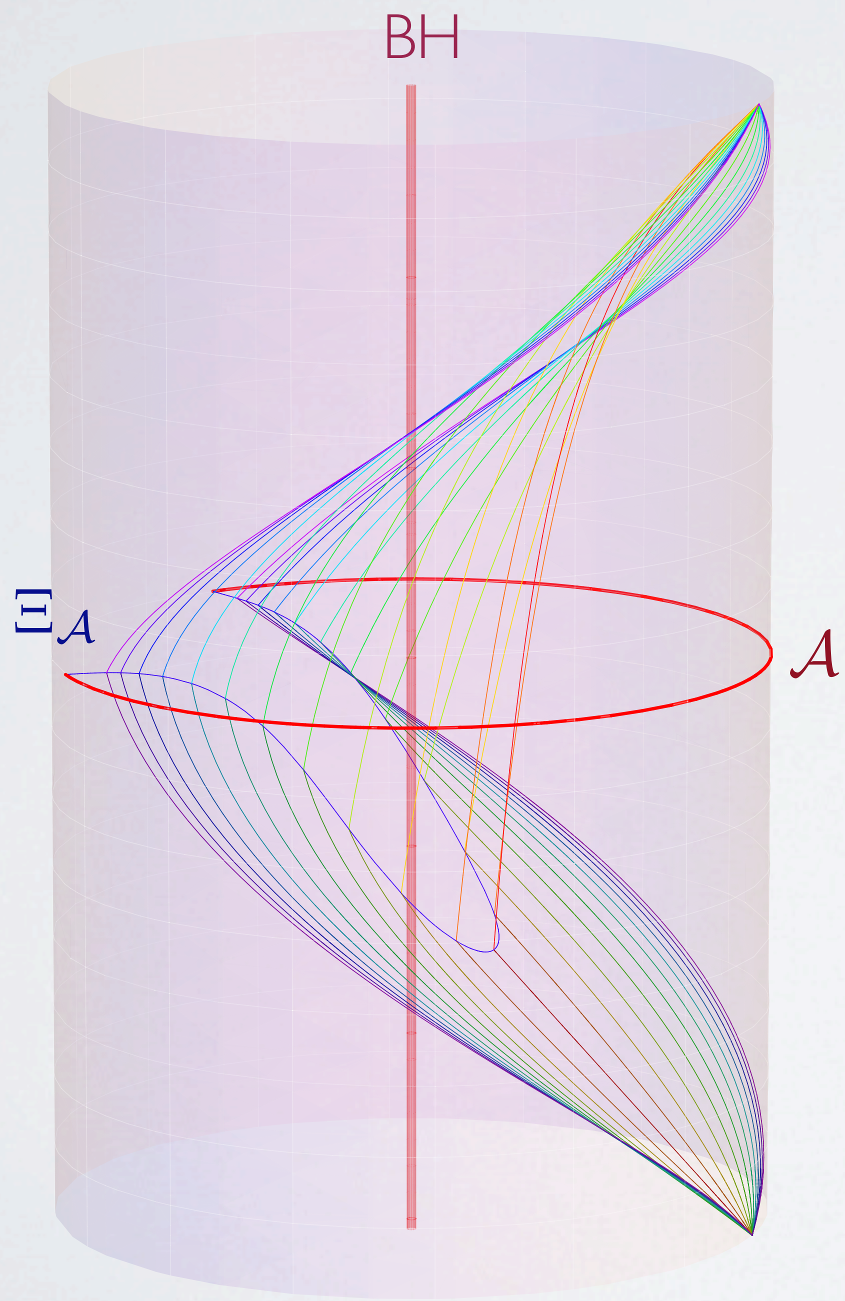
(a) (b) (c)

General properties:

