Varieties of commuting matrices

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Introduction

**The problem**

\(\mathbb{F}\) algebraically closed field.

**Notation**

\[ C(d, n) = \{(A_1, A_2, \ldots, A_d) \in M_n(\mathbb{F})^d; A_iA_j = A_jA_i \text{ for each } i \text{ and } j\} \]

\[ N(d, n) = \{(A_1, A_2, \ldots, A_d) \in C(d, n); A_1, \ldots, A_d \text{ are nilpotent}\} \]

Both sets are affine varieties in \(\mathbb{F}^{dn^2}\).

**The problem**

For which positive integers \(d\) and \(n\) are the varieties \(C(d, n)\) and \(N(d, n)\) irreducible?
Applications

- biology
- physics
- algebraic geometry: Hilbert schemes
- commutative algebra: theory of modules over artinian commutative rings
- linear algebra:
  - stability of invariant subspaces of commuting matrices
  - dimension of commutative matrix subalgebras
Generic and $r$-regular matrices

Definitions

A matrix $A \in M_n(\mathbb{F})$ is:

- **$r$-regular**, if each its eigenspace is at most $r$-dimensional;
- **generic**, if it has $n$ distinct eigenvalues.

Notations

- $R^r(d, n) = \{(A_1, A_2, \ldots, A_d) \in C(d, n); \text{ some linear combination of } A_1, \ldots, A_d \text{ is } r\text{-regular}\}$
- $R^r_N(d, n) = R^r(d, n) \cap N(d, n)$
- $G(d, n) = \{(A_1, A_2, \ldots, A_d) \in C(d, n); \text{ some linear combination of } A_1, \ldots, A_d \text{ is generic}\}$. 
**Proposition**

- The variety $C(d, n)$ is irreducible if and only if $C(d, n) = R^1(d, n) = G(d, n)$.
- The variety $N(d, n)$ is irreducible if and only if $N(d, n) = R^1_N(d, n)$.

**Proof**

- The set $G(d, n)$ and all sets $R^r(d, n)$ are open in $C(d, n)$ in the Zariski topology, and $R^r_N(d, n)$ are open in $N(d, n)$.
- A matrix that commutes with 1-regular matrix is a polynomial in that matrix.
- $R^1(d, n)$ and $G(d, n)$ are irreducible and of dimension $n^2 + (d - 1)n$.
- $R^1_N(d, n)$ is irreducible and of dimension $n^2 - n + (d - 1)(n - 1)$.

**Corollary**

*If $N(d, m)$ is irreducible for each $m \leq n$, then $C(d, n)$ is irreducible.*
Dimension of $d$-generated commutative matrix algebra

Theorem (Gerstenhaber, 1961)
If the variety $C(d, n)$ is irreducible, then each unital algebra generated by $d$-tuple of commuting $n \times n$ matrices is at most $n$-dimensional.

Idea of the proof
- $\dim \mathbb{F}[A_1, A_2, \ldots, A_d] \leq n$ is closed condition in the Zariski topology.
- The variety
  $$V = \{ (A_1, A_2, \ldots, A_d) \in C(d, n); \dim \mathbb{F}[A_1, A_2, \ldots, A_d] \leq n \}$$
  contains $G(d, n)$.
- Irreducibility of $C(d, n)$ implies $C(d, n) = \overline{G(d, n)}$, hence $V = C(d, n)$. 
Proposition (Guralnick, 1992)

For \( n \geq 4 \) the commutative algebra of all \( n \times n \) matrices of the form

\[
\begin{bmatrix}
a_1 & & & \\
& a_2 & & \\
& & \ddots & \\
& & & a_{n-4}
\end{bmatrix}
\begin{bmatrix}
b \\
c \\
d \\
b \\
b \\
b
\end{bmatrix}
\]

is \((n + 1)\)-dimensional and generated by 4 matrices as a unital algebra.
Corollary

If $d \geq 4$ and $n \geq 4$, then $C(d, n)$ is reducible.

Proposition (Schur, 1905)

The commutative algebra of all $n \times n$ matrices of the form

$$\begin{bmatrix}
\lambda I & A \\
0 & \lambda I
\end{bmatrix}$$

where $\lambda \in \mathbb{F}$, $A \in M_{m \times (n-m)}(\mathbb{F})$, $m = \left\lfloor \frac{n}{2} \right\rfloor$ is $(m(n-m) + 1)$-dimensional.

