Workshop and Conference on Geometrical Aspects of Quantum States in Condensed Matter

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Momentum space topology in Standard Model and in condensed matter

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Momentum space topology in Standard Model & condensed matter

1. Gapless & gapped topological media
2. Fermi surface as topological object
3. Fermi points (Weyl, Majorana & Dirac points) & nodal lines
   * superfluid 3He-A, topological semimetals, cuprate superconductors, graphene
     vacuum of Standard Model of particle physics in massless phase
   * gauge fields, gravity, chiral anomaly as emergent phenomena; quantum vacuum as 4D graphene
   * exotic fermions: quadratic, cubic & quartic dispersion; dispersionless fermions; Horava gravity
4. Flat bands & Fermi arcs in topological matter
   * surface flat bands: 3He-A, semimetals, cuprate superconductors, graphene, graphite
   * towards room-temperature superconductivity
   * 1D flat band in the vortex core
   * Fermi-arc on the surface of topological matter with Weyl points
5. Fully gapped topological media
   * superfluid 3He-B, topological insulators, chiral superconductors, quantum spin Hall insulators,
     vacuum of Standard Model of particle physics in present massive phase, vacua of lattice QCD
   * Majorana edge states & zero modes on vortices (planar phase, topological insulator & 3He-B)
3+1 sources of effective Quantum Field Theories in many-body system & quantum vacuum

I think it is safe to say that no one understands Quantum Mechanics

Richard Feynman

Thermodynamics is the only physical theory of universal content

Albert Einstein

Symmetry: conservation laws, translational invariance, spontaneously broken symmetry, Grand Unification, ...

Effective theories of quantum liquids:
- Two-fluid hydrodynamics of superfluid $^4$He
- Fermi liquid theory of liquid $^3$He

Topology: winding number
one can’t comb hair on a ball smooth, anti-Grand-Unification

Lev Landau

missing ingredient in Landau theories
Topological classes as defects in momentum space

Fermi surface: vortex ring in $\mathbf{p}$-space

metals, normal $^{3}\text{He}$

$\omega$

$p_y (p_z)$

$p_F$

$p_x$

$\Delta \Phi = 2\pi$

Weyl point - hedgehog in $\mathbf{p}$-space

$^{3}\text{He}$-A, vacuum of Standard Model, topological semimetals (Abrikosov)

fully gapped topological matter:

$^{3}\text{He}$-B, topological insulators, quantum-Hall states, $^{3}\text{He}$-A film,
vacuum of Standard Model

Khodel-Shaginyan flat band: $\pi$-vortex in $\mathbf{p}$-space

$H = + c \sigma \cdot \mathbf{p}$

Fermi arc on surface of Weyl materials

Dirac strings in $\mathbf{p}$-space terminating on monopole
bulk - edge correspondence:
topology in bulk protects
gapless fermions on surface of fully gapped systems;
higher order nodes in nodal topological materials

2D Quantum Hall insulator & 3He-A film  
gapless chiral edge states (GV 1992)
3D topological insulator  
gapless Dirac fermions (Volkov-Pankratov 1985)
superfluid 3He-B  
gapless Majorana fermions (Salomaa-GV 1988)
3He-A, Weyl semimetal with point nodes  
Fermi arc (nodal line) on surface (Tutsumi et al 2011)
graphene with point nodes  
dispersionless 1D flat band (Ryu-Hatsugai 2002)
semimetal with Fermi lines  
2D flat band on the surface (Heikkila-GV 2010)

bulk - defect correspondence:
topology in bulk protects gapless fermions inside topological defect

relativistic string  
fermion zero modes in core (Jackiw-Rossi 1981)
3He-A with point nodes  
1D flat band in the core (Kopnin-Salomaa 1991)
2D p+ip superconductor  
Majorana fermions in the core (GV 1999)
two major universality classes of gapless fermionic vacua

Landau theory of Fermi liquid

standard Model + gravity

vacuum with Fermi surface:
metals, normal $^3\text{He}$

vacuum with Fermi (Weyl) point:
$^3\text{He}$-A, planar phase, Weyl semimetal, vacuum of SM

gravity emerges from
Fermi (Weyl) point
analog of
Fermi surface

$g^{\mu\nu} (p_\mu - eA_\mu - e\tau \cdot W_\mu)(p_\nu - eA_\nu - e\tau \cdot W_\nu) = 0$

Theory of topological matter:
Nielsen, So, Ishikawa, Matsuyama, Haldane, Yakovenko, Horava, Kitaev,
Ludwig, Schnyder, Ryu, Furusaki, S-C Zhang, Kane, Liang Fu, ...
Fermi surface as topological object

Energy spectrum of non-interacting gas of fermionic atoms

\[ \varepsilon(p) = \frac{p^2}{2m} - \mu = \frac{p^2}{2m} - \frac{p_F^2}{2m} \]

- \( \varepsilon > 0 \): empty levels
- \( \varepsilon < 0 \): occupied levels: Fermi sea
- \( \varepsilon = 0 \): Fermi surface

\[ p = p_F \]

is Fermi surface a domain wall in momentum space?

Is it a vortex ring?

Green’s function

\[ G^{-1} = i\omega - \varepsilon(p) \]

\[ \Delta \Phi = 2\pi \]

Fermi surface: vortex ring in \( p \)-space

Phase of Green’s function

\[ G(\omega,p) = |G|e^{i\Phi} \]

has winding number \( N = 1 \)

No! it is a vortex ring
Migdal jump, non-Fermi liquids & p-space topology

* Singularity at Fermi surface is robust to perturbations:
  * Winding number $N=1$ cannot change continuously,
  * Interaction (perturbative) cannot destroy singularity

* Typical singularity: Migdal jump

* Other types of singularity with the same winding number:
  Luttinger Fermi liquid,
  marginal Fermi liquid, pseudo-gap ...

* Example: zeroes in $G(\omega,p)$ have the same $N=1$ as poles

$$G(\omega,p) = \frac{Z(p,\omega)}{i\omega - \varepsilon(p)}$$

$$Z(p,\omega) = (\omega^2 + \varepsilon^2(p))^\gamma$$

* Important for interacting systems, where quasiparticles are ill defined

* Fermi surface exists in superfluids/superconductors, examples: 3He-A in flow &
  Gubankova-Schmitt-Wilczek, PRB74 (2006) 064505,
  but Luttinger theorem is not applied
quantized vortex in $r$-space $\equiv$ Fermi surface in $p$-space
homotopy group $\pi_1$

Topology in $r$-space

$\Psi(r) = |\Psi| e^{i\Phi}$
scalar order parameter
of superfluid & superconductor

vortex ring

$\Delta \Phi = 2\pi$

$N_1 = 1$

Topology in $p$-space

$G(\omega, p) = |G| e^{i\Phi}$
Green’s function (propagator)

$\omega$

$\omega$

$N_1 = 1$

$\Delta \Phi = 2\pi$

Fermi surface

classes of mapping $S^1 \rightarrow U(1)$
manifold of broken symmetry vacuum states

classes of mapping $S^1 \rightarrow GL(n, \mathbb{C})$
space of non-degenerate complex matrices

how is it in $p$-space?
non-topological flat bands due to interaction

*Khodel-Shaginyan fermion condensate*

\[ E\{n(p)\} \]
\[ \delta E\{n(p)\} = \int \epsilon(p) \delta n(p) d^d p = 0 \]
solutions: \( \epsilon(p) = 0 \quad \text{or} \quad \delta n(p) = 0 \)

\[ \delta n(p) = 0 \]
\[ \epsilon(p) = 0 \quad \delta n(p) = 0 \quad \text{splitting of Fermi surface to Fermi ball (flat band)} \]

