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How to use wave collapse to get short pulses in fibers

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OUTLINE

- Motivation: solitons and collapses
- Why collapse is absent in the 1D NLS
- Main model
- Hamiltonian formulation and two soliton families
- Stability analysis of solitons
- Instability of solitons and collapse
- Numerical experiment

Motivation: solitons and collapses

- In this talk we discuss a possibility of light pulse compression in fiber optics using the mechanism of wave collapse.
- It is well known that wave collapse in optics provides the giant pulse shortening in two- and three-dimensional geometry due to the Kerr nonlinearity.
- However, in the one-dimensional case, for instance, in fibers the cubic Kerr nonlinearity is compensated by wave dispersion responsible for the optical pulse broadening that results in stable optical envelope solitons. By this reason, the wave collapse in such a case is impossible.

Motivation: solitons and collapses

- We consider two mechanisms how it is possible to diminish the Kerr constant in order to provide wave collapse appearance due to higher nonlinearities.
- The first mechanism weaken the Kerr constant is connected with the interaction of light and acoustic waves (or Mandelshtam-Brillouin scattering) (Kuznetsov [1999]).
- Another idea was suggested by Gabitov & Lushnikov [2002] to use a nonlinear phase-shift interferometric converter (the analog of the Mach-Zehnder interferometer) which provides a way to control the sign of the nonlinear phase shift.

Motivation: solitons and collapses

- In the case of a small Kerr constant it is necessary to take into account the higher nonlinearities: nonlinear correction to the group velocity, responsible for pulse steepening, Raman scattering and six-wave interaction as well. In this situation depending on the sign of the renormalized Kerr constant χ there exist two soliton families (Agafontsev, Dias & Kuznetsov [2008]).

Motivation: solitons and collapses

- For the first family with positive χ for very small amplitudes these solitons are nothing more than the NLS solitons of the sech form. For larger amplitudes these solitons transform into pulses with a chirp and an almost constant pulse energy. This soliton family is shown to be stable in the Lyapunov sense.
- The second soliton family with $\chi < 0$ has the same behavior at small pulse durations but occurs to be unstable. The nonlinear development of this instability results in the collapse of solitons. Near the collapse time, the pulse amplitude and its width exhibit a self-similar behavior with a small asymmetry in the pulse tails due to self-steepening.

Motivation: solitons and collapses

- The problem discussed here has a similarity with the description of behavior of solitons near bifurcations, both supercritical and subcritical. First time this staff was investigated by Longuet-Higgins [1989] for water waves and later in many details by Akylas [1993], Longuet-Higgins [1993] and many others. The bifurcation parameter is the soliton velocity V. When V approaches the critical velocity V_{cr} which coincides with minimal (or maximal) phase velocity of linear waves soliton undergoes a bifurcation. What kind of bifurcations depends on the nonlinear media characteristics.

Motivation: solitons and collapses

- For nonlinear optics the question about supercritical bifurcations for optical solitons was considered in Zakharov & Kuznetsov [1998] from the Hamiltonian point of view and later for subcritical bifurcations (Kuznetsov [1999]).
- More reach situation takes place for bifurcations in the system of interactive basic and second harmonics in nonlinear optical media (Grimshaw, Kuznetsov, Shapiro [2001]).

Why collapse is absent in the 1D NLS

Consider the 1D NLS for dimensionalless envelope ψ of the electric field

$$i\psi_z + \psi_{tt} + \chi|\psi|^2)\psi = 0.$$

Here $\chi(= \pm 2)$ is the normalized Kerr constant.
In this equation the second term is responsible for dispersion broadening of the pulse that can be stopped by the nonlinear attraction between particles when χ is positive (focusing nonlinearity). In this case we can expect existence of the localized structures , i.e. solitons. (Evidently that solitons are absent for $\chi < 0$, i.e. for defocussing nonlinearity).

Why collapse is absent in the 1D NLS

The simplest soliton solution $\psi = \psi_s(t)e^{i\lambda^2 z}$ is defined from the "stationary" NLS equation,

$$-\lambda^2\psi_s + \partial_t^2\psi_s + 2|\psi_s|^2\psi_s = 0,$$

where $-\lambda^2$ has the meaning of the "energy" of the bound state (i.e. soliton), and

$$\psi_s = \lambda\,\mathrm{sech}\,(\lambda t).$$

It is easy to check that $\psi_s(t)$ represents a stationary point of the Hamiltonian

$$H = \int |\partial_t\psi|^2 dt - \int |\psi|^4 dt \equiv I_1 - I_2$$

for fixed number of "particles" $N = \int |\psi|^2 dt$:

$$\delta(H + \lambda^2 N) = 0.$$

Why collapse is absent in the 1D NLS

Hence we can conclude that such soliton solution will be stable if it realizes the minimum of H. This is the stability criterion in the Lyapunov sense.