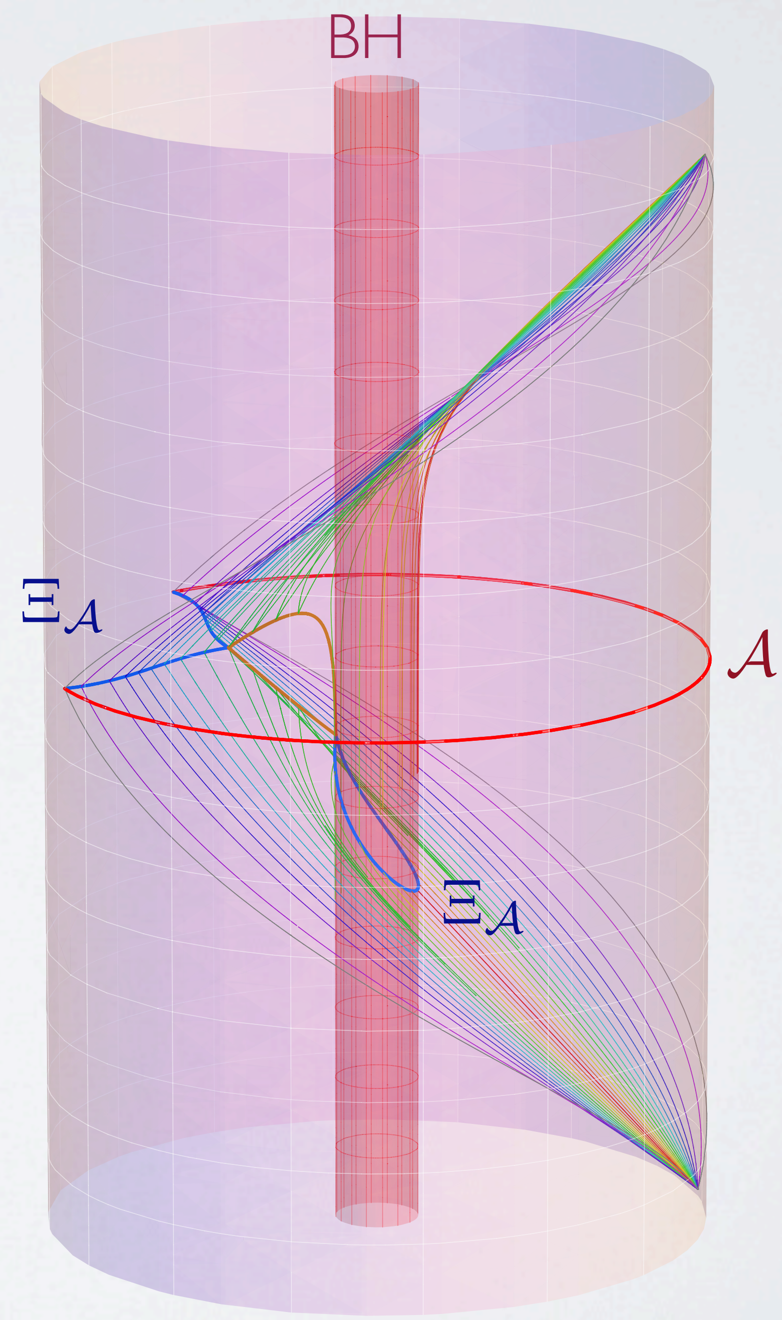
- In general $\Xi_{\mathcal{A}}$ does not penetrate as far into the bulk as the bulk extremal surface $\mathfrak{E}_{\mathcal{A}}$ associated with $\mathcal{A}$
- The causal wedge construction is mildly "teleological" (but only on light-crossing timescale)
- Causal Holographic Information $\chi_{\mathcal{A}}$ in general bounds EE $S_{\mathcal{A}}$ (and coincides in special cases)
- However $\chi_{\mathcal{A}}$ can violate strong subadditivity -- hence it cannot be a von Neumann entropy...
- While $\Xi_{\mathcal{A}}$ cannot penetrate black hole event horizons, $\mathfrak{E}_{\mathcal{A}}$ can penetrate dynamical black hole horizons (by a limited amount).
- Even if $\mathcal{A}$ is simply connected region, the causal wedge $\blacklozenge_{\mathcal{A}}$ can be topologically complicated.

Causal wedge can have "holes"

In 3 dimensions:

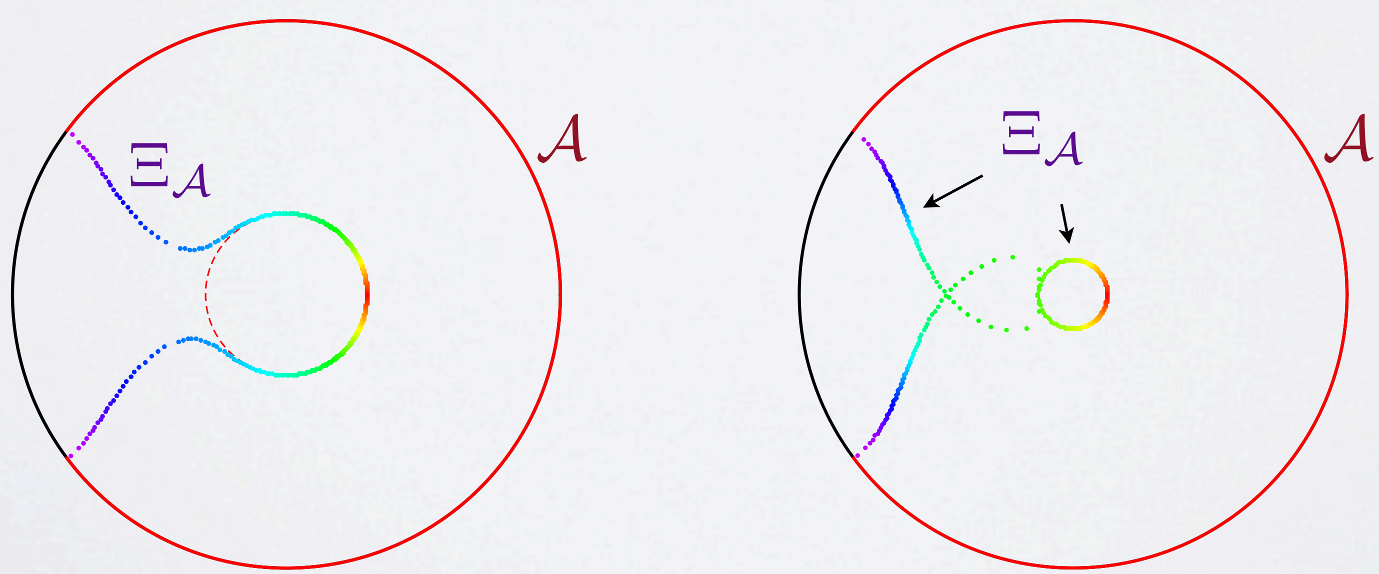


In 5 dimensions:



Causal wedge can have "holes"

- Even if $\mathcal{A}$ is simply connected region, the causal wedge $\blacklozenge_{\mathcal{A}}$ can be topologically complicated.
- e.g. in Schw-AdS$_d$ with d>3, for sufficiently large region (and fixed BH size), the causal wedge `wraps around' the BH.
- conversely, for fixed region $\mathcal{A}$ > hemisphere, $\exists$ small enough BH s.t. the causal wedge has a hole
- $\Rightarrow$ $\Xi_{\mathcal{A}}$ can have two disconnected pieces. [VH, Rangamani, Tonni]



Implication for entanglement entropy

- Important implication: whenever $\mathcal{A}$ is large enough for $\Xi_{\mathcal{A}}$ to have two disconnected pieces, there cannot exist a single connected extremal (minimal) surface $\mathfrak{E}_{\mathcal{A}}$ homologous to $\mathcal{A}$!
  - However, by the homology constraint, part of $\mathfrak{E}_{\mathcal{A}}$ must reach around the BH.
  - So $\mathfrak{E}_{\mathcal{A}}$ must likewise have two disconnected pieces, one on the horizon and one homologous to $\mathcal{A}^c$ (=complement of $\mathcal{A}$)
- Hence we have the universal formula for the entanglement entropy, whenever $\mathcal{A}$ is large enough:

$$S_{\mathcal{A}} = S_{\mathcal{A}^c} + S_{BH}$$

  - Automatically saturates the Araki-Lieb inequality
- So we can extract BH (thermal) entropy from entanglement entropy [cf. Azeyanagi, Nishioka, Takayanagi]

EE is fine-grained observable!

- In contrast to the static (i.e. eternal) black hole, for a collapsed black hole, there is no non-trivial homology constraint on extremal surfaces.
- Hence we always have $S_{\mathcal{A}} = S_{\mathcal{A}^c}$ as for a pure state.
- Correspondingly no extremal surface can actually sit precisely at the horizon since there is no 'neck' (bifurcation surface).
- Hence entanglement entropy is sensitive to very 'fine-grained' information: it can tell whether the black hole is eternal or collapsed, arbitrarily late after the collapse (when all 'coarse-grained' observables have thermalized).
- And this in spite of its classical geometrical nature...
- Other diagnostics of thermal vs. pure state (e.g. periodicity in imaginary time appear much more subtle).

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Summary for entanglement entropy

- The extremal surfaces $\mathfrak{E}_{\mathcal{A}}$
  - can exist in large multiplicities
  - need not exist for static black hole in single homologous piece
  - can penetrate horizon of a collapsed black hole

- The entanglement entropy $S_{\mathcal{A}}$
  - always behaves causally
  - need not be continuous in region size
  - allows us to extract full thermal entropy
  - need not increase monotonically during thermalization
  - distinguishes between pure and thermal states (collapsed versus eternal black holes), arbitrarily long after 'themalization'
  - hence is a 'fine-grained' observable

Summary for causal wedge & CHI

- The causal wedge $\blacklozenge_{\mathcal{A}}$
  - is the most natural (minimal nontrivial) bulk spacetime region related to $\mathcal{A}$
  - corresponds to bulk region most easily reconstructed from $\rho_{\mathcal{A}}$
  - cannot penetrate event horizon of a black hole, but can have 'holes'
- The causal holographic information $\chi_{\mathcal{A}}$
  - coincides with entanglement entropy $S_{\mathcal{A}}$ in certain special cases (when DoFs in $\mathcal{A}$ are maximally entangled with those outside)
  - in general provides an upper bound on entanglement entropy
  - monotonically increases during thermalization
  - may behave quasi-teleologically, but only on light-crossing timescales
  - remains smooth as a function of time and the size of $\mathcal{A}$

Future directions

Most important questions still remain:

- What is the direct boundary interpretation/construction of the causal holographic surface $\Xi_{\mathcal{A}}$ and 'information' $\chi_{\mathcal{A}}$?
- What is the bulk dual of the reduced density matrix $\rho_{\mathcal{A}}$?
- Given a bulk location, how do we extract the geometry there from the CFT?
- How does the CFT encode bulk locality and causality?

THE END...?