S.-S. Lee
Non-Fermi liquid from a charged black hole: A critical Fermi ball
PRD 79, 086006 (2009)

anti-de Sitter/conformal field theory correspondence
Flat band as momentum-space dark soliton terminated by half-quantum vortices

phase of Green's function changes by \( \pi \) across the "dark soliton"
3. Classes of Fermi points & nodal lines:
superfluid $^3$He-A, Standard Model, semimetals, graphene, cuprate SC, ...
surface of $^3$He-B & topological insulators

- Magnetic hedgehog
- Weyl point
- or Berry phase Dirac monopole

Right-handed and left-handed massless quarks and leptons are elementary particles in Standard Model

Landau CP symmetry is emergent

Close to Fermi point

$$H = + c \sigma \cdot p$$

Right-handed Weyl electron = hedgehog in $p$-space with spines = spins
Topological invariant for right and left elementary particles

hedgehog with spines (spins) outward ($N_3 = +1$)

$h = + c \sigma \cdot p$

$g(p) = + cp$

hedgehog with spines (spins) inward ($N_3 = -1$)

$h = - c \sigma \cdot p$

$g(p) = - cp$

$N_3 = \frac{1}{8\pi} e_{ijk} \int dS^i \hat{g} \cdot (\partial^j \hat{g} \times \partial^k \hat{g})$

Berry phase

$\gamma(C) = \Phi_0 = 2\pi$

Berry 'magnetic' field

$H(p) = \frac{\dot{p}}{2p^2}$
Weyl fermions in 3+1 gapless topological cond-mat

Topologically protected Weyl points in:

Topological semi-metal or Weyl metals (Abrikosov-Beneslavskii 1971), \( ^3\text{He-A} \) (1982), triplet Fermi gases, CoSb\(_3\) (arXiv:1204.5905)

\[
N_3 = \frac{1}{8\pi} e^{ijk} \int dS^k \hat{g} \cdot (\bigtriangledown p_i \hat{g} \times \bigtriangledown p_j \hat{g})
\]

over 2D surface \( S \)
in 3D p-space

\[
p^2 = p_x^2 + p_y^2 + p_z^2
\]

\[
H = \left( \begin{array}{ccc}
\frac{p^2}{2m} - \mu & c(p_x + ip_y) \\
c(p_x - ip_y) - \frac{p^2}{2m} + \mu \\
\end{array} \right) = \left( \begin{array}{ccc}
g_3(p) & g_1(p) + ig_2(p) \\
g_1(p) - ig_2(p) & -g_3(p) \\
\end{array} \right)
\]
emergence of relativistic chiral Weyl fermions near Fermi points

original non-relativistic Hamiltonian

\[
H = \left( \frac{p^2}{2m} - \mu \begin{pmatrix} c(p_x + i p_y) \\ c(p_x - i p_y) - \frac{p^2}{2m} + \mu \end{pmatrix} \right) = \begin{pmatrix} g_3(p) & g_1(p) + ig_2(p) \\ g_1(p) - ig_2(p) & -g_3(p) \end{pmatrix} = \tau \cdot g(p)
\]

close to nodes, i.e. in low-energy corner
relativistic chiral fermions emerge

\[
H = N_3 c \tau \cdot (p - p_0)
\]

\[
E = -c\tilde{p}
\]

chirality is emergent ??

what else is emergent?

top. invariant determines chirality in low-energy corner

relativistic invariance as well
bosonic collective modes in two generic fermionic vacua

Landau theory of Fermi liquid

Fermi surface

collective Bose modes of fermionic vacuum:
propagating oscillation of shape of Fermi surface

Landau, ZhETF 32, 59 (1957)

Standard Model + gravity

collective Bose modes:
propagating oscillation of position of Fermi point

\[ p \rightarrow p - eA \]
form effective dynamic electromagnetic field

\[ E^2 = c^2 p^2 \rightarrow g^{ik} p_i p_k \]
form effective dynamic gravity field

two generic quantum field theories of interacting bosonic & fermionic fields
relativistic quantum fields & gravity emerging near Weyl point

Atiyah-Bott-Shapiro construction:
linear expansion of Hamiltonian near the nodes in terms of Dirac $\Gamma$-matrices

$$E = v_F (p - p_F)$$  emergent relativity
linear expansion near Fermi surface

$$H = e_a^k \Gamma^a \cdot (p_k - p_k^0)$$  linear expansion near Weyl point

all ingredients of Standard Model:
chiral fermions & gauge fields
emerge in low-energy corner
together with spin, Dirac $\Gamma$—matrices,
gravity & physical laws:
Lorentz & gauge invariance, equivalence principle, etc

effective metric:
emergent gravity

$$g^{uv}(p_u - eA_u - e\tau \cdot W_u)(p_v - eA_v - e\tau \cdot W_v) = 0$$

effective metric: emergent gravity

effective $SU(2)$ gauge field

Einstein-Cartan-Sciama-Kibble theory
with tetrads, spin connection & torsion
Chiral Weyl fermions in Standard Model

Left particles

\[ \begin{array}{c|c}
+2/3 & -1/3 \\
\hline
u_L & d_L \\
+1/6 & +1/6 \\
\end{array} \]

Quarks

\[ \begin{array}{c|c}
+2/3 & -1/3 \\
\hline
u_R & d_R \\
+2/3 & -1/3 \\
\end{array} \]

Right particles

\[ \begin{array}{c|c}
0 & -1 \\
\hline
\nu_L & e_L \\
-1/2 & -1/2 \\
\end{array} \]

Leptons

\[ \begin{array}{c|c}
0 & -1 \\
\hline
\nu_R & e_R \\
0 & -1 \\
\end{array} \]

\[ H = - c \sigma \cdot p \]

\[ N_3 = -1 \]

\[ H = + c \sigma \cdot p \]

\[ N_3 = +1 \]

Hedgehog with spines (spins) inward \((N_3=-1)\)

Hedgehog with spines (spins) outward \((N_3=+1)\)

\[ N_3 = \frac{1}{24\pi^2} \epsilon_{\mu\nu\lambda\gamma} \text{tr} \int dS^\gamma G^\mu G^{-1} G^\nu G^{-1} G^\lambda G^{-1} \]

Over 3D surface \(S\) in 4D momentum space

General topological invariant in terms of Green's function for interacting systems
Standard Model topological invariant

Topological invariant protected by symmetry

\[ N_K = \frac{1}{24\pi^2} e^{\mu\nu\lambda} \text{tr} \int dV K G^{\mu} G^{-1} G^{\nu} G^{-1} G^{\lambda} G^{-1} \]

over \( S^3 \)

\( G \) is Green’s function, \( K \) is symmetry operator

\[ GK = +/- KG \]

for Standard Model vacuum \( K \) is \( Z_2 \) center group

\[ K = \exp 2\pi i \tau_3 \]

\( \tau_3 \) weak isotopic spin of SU(2)

\[ N_K = 16 n_g \]

16 massless Weyl particles in one generation are protected by combined symmetry and topology
Weyl point (Dirac monopole) as origin of chiral anomaly
example:
baryon production from vacuum by hypermagnetic field

chiral anomaly equation
(Adler, Bell, Jackiw)

\[
\dot B = \frac{1}{4\pi^2} N_B \mathbf{B}_Y \cdot \mathbf{E}_Y
\]

\[
N_B = \frac{1}{24\pi^2} e_{\mu\nu\lambda} \text{tr} \int dV \mathbf{B} Y^2 \mathbf{G} \nabla^\mu \mathbf{G}^{-1} \mathbf{G} \nabla^\nu \mathbf{G}^{-1} \mathbf{G} \nabla^\lambda \mathbf{G}^{-1}
\]

matrix of baryonic charge
matrix of hypecharge

\[
\dot B = \frac{1}{4\pi^2} \mathbf{B}_Y \cdot \mathbf{E}_Y \sum_a \mathbf{B}_a \mathbf{C}_a Y^2
\]

\[B_a \text{ -- baryonic charge} \]
\[Y_a \text{ -- hypercharge} \]
\[C_a = +1 \text{ for right} \]
\[-1 \text{ for left} \]

dirac monopole in p-space

Berry phase
\[\gamma(C) = \Phi_0 = 2\pi\]

