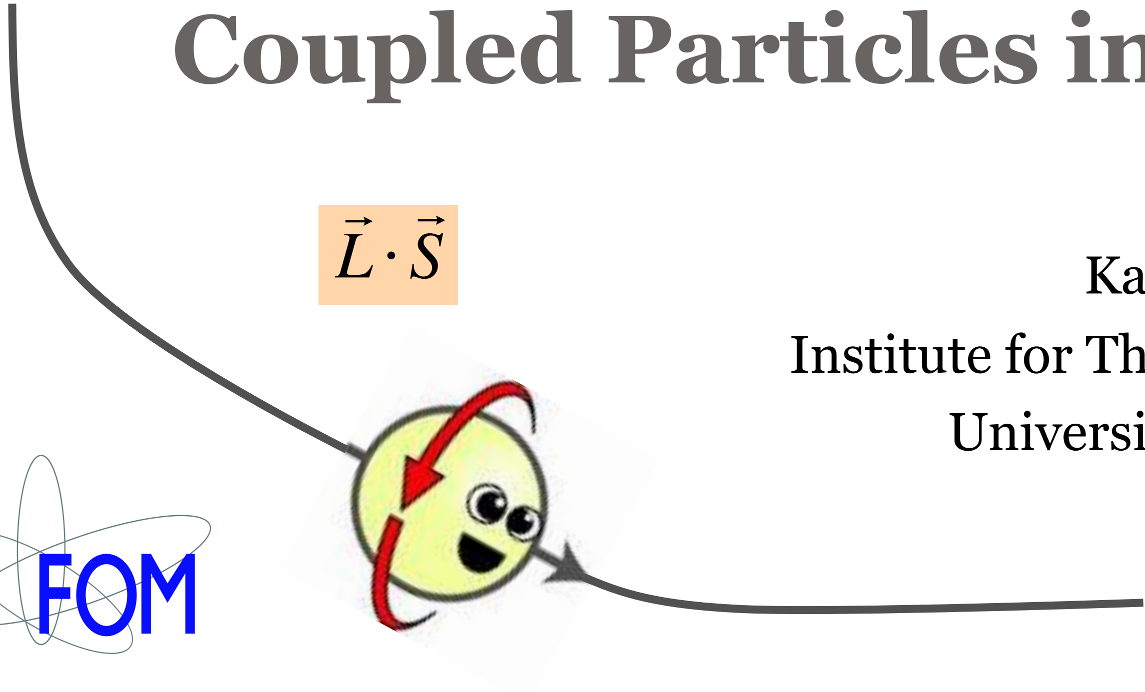
Symmetries of Spin-Orbit Coupled Particles in 3D



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manuscript in preparation with
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How Dirac's singleton organizes the spectrum of a 3D particle moving in a non-Abelian gauge potential

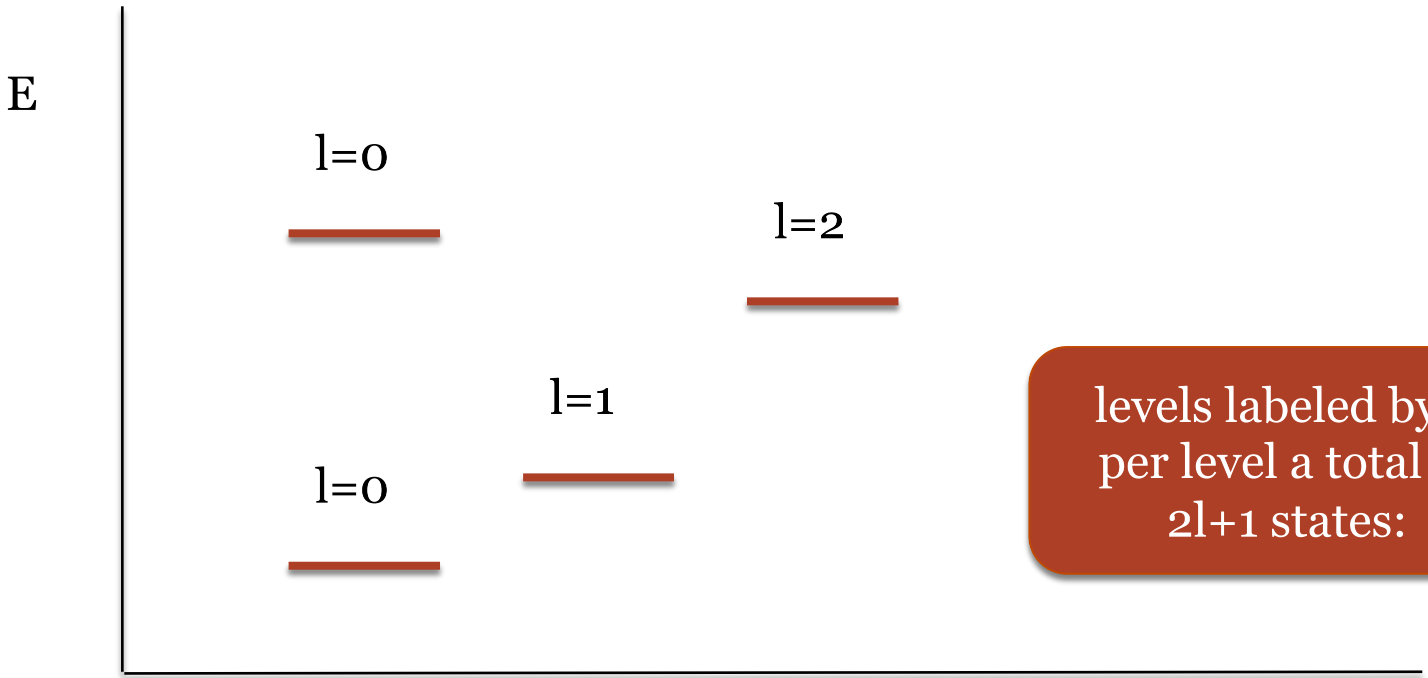
Towards a geometrical understanding of 3D Landau Levels and Topological Insulators

Radially symmetric problems in 3D

$$V(\vec{r}) = V(r)$$

QM 101:

Radially symmetric potential V(r) leads to a manifest SO(3) symmetry in the 1-particle spectrum