Corollary (Gerstenhaber, 1961)

If $d \geq n \geq 4$, then $N(d, n)$ is reducible.

Proposition (Kirillov, Neretin, 1984; Guralnick, 1992)

If $n \leq 3$, then $C(d, n)$ and $N(d, n)$ are irreducible for each $d$. 
Known results

Pairs

Theorem (Motzkin, Taussky, 1955)

For each positive integer $n$ the variety $C(2, n)$ is irreducible.

Proof

- $(A, B) \in C(2, n)$ arbitrary pair.
- $B$ commutes with some 1-regular matrix $C$.
- The line $\mathcal{L} = \{(\lambda A + (1 - \lambda)C, B); \lambda \in \mathbb{F}\}$ intersects $R^1(2, n)$.
- Line is irreducible, therefore $\mathcal{L} \subseteq R^1(2, n)$. In particular, $(A, B) \in R^1(2, n)$.

Generalization (Richardson, 1979)

If $\mathfrak{g}$ is a reductive Lie algebra over $\mathbb{F}$ and $\text{char} \, \mathbb{F} = 0$, then the commuting variety $C(\mathfrak{g}) = \{(x, y) \in \mathfrak{g} \times \mathfrak{g}; [x, y] = 0\}$ is irreducible.
**Theorem (Baranovsky, 2001; Basili, 2003)**

*If* $\text{char } F = 0$ *or* $\text{char } F \geq \frac{n}{2}$, then the variety $N(2, n)$ *is irreducible.*

**Generalization (Premet, 2003)**

Let $\mathfrak{g}$ be a semisimple Lie algebra over $F$ and assume that $\text{char } F$ is good for the root system of $\mathfrak{g}$. Then the nilpotent commuting variety

$$C_{\text{nil}}(\mathfrak{g}) = \{(x, y) \in \mathfrak{g} \times \mathfrak{g}; [x, y] = 0, x, y \text{ nilpotent}\}$$

is equidimensional and its irreducible components are parametrized by the distinguished nilpotent orbits in $\mathfrak{g}$.

In particular, $N(2, n)$ is irreducible for each $n$ in any characteristic.
Known results

**Triples**

**Theorem (Holbrook, Omladič, 2001)**

The variety $C(3, n)$ is reducible for $n \geq 29$.

**Idea of the proof**

The set of all triples $(A, B, C) \in C(3, 29)$ with $A$ similar to the matrix

$$
\lambda I_{29} + \begin{bmatrix}
0 & I_8 & 0 & 0 \\
0 & 0 & I_8 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

for some $\lambda \in F$ is contained in a proper subvariety of $C(3, 29)$ of dimension at least $29^2 + 2 \cdot 29$.

**Theorem (Guralnick, Sethuraman, Holbrook, Omladič, Han, K.Š.)**

If $\text{char } F = 0$, then $C(3, n)$ is irreducible for $n \leq 8$. 
Known results

Nilpotent triples

Theorem (Clark, O’Meara, Vinsonhaler, 2011)

The variety $N(3, n)$ is reducible for $n \geq 13$.

Theorem (Young, 2010)

The variety $N(3, 4)$ is irreducible.
Aproximation by 1-regular commuting matrices

Reductions

Lemma

If \( \varphi : C(3, n) \to C(3, n) \) is a polynomial map that maps \( R^1(3, n) \) to itself, then \((A, B, C) \in R^1(3, n)\) implies \( \varphi(A, B, C) \in R^1(3, n) \).

Corollaries

- The property that a triple from \( C(3, n) \) belongs to \( R^1(3, n) \) depends only on the algebra generated by the triple.
- While proving that a triple belongs to \( R^1(3, n) \) we can simultaneously conjugate the matrices in the triple by any invertible matrix.
- We can assume that one of the matrices is in the Jordan canonical form.

The same reductions can be made in the nilpotent case.
Lemma

If $C(3, m)$ is irreducible for each $m < n$ and if all matrices in a triple from $C(3, n)$ commute with a matrix having two distinct eigenvalues, then the triple belongs to $G(3, n)$.