Berry 'magnetic' field
\[H(p) = \frac{\dot p}{2p^2}\]

earlier indications of axial anomaly in 3He-A from Weyl point:
anomaly in orbital dynamics & paradox of angular & linear momenta
Symmetry protected invariants and chiral anomaly

Prefactors of $r$-space topological terms are $p$-space topological invariant

Chiral anomaly from 3+1 Weyl points

\[
K_{abc} = \frac{1}{24\pi^2} e_{\mu \nu \lambda} \text{tr} \int dV K_a K_b K_c \mathbf{G} \nabla^\mu \mathbf{G}^{-1} \mathbf{G} \nabla^\nu \mathbf{G}^{-1} \mathbf{G} \nabla^\lambda \mathbf{G}^{-1}
\]

Over $S^3$

Prefactor of Chern-Simons term in 2+1 gapped topological matter

\[
K_{ab} = \frac{1}{24\pi^2} e_{\mu \nu \lambda} \text{tr} \int dV K_a K_b \mathbf{G} \nabla^\mu \mathbf{G}^{-1} \mathbf{G} \nabla^\nu \mathbf{G}^{-1} \mathbf{G} \nabla^\lambda \mathbf{G}^{-1}
\]

Over 2+1 p-space
experimental verification of chiral anomaly equation
measurement of Kopnin force

momentum from vacuum of fermion zero modes

$\dot{P} = \sum_a P_a \dot{n}_a$

$P_a$ -- momentum of Weyl point (fermionic charge)
$e_a$ -- effective electric charge

$\dot{P} = (1/4\pi^2) B \cdot E \sum_a P_a C_a e_a$

applied to $^3$He-A

$C_a = +1$ for right
$-1$ for left

baryons from vacuum

$\dot{B} = \sum_a B_a \dot{n}_a$

$B_a$ -- baryonic charge
$Y_a$ -- hypercharge

$\dot{B} = (1/4\pi^2) B_Y \cdot E_Y \sum_a B_a C_a Y_a^2$

applied to Standard Model

$C_a = +1$ for right
$-1$ for left

quasiparticles move from vacuum to the positive energy world,
where they are scattered by quasiparticles in bulk
and transfer momentum from vortex to normal component

this is the source of Kopnin spectral flow force

Bevan, Manninen, Cook, Hook, Hall, Vachaspati & GV
Momentum creation by vortices in superfluid 3He as a model of primordial baryogenesis, Nature 386, 689 (1997)
Experimental chiral anomaly: spectral flow force on a vortex

superfluid Reynolds number

\[ \text{Re}_\alpha(T) = \frac{1 - d_\perp}{d_\parallel} \approx \omega_0 \tau \]

\[ \text{Re}_\alpha(T_c) = 0 \quad \text{Re}_\alpha(T \sim 0.6 T_c) \sim 1 \quad \text{Re}_\alpha(0) = \infty \]

- laminar vortex flow
- turbulent vortex flow
Chiral magnetic effect (CME) leads to observed helical instability

\[ \mathbf{L} = \mathbf{A} \cdot \mathbf{B} (\mu_L - \mu_R) / 8\pi^2 \quad \mathbf{J} = \mathbf{B} (\mu_L - \mu_R) / 8\pi^2 \]

counterflow:
fermionic charge \( \mathbf{P} \sim (\mathbf{v}_n - \mathbf{v}_s) \neq 0; \)
no vortices (no magnetic field)

no counterflow
\( \mathbf{P} \sim (\mathbf{v}_n - \mathbf{v}_s) = 0; \)
vertices (magnetic field)

helical instability in \(^3\text{He}-\text{A}\): onset of magnetogenesis

Helsinki 1996
From massless Weyl particles to massive Dirac particles

where are massive Dirac particles?

Dirac particle - composite object: mixture of left and right Weyl particles

$E = cp$

$E^2 = c^2p^2 + m^2$

is Dirac vacuum topologically trivial?

fully gapped vacua can be also topologically nontrivial (3He-B, topological insulators, ...)

$T_{ew} \sim 1 \text{ TeV} \sim 10^{16} \text{K}$
p-space analogs of graphene
emergence of 2+1 gapless relativistic fermions in 2D graphene

quantum vacuum as crystal

Weyl/Dirac point $N = +1$

Weyl/Dirac point $N = -1$

p-space analog of 3D graphene: superconductor $\alpha$ - phase

p-space analog of 4D graphene in lattice QCD
emergence of 2+1 relativistic fermions due to topology of graphene nodes

\[ N = \frac{1}{4\pi i} \text{tr} \left[ K \oint dl \ H^{-1} \nabla_l \ H \right] \]

\( K \) - symmetry operator, commuting or anti-commuting with \( H \)

close to nodes:

\[ H_N = +1 = \tau_x p_x + \tau_y p_y \]

\[ H_N = -1 = \tau_x p_x - \tau_y p_y \]

\[ K = \tau_z \]

for real interacting systems

the Hamiltonian \( H(p) \) is substituted by inverse Green’s function at zero frequency \( G^{-1}(\omega=0,p) \)
SU(2) gauge fields emerging near Weyl & Dirac points

Atiyah-Bott-Shapiro construction:
linear expansion of Hamiltonian near the nodes in terms of Dirac $\Gamma$-matrices

$$H = e_a^k \Gamma^a \cdot (p_k - eA_k - e\tau \cdot W_k)$$

effective tetrads: emergent gravity

effective isotopic spin

effective electromagnetic field

effective SU(2) gauge field

SU(2) gauge field is collective mode of vacuum with Weyl point

isotopic spin comes from spin or from band indices

SU(2) field near Dirac points in graphene

$$H_N = +1 = \tau_x(p_x - A_x - \sigma \cdot W_x) + \tau_y(p_y - A_y - \sigma \cdot W_y)$$
Summation of topological charges in action

exotic fermions:
massless fermions with quadratic, cubic & quartic dispersion
semi-Dirac fermions

\[ N = \frac{1}{4\pi i} \text{tr} \left[ K \oint dl \ H^{-1} \nabla_l H \right] \]

bilayer graphene
double cuprate layer

N=+1
\[ E = cp \]
relativistic massless fermions

N=+1
\[ E = cp \]

N=+1
\[ E = cp \]

N=0
\[ E = cp \]
Dirac fermion

N=+2
\[ E^2 = \left( \frac{p^2}{2m} \right)^2 \]
nonrelativistic Dirac fermion with quadratic dispersion

re-entrant violation of Lorentz invariance & neutrino physics

Klinkhamer & GV arXiv:1109.6624
**nonlinear Dirac fermions**

N=1: Dirac fermions with linear dispersion

\[
H = \begin{pmatrix}
0 & p_x + ip_y \\
p_x - ip_y & 0
\end{pmatrix} = \sigma_x p_x + \sigma_y p_y
\]

N=2: Dirac fermions with quadratic dispersion

\[
H = \begin{pmatrix}
0 & (p_x + ip_y)^2 \\
(p_x - ip_y)^2 & 0
\end{pmatrix}
\]

N=3: Dirac fermions with cubic dispersion

\[
H = \begin{pmatrix}
0 & (p_x + ip_y)^3 \\
(p_x - ip_y)^3 & 0
\end{pmatrix}
\]