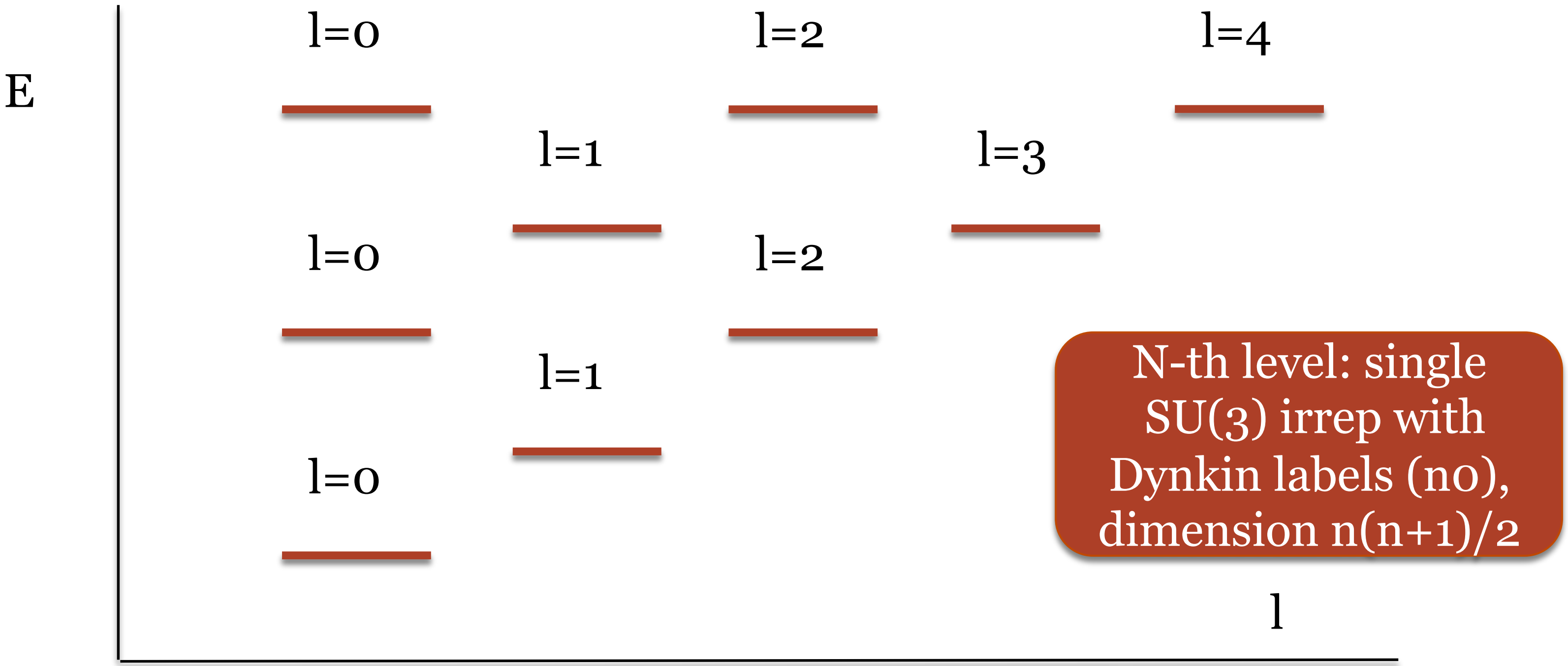
levels labeled by l; per level a total of 2l+1 states:

Enhanced symmetry (i) – Harmonic Osc.

$$V(\vec{r}) = r^2/2$$

QM 102:

Spin-less particle in harmonic potential displays degeneracies organized by an SU(3) symmetry

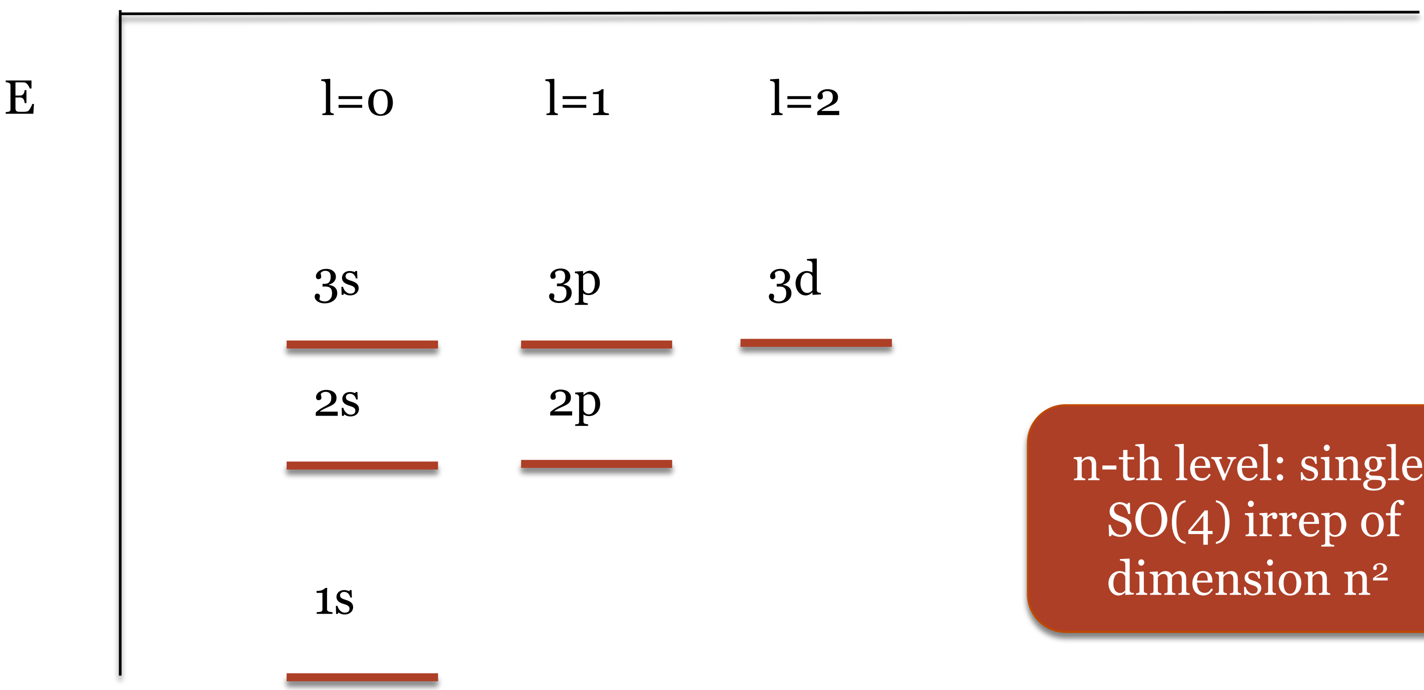


Enhanced symmetry (ii) – Runge Lenz

$$V(\vec{r}) = -q^2 / r$$

QM 103:

Spin-less particle in 1/r potential displays degeneracies organized by an SO(4) symmetry

