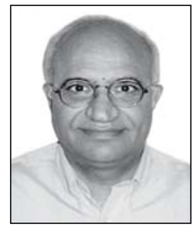
The Logarithmic Mean

Rajendra Bhatia

The inequality between the arithmetic mean (AM) and geometric mean (GM) of two positive numbers is well known. This article introduces the logarithmic mean, shows how it leads to refinements of the AM–GM inequality. Some applications and properties of this mean are shown. Some other means and related inequalities are discussed.



Rajendra Bhatia is a Professor at the Indian Statistical Institute, New Delhi. Some of his recent work is on means of matrices. But he is not a mean theorist.

One of the best known and most used inequalities in mathematics is the inequality between the harmonic, geometric, and arithmetic means. If a and b are positive numbers, these means are defined, respectively, as

$$\begin{aligned} H(a,b) &= \left(\frac{a^{-1}+b^{-1}}{2}\right)^{-1}, \\ G(a,b) &= \sqrt{ab}, \quad A(a,b) = \frac{a+b}{2}, \end{aligned} \tag{1}$$

and the inequality says that

$$H(a,b) \le G(a,b) \le A(a,b). \tag{2}$$

Means other than the three 'classical' ones defined in (1) are used in different problems. For example, the *root mean square*

$$B_2(a,b) = \left(\frac{a^2+b^2}{2}\right)^{1/2}, \tag{3}$$

is often used in various contexts. Following the mathematician's penchant for generalisation, the four means mentioned above can be subsumed in the family

$$B_p(a,b) = \left(\frac{a^p+b^p}{2}\right)^{1/p}, \quad -\infty < p < \infty, \tag{4}$$

Keywords
Inequalities, logarithmic mean, power mean, Holder mean, arithmetic-geometric mean (AGM), arithmetic-mean geometric-mean inequality (AM-GM inequality), Heinz mean.

A substantial part of the mathematics classic *Inequalities* by Hardy, Littlewood and Pölya is devoted to the study of these means and their applications.

variously known as *binomial means, power means,* or *Hölder means*. When $p = -1, 1$, and 2, respectively, $B_p(a, b)$ is the harmonic mean, the arithmetic mean, and the root mean square. If we understand $B_0(a, b)$ to mean

$$B_0(a, b) = \lim_{p\to 0} B_p(a, b),$$

then

$$B_0(a, b) = G(a, b). \tag{5}$$

In a similar vein we can see that

$$\begin{aligned} B_\infty(a, b) &:= \lim_{p\to\infty} \left(\frac{a^p + b^p}{2}\right)^{1/p} = \max(a, b), \\ B_{-\infty}(a, b) &:= \lim_{p\to-\infty} \left(\frac{a^p + b^p}{2}\right)^{1/p} = \min(a, b). \end{aligned}$$

A little calculation shows that

$$B_p(a, b) \le B_q(a, b) \qquad \text{if} \quad p \le q. \tag{6}$$

This is a strong generalization of the inequality (2). We may say that for $-1 \le p \le 1$, the family B_p *interpolates* between the three means in (1) as does the inequality (6) with respect to (2).

A substantial part of the mathematics classic *Inequalities* by G Hardy, J E Littlewood and G Pölya is devoted to the study of these means and their applications. The book has had quite a few successors, and yet new properties of these means continue to be discovered.

The purpose of this article is to introduce the reader to the *logarithmic mean,* some of its applications, and some very pretty mathematics around it.

The logarithmic mean of two positive numbers a and b is the number $L(a, b)$ defined as

$$L(a, b) = \frac{a - b}{\log a - \log b} \quad \text{for} \quad a \neq b, \tag{7}$$

with the understanding that

$$L(a,a) = \lim_{b\to a} L(a,b) = a.$$

There are other interesting representations for this object, and the reader should check the validity of these formulas:

$$L(a,b) = \int_0^1 a^t b^{1-t} \mathrm{d}t, \tag{8}$$

$$\frac{1}{L(a,b)} = \int_0^1 \frac{\mathrm{d}t}{ta+(1-t)b}, \tag{9}$$

$$\frac{1}{L(a,b)} = \int_0^\infty \frac{\mathrm{d}t}{(t+a)(t+b)}. \tag{10}$$

The logarithmic mean always falls between the geometric and the arithmetic means; i.e.,

$$G(a,b) \leq L(a,b) \leq A(a,b). \tag{11}$$

We indicate three different proofs of this and invite the reader to find more.

The logarithmic mean always falls betwen the geometric mean and the arithmetic mean.