Dirac fermions with nonlinear dispersion

\[
H = \begin{pmatrix}
0 & (p_x + ip_y)^N \\
(p_x - ip_y)^N & 0
\end{pmatrix}
\]
multiple Fermi point

cubic spectrum in trilayer graphene

\[ N = 1 + 1 + 1 = 3 \]

\[ E = cp \]

multilayered graphene

\[ N = 1 + 1 + 1 + ... \]

spectrum in the outer layers

\[ E = p^N \]
\[ E = -p^N \]

what kind of induced gravity emerges near degenerate Fermi point?

route to topological flat band on the surface of 3D material
Splitting of Dirac and Weyl points

Splitting of quadratic point or trigonal warping

\[ N = 2 = 1 + 1 + 1 - 1 \]

\[ N = +1 \quad N = -1 \quad N = +1 \]

\[ E \quad p_y \quad p_x \]
Horava anisotropic scaling gravity

anisotropic \(z=3\) scaling: \(x = b \, x', \; t = b^3 t'\)

\[ S_{\text{grav}} = \int d^3 x \, dt \, R^3 \]

Horava anisotropic \(z=2\) scaling in bilayered graphene

\[ N = \frac{1}{4\pi i} \text{tr} \left[ K \oint dl \, H^{-1} \nabla_l H \right] \]

\(N=+1\)

\begin{align*}
E & = cp \\
p_x & \\
p_y &
\end{align*}

2+1 massless Dirac fermions

\(N=+1\)

\begin{align*}
E & = cp \\
p_x & \\
p_y &
\end{align*}

\(N=0\)

\begin{align*}
E^2 & = 2c^2 p^2 + 4m^2 \\
p_x & \\
p_y &
\end{align*}

massive fermions

\(N=+2\)

\begin{align*}
E & = (p^2 / 2m)^2 \\
p_x & \\
p_y &
\end{align*}

massless Dirac fermions with quadratic dispersion
Fermions in 2+1 bylayer graphene

single layer

\[ H = \begin{pmatrix} 0 & p_x + ip_y \\ p_x - ip_y & 0 \end{pmatrix} = \sigma_x p_x + \sigma_y p_y = \begin{pmatrix} 0 & (\mathbf{e}_1 + i \mathbf{e}_2) \cdot (\mathbf{p} - \mathbf{A}) \\ (\mathbf{e}_1 - i \mathbf{e}_2) \cdot (\mathbf{p} - \mathbf{A}) & 0 \end{pmatrix} \]

double layer

\[ H = \begin{pmatrix} 0 & (p_x + ip_y)^2 \\ (p_x - ip_y)^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & [(\mathbf{e}_1 + i \mathbf{e}_2) \cdot (\mathbf{p} - \mathbf{A})]^2 \\ [(\mathbf{e}_1 - i \mathbf{e}_2) \cdot (\mathbf{p} - \mathbf{A})]^2 & 0 \end{pmatrix} \]

anisotropic scaling: \( x = b \ x' \), \( t = b^2 \ t' \)

Horava-Lifshitz gravity:
Horava, Quantum gravity at a Lifshitz point
PRD 79, 084008 (2009)
Heisenberg-Euler action in massive QED
non-linear corrections to Maxwell equations due to vacuum polarization

\[ S_{HE} = \int d^3x \, dt \left[ \frac{(B^2 - E^2)^2}{M^4} + \frac{(B \cdot E)^2}{M^4} \right] \]

\( M \) is rest energy of electrons \( B,E \ll M^2 \)

What is the Heisenberg-Euler action for relativistic massless QED emerging in condensed matter?
What is the Heisenberg-Euler action for massless QED with exotic Dirac fermions with quadratic and cubic spectrum?

relation to Horava quantum gravity with anisotropic scaling
isotropic QED emerging in 3He-A and single layer graphene

isotropic scaling: \( x = b \, x' \), \( t = b \, t' \), \( B = b^{-2} \, B' \), \( E = b^{-2} \, E' \), \( S = S' \)

3+1 isotropic QED & emerging in Weyl superfluids & semimetals

\[
S_{\text{QED}} = \int d^3x \, dt \left( B^2 - E^2 \right) \ln 1/(B^2 - E^2) 
\]

imaginary action (Schwinger pair production) at \( B^2 < E^2 \)

2+1 isotropic QED emerging in single layer graphene

\[
S_{\text{QED}} = \int d^2x \, dt \left( B^2 - E^2 \right)^{3/4} 
\]

imaginary action (Schwinger pair production) at \( B^2 < E^2 \)
2+1 anisotropic QED emerging in bylayer graphene

\[
H = \begin{pmatrix}
(p_x + ip_y)^2/2m & 0 \\
(p_x - ip_y)^2/2m & 0
\end{pmatrix} = \begin{pmatrix}
0 & [(e_1+i e_2)(\mathbf{p} - \mathbf{A})]^2/2m \\
[(e_1-i e_2)(\mathbf{p} - \mathbf{A})]^2/2m & 0
\end{pmatrix}
\]

Heisenberg-Euler action

anisotropic scaling: \( x = bx' \), \( t = b^2 t' \), \( B = b^2 B' \), \( E = b^{-3} E' \), \( S = S' \)

\[
S = \frac{1}{m} \int \frac{d^2x}{b^2} \frac{d^2t}{b^2} \frac{B^2}{b^{-4}} g(\mu)
\]

\( g(\mu) \) – dimensionless function of dimensionless parameter \( \mu = \frac{m^2E^2}{B^3} \)

magnetic field asymptote

\[
S_B = \frac{1}{m} \int \frac{d^2x}{b^2} \frac{d^2t}{b^2} \ln \frac{1}{B^2}
\]

Schwinger pair production \( \sim E^{4/3} \)

electric field asymptote

\[
S_E = \frac{1}{m} \int \frac{d^2x}{b^2} \frac{d^2t}{b^2} \left(-\frac{m^2E^2}{b^{-4}}\right)^{2/3}
\]

pair production mainly occurs at \( \mu > 1 \) i.e at \( E^2 > B^3/m^2 \)
Schwinger pair production

\[ \text{Im } S = \pi^{-2} B^2 \mu \int_0^1 dx \exp(-\mu f(x)) \]

\[ \mu = m^2 E^2 / B^3 \]

\[ f(x) = x - (1+x)/2 \ln(1+x) - (1-x)/2 \ln(1-x) \]

pair production mainly occurs at \( \mu > 1 \) i.e at \( E^2 > B^3 / m^2 \)

at \( \mu >> 1 \)

\[ f(x) = x^3 / 6 \]

Schwinger pair production \( \text{Im } S \sim E^{4/3} m^{1/3} \)

M.I. Katsnelson & G.E. Volovik,
Quantum electrodynamics with anisotropic scaling:
Heisenberg-Euler action and Schwinger pair production in the bilayer graphene,
Pis’ma ZhETF 95, 457 (2012); arXiv:1203.1578
4. Flat bands & Fermi arcs in topological matter

nodal spiral in multilayered graphene generates flat band with zero energy in the top and bottom layers
Hekilla, Kopnin, GV

nodes in graphene generate flat band on zigzag edge

nodal lines in cuprate superconductors generate flat band on side surface
Shinsei Ryu

approximate flat band on side surface of graphite
formation of nodal spiral in bulk (together with flat band on the surface) by stacking of graphene layers

\[ \sigma_+ + \sigma_- \]

\[ H_{i,j} = (\sigma_x p_x + \sigma_y p_y) \delta_{i,j} + (\sigma_+ t_+ + \sigma_- t_-) \delta_{i,j+1} \]

\[ N_1 = \frac{1}{4\pi i} \text{tr} \left[ \mathbf{K} \oint dl \mathbf{H}^{-1} \nabla_l \mathbf{H} \right] \]
Emergence of nodal line from gapped branches by stacking graphene layers
example of topological bulk-surface correspondence:
Nodal spiral generates flat band on the surface
projection of spiral on the surface determines boundary of flat band

\[ N = \frac{1}{4\pi i} \text{tr} \left[ K \oint_C dl \mathbf{H}^{-1} \nabla \mathbf{H} \right] \]

at each \((p_x, p_y)\) except the boundary of circle one has 1D fully gapped state (insulator)

\[ N_{\text{out}}(p_x, p_y) = 0 \quad \text{trivial 1D insulator} \]

\[ N_{\text{in}}(p_x, p_y) = 1 \quad \text{topological 1D insulator} \]