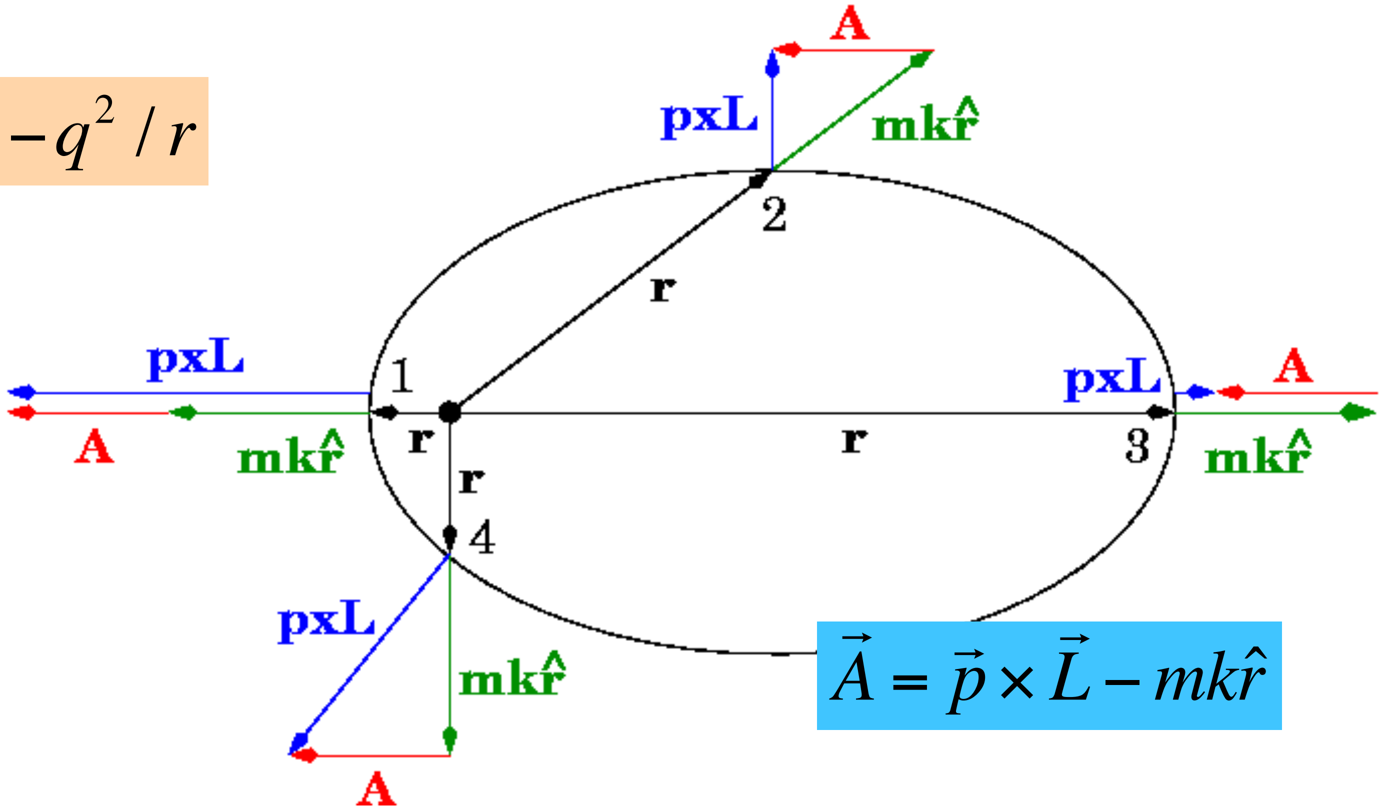


n-th level: single SO(4) irrep of dimension n^2

Enhanced symmetry (ii) – Runge Lenz

QM SO(4) degeneracies can be traced back to an higher integral of motion (Laplace-Runge-Lenz vector) of the classical Kepler orbits

$$V(\vec{r}) = -q^2 / r$$



$$\vec{A} = \vec{p} \times \vec{L} - mk\hat{r}$$

Enhanced symmetry (iii) – SO coupled HO

$$H_{\mathrm{HO,SO}} = \frac{p^2}{2} + \frac{r^2}{2} - 2\vec{L}\cdot\vec{S}$$

today:

Spin-1/2 particles in harmonic potential and with SO coupling of specific strength displays degeneracies organized by a `stretched SO(3,2)' non-compact symmetry.

Physical states grouped into (truncations of) singleton representation of SO(3,2).

Enhanced symmetry (iii) – SO coupled HO

$$H_{\mathrm{HO,SO}} = \frac{p^2}{2} + \frac{r^2}{2} - 2\vec{L}\cdot\vec{S}$$

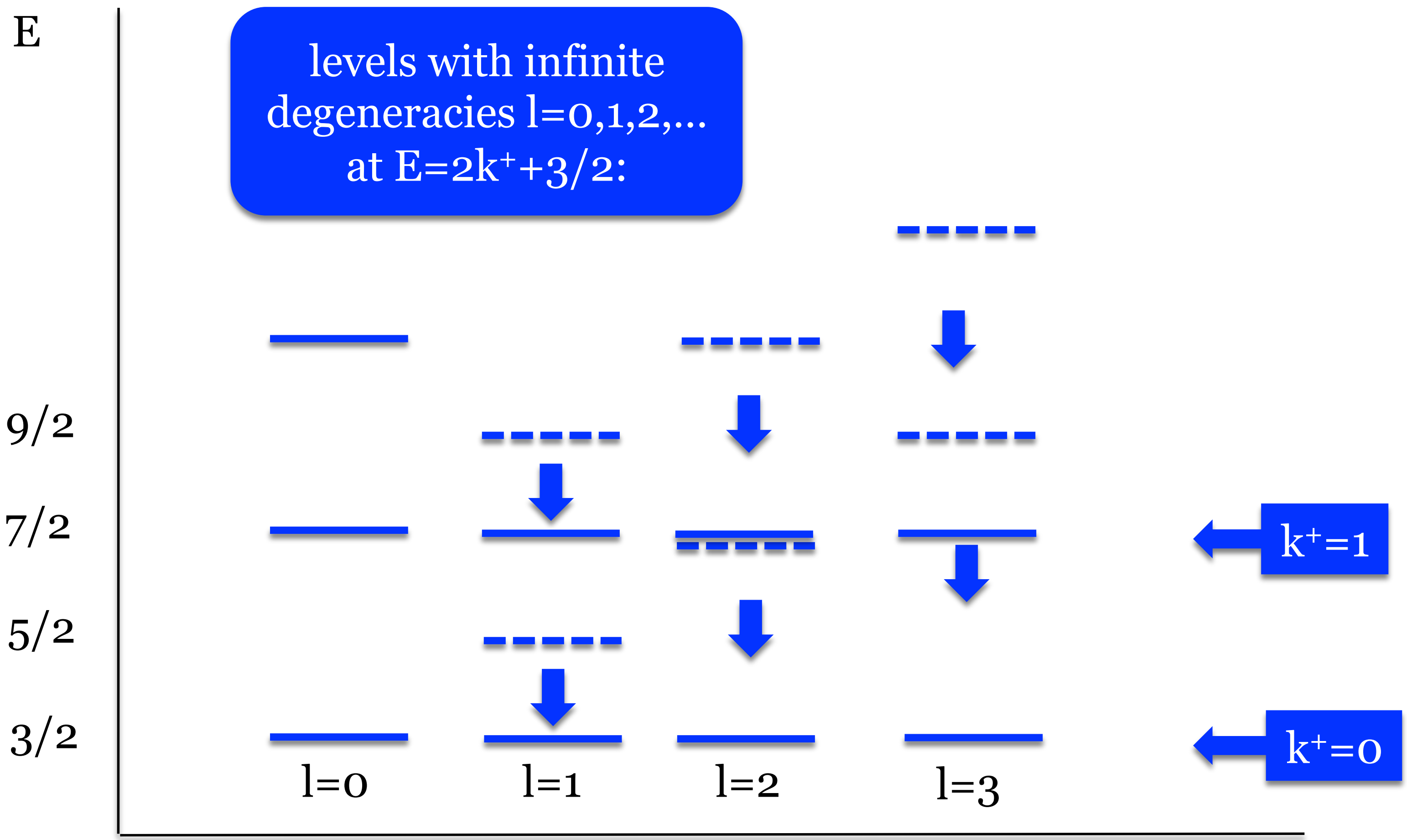
quick determination spectrum:

SO term shifts HO energies according to

$$\begin{aligned} 2\vec{L}\cdot\vec{S} &= +l \quad \text{for} \quad j = l + \frac{1}{2} \\ &= -l - 1 \quad \text{for} \quad j = l - \frac{1}{2} \end{aligned}$$

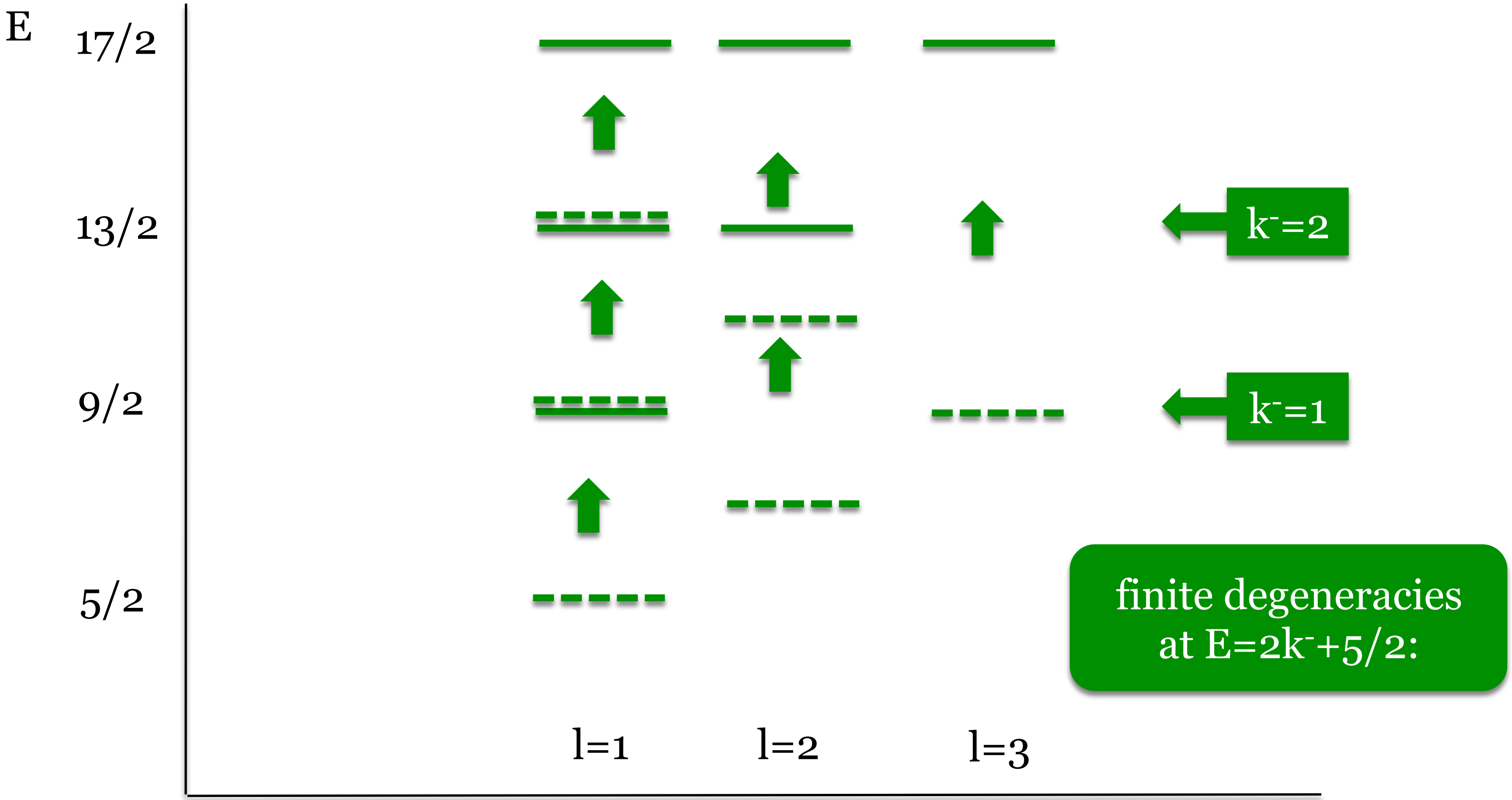
Enhanced symmetry (iii) – SO coupled HO

+ branch: $j=l+1/2$



Enhanced symmetry (iii) – SO coupled HO

- branch: j=l-1/2



spin-1/2 particle in non-Abelian gauge connection

[Yi Li and Congjun Wu, arXiv:1103.5422]

$$A^a_{\alpha\beta} = \frac{1}{2} g \varepsilon_{abc} \sigma^b_{\alpha\beta} r^c$$

$$H = \frac{1}{2m}(\vec{p} - \frac{q}{c}\vec{A})^2 + V(r)$$

$$V(r) = -\frac{1}{2} m\omega_0^2 r^2$$

$$\omega_0 = \frac{qg}{2mc}$$

H reduces to $H_{\mathrm{HO,SO}}$ by direct expansion

spin-1/2 particle in non-Abelian gauge connection

[Yi Li and Congjun Wu, arXiv:1103.5422]

$$H = \frac{1}{2m}(\vec{p} - \frac{q}{c}\vec{A})^2 + V(r)$$

- Aharanov-Casher potential, TR symmetry unbroken
- 3D generalization of quantum spin Hall hamiltonian
- possible physical realization in strained semiconductors or with ultracold atoms with synthetic spin-orbit coupling