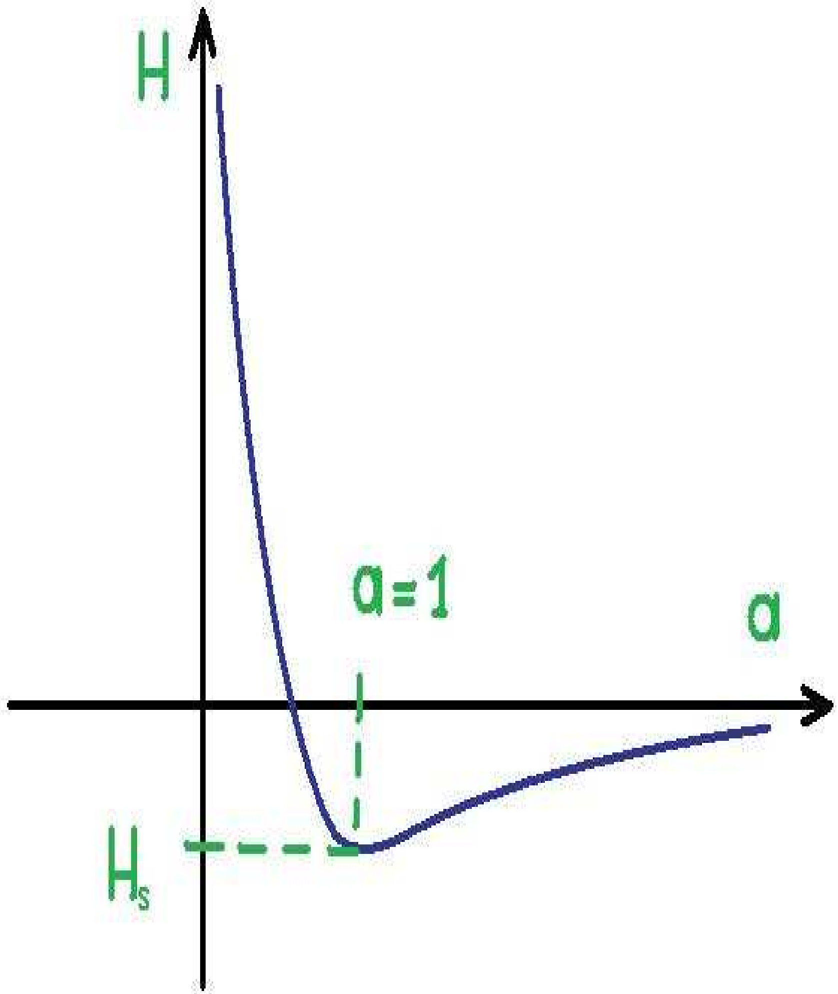
To verify that NLS soliton is stable it is enough to consider the simplest scaling transformation remaining N:

$$\psi_s(t) \longrightarrow a^{-1/2}\psi_s(t/a).$$

Under these transforms H becomes function of the scaling parameter a:

$$H(a) = \frac{I_1}{a^2} - \frac{I_2}{a}$$

Why collapse is absent in the 1D NLS



Why collapse is absent in the 1D NLS

Hence we can see that the NLS soliton realizes the minimum of H under scaling transformations. It is possible to prove that in the general case (under any transform with fixed N) the Hamiltonian is unbounded from below ant it's minimum corresponds to the soliton solution. The proof is based on the integral inequalities following from the so-called Sobolev imbedding theorem (for detail see, e.g., V.E. Zakharov and E.A. Kuznetsov, *Solitons and collapses - two scenarios of the evolution of nonlinear wave systems* , Physics Uspekhi **55**, 535 - 556 (2012)). Stability of solitons based on the Lyapunov theorem can be considered as the energetical principle for the NLS equation.