Corollary

Only triples of commuting matrices that span a vector space of nilpotent matrices have to be considered.
Simultaneous commutative approximation

- $(A, B, C)$ any triple of commuting (nilpotent) matrices.
- $X, Y, Z \in M_n(\mathbb{F})$ such matrices that $A + \lambda X$, $B + \lambda Y$ and $C + \lambda Z$ commute for each $\lambda \in \mathbb{F}$ (and they are nilpotent if we are proving irreducibility of $N(3, n)$).
- Suppose that for each $\lambda$ from some open subset of $\mathbb{F}$ the triple $(A + \lambda X, B + \lambda Y, C + \lambda Z)$ belongs to $\overline{R_1^{1}(3, n)}$ (respectively $\overline{R_1^{1}_N(3, n)}$).

Typically this happens if some linear combination of $A + \lambda X$, $B + \lambda Y$ and $C + \lambda Z$ either has higher rank than $A$, $B$ and $C$ or (in the case of $C(3, n)$) it has two distinct eigenvalues.
- Then $(A, B, C) \in \overline{R_1^{1}(3, n)}$ (respectively $\overline{R_1^{1}_N(3, n)}$).
Variety of commuting pairs in the centralizer of a matrix

Notations

For \( A \in M_n(\mathbb{F}) \) let:
- \( C(A) = \{ B \in M_n(\mathbb{F}); AB = BA \} \): the centralizer of \( A \).
- \( N(A) = \{ B \in C(A); B \text{ nilpotent} \} \): the nilpotent centralizer.
- \( C_2(A) = \{ (B, C) \in C(A) \times C(A); BC = CB \} \).
- \( N_2(A) = C_2(A) \cap (N(A) \times N(A)) \).

Proposition

If \( C_2(A) \) is irreducible for each \( n \times n \) matrix \( A \), then \( C(3, n) \) is irreducible.

Theorem (Neubauer, Sethuraman, 1999)

If \( A \) is 2-regular matrix, then \( C_2(A) \) is irreducible.
Commuting pairs in the centralizer of the given matrix

Pairs of 1-regular matrices in $C_2(A)$

Notation

$R_2(A) = \{(B, C) \in C_2(A); B \text{ or } C \text{ 1-regular}\}$.

Proposition

For each $A \in M_n(\mathbb{F})$ the closure $\overline{R_2(A)}$ is irreducible variety of dimension $\dim C(A) + n$.

Corollary

$C_2(A)$ is irreducible if and only if $\overline{R_2(A)} = C_2(A)$.
Irreducibility of $C_2(A)$ in 3-regular case

Using simultaneous commutative approximation by pairs of 1-regular commuting matrices in the centralizer of $A$ we obtain:

**Theorem**

*If $A$ is 3-regular matrix, then $C_2(A)$ is irreducible.*
Proposition

For \( A = \begin{pmatrix} 0 & I_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \) the variety \( C_2(A) \) is reducible.
Commuting pairs in the centralizer of the given matrix

Proof of the proposition

The variety of all pairs \((\lambda I_{14} + B, \mu I_{14} + C) \in C_2(A)\) with \(B\) and \(C\) of the form

\[
B = \begin{bmatrix}
0 & B_1 & B_2 & B_3 & B_4 & B_5 \\
0 & 0 & B_1 & B_2 & 0 & B_4 \\
0 & 0 & 0 & B_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & B_6 & B_7 & 0 & B_8 \\
0 & 0 & 0 & B_6 & 0 & 0
\end{bmatrix}, \quad
C = \begin{bmatrix}
0 & C_1 & C_2 & C_3 & C_4 & C_5 \\
0 & 0 & C_1 & C_2 & 0 & C_4 \\
0 & 0 & 0 & C_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & C_6 & C_7 & 0 & C_8 \\
0 & 0 & 0 & C_6 & 0 & 0
\end{bmatrix}
\]

has dimension at least \(72 = 14 + \dim C(A)\).
Commuting pairs in the centralizer of the given matrix

4-regular cases

Theorem

Let $\text{char } F = 0$ and let $A$ be a 4-regular matrix whose Jordan canonical form has at most two nonzero Jordan blocks for each eigenvalue. Then $C_2(A)$ is irreducible.

Corollary

Let $\text{char } F = 0$. If $A \in M_n(F)$ is either 3-regular or 4-regular and its Jordan canonical form has at most two nonzero Jordan blocks for each eigenvalue, then for each pair $(B, C) \in C_2(A)$ the triple $(A, B, C)$ belongs to $G(3, n)$.