Symmetries of SO coupled HO

$$H_{\mathrm{HO,SO}} = \frac{p^2}{2} + \frac{r^2}{2} - 2\vec{L}\cdot\vec{S}$$

Conserved charges: J^2, L^2, J_z, A_3, with

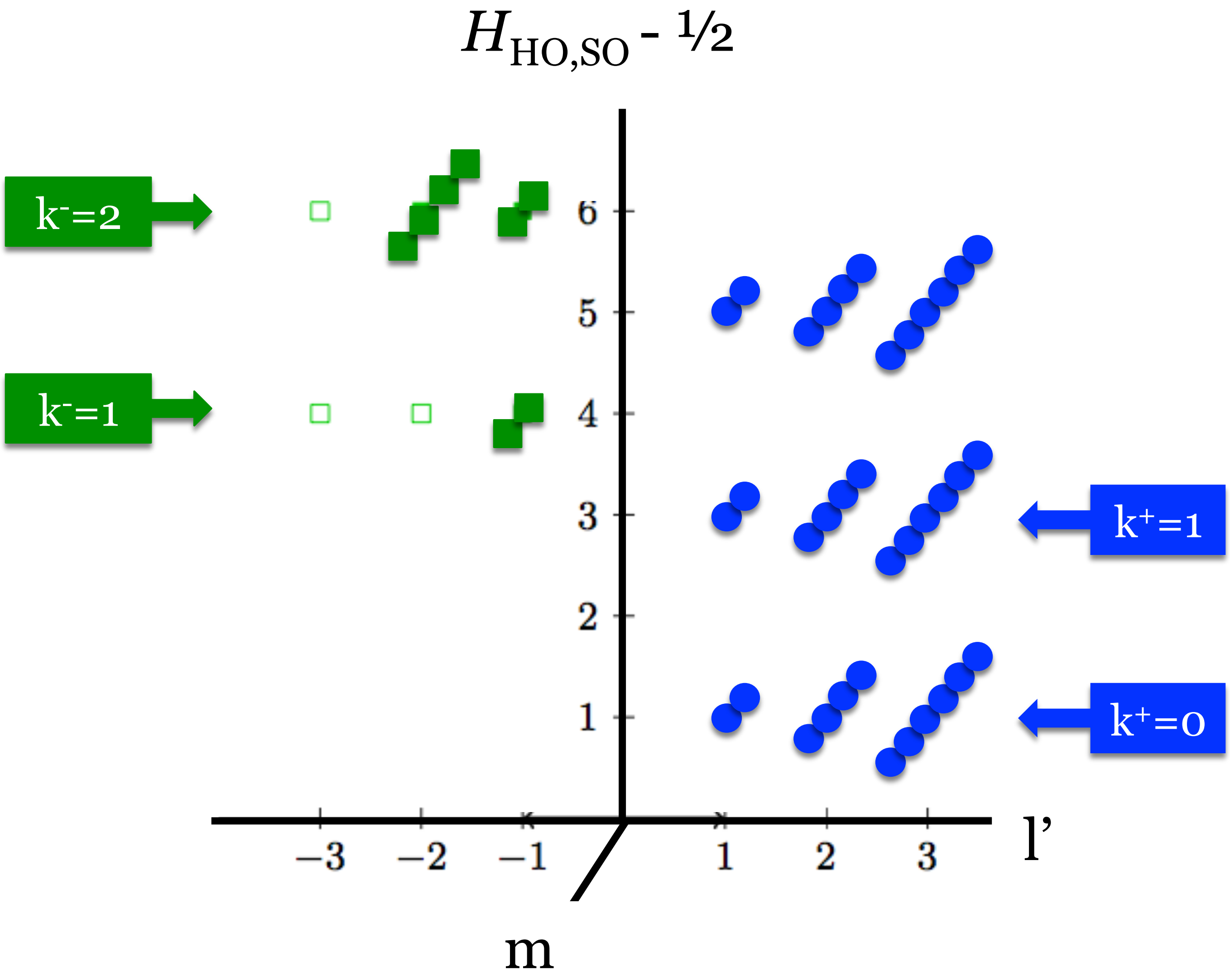
$$A_3 = \vec{L}\cdot\vec{\sigma} + 1$$

$$L^2\psi_{klm} = l(l+1)\psi_{klm} \qquad J^2\psi_{klm} = j(j+1)\psi_{klm}$$
$$A_3\psi_{klm} = l'\psi_{klm} \qquad J_z\psi_{klm} = m\psi_{klm}$$

Energies of +/- branches

$$E^+ = 2k^+ + \frac{3}{2}, \quad k^+ = 0,1,\ldots \qquad E^- = 2k^- + \frac{5}{2}, \quad k^- = 1,2,\ldots$$

Spectrum of $H_{\mathrm{HO,SO}}$



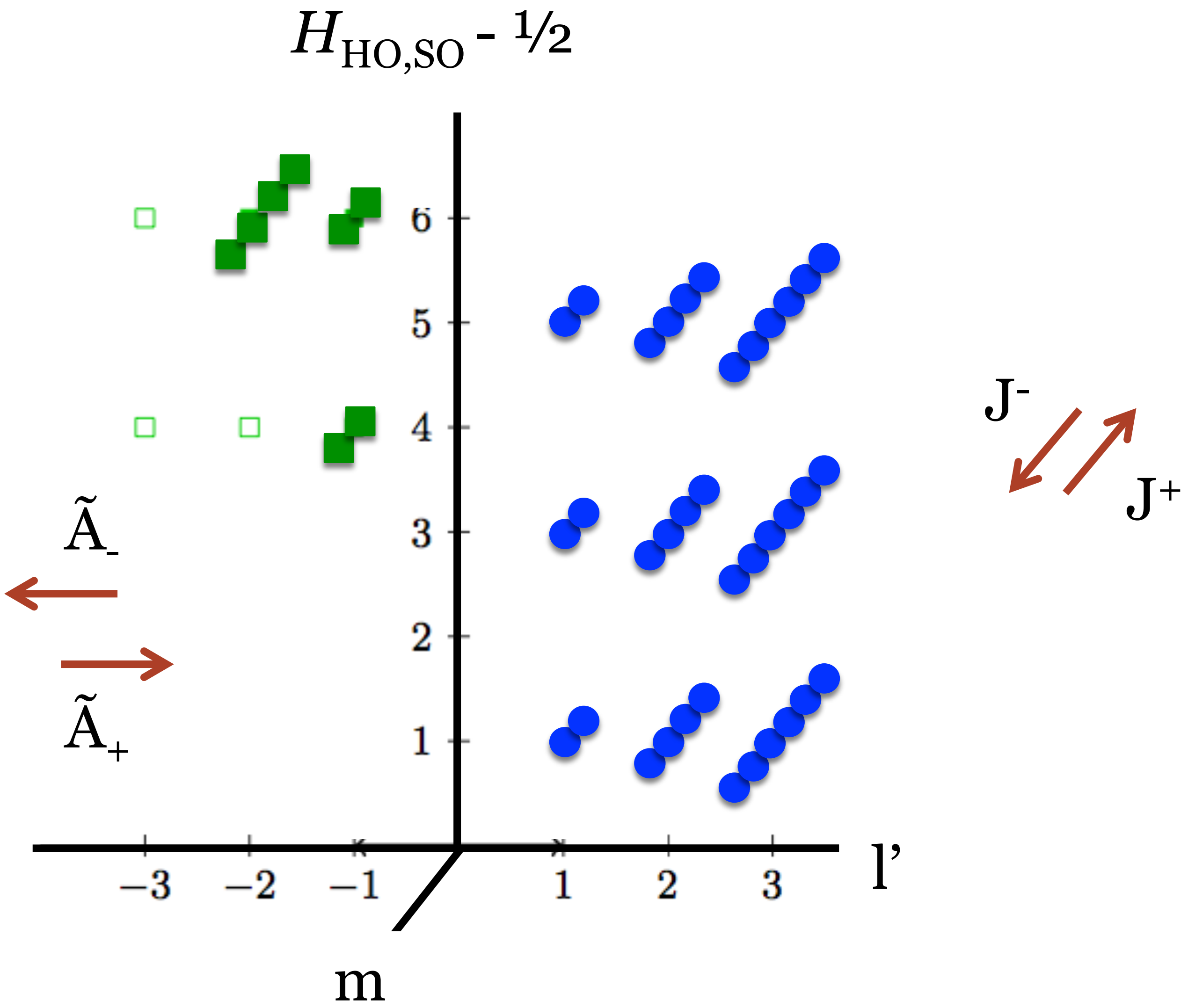
Symmetry operators

angular momentum J^+, J^-, J_z

step-operators for A_3 :

$$\boxed{\begin{aligned}\tilde{A}_+ &= \frac{1}{\sqrt{2}}(\vec{L}\cdot\vec{\sigma}a_3^\dagger - L_+a_+^\dagger - L_-a_-^\dagger)\\ \tilde{A}_- &= \frac{1}{\sqrt{2}}(a_3\vec{L}\cdot\vec{\sigma} - a_+L_- - a_-L_+)\end{aligned}}$$