Main model

If the Hamiltonian is bounded from below then we can expect formation of solitons but never can get collapse, i.e. formation of singularity in a finite time. To obtain collapse, thus, we should have the unbounded Hamiltonians. Collapse in such a case can be considered as a process of falling down of a particle in a self-consistent unbounded potential.
Thus, for fibers (1D system) collapse is possible if by some reason the Kerr constant occurs small enough constant. In this case one needs to keep next order terms beyond the Kerr nonlinearity.

Main model

The NLS equation in such case has the form

$$i\psi_z + \psi_{tt} + f(|\psi|^2)\psi + 4i\beta|\psi|^2\psi_t + i\gamma(|\psi|^2)_t\psi = 0.$$

Here $f(|\psi|^2) = \chi|\psi|^2 + C|\psi|^4 + ...$, the term $\sim \beta$ is responsible for the nonlinear correction to the group velocity (NLG), another term of the same order $\sim \gamma$ is for the Raman scattering.

Main model

The Raman term can be excluded by means of the transformation

$$\psi = \widetilde{\psi} \exp\left(-\frac{i\gamma}{2} \int_{-\infty}^{t} |\widetilde{\psi}|^2 dt' \right)$$

under which the NLS takes the form (tilde is omitted)

$$i\psi_z + \psi_{tt} + f(|\psi|^2)\psi + i4\beta|\psi|^2\psi_t + \alpha|\psi|^4\psi = 0$$

where

$$\alpha = -\gamma\frac{(\gamma - 8\beta)}{8}.$$

For $\beta = 0$ $\alpha < 0$, i.e. the Raman scattering provides defocussing.

Hamiltonian formulation

The extended NLS belongs to the Hamiltonian equations,

$$i\psi_z = \frac{\delta H}{\delta \psi^*}, \;\; H = \int \left[|\psi_t|^2 - \frac{\chi}{2}|\psi|^4 + i\beta(\psi_t^* \psi - \psi_t \psi^*)|\psi|^2 - C|\psi|^6 \right] dt$$

where we assume the constant χ being small and $C > 0$. The β- term appears as the result of expansion of 4-wave matrix element near $\omega = \omega_0$:

$$T_{\omega\omega_1\omega_2\omega_3} = T_{\omega_0\omega_0\omega_0\omega_0} + \left.\frac{\partial T}{\partial \omega}\right|_{\omega_i} (\Omega + \Omega_1 + \Omega_2 + \Omega_3)$$

where $\Omega = \omega - \omega_0$,

$$T_{\omega_0\omega_0\omega_0\omega_0} = -\frac{\chi}{2\pi}, \;\; \left.\frac{\partial T}{\partial \omega}\right|_{\omega_i = \omega_0} = -\frac{\beta}{2\pi}.$$

Solitons

Soliton solutions of the extended NLS can be found from the variational problem

$$\delta(H + \lambda^2 N) = 0 \quad \left(N = \int |\psi|^2 dt\right).$$

The corresponding soliton solutions can be found explicitly in terms of amplitude r and phase ϕ ($\psi = re^{i\phi}$):

$$\begin{aligned} r^2 &= \frac{4\lambda^2}{\sqrt{16\lambda^2 C_1 + \chi^2}\cosh(2\lambda t) - \chi}, \\ \phi &= -\frac{\beta^2}{\sqrt{C_1}}\tan^{-1}\left[\frac{\sqrt{16\lambda^2 C_1 + \chi^2}\, e^{2\lambda t} - \chi}{4\lambda\sqrt{C_1}}\right] \quad \text{(chirp)} \end{aligned}$$

where C is renormalized as $C_1 = C + \beta^2$.