When $a = b$, all the three means in (11) are equal to a. Suppose $a > b$, and put $w = a/b$. The first inequality in (11) is equivalent to saying

$$\sqrt{w} \leq \frac{w-1}{\log w} \quad \text{for} \quad w > 1.$$

Replacing w by u^2, this is the same as saying

$$2 \log u \leq \frac{u^2-1}{u} \quad \text{for} \quad u > 1. \tag{12}$$

The two functions $f(u) = 2 \log u$, and $g(u) = (u^2-1)/u$ are equal to 0 at $u = 1$, and a small calculation shows that $f'(u) < g'(u)$ for $u > 1$. This proves the desired inequality (12), and with it the first inequality in (11). In the same way, the second of the inequalities (11) can be reduced to

$$\frac{u-1}{u+1} \leq \frac{\log u}{2} \quad \text{for} \quad u \geq 1.$$

In this form we recognize it as one of the fundamental inequalities of analysis: $t \leq \sinh t$ for all $t \geq 0$.

and proved by calculating derivatives.

A second proof goes as follows. Two applications of the arithmetic-geometric mean inequality show that

$$t^2 + 2t\sqrt{ab} + ab \leq t^2 + t(a+b) + ab$$

$$\leq t^2 + t(a+b) + \left(\frac{a+b}{2}\right)^2$$

for all $t \geq 0$. Using this one finds that

$$\int_0^\infty \frac{\mathrm{d}t}{(t+\frac{a+b}{2})^2} \leq \int_0^\infty \frac{\mathrm{d}t}{(t+a)(t+b)} \leq \int_0^\infty \frac{\mathrm{d}t}{(t+\sqrt{ab})^2}.$$

Evaluation of the integrals shows that this is the same as the assertion in (11).

Since a and b are positive, we can find real numbers x and y such that $a = e^x$ and $b = e^y$. Then the first inequality in (11) is equivalent to the statement

$$e^{(x+y)/2} \leq \frac{e^x - e^y}{x-y},$$

or

$$1 \leq \frac{e^{(x-y)/2} - e^{(y-x)/2}}{x-y}.$$

This can be expressed also as

$$1 \leq \frac{\sinh(x-y)/2}{(x-y)/2}.$$

In this form we recognise it as one of the fundamental inequalities of analysis: $t \leq \sinh t$ for all $t \geq 0$. Very similar calculations show that the second inequality in (11) can be reduced to the familiar fact $\tanh t \leq t$ for all $t \geq 0$.

Each of our three proofs shows that if $a \neq b$, then $G(a,b) < L(a,b) < A(a,b)$. One of the reasons for the interest in (11) is that it provides a refinement of the fundamental inequality between the geometric and the arithmetic means.

The logarithmic mean plays an important role in the study of conduction of heat in liquids flowing in pipes. Let us explain this briefly. The flow of heat by steady unidirectional conduction is governed by *Newton's law of cooling*: if q is the rate of heat flow along the x-axis across an area A normal to this axis, then

$$q = k\,A\,\frac{\mathrm{d}T}{\mathrm{d}x}, \tag{13}$$

where $\mathrm{d}T/\mathrm{d}x$ is the temperature gradient along the x direction and k is a constant called the thermal conductivity of the material. (See, for example, R Bhatia, *Fourier Series*, Mathematical Association of America, p.2, 2004.) The cross-sectional area A may be constant, as for example in a cube. More often (as in the case of a fluid travelling in a pipe) the area A is a variable. In engineering calculations, it is then more convenient to replace (13) by

$$q = k\,A_m\,\frac{\triangle T}{\triangle x}, \tag{14}$$

where $\triangle T$ is the difference of temperatures at two points at distance $\triangle x$ along the x-axis, and A_m is the *mean cross section* of the body between these two points. For example, if the body has a uniformly tapering rectangular cross section, then A_m is the arithmetic mean of the two boundary areas A_1 and A_2.