**topological insulator** has 1D gapless edge state
manifold \((p_x, p_y)\) of edge states forms **flat band**
Nodal spiral generates flat band on the surface

projection of nodal spiral on the surface determines boundary of flat band

lowest energy states:

surface states form the flat band

energy spectrum in bulk (projection to $p_x, p_y$ plane)

nodal line

bulk states
Modified nodal spiral in rhombohedral graphite: spiral of Fermi surfaces (McClure 1969)

Fig. 1. The crystal lattice of rhombohedral graphite. The numbering of the groups of neighbors of the central A atom is explained in the text.

for conventional graphite: approximate flat band on the lateral surface

Fig. 2. The Fermi surface of rhombohedral graphite. The surface is centered on one of the six vertical zone edges. The widths of the surfaces have been magnified by more than an order of magnitude.

projections on surfaces determine approximate flat band
Nodal lines in graphite transformed to chain of electron and hole FS

for conventional graphite:
approximate flat band
on the lateral surface
Flat band on the graphene edge

\[ N = \frac{1}{4\pi i} \text{tr} \left[ \mathbf{K} \oint dl \mathbf{H}^{-1} \nabla_l \mathbf{H} \right] \]

a. flat band: half-quantum vortex in p-space
b. flat band in the vortex core
c. Dirac point

d. Dirac point

\[ N = 0 \]
\[ N = +1 \]
\[ N = +1 \]
\[ N = 0 \]
Surface superconductivity in topological semimetals: route to room temperature superconductivity

Extremely high DoS of flat band gives high transition temperature:

- Normal superconductors: exponentially suppressed transition temperature
  \[ T_c = T_F \exp(-1/g\nu) \]

- Flat band superconductivity: linear dependence of \( T_c \) on coupling \( g \)
  \[ T_c \sim g \]  

\[ 1 = g \int \frac{d^2p}{2\hbar^2} \frac{1}{E(p)} \]

DoS = \( \nu(\varepsilon) \sim \varepsilon^{2/N - 1} \)

\( N \) is number of layers

\( N = 4: \nu(\varepsilon) \sim \varepsilon^{-1/2} \) Kopaev (1970); Kopaev-Rusinov (1987)
evidence of room-temperature superconductivity?

2000: Kopelevich Y, Esquinazi P, Torres J H S and Moehlecke S
J. of Low Temp. Phys. 11, 691–702

2002: Kempa H, Esquinazi P and Kopelevich Y
Phys. Rev. B 65 241101

2007: Kopelevich Y and Esquinazi P
J. of Low Temp. Phys. 146 629

Advanced Materials 24 5826


Ballestar A, Barzola-Quiquia J, Scheike T and Esquinazi P
New J. Phys. 15 023024

From Weyl point to quantum Hall topological insulators

\[
N_3 = \frac{1}{8\pi} e_{ijk} \int dS^k \hat{\mathbf{g}} \cdot (\nabla_{p_i} \hat{\mathbf{g}} \times \nabla_{p_j} \hat{\mathbf{g}})
\]
over 2D surface S in 3D momentum space

top. invariant for Weyl point in 3+1 system

\[
\widetilde{N}_3(p_z) = \begin{cases} 
0 & \text{2D trivial insulator} \\
1 & \text{2D topological Hall insulator}
\end{cases}
\]

at each \(p_z\) one has 2D insulator or fully gapped 2D superfluid

\[
\widetilde{N}_3(p_z) = \frac{1}{8\pi} e_{ijk} \int dp_x dp_y \hat{\mathbf{g}} \cdot (\nabla_{p_x} \hat{\mathbf{g}} \times \nabla_{p_y} \hat{\mathbf{g}})
\]
over the whole 2D momentum space or over 2D Brillouin zone

top. invariant for fully gapped 2+1 system
3D matter with Weyl points:
Topologically protected flat band in vortex core

\[ \tilde{N}_3(p_z) = 0 \]

at each \( p_z \) between two values:
2D topological Hall insulator:
zero energy states \( E(p_z)=0 \),
in the vortex core
(1D flat band \( 1011.4665 \))

\[ \tilde{N}_3(p_z) = 1 \]

\[ \tilde{N}_3(p_z) = 0 \]

\[ \tilde{N}_3(p_z) = \frac{1}{4\pi^2} \text{tr} \int dp_x \, dp_y \, d\omega \, G \nabla_\omega G^{-1} G \nabla_{p_x} G^{-1} G \nabla_{p_y} G^{-1} \]

GV & Yakovenko (1989)
Topologically protected flat band in vortex core of superfluids with Weyl points

flat band in spectrum of fermions bound to core of 3He-A vortex (Kopnin-Salomaa 1991)

E\left(p_z, Q\right)

flat band of bound states terminates on zeroes of continuous spectrum (i.e. on Weyl points)
3He-A with Weyl points:
Topologically protected
Dirac valley (Fermi arc) on surface

for each $|p_z| < p_F \cos \lambda$
one has 2D topological Hall insulator with
zero energy edge states $E(p_z) = 0$

(Dirac valley PRB 094510 or Fermi arc PRB 205101)
Edge states at interface between effective two 2+1 topological insulators & Fermi arc

$\tilde{N}_3 = N_-$

$\tilde{N}_3 = N_+$

Index theorem:
number of fermion zero modes at interface:

$\nu = N_+ - N_-$

GV JETP Lett. 55, 368 (1992)

Fermi arc in 2D:
Fermi surface which terminates on two points:
projections of Weyl points

$E(p_y, |p_z| < p_F)$

left moving edge states

empty

occupied

on the edge of insulator with

$\tilde{N}_3 = 1$

one fermion zero mode
$\nu = 1$
**Fermi arc:**

Fermi surface separates positive and negative energies, but has boundaries.

\[
E(p_y, p_z) > 0 \quad \text{Fermi surface}
\]

\[
E(p_y = 0, |p_z| < p_F) = 0
\]

\[
E(p_y, p_z) < 0
\]

**Fermi surface of localized states is terminated by projections of Weyl points when localized states merge with continuous spectrum.**

Tsutsumi, et al PRB 094510

---

**Fermi surface of bound states may leak to higher dimension.**

[L] spectrum of edge states on left wall

[R] spectrum of edge states on right wall
5. Fully gapped topological matter

skyrmions in p-space
emergent relativistic fermions as edge states

3+1 vacuum with massless fermions

\[ N_3 = \frac{1}{8\pi} \int e^{ijk} \left( \nabla_{p_i} \hat{g} \times \nabla_{p_j} \hat{g} \right) \]

around Fermi point

Fully gapped 2+1 vacuum

dimensional reduction

gapless 1+1 edge states

dimensional reduction

Fully gapped 4+1 vacuum gives 3+1 relativistic fermions (Kaplan, arXiv:1112.0302)

\[ \tilde{N}_3 = \frac{1}{8\pi} \int dp_x dp_y \left( \nabla_{p_x} \hat{g} \times \nabla_{p_y} \hat{g} \right) \]

over the whole 2D momentum space or over 2D Brillouin zone
topological insulators & gapped superconductors in 2+1

topological insulator =
bulk insulator
with topologically protected
gapless states on the boundary

topological gapped superconductor =
superconductor with gap in bulk
but with topologically protected
gapless states on the boundary