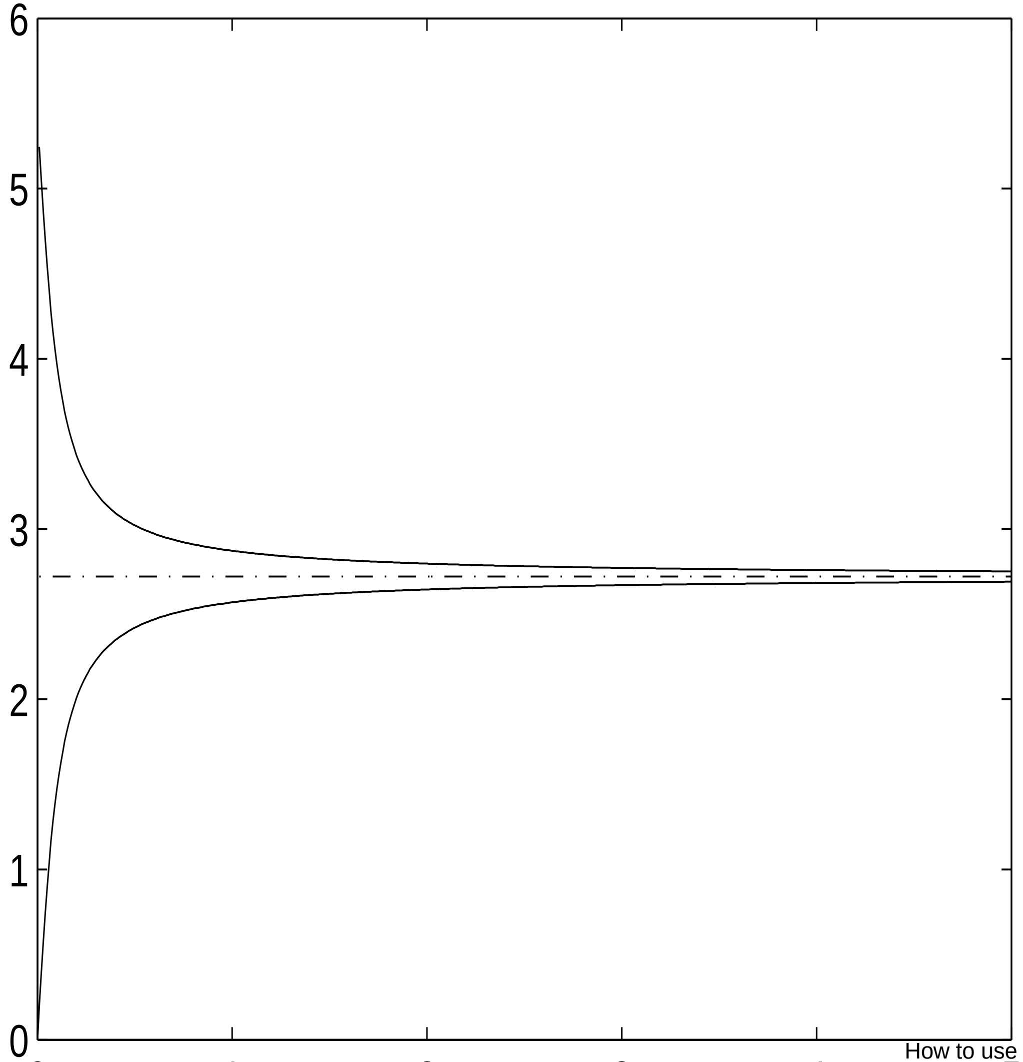
Solitons

Important: This soliton-type solution exists only if $C_1 > 0$.
As the result depending on the sign of χ we have two solution families. For the first family ($\chi > 0$) these solitons degenerate to usual NLS solitons of the sech type as $\lambda \to 0$, respectively, the pulse power ($\sim N$) vanishes proportionally to λ. At large λ N saturates approaching from below its maximal value $N_{cr} = \pi/(2\sqrt{C_1})$. For this family the derivative

$$\frac{\partial N}{\partial \lambda^2} > 0.$$

Solitons

Dependences of N on λ for two soliton families. Dashed line corresponds to the the critical value $N_{cr} = \pi/(2\sqrt{C_1})$.



Solitons

The upper soliton family ($\chi < 0$, defocussing nonlinearity) demonstrates familiar behavior at large λ with exponential vanishing at the infinity and very different at small λ. For $\lambda = 0$ the soliton decays in a power-law fashion:

$$r_{lim}^2 = \frac{2|\chi|}{\chi^2 t^2 + 4C_1}$$

with its maximum value in amplitude $A = \sqrt{|\chi|/(2C_1)}$. Respectively, N gets its maximal value at $\lambda = 0$ equal to $2N_{cr}$ and approaches from above to N_{cr} at large λ; derivative

$$\frac{\partial N}{\partial \lambda^2} < 0(!)$$

Stability criterion

- The dependence $N(\lambda)$ gives possibility to make some predictions about soliton stability based on the Vakhitov-Kolokolov criterion.
- If the derivative $\partial N_s/\partial\lambda^2$ on the soliton family is negative then solitons are unstable and they are stable in opposite situation.
- For the lower soliton branch this criterion gives stability and instability for solitons with $\chi < 0$.
- However, the Vakhitov-Kolokolov criterion, derived for the classical NLSE, can not be applied to our system strictly speaking.

Lyapunov stability for the lower