Consider heat flow in a long hollow cylinder where end effects are negligible. Then the heat flow can be taken to be essentially radial. (See, for example, J Crank: *The Mathematics of Diffusion,* Clarendon Press, 1975). The cross-sectional area in this case is proportional to the distance from the centre of the pipe. If L is the length of the pipe, the area of the cylindrical surface at distance

The logarithmic mean plays an important role in the study of conduction of heat in liquids flowing in pipes.

x from the axis is $2\pi xL$. So, the total heat flow q across the section of the pipe bounded by two coaxial cylinders at distance x_1 and x_2 from the axis, using (13), is seen to satisfy the equation

$$q\int_{x_1}^{x_2}\frac{\mathrm{d}x}{2\pi xL}=k\triangle T, \tag{15}$$

or,

$$q=\frac{k\,2\pi L\ \triangle T}{\log\,x_2-\log\,x_1}.$$

If we wish to write this in the form (14) with $x_2-x_1=\triangle x$, then we must have

$$A_m=2\pi L\,\frac{x_2-x_1}{\log\,x_2-\log\,x_1}=\frac{2\pi Lx_2-2\pi Lx_1}{\log\,2\pi Lx_2-\log\,2\pi Lx_1}.$$

In other words,

$$A_m=\frac{A_2-A_1}{\log\,A_2-\log\,A_1},$$

the logarithmic mean of the two areas bounding the cylindrical section under consideration. In the engineering literature this is called the *logarithmic mean area.*

If instead of two coaxial cylinders we consider two concentric spheres, then the cross-sectional area is proportional to the square of the distance from the centre. In this case we have, instead of (15),

$$q\int_{x_1}^{x_2}\frac{\mathrm{d}x}{4\,\pi\,x^2}=k\triangle T.$$

A small calculation shows that in this case

$$A_m=\sqrt{A_1A_2},$$

the geometric mean of the two areas bounding the annular section under consideration.

Thus the geometric and the logarithmic means are useful in calculations related to heat flow through spherical and cylindrical bodies, respectively. The latter relates to the more common phenomenon of flow through pipes.

Let us return to inequalities related to the logarithmic mean. Let t be any nonzero real number. In the equality (11) replace a and b by a^t and b^t, respectively. This gives

$$(ab)^{t/2} \leq \frac{a^t - b^t}{t(\log\, a - \log\, b)} \leq \frac{a^t + b^t}{2},$$

from which we get

$$t(ab)^{t/2}\,\frac{a-b}{a^t - b^t} \leq \frac{a-b}{\log\, a - \log\, b} \leq t\frac{a^t + b^t}{2}\,\frac{a-b}{a^t - b^t}.$$

The middle term in this equality is the logarithmic mean. Let G_t and A_t be defined as

$$\begin{aligned} G_t(a,b) &= t(ab)^{t/2}\frac{a-b}{a^t - b^t}, \\ A_t(a,b) &= t\frac{a^t + b^t}{2}\frac{a-b}{a^t - b^t}. \end{aligned}$$

We have assumed in these definitions that $t \neq 0$. If we define G_0 and A_0 as the limits

$$\begin{aligned} G_0(a,b) &= \lim_{t \to 0} G_t(a,b), \\ A_0(a,b) &= \lim_{t \to 0} A_t(a,b), \end{aligned}$$

then

$$G_0(a,b) = A_0(a,b) = L(a,b).$$

The reader can verify that

$$\begin{aligned} G_1(a,b) &= \sqrt{ab}, \quad A_1(a,b) = \frac{a+b}{2}, \\ G_{-t}(a,b) &= G_t(a,b), \quad A_{-t}(a,b) = A_t(a,b). \end{aligned}$$

We have an infinite family of inequalities that includes the AM–GM inequality, and other interesting inequalities.

For fixed a and b, $G_t(a,b)$ is a decreasing function of $|t|$, while $A_t(a,b)$ is an increasing function of $|t|$. (One proof of this can be obtained by making the substitution $a = e^x$, $b = e^y$.) The last inequality obtained above can be expressed as

$$G_t(a,b) \leq L(a,b) \leq A_t(a,b), \tag{16}$$

for all t. Thus we have an infinite family of inequalities that includes the arithmetic–geometric mean inequality, and other interesting inequalities. For example, choosing $t = 1$ and $1/2$, we see from the information obtained above that

$$\sqrt{ab} \leq \frac{a^{3/4}b^{1/4} + a^{1/4}b^{3/4}}{2} \leq L(a,b)$$

$$\leq \left(\frac{a^{1/2} + b^{1/2}}{2}\right)^2 \leq \frac{a+b}{2}. \tag{17}$$

This is a refinement of the fundamental inequality (11). The second term on the right is the binomial mean $B_{1/2}(a,b)$. The second term on the left is one of another family of means called *Heinz means* defined as

$$H_\nu(a,b) = \frac{a^\nu b^{1-\nu} + a^{1-\nu}b^\nu}{2}, \quad 0 \leq \nu \leq 1. \tag{18}$$

Clearly

$$\begin{aligned} H_0(a,b) &= H_1(a,b) = \frac{a+b}{2}, \\ H_{1/2}(a,b) &= \sqrt{ab}, \\ H_{1-\nu}(a,b) &= H_\nu(a,b). \end{aligned}$$

Thus the family H_ν is yet another family that interpolates between the arithmetic and the geometric means. The reader can check that

$$H_{1/2}(a,b) \leq H_\nu(a,b) \leq H_0(a,b), \tag{19}$$

for $0 \leq \nu \leq 1$. This is another refinement of the arithmetic–geometric mean inequality.