$p$-wave 2D superconductor (Sr$_2$RuO$_4$?), $^3$He-A thin film, CdTe/HgTe/Cd insulator quantum well, planar phase film

who protects gapless states?

generic example:

\[
H = \begin{pmatrix}
\frac{p_x^2}{2m} & -\mu & c(p_x + ip_y)\\
-c(p_x - ip_y) & -\frac{p_y^2}{2m} + \mu
\end{pmatrix}
\]
\[
p^2 = p_x^2 + p_y^2
\]

How to extract useful information on energy states from this Hamiltonian without solving equation

\[
H\psi = E\psi
\]
Topological invariant in momentum space

\[ H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix} \]

\[ p^2 = p_x^2 + p_y^2 \]

\[ \tilde{N}_3 = \frac{1}{8\pi} e^{ijk} \int dp_x dp_y \hat{g} \cdot (\nabla_{p_x} \hat{g} \times \nabla_{p_y} \hat{g}) \]

Skyrmion (coreless vortex) in momentum space at \( \mu > 0 \)

unit vector \( \hat{g}(p_x, p_y) \)

sweeps unit sphere

\[ \tilde{N}_3 (\mu > 0) = 1 \]

GV, JETP 67, 1804 (1988)
quantum phase transition:
from topological to non-topological insulator/superconductor

$$H = \left( \begin{array}{cc}
\frac{p^2}{2m} - \mu & c(p_x + ip_y) \\
c(p_x - ip_y) & -\frac{p^2}{2m} + \mu
\end{array} \right) = \left( \begin{array}{cc}
g_3(p) & g_1(p) + ig_2(p) \\
g_1(p) - ig_2(p) & g_3(p)
\end{array} \right) = \tau \cdot g(p)$$

Topological invariant in momentum space

$$N_3 = \frac{1}{8\pi} \ e_{ijk} \int dp_x dp_y \ \hat{g} \cdot (\nabla_{p_x} \hat{g} \times \nabla_{p_y} \hat{g})$$

intermediate state at $\mu = 0$ must be gapless

$\Delta \tilde{N}_3 \neq 0$ is origin of fermion zero modes at the interface between states with different $\tilde{N}_3$
Quantum phase transitions in thin $^3$He-A film

$p$-space invariant in terms of Green’s function & topological QPT

\[ \tilde{N}_3 = \frac{1}{24\pi^2} \epsilon_{\mu\nu\lambda} \text{tr} \int d^2p \, d\omega \, G^\mu G^{-1} G^\nu G^{-1} G^\lambda G^{-1} \]

GV & Yakovenko

\[ \tilde{N}_3 = 0 \]

\[ \tilde{N}_3 = 2 \]

\[ \tilde{N}_3 = 4 \]

\[ \tilde{N}_3 = 6 \]

Transition between plateaus must occur via gapless state!
topological quantum phase transitions

transitions between ground states (vacua) of the same symmetry, but different topology in momentum space

equation: QPT between gapless & gapped matter

\( T \) (temperature)

no change of symmetry along the path

different asymptotes when \( T \to 0 \)

\( T^n \)

\( e^{-\Delta/T} \)

quantum phase transition at \( q=q_c \)

topological semi-metal

topological insulator

other topological QPT:
Lifshitz transition,
transition between topological and non-topological superfluids,
plateau transitions,
confinement-deconfinement transition, ...

QPT interrupted by thermodynamic transitions

no symmetry change along the path

line of 1-st order transition

line of 2-nd order transition

broken symmetry

line of 1-st order transition

line of 2-nd order transition
interface between two 2+1 topological insulators or gapped superfluids

\[ H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + i p_y \tanh x) \\ c(p_x - i p_y \tanh x) - \frac{p^2}{2m} + \mu \end{pmatrix} \]

* domain wall in 2D chiral superconductors:
Edge states at interface between two 2+1 topological insulators or gapped superfluids

Index theorem:
number of fermion zero modes at interface:

\[ \nu = N_+ - N_- \]
Edge states and currents

2D non-topological insulator or vacuum

2D topological insulator

2D non-topological insulator or vacuum

\[ \tilde{N}_3 = 0 \]

\[ \tilde{N}_3 = 1 \]

\[ \tilde{N}_3 = 0 \]

\[ J_y = J_{\text{left}} + J_{\text{right}} = 0 \]
Edge states & intrinsic QHE: topological invariant determines Hall quantization

2D non-topological insulator or vacuum

2D topological insulator

2D non-topological insulator or vacuum

\[ \tilde{N} = 0 \]

apply voltage \( V \)

apply voltage \( V \)

\[ \tilde{N} = 1 \]

\[ \tilde{N} = 0 \]

\[ E(p_y) - V/2 \]

\[ E(p_y) + V/2 \]

\[ J_y = J_{\text{left}} + J_{\text{right}} = \sigma_{xy} E_y \]

\[ \sigma_{xy} = \frac{e^2}{4\pi} \tilde{N} \]

left moving edge states

right moving edge states

extra number of left moving states

deficit of right moving states

\[ \tilde{N} = 0 \]

current

left moving edge states

extra number of left moving states

deficit of right moving states

\[ \tilde{N} = 1 \]

\[ \tilde{N} = 0 \]

\[ m_{xy} \]

\[ E_y \]
Intrinsic quantum Hall effect & momentum-space invariant

\[ S_{CS} = \frac{e^2}{16\pi} \tilde{N} e^{\mu\nu\lambda} \int d^2x \, dt \, A_\mu \, F_{\nu\lambda} \]

p-space invariant \hspace{2cm} r-space invariant

electric current \hspace{1cm} \[ J_x = \frac{\delta S_{CS}}{\delta A_x} = \frac{e^2}{4\pi} \tilde{N}_3 E_y \]

quantized intrinsic Hall conductivity (without external magnetic field)

\[ \sigma_{xy} = \frac{e^2}{4\pi} \tilde{N}_3 \]

GV & Yakovenko
general Chern-Simons terms & momentum-space invariant

(interplay of \(r\)-space and \(p\)-space topologies)

\[
S_{CS} = \frac{1}{16\pi} \tilde{N}_{3I} \epsilon^{\mu\nu\lambda} \int d^2x \, dt \, A_\mu^I \, F_{\nu\lambda}^J
\]

\(p\)-space invariant protected by symmetry
\(r\)-space invariant

dimensional reduction of chiral anomaly in 3+1

\[
\tilde{N}_{3I} = \frac{1}{24\pi^2} \epsilon_{\mu\nu\lambda} \text{tr} \left[ \int d^2p \, d\omega \, K_I K_J \, \nabla^\mu G^{-1} \nabla^\nu G^{-1} \nabla^\lambda G^{-1} \right]
\]

\(K_I\) - charge interacting with gauge field \(A_\mu^I\)

\(K = e\) for electromagnetic field \(A_\mu^I\)

\(K = \hat{\sigma}_z\) for effective spin-rotation field \(A_\mu^Z\) \((A_0^Z = \gamma H^Z)\)

\[id/dt - \gamma \hat{\sigma} \cdot H = id/dt - \hat{\sigma} \cdot A^i_0\]

applied Pauli magnetic field plays the role of components of effective SU(2) gauge field \(A_\mu^i\)
Intrinsic spin-current quantum Hall effect & momentum-space invariant

\[ J^z_x = \frac{1}{4\pi} (\gamma N_{ss} \frac{dH^z}{dy} + N_{se} E_y) \]

spin-spin QHE

spin-charge QHE

2D singlet superconductor:

\[ \sigma_{xy}^{\text{spin/spin}} = \frac{N_{ss}}{4\pi} \]