For general 4-regular matrices $A$ the problem of (ir)reducibility of $C_2(A)$ is still open.
The set $D_2(A)$

**Notations**

For a nilpotent $n \times n$ matrix $A$ let:

- $D(A) = \{ B \in N(A); \dim \mathbb{F}[A, B] = n \}$.
- $D_2(A) = \{ (B, C) \in N_2(A); \dim \mathbb{F}[A, B] = n \}$.

**Lemma**

*The sets $D(A)$ and $D_2(A)$ are Zariski open in $N(A)$ and $N_2(A)$, respectively.*

**Lemma (Basili, 2003)**

*The variety $N(A)$ is irreducible and of dimension $\dim C(A) - \dim \ker A$.***
Proposition

$D_2(A)$ is irreducible and of dimension $\dim C(A) - \dim \ker A + n - 1$.

Proof

- Let $A$, $B$ be nilpotent $n \times n$ matrices with $\dim \mathbb{F}[A, B] = n$.
- If $C$ commutes with $A$ and $B$, then $C \in \mathbb{F}[A, B]$ by the theorem of Neubauer and Saltman.
- If the Jordan canonical form of $A$ has Jordan blocks of orders $n_1 \geq n_2 \geq \cdots \geq n_k \geq 1$, then

$$\{ A^i B^j; 0 \leq j \leq k - 1, 0 \leq i \leq n_{j+1} - 1 \}$$

is a basis of $\mathbb{F}[A, B]$ by the theorem of Barria and Halmos.
The polynomial map $\varphi: D(A) \times \mathbb{F}^{n_1-1} \times \mathbb{F}^{n_2} \times \cdots \times \mathbb{F}^{n_k} \rightarrow D_2(A)$ defined by

$$\varphi(B, (c_{21}, \ldots, c_{n_11}), (c_{12}, \ldots, c_{n_22}) \ldots, (c_{1k}, \ldots, c_{n_kk})) =$$

$$= (B, \sum_{i=2}^{n_1} c_{i1}A^{i-1} + \sum_{j=2}^{k} \sum_{i=1}^{n_j} c_{ij}A^{i-1}B^{j-1})$$

is bijective and a birational equivalence.

**Corollary**

$N_2(A)$ is irreducible if and only if $\overline{D_2(A)} = N_2(A)$. 
Irreducibility of $N_2(A)$ in 2-regular case

**Theorem**

If $A$ is 2-regular nilpotent matrix, then $N_2(A)$ is irreducible.

**Proof in the case when the Jordan blocks of $A$ are of the same size**

- $A = \begin{bmatrix} 0 & I_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & I_2 & \cdots & 0 \\ 0 & 0 \\ \end{bmatrix}$

- $B = \begin{bmatrix} B_1 & B_2 & \cdots & B_k \\ \vdots & \ddots & \ddots & \vdots \\ B_1 & B_2 & \cdots & B_k \\ \end{bmatrix}$, $C = \begin{bmatrix} C_1 & C_2 & \cdots & C_k \\ \vdots & \ddots & \ddots & \vdots \\ C_1 & C_2 & \cdots & C_k \\ \end{bmatrix}$. 
If $B_1 \neq 0$, then $\dim(\ker A \cap \ker B) = 1$ or $\dim(\ker A^T \cap \ker B^T) = 1$, and $\dim \mathbb{F}[A, B] = n$ by the theorem of Košir.

Let $B_1 = C_1 = 0$ and assume that $B_2$ is not scalar. Then $B_2C_2 = C_2B_2$, therefore $C_2 = \alpha I_2 + \beta B_2$ for some $\alpha, \beta \in \mathbb{F}$. We can add polynomials in $A$ and $B$ to $C$, so we can assume that $C_2 = 0$. Similarly $C_3 = \cdots = C_{k-1} = 0$.

The set of all $n \times n$ matrices that are not generic is $(n^2 - 1)$-dimensional variety defined by $\det p'_X(X) = 0$, where $p'_X$ is the derivative of the characteristic polynomial of $X$.