Spectrum of $H_{\mathrm{HO,SO}}$



Symmetry operators

$$\boxed{\begin{aligned}\tilde{A}_+ &= \tfrac{1}{\sqrt{2}}(\vec{L}\cdot\vec{\sigma}a_3^\dagger - L_+a_+^\dagger - L_-a_-^\dagger)\\ \tilde{A}_- &= \tfrac{1}{\sqrt{2}}(a_3\vec{L}\cdot\vec{\sigma} - a_+L_- - a_-L_+)\end{aligned}}$$

Commuting with J^+, J^- leads to

$$\boxed{\begin{aligned}\tilde{B}_+ &= \tfrac{1}{\sqrt{2}}(\vec{L}\cdot\vec{\sigma}a_-^\dagger + L_3a_-^\dagger - L_+a_3^\dagger)\\ \tilde{B}_- &= \tfrac{1}{\sqrt{2}}(a_-\vec{L}\cdot\vec{\sigma} + a_-L_3 - a_3L_-)\\ \tilde{C}_+ &= \tfrac{1}{\sqrt{2}}(\vec{L}\cdot\vec{\sigma}a_+^\dagger - L_3a_+^\dagger - L_-a_3^\dagger)\\ \tilde{C}_- &= \tfrac{1}{\sqrt{2}}(a_+\vec{L}\cdot\vec{\sigma} - a_+L_3 - a_3L_+)\end{aligned}}$$

Further commutators give more complicated expressions -> not a closed Lie algebra … ?!

Symmetry operators - ctd

after rescaling (with $F=H_{HO}+A_3$)

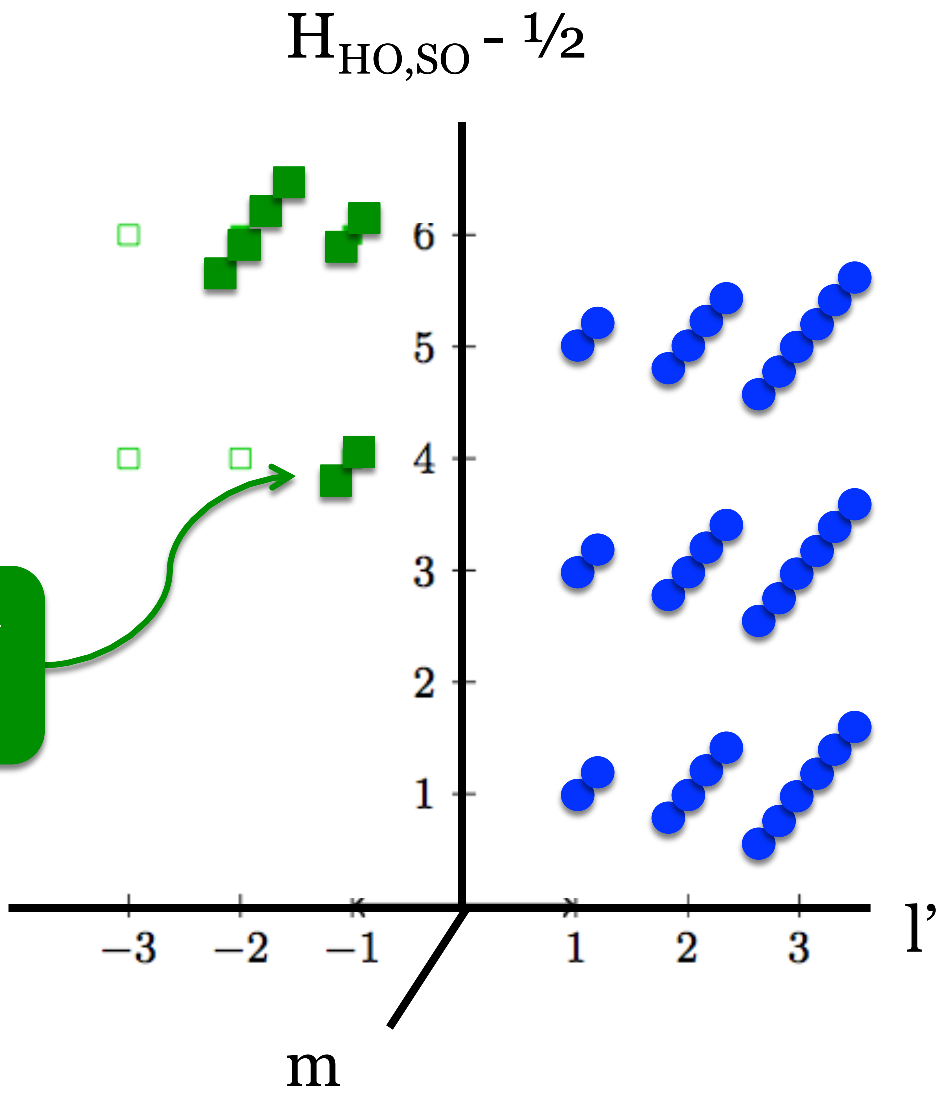
$$\tilde{A}_+ = \left(F - \tfrac{3}{2}\right)^{\frac{1}{2}} A_+$$
$$\tilde{A}_- = A_- \left(F - \tfrac{3}{2}\right)^{\frac{1}{2}}$$

obtain commutators of an SO(2,1) Lie algebra

$$[A_3, A_\pm] = \pm A_\pm \qquad [A_+, A_-] = -A_3$$

Symmetry operators - ctd

Rescaling explains that physical symmetry operators $\tilde{A}_{\pm}$ can give **finite** representations whereas those of SO(2,1) generators $A_{\pm}$ are necessarily **infinite**



k^{-}=1 states in – branch annihilated by $\tilde{A}_{-}$ since F=3/2

$$\boxed{\begin{aligned} \tilde{A}_+ &= \left(F - \tfrac{3}{2}\right)^{\frac{1}{2}} A_+ \\ \tilde{A}_- &= A_- \left(F - \tfrac{3}{2}\right)^{\frac{1}{2}} \end{aligned}}$$