If we choose $t = 2^{-n}$, for any natural number n, then we get from the first inequality in (16)

$$2^{-n}(ab)^{2^{-(n+1)}} \frac{a-b}{a^{2^{-n}} - b^{2^{-n}}} \;\leq\; L(a,b).$$

Using the identity

$$a - b = \left(a^{2^{-n}} - b^{2^{-n}}\right)\left(a^{2^{-n}} + b^{2^{-n}}\right)\left(a^{2^{-n+1}} + b^{2^{-n+1}}\right)\cdots \left(a^{2^{-1}} + b^{2^{-1}}\right),$$

we get from the inequality above

$$(ab)^{2^{-(n+1)}} \prod_{m=1}^{n} \frac{a^{2^{-m}} + b^{2^{-m}}}{2} \;\leq\; L(a,b). \tag{20}$$

Similarly, from the second inequality in (16) we get

$$L(a,b) \;\leq\; \frac{a^{2^{-n}} + b^{2^{-n}}}{2} \prod_{m=1}^{n} \frac{a^{2^{-m}} + b^{2^{-m}}}{2}. \tag{21}$$

If we let $n \to \infty$ in the two formulas above, we obtain a beautiful product formula:

$$L(a,b) = \prod_{m=1}^{\infty} \frac{a^{2^{-m}} + b^{2^{-m}}}{2}. \tag{22}$$

This adds to our list of formulas (7)–(10) for the logarithmic mean.

Choosing $b = 1$ in (22) we get after a little manipulation the representation for the logarithm function

$$\log\, x = (x-1) \prod_{m=1}^{\infty} \frac{2}{1 + x^{2^{-m}}}, \tag{23}$$

for all $x > 0$.

There are more analytical delights in store; the logarithmic mean even has a connection with the fabled Gauss arithmetic-geometric mean.

We can turn this argument around. For all $x > 0$ we have

$$\log x = \lim_{n\to\infty} n\left(x^{1/n} - 1\right). \tag{24}$$

Replacing n by 2^n, a small calculation leads to (23) from (24). From this we can obtain (22) by another little calculation.

There are more analytical delights in store; the logarithmic mean even has a connection with the fabled Gauss arithmetic–geometric mean that arises in a totally different context. Given positive numbers a and b, inductively define two sequences as

$$\begin{aligned} a_0 &= a, & b_0 &= b \\ a_{n+1} &= \frac{a_n + b_n}{2}, & b_{n+1} &= \sqrt{a_n b_n}. \end{aligned}$$

Then $\{a_n\}$ is a decreasing, and $\{b_n\}$ an increasing, sequence. All a_n and b_n are between a and b. So both sequences converge. With a little work one can see that $a_{n+1} - b_{n+1} \le \frac{1}{2}(a_n - b_n)$, and hence the sequences $\{a_n\}$ and $\{b_n\}$ converge to a common limit. The limit $AG(a,b)$ is called the Gauss arithmetic–geometric mean. Gauss showed that

$$\begin{aligned} \frac{1}{AG(a,b)} &= \frac{2}{\pi}\int_0^\infty \frac{dx}{\sqrt{(a^2+x^2)(b^2+x^2)}} \\ &= \frac{2}{\pi}\int_0^{\pi/2} \frac{d\varphi}{\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}}. \end{aligned} \tag{25}$$