There exist $\mu, \nu \in \mathbb{F}$ such that $\mu B_2 - \nu C_k$ is not generic.
If $\nu = 0$, then we can assume that $B_2$ is nilpotent. Then

$$X = \begin{bmatrix} B_2 & B_3 & \cdots & B_k & 0 \\ B_2 & B_3 & \cdots & B_k \\ \vdots & \vdots & \ddots & \vdots \\ B_2 & B_3 & \cdots & B_k \end{bmatrix}$$

is nilpotent, it commutes with $B$

and $(B, C + \lambda X) \in D_2(A)$ for $\lambda \neq 0$.

If $\nu \neq 0$, then we can subtract $\frac{\mu}{\nu} A^{k-2} B$ from $C$, so we can assume that $C_k$ is not generic and we can proceed similarly as above.
Suppose that $B_2$ is a scalar. There exists $C_l$ that is not a scalar. Then $X = \begin{bmatrix} 0 & C_l & \cdots & C_k & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & C_l & \cdots & C_k & 0 \\ 0 & C_l & \cdots & C_k \\ \vdots & \vdots & \ddots & \vdots \\ 0 & C_l \\ 0 \end{bmatrix}$ is nilpotent, it commutes with $C$ and $(B + \lambda X, C) \in D_2(A)$ for $\lambda \neq 0$.

**Corollary**

If $(A, B, C) \in N(3, n)$ and $A$ is 2-regular, then $(A, B, C) \in \overline{R_1^N(3, n)}$. 
Commuting pairs in the centralizer of the given matrix

Reducibility of $N_2(A)$ in 3-regular case

**Proposition**

For $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, the variety $N_2(A)$ is reducible.
Proposition

Let \( k < n \) and assume that each triple of commuting nilpotent \( n \times n \) matrices whose linear span contains a matrix of rank at least \( k \) belongs to \( R_{N}^{1}(3, n) \). If \((A, B, C) \in N(3, n)\) and the Jordan canonical form of \( A \) has one Jordan block of order \( k \) and \( n - k \) zero Jordan blocks, then \((A, B, C) \in R_{N}^{1}(3, n)\).

Remark

The same result for triples from \( C(3, n) \) was proved by Holbrook and Omladič.

Remark

Additional assumptions have no influence on the irreducibility of \( N(3, n) \).
Proposition

Let $k < n$ and assume that $C(3, m)$ is irreducible for each $m < n$ and that each triple of commuting $n \times n$ matrices whose linear span contains a matrix of rank at least $k$ belongs to $G(3, n)$. If $(A, B, C) \in C(3, n)$, $\text{rank } A = k - 1$ and the Jordan canonical form of $A$ has more zero than nonzero Jordan blocks, then $(A, B, C) \in G(3, n)$. 
Theorem (Holbrook, Omladič, 2001)

If commuting $n \times n$ matrices $A$, $B$ and $C$ generate an algebra with radical of square zero, then $(A, B, C) \in G(3, n)$.

Generalizations

- If commuting $n \times n$ matrices $A$, $B$ and $C$ generate an algebra with radical of square zero, then $(B, C) \in R_2(A)$.

- Let $l > 0$, $m \geq 0$ and $n = 2l + m$, and assume that each triple of commuting nilpotent $n \times n$ matrices whose linear span contains either a matrix of rank at least $l + 1$ or a matrix with nonzero square belongs to $R_1^N(3, n)$. If $(A, B, C) \in N(3, n)$ and the Jordan canonical form of $A$ has $l$ Jordan blocks of order 2 and $m$ zero Jordan blocks, then $(A, B, C) \in R_1^N(3, n)$.
Triples of matrices with small Jordan blocks

Only one Jordan block of order exceeding 2

If $A$, $B$ and $C$ generate vector space of nilpotent matrices, then the square of the algebra $\mathbb{F}[A, B, C]$ is zero if and only if the Jordan canonical form of each linear combination of $A$, $B$ and $C$ has Jordan blocks of orders at most 2 only.