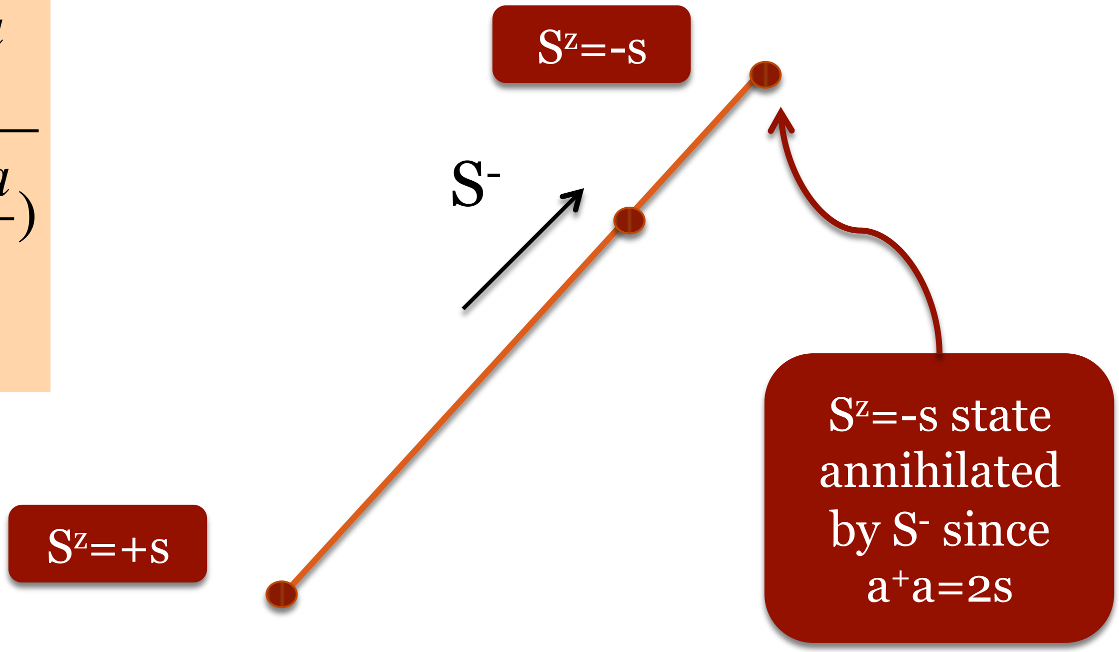
Holstein-Primakoff

non-linear transformation maps canonical boson (a^+, a) with semi-infinite representations, into Lie algebra SU(2) with finite (dimension 2s+1) representation

$$S^+ = \hbar\sqrt{2s}\sqrt{(1-\frac{a^+a}{2s})}\,a$$

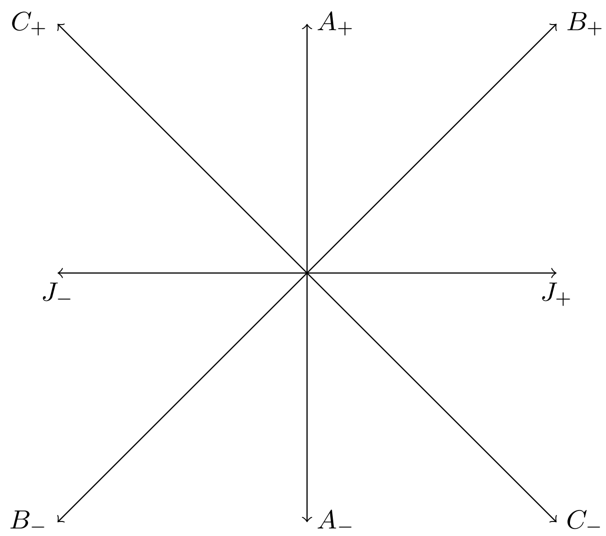
$$S^- = \hbar\sqrt{2s}\,a^+\sqrt{(1-\frac{a^+a}{2s})}$$

$$S^z = \hbar(s - a^+a)$$



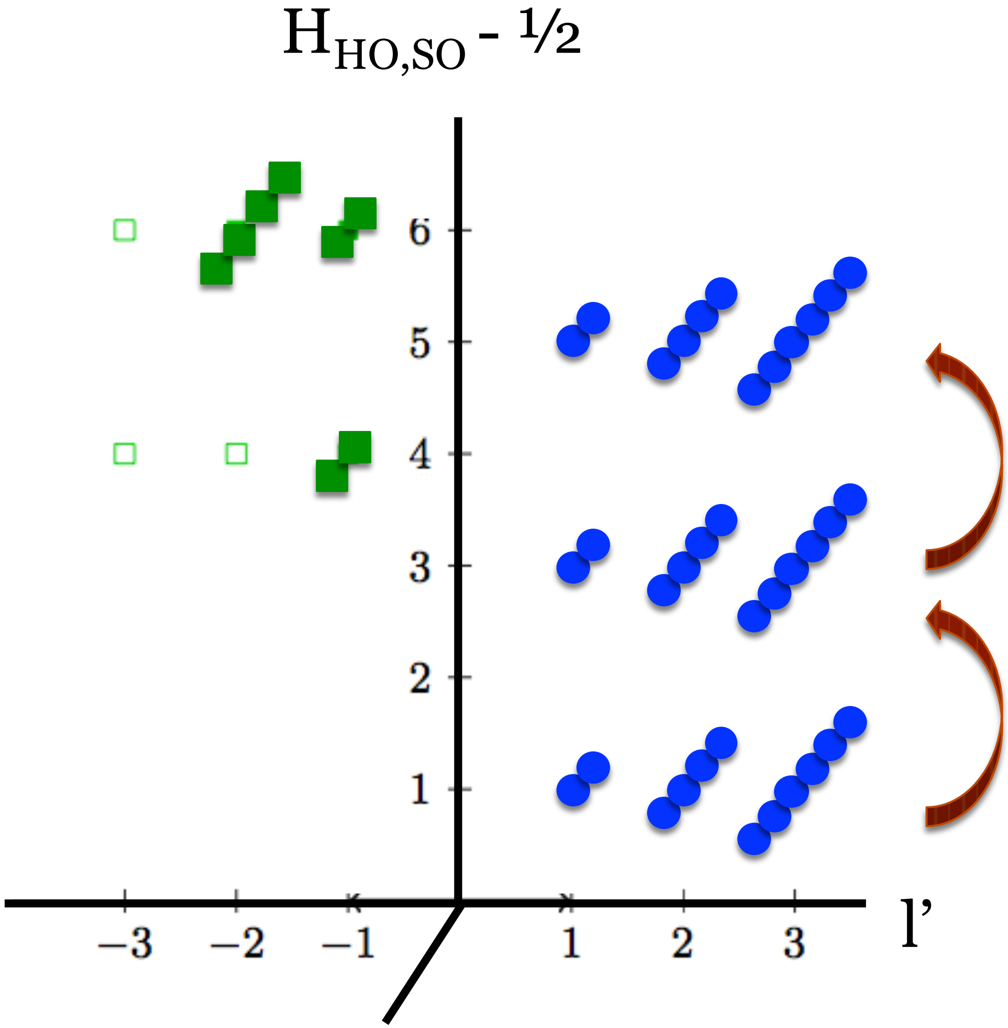
Algebra of rescaled symmetry operators

$$
\begin{aligned}
[A_3, B_\pm] &= \pm B_\pm & [A_3, C_\pm] &= \pm C_\pm \\
[J_3, B_\pm] &= \pm B_\pm & [J_3, C_\pm] &= \mp C_\pm \\
[J_\pm, A_\pm] &= \pm B_\pm & [J_\mp, B_\pm] &= \pm A_\pm \\
[J_\mp, A_\pm] &= \pm C_\pm & [J_\pm, C_\pm] &= \pm A_\pm \\
[A_\mp, B_\pm] &= \pm J_\pm & [B_+, B_-] &= -A_3 - J_3 \\
[A_\pm, C_\mp] &= \mp J_\pm & [C_+, C_-] &= -A_3 + J_3
\end{aligned}
$$



this is non-compact Lie algebra SO(3,2)