\( s \)-wave: \( N_{ss} = 0 \)

\( p_x + ip_y \): \( N_{ss} = 2 \)

\( d_{xx-yy} + id_{xy} \): \( N_{ss} = 4 \)

film of planar phase of superfluid \(^3\)He

\[ \sigma_{xy}^{\text{spin/charge}} = \frac{N_{se}}{4\pi} \]

GV & Yakovenko
spin quantum Hall effect: planar phase film of $^3$He & 2D topological insulator

$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + i p_y \sigma_z) \\ c(p_x - i p_y \sigma_z) & -\frac{p^2}{2m} + \mu \end{pmatrix}$$

$$\tilde{N} = \frac{1}{24\pi^2} \epsilon_{\mu \nu \lambda} \text{tr} \left[ \int d^2 p \ d\omega \ G^{\mu} G^{-1} G^{\nu} G^{-1} G^{\lambda} G^{-1} \right] = 0$$

$$\tilde{N}_{se} = \frac{1}{24\pi^2} \epsilon_{\mu \nu \lambda} \text{tr} \left[ \int d^2 p \ d\omega \ \sigma_z G^{\mu} G^{-1} G^{\nu} G^{-1} G^{\lambda} G^{-1} \right]$$

$$\tilde{N}_3 = \tilde{N}_3^+ - \tilde{N}_3^- = 2$$

spin quantum Hall effect

spin current $$J^z_x = \frac{1}{4\pi} N_{se} E_y$$ spin-charge QHE

$$\sigma_{xy} = \frac{N_{se}}{4\pi} N_{se} = 2$$

GV & Yakovenko
Intrinsic spin-current quantum Hall effect & edge state

spin current \( J^z_x = \frac{1}{4\pi} (\gamma N_{ss} dH^z/dy + N_{se} E_y) \)

spin-charge QHE

\( N_{se} = 0 \) spin current
\( N_{se} = 2 \) spin current
\( N_{se} = 0 \) spin current

left moving spin up
right moving spin down

\( E(p_y) - V/2 \)
\( p_y \)

left moving spin up
right moving spin down

\( E(p_y) + V/2 \)
\( p_y \)

right moving spin up
left moving spin down

spin/charge \( \sigma_{xy} = \frac{N_{se}}{4\pi} \)

electric current is zero
spin current is nonzero
3D topological superfluids / insulators / semiconductors / vacua

- Gapless topologically nontrivial vacua
  - 3He-A, Standard Model above electroweak transition, semimetals, 4D graphene (cryocrystalline vacuum)

- Fully gapped topologically nontrivial vacua
  - 3He-B, Standard Model below electroweak transition, topological insulators, triplet & singlet chiral superconductor, ...

\[ \text{Bi}_2\text{Te}_3 \]
Present vacuum as semiconductor or insulator

3 conduction bands of d-quarks
electric charge $q = -1/3$

3 conduction bands of u-quarks, $q = +2/3$

conduction electron band, $q = -1$

neutrino band, $q = 0$

valence electron band, $q = -1$

3 valence u-quark bands
$q = +2/3$

3 valence d-quark bands
$q = -1/3$

Quantum vacuum: Dirac sea

Dielectric and magnetic properties of vacuum (running coupling constants)

Electric charge of quantum vacuum
$$Q = \sum_a q_a = N \left[-1 + 3(-1/3) + 3(2/3)\right] = 0$$
fully gapped 3+1 topological matter
superfluid $^3$He-B, topological insulator $\text{Bi}_2\text{Te}_3$, present vacuum of Standard Model

* Standard Model vacuum as topological insulator

**Topological invariant protected by symmetry**

\[
N_K = \frac{1}{24\pi^2} e_{\mu\nu\lambda} \text{tr} \int d\mathbf{v} \, K \, G^{-\mu} G^{-1} G^{-\nu} G^{-1} G^{-\lambda} G^{-1}
\]

over 3D momentum space

$G$ is Green’s function at $\omega=0$, $K$ is symmetry operator $G K = +/- K G$

Standard Model vacuum: $K = \gamma_5$ $G \gamma_5 = -\gamma_5 G$

\[
N_K = 8n_g
\]

8 massive Dirac particles in one generation
topological superfluid $^3$He-B

$$H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu & c_B \sigma \cdot \mathbf{p} \\ c_B \sigma \cdot \mathbf{p} & -\frac{p^2}{2m^*} + \mu \end{pmatrix} = \left( \frac{p^2}{2m^*} - \mu \right) \tau_3 + c_B \sigma \cdot \mathbf{p} \tau_1$$

$$\mu$$

$$1/m^*$$

non-topological superfluid

$$N_K = 0$$

$$N_K = -1$$

Dirac

$$N_K = -2$$

topological superfluid

$$N_K = +2$$

$$N_K = +1$$

$$N_K = 0$$

$$N_K = 0$$

topological $^3$He-B

topological superfluid

$$H \tau_2 = -\tau_2 H$$

$$K = \tau_2$$

Dirac vacuum

$$1/m^* = 0$$

$$\mu$$

GV JETP Lett. 90, 587 (2009)
Boundary of 3D gapped topological superfluid

\[ H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu + U(z) & c_B \mathbf{\sigma} \cdot \mathbf{p} \\ c_B \mathbf{\sigma} \cdot \mathbf{p} & \frac{p^2}{2m^*} + \mu - U(z) \end{pmatrix} \]

spectrum of Majorana fermion zero modes

\[ H_{zm} = c_B \mathbf{z} \cdot \mathbf{\sigma} \mathbf{x} \mathbf{p} = c_B \left( \sigma_x p_y - \sigma_y p_x \right) \]
fermion zero modes on Dirac wall

\[ H = \begin{pmatrix} -M(z) & c\sigma \cdot p \\ c\sigma \cdot p & +M(z) \end{pmatrix} \]

Volkov-Pankratov, 2D massless fermions in inverted contacts, JETP Lett. 42, 178 (1985)

in Bi\textsubscript{2}Te\textsubscript{3} Dirac point is below FS: nodal line on surface of topological insulator
Majorana fermions: edge states on the boundary of 3D gapped topological matter

* boundary of topological superfluid $^3$He-B

\[
\begin{align*}
N_K &= 0 & N_K &= +2 \\
\text{vacuum} & & \text{$^3$He-B}
\end{align*}
\]

\[
H = \begin{pmatrix}
\frac{p^2}{2m^*} - \mu + U(z) & c\sigma \cdot p \\
c\sigma \cdot p & \frac{p^2}{2m^*} + \mu - U(z)
\end{pmatrix}
\]

* Dirac domain wall

spectrum of fermion zero modes
\[
H_{zm} = c (\sigma_x p_y - \sigma_y p_x)
\]

Volkov-Pankratov, 2D massless fermions in inverted contacts
JETP Lett. 42, 178 (1985)

helical fermions
**Majorana fermions** on interface in topological superfluid $^3$He-B

$$H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu & \sigma_x c_x p_x + \sigma_y c_y p_y + \sigma_z c_z p_z \\ \sigma_x c_x p_x + \sigma_y c_y p_y + \sigma_z c_z p_z & \frac{p^2}{2m^*} + \mu \end{pmatrix}$$

one of 3 "speeds of light" changes sign across wall

**spectrogram of fermion zero modes**

$$H_{zm} = c (\sigma_x p_y - \sigma_y p_x)$$
Zero energy states in the core of vortices in topological superfluids

vortices in fully gapped 3+1 system

fermion zero modes in vortex core
Bound states of fermions on cosmic strings and vortices

Spectrum of quarks in core of electroweak cosmic string

quantum numbers: Q - angular momentum & \( p_z \) - linear momentum

\[ E(p_z) = -cp_z \] for d quarks

\[ E(p_z) = cp_z \] for u quark

asymmetric branches cross zero energy

Index theorem: Number of asymmetric branches = N
N is vortex winding number

Jackiw & Rossi
Bound states of fermions on vortex in s-wave superconductor

Carolì, de Gennes & J. Matricon, Phys. Lett. 9 (1964) 307

$$E(Q, p_z = 0)$$

Number of asymmetric Q-branches = 2N
N is vortex winding number

Index theorem for approximate fermion zero modes:

$$E(Q, p_z) = -Q \omega_0 (p_z)$$

$$\omega_0 = \Delta^2 / E_F \ll \Delta$$

Angular momentum Q is half-odd integer in s-wave superconductor

Index theorem for true fermion zero modes?

no true fermion zero modes:
no asymmetric branch as function of $$p_z$$

is the existence of fermion zero modes related to topology in bulk?