**Theorem**

Let $k \geq 3$, $l \geq 0$ and $n \geq k + 2l$, and assume that $C(3, m)$ is irreducible for each $m < n$ and that each triple of commuting $n \times n$ matrices whose linear span contains either a matrix of rank at least $k + l$ or a matrix with square of rank at least $k - 1$ belongs to $G(3, n)$. If $(A, B, C) \in C(3, n)$ and the Jordan canonical form of $A$ has one Jordan block of order $k$, $l$ Jordan blocks of order 2 and $n - 2l - k$ zero Jordan blocks, then $(A, B, C) \in G(3, n)$.
Sketch of the proof for $k \geq 4$

$$A = \begin{bmatrix} J_k & 0 & 0 & 0 \\ 0 & 0 & I_\ell & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & e_1 a^T & e_2 a^T + e_1 b^T & e_1 c^T \\ d e_{k-1}^T + f e_k^T & D & E & F \\ d e_k^T & 0 & D & 0 \\ g e_k^T & 0 & G & H \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & e_1 a'^T & e_2 a'^T + e_1 b'^T & e_1 c'^T \\ d' e_{k-1}^T + f' e_k^T & D' & E' & F' \\ d' e_k^T & 0 & D' & 0 \\ g' e_k^T & 0 & G' & H' \end{bmatrix}. $$
We can assume \( \text{rank}(\alpha A + \beta B + \gamma C) \leq k + l - 1 \) and  
\( \text{rank}(\alpha A + \beta B + \gamma C)^2 \leq k - 2 \) for all \( \alpha, \beta, \gamma \in \mathbb{F} \), therefore  
\( D = D' = 0, \ H = H' = 0, \) and  
\( (\beta F + \gamma F')(\beta G + \gamma G') = 0 \) and  
\( (\beta G + \gamma G')(\beta F + \gamma F') = 0 \) for all \( \beta, \gamma \in \mathbb{F} \).

The only nontrivial case to consider is \( G = G' = 0 \).

It turns out that we can assume that \( a \) and \( a' \) are linearly independent, therefore by conjugation we can get  
\( b = b' = 0 \) and  
\( c = c' = 0 \). Similarly,  
\( f = f' = 0 \).
It suffices to prove the existence of the matrices

\[
X = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & Z & 0 & 0 \\
0 & 0 & Z & 0 \\
0 & 0 & U & T
\end{bmatrix}
\] and

\[
Y = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & Z' & 0 & 0 \\
0 & 0 & Z' & 0 \\
0 & 0 & U' & T'
\end{bmatrix},
\]
not both of them zero, such that \(XY = YX\) and \(BY + XC = CX + YB\).

The equations \(XY = YX\) and \(BY + XC = CX + YB\) are equivalent to

\[
a^T Z' = a'^T Z, \quad Zd' = Z'd,
\] (1)

\[
ZE' + EZ' + FU' = Z'E + E'Z + F'U,
\] (2)

\[
ZF' + FT' = Z'F + F'T,
\] (3)

\[
Ud' + Tg' = U'd + T'g,
\] (4)

\[
ZZ' = Z'Z,
\] (5)

\[
UZ' + TU' = U'Z + T'U, \quad TT' = T'T.
\] (6)
If $A$ has $p = n - k - 2l > 0$ zero Jordan blocks, then the dimension of
\[(Z, Z', U, U', T, T') \in M_l(F)^2 \times M_{p \times l}(F)^2 \times M_p(F)^2;\]
\[a^T Z' = a'^T Z, Zd' = Z'd, Ud' + Tg' = U'd + T'g\]
is at least $2l^2 + 2pl + 2p^2 - 2l - p$, and the dimension of
\[(Z, Z', U, U', T, T') \in M_l(F)^2 \times M_{p \times l}(F)^2 \times M_p(F)^2;\]
\[ZZ' = Z'Z, UZ' + TU' = U'Z + T'U, TT' = T'T,\]
\[ZE' + EZ' + FU' = Z'E + E'Z + F'U, ZF' + FT' = Z'F + F'T\]
is at least $p^2 + p + 2l$, therefore their intersection has dimension at least $p^2$ and it contains some nonzero point.
If $A$ has no zero Jordan blocks, then using the theorem on fibres of polynomial map we show that the projection from

$$\{(Z, Z', W, W', x, x', y, y') \in M_l(\mathbb{F})^4 \times (\mathbb{F}^l)^4; Z\text{ generic and invertible},$$

$$ZZ' = Z'Z, ZW' + WZ' = Z'W + W'Z,$$

$$x^T Z' = x'^T Z, Zy' = Z'y, x^T y' = x'^T y$$

to

$$\{(W, W', x, x', y, y') \in M_l(\mathbb{F})^2 \times (\mathbb{F}^l)^4; x^T y' = x'^T y\},$$

is dominant. The first set is of dimension $2l^2 + 4l$ and the second one of the dimension $2l^2 + 4l - 1$, therefore the set of all pairs $(Z, Z') \in (M_l(\mathbb{F}))^2$ satisfying $ZZ' = Z'Z$, $a^T Z' = a'^T Z$, $Zd' = Z'd$ and $ZE' + EZ' = Z'E + E'Z$ has dimension at least 1 by the theorem on fibres.
Irreducibility of $C(3, n)$ for $n \leq 10$