Algebraic structure – further results

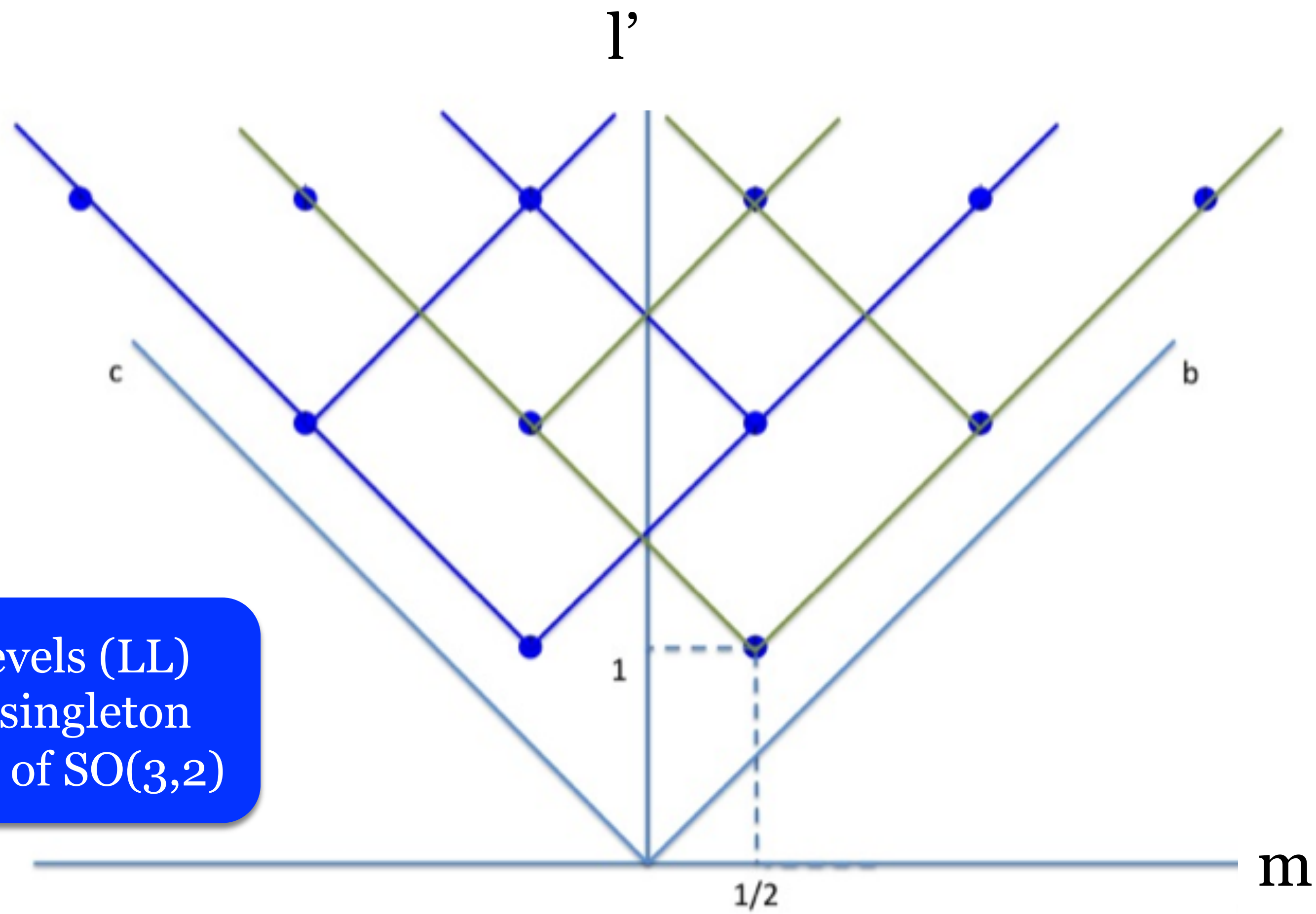


spectrum generating operators sending $H_{\mathrm{HO,SO}} \rightarrow H_{\mathrm{HO,SO}} \pm 2$;

full dynamical algebra takes form SO(3,2) x SO(2,1)

Basic SO(3,2) irrep – the singleton

[PAM Dirac, J Math Phys 1963]



3D Landau levels (LL) organized as singleton representation of SO(3,2)

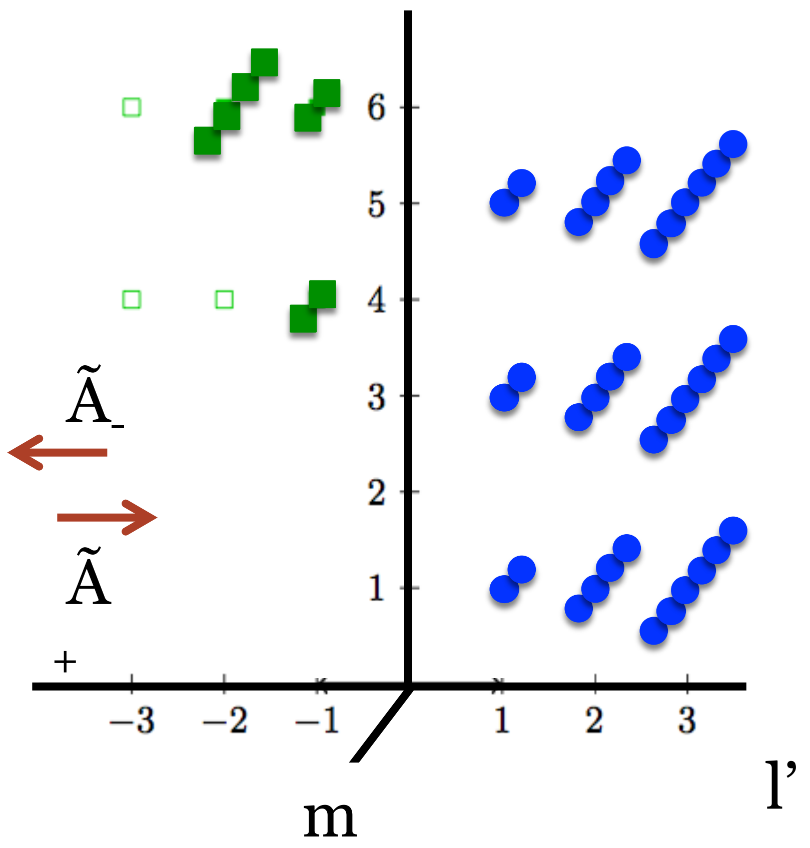
3D Landau levels and topological insulator

[Yi Li and Congjun Wu, arXiv:1103.5422]

$$l_g = \frac{\hbar c}{qg}$$

- Lowest LL: quaternionic analyticity
- density profile peaks near $r = l_g \sqrt{(2l')}$
- with open boundary conditions in radial direction states with $l' > l_{\text{crit}}$ are located near surface – these take form of gapless helical Dirac fermions
- bulk states with odd number of LL filled represent nontrivial topological insulator (TI)

Perspective



- implications of `stretched SO(3,2)' for LL structure
- geometrical picture in relation to AdS4
- 3D LL in finite geometry (3-sphere or 3-torus) to allow for numerics on fractional TI's
- …