GV JETP Lett. 57, 244 (1993)
fermions zero modes on symmetric vortex in 3He-B

topological $^3$He-B at $\mu > 0$ : $N_K = 2$

$E(Q, p_z = 0)$

$E(p_z, Q)$

$E_Q = -Q\omega_0$

$\omega_0 = \Delta^2/E_F \ll \Delta$

$Q$ is integer for p-wave superfluid $^3$He-B

Misirpashaev & GV
Fermion zero modes in symmetric vortices in superfluid 3He,
fermions zero modes on symmetric vortex in 3He-B
topological $^3$He-B at $\mu > 0$: $N_K = 2$

Misirpashaev & GV
Fermion zero modes in symmetric vortices in superfluid 3He,
topological quantum phase transition in bulk & in vortex core

\[ E(p_z, Q) \]

- Non-topological superfluid \( N_K = 0 \)
- Topological superfluid \( ^3\text{He-B} \) \( N_K = +2 \)

\[ \frac{1}{m^*} \]

\[ \mu > 0 \]
\[ \mu < 0 \]

\[ \mu > 0 \]
\[ \mu < 0 \]
superfluid $^3$He-B as non-relativistic limit of relativistic triplet superconductor

$$H = \begin{pmatrix} c\alpha \cdot \mathbf{p} + \beta M - \mu_R & \gamma_5 \Delta \\ \gamma_5 \Delta & -c\alpha \cdot \mathbf{p} - \beta M + \mu_R \end{pmatrix}$$

relativistic triplet superconductor

\[ c p \ll M \]
\[ \mu \ll M \]

$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c_B \sigma \cdot \mathbf{p} \\ c_B \sigma \cdot \mathbf{p} & \frac{p^2}{2m} + \mu \end{pmatrix}$$

superfluid $^3$He-B

\[ c_B = c \Delta / M \]
\[ m = M / c^2 \]

\[ (\mu + M)^2 = \mu_R^2 + \Delta^2 \]
phase diagram of topological states of relativistic triplet superconductor

\[ H = \begin{pmatrix} \alpha \cdot p + \beta M - \mu_R & \gamma_5 \Delta \\ \gamma_5 \Delta & -c \alpha \cdot p - \beta M + \mu_R \end{pmatrix} \]

\[ \begin{pmatrix} \frac{p^2}{2m} - \mu & c_B \sigma \cdot p \\ c_B \sigma \cdot p & -\frac{p^2}{2m} + \mu \end{pmatrix} \]
energy spectrum in relativistic triplet superconductor

\[ \mu_R^2 = M^2 - \Delta^2 \]

gapless spectrum at topological quantum phase transition

\[ |\mu_R| < \mu_R^* \]

soft quantum phase transition: Higgs transition in p-space

\[ |\mu_R| > \mu_R^* \]

\[ H = \begin{pmatrix} c \alpha \cdot p + \beta M - \mu_R & \gamma_5 \Delta \\ \gamma_5 \Delta & - c \alpha \cdot p - \beta M + \mu_R \end{pmatrix} \]
spectrum of non-relativistic $^3$He-B

\[
H = \begin{pmatrix}
\frac{p^2}{2m^*} - \mu & c_B \sigma \cdot p \\
-c_B \sigma \cdot p & \frac{-p^2}{2m^*} + \mu
\end{pmatrix}
\]

- $\mu < 0$
- $\mu = 0$
- $\mu < \Delta^2/M$
- $\mu > \Delta^2/M$

gapless spectrum at topological quantum phase transition

soft quantum phase transition: Higgs transition in p-space

$N_K = 0$
$N_K = +2$
$N_K = +1$
$N_K = 0$
$N_K = +2$
$N_K = 0$
$N_K = -1$
$N_K = -2$
$N_K = -1$
$N_K = 0$
$N_K = +1$
$N_K = 0$
$N_K = -2$
fermion zero modes in relativistic triplet superconductor

\[
H = \begin{pmatrix}
  c\alpha \cdot p + \beta M - \mu_R & \gamma_5 \Delta \\
  \gamma_5 \Delta & - c\alpha \cdot p - \beta M + \mu_R
\end{pmatrix}
\]

vortices in topological superconductors have fermion zero modes

generalized index theorem?
index theorem for fermion zero modes on vortices

(interplay of \( r \)-space and \( p \)-space topologies)

\[
N_5 = \frac{1}{4\pi^3 i} \text{tr} \left[ \int d^3p \, d\omega \, d\phi \ G \nabla_\omega G^{-1} G \nabla_\phi G^{-1} G \nabla_{px} G^{-1} G \nabla_{py} G^{-1} G \nabla_{pz} G^{-1} \right]
\]

for vortices in Dirac vacuum

\[ N_5 = N \quad \text{winding number} \]

\( N_5 \) invariant was introduced by Golterman, Jansen & Kaplan for lattice fermions


see also M.A. Zubkov & GV
Momentum space topological invariants for the 4D relativistic vacua with mass gap

Relaxation rate: $1/\tau = 1/\tau_0(\Omega) + C(\Omega) \exp(-\Delta/T)$
BROKEN SYMMETRY OF VORTEX CORES IN $^3$He-B

\begin{align*}
P, \text{ bar} & \quad \text{Solid} \\
0 & \quad 30 \\
1 & \quad 20 \\
2 & \quad 10 \\
\text{T, mK} & \\
0 & \quad 1 \quad 2 \\
\end{align*}

\begin{align*}
\| A \| & \sim \text{few } \xi \\
\text{Broken symmetry core} & \quad \text{Axisymmetric core} \\
\end{align*}

Ikkala, Hakonen, Bunkov, Krusius et al 1982-
Salomaa, Volovik, Thuneberg et al
DAMPING OF SPIN PRECESSION VIA VORTEX CORES

Torque from precessing magnetic moment puts vortex core in twisting motion (oscillations / precession) 

Transitions between the core-bound fermion states are triggered and the core gets overheated 

Dissipation

(Kopnin and Volovik, 1998)
Conclusion

**Momentum-space topology** determines:

- universality classes of quantum vacua
- effective field theories in these quantum vacua
- topological quantum phase transitions (Lifshitz, plateau, etc.)
- quantization of Hall and spin-Hall conductivity
- topological Chern-Simons & Wess-Zumino terms
- quantum statistics of topological objects
- edge states (bulk-surface correspondence)
- fermion zero modes on quantum vortices (bulk-vortex correspondence)
- chiral anomaly, chiral magnetic effect, spectral flow force in vortex dynamics ...
- exotic fermions: Dirac fermions with quadratic, cubic, quartic ... spectrum, flat band, Fermi arc, Majorana fermions, etc.
- effective gravity, where tetrad $e_a{}^\mu$ is more fundamental than metric $g^{\mu\nu}$
- new type of gravity (Horava gravity with anisotropic scaling)