Let $\text{char } \mathbb{F} = 0$. The following cases of sizes of Jordan blocks remain to be considered:

- $n = 9$: $3, 3, 2, 1$.
- $n = 10$: $4, 3, 2, 1$, $3, 3, 3, 1$, $3, 3, 2, 2$, $3, 3, 2, 1, 1$.

Using simultaneous commutative approximation we prove that such triples belong to $G(3, n)$.

**Theorem**

For $n \leq 10$ the variety $C(3, n)$ is irreducible.

**Corollary**

Each unital algebra generated by three commuting $n \times n$ matrices is at most $n$-dimensional for $n \leq 10$. 
Irreducibility of $N(3, n)$ for $n \leq 6$

**Proposition**

If $(A, B, C) \in N(3, 6)$ and the Jordan canonical form of $A$ has Jordan blocks of orders 3, 2 and 1, then $(A, B, C) \in R^1_N(3, n)$.

**Theorem**

For $n \leq 6$ the variety $N(3, n)$ is irreducible.
Reducibility of $N(d, n)$ for $d, n \geq 4$ in almost all cases

We already know that $N(d, n)$ is reducible for $d \geq n \geq 4$.

**Theorem**

*If $d \geq 5$ and $n \geq 4$ or if $d = 4$ and $n \geq 6$, then $N(d, n)$ is reducible.*

**Proof**

- $m = \lfloor \frac{n}{2} \rfloor$.

- Let $\mathcal{V}$ be the variety of all $d$-tuples of the form
  \[
  \left( \begin{bmatrix}
  0 & I_m & 0 \\
  0 & B_2 & C_2 \\
  0 & 0 & 0
  \end{bmatrix}, \begin{bmatrix}
  0 & B_2 & C_2 \\
  0 & 0 & 0 \\
  0 & 0 & 0
  \end{bmatrix}, \ldots, \begin{bmatrix}
  0 & B_d & C_d \\
  0 & 0 & 0 \\
  0 & 0 & 0
  \end{bmatrix} \right)
  \]
  where $B_2, \ldots, B_d \in M_m(\mathbb{F})$ and $C_2, \ldots, C_d \in M_{m \times (n-2m)}(\mathbb{F})$.

- $\mathcal{V}$ is a subvariety of $N(d, n)$, it is irreducible and of dimension $(d - 1)m(n - m)$. 
Define a polynomial map \( \varphi: \text{SL}_n(\mathbb{F}) \times V \to N(d, n) \) by
\[
\varphi(P, (A_1, A_2, \ldots, A_d)) = (P^{-1}A_1P, P^{-1}A_2P, \ldots, P^{-1}A_dP).
\]
All fibres are birationally equivalent to \( C(A_1) \cap \text{SL}_n(\mathbb{F}) \), where
\[
A_1 = \begin{bmatrix} 0 & l_m & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]
therefore they are of dimension \( m^2 + (n - m)^2 - 1 \).

Let \( W \) be the closure of the image of the map \( \varphi \). Then
\[
\dim W = (d + 1)m(n - m)
\]
by the theorem on fibres.

If \( d \geq 5 \) and \( n \geq 4 \) or if \( d = 4 \) and \( n \geq 6 \), then
\[
\dim W = (d + 1)m(n - m) \geq (d + n - 1)(n - 1) = \dim \overline{R^1_N(d, n)}.
\]

\( W \) is a proper subvariety of \( N(d, n) \), so \( N(d, n) \) is reducible.
Irreducibility of $N(4, 5)$

**Theorem**

$N(4, 5)$ is irreducible.

**Proof**

- For any $(A_1, A_2, A_3, A_4) \in N(4, 5)$ we have to show $(A_1, A_2, A_3, A_4) \in R^1_N(4, 5)$.