These integrals called 'elliptic integrals' are difficult ones to evaluate, and the formula above relates them to the mean value $AG(a,b)$. Clearly

$$G(a,b) \le AG(a,b) \le A(a,b). \tag{26}$$

Somewhat unexpectedly, the mean $L(a,b)$ can also be realised as the outcome of an iteration closely related to the Gauss iteration. Let A_t and G_t be the two families defined earlier. A small calculation, that we leave to the reader, shows that

$$\frac{A_t + G_t}{2} = A_{t/2}, \qquad \sqrt{A_{t/2} G_t} = G_{t/2}. \tag{27}$$

For $n = 1, 2, \ldots$, let $t = 2^{1-n}$, and define two sequences a'_n and b'_n as $a'_n = A_t$, $b'_n = G_t$; i.e.,

$$\begin{aligned} a'_1 &= A_1 = \frac{a+b}{2}, \quad b'_1 = G_1 = \sqrt{ab}, \\ a'_2 &= A_{1/2} = \frac{a'_1 + b'_1}{2}, \; b'_2 = G_{1/2} = \sqrt{A_{1/2} G_1} = \sqrt{a'_2 b'_1}, \\ &\vdots \\ a'_{n+1} &= \frac{a'_n + b'_n}{2}, \quad b'_{n+1} = \sqrt{a'_{n+1} b'_n}. \end{aligned}$$

Somewhat unexpectedly, the logarithmic mean can also be realized as the outcome of an iteration closely related to the Gauss iteration.

We leave it to the reader to show that the two sequences $\{a'_n\}$ and $\{b'_n\}$ converge to a common limit, and that limit is equal to $L(a, b)$. This gives one more characterisation of the logarithmic mean. These considerations also bring home another interesting inequality

$$L(a, b) \leq AG(a, b). \tag{28}$$

Finally, we indicate yet another use that has recently been found for the inequality (11) in differential geometry. Let $\|T\|_2$ be the Euclidean norm on the space of $n \times n$ complex matrices; i.e.,

$$\|T\|_2^2 = \operatorname{tr} T^* T = \sum_{i,j=1}^{n} |t_{ij}|^2 .$$

A matrix version of the inequality (11) says that for all positive definite matrices A and B and for all matrices X, we have

$$\left|\left|A^{1/2} X B^{1/2}\right|\right|_2 \leq \left\|\int_0^1 A^t X B^{1-t} \mathrm{d}t\right\|_2 \leq \left\|\frac{AX + XB}{2}\right\|_2 . \tag{29}$$

The space $\mathbb{H}_n$ of all $n \times n$ Hermitian matrices is a real vector space, and the exponential function maps this onto the space $\mathbb{P}_n$ consisting of all positive definite matrices. The latter is a Riemannian manifold. Let $\delta_2(A, B)$ be the natural Riemannian metric on $\mathbb{P}_n$. A very fundamental inequality called the *exponential metric increasing property* says that for all Hermitian matrices H and K

$$\delta_2\left(e^H, e^K\right) \geq ||H - K||_2. \qquad (30)$$

A short and simple proof of this can be based on the first of the inequalities in (29). The inequality (30) captures the important fact that the manifold $\mathbb{P}_n$ has nonpositive curvature. For more details see the Suggested Reading.

Address for Correspondence
Rajendra Bhatia
Theoretical Statistics and Mathematics Unit
Indian Statistical Institute
Delhi Centre
7, SJS Sansanwal Marg
New Delhi 110 016, India.
Email: rbh@isid.ac.in

Suggested Reading

[1] G Hardy, J E Littlewood and G Pölya, *Inequalities*, Cambridge University Press, Second edition, 1952. (This is a well-known classic. Chapters II and III are devoted to 'mean values'.)

[2] P S Bullen, D S Mitrinovic and P M Vasic, *Means and their Inequalities*, D Reidel, 1998. (A specialised monograph devoted exclusively to various means.)

[3] W H McAdams, *Heat Transmission*, Third edition, McGraw Hill, 1954. (An engineering text in which the logarithmic mean is introduced in the context of fluid flow.)

[4] B C Carlson, The logarithmic mean, *American Mathematical Monthly*, Vol.79, pp.615–618, 1972. (A very interesting article from which we have taken some of the material presented here.)

[5] R Bhatia, *Positive Definite Matrices*, Princeton Series in Applied Mathematics, 2007, and also TRIM 44, Hindustan Book Agency, 2007. (Matrix versions of means, and inequalities for them, can be found here. The role of the logarithmic mean in this context is especially emphasized in Chapters 4-6.)

[6] R Bhatia and J Holbrook, Noncommutative Geometric Means, *Mathematical Intelligencer*, Vol.28, pp.32–39, 2006. (A quick introduction to some problems related to matrix means, and to the differential geometric context in which they can be placed.)

[7] Tung-Po Lin, The Power Mean and the Logarithmic Mean, *American Mathematical Monthly*, Vol.81, pp.879–883, 1974.

[8] S Chakraborty, A Short Note on the Versatile Power Mean, *Resonance*, Vol. 12, No. 9, pp.76–79, September 2007.