- The only nontrivial case is when $A_1, A_2, A_3, A_4$ are of rank 2 and square zero.

\[
A_1 = \begin{bmatrix} 0 & I_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_i = \begin{bmatrix} 0 & B_i & a_i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ for } i = 2, 3, 4 \text{ where }
\]

$B_i \in M_2(F), a_i \in F^2$. 

Klemen Šivic (ETH Zürich) Varieties of commuting matrices ICTP, Trieste, 8th July 2013
Let $\mathcal{U}$ be the set of all quadruples $(X_1, X_2, X_3, X_4) \in N(4, 5)$ such that $X_1$ is 1-regular and for each $i \leq 4$, $X_i = \begin{bmatrix} Y_i & Z_i \\ 0 & W_i \end{bmatrix}$ for some strictly upper triangular $Y_i \in M_2(\mathbb{F})$, some $Z_i \in M_{2 \times 3}(\mathbb{F})$ and some $W_i \in M_{3}(\mathbb{F})$.

If $(X_1, X_2, X_3, X_4) \in \mathcal{U}$, then $X_2, X_3, X_4$ are polynomials in $X_1$, therefore $\overline{\mathcal{U}}$ is birationally equivalent to $\mathbb{F} \times N_3 \times M_{2 \times 3}(\mathbb{F}) \times (\mathbb{F}_3[t])^3$, where $N_n$ denotes the variety of all nilpotent $n \times n$ matrices.

It is known that $N_n$ is irreducible and of dimension $n^2 - n$ for each $n$, therefore $\overline{\mathcal{U}}$ is irreducible and of dimension 25.
Let $\pi: \overline{U} \to M_{2\times 3}(\mathbb{F})^4$ be the projection defined by

$$
\pi \left( \begin{bmatrix} Y_1 & Z_1 \\ 0 & W_1 \end{bmatrix}, \begin{bmatrix} Y_2 & Z_2 \\ 0 & W_2 \end{bmatrix}, \begin{bmatrix} Y_3 & Z_3 \\ 0 & W_3 \end{bmatrix}, \begin{bmatrix} Y_4 & Z_4 \\ 0 & W_4 \end{bmatrix} \right) = 
$$

$$
= (Z_1, Z_2, Z_3, Z_4).
$$

Assume that $\pi$ is not dominant. Then $\overline{\pi(U)}$ is a proper subvariety of $M_{2\times 3}(\mathbb{F})^4$, therefore $\dim \overline{\pi(U)} \leq 23$.

By the theorem on fibres all fibres of $\pi$ have dimension at least

$$
\dim \overline{U} - \dim \overline{\pi(U)} \geq 2.
$$

If

$$
Z_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},
Z_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},
$$

$$
Z_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
Z_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},
$$

then $\dim \pi^{-1}(Z_1, Z_2, Z_3, Z_4) = 1$, a contradiction.
Therefore $\pi(U)$ is dense in $M_{2 \times 3}(\mathbb{F})^4$.

If $(Z_1, Z_2, Z_3, Z_4) \in \pi(U)$ is arbitrary quadruple, then there exist strictly upper triangular matrices $Y_1, Y_2, Y_3, Y_4 \in M_2(\mathbb{F})$ and nilpotent matrices $W_1, W_2, W_3, W_4 \in M_3(\mathbb{F})$ such that

$$
\begin{pmatrix}
Y_1 & Z_1 \\
0 & W_1
\end{pmatrix}, \begin{pmatrix}
Y_2 & Z_2 \\
0 & W_2
\end{pmatrix}, \begin{pmatrix}
Y_3 & Z_3 \\
0 & W_3
\end{pmatrix}, \begin{pmatrix}
Y_4 & Z_4 \\
0 & W_4
\end{pmatrix}
$$

belongs to $R_N^1(4, 5)$.

Identify $M_{2 \times 3}(\mathbb{F})$ with the upper right corners of $5 \times 5$ matrices of the form

$$
\begin{bmatrix}
0 & * \\
0 & 0
\end{bmatrix}
$$

Then $\pi(U) \subseteq \overline{R_N^1(4, 5)}$, therefore

$M_{2 \times 3}(\mathbb{F})^4 = \pi(U) \subseteq \overline{R_N^1(4, 5)}$.

In particular, $(A_1, A_2, A_3, A_4) \in \overline{R(4, 5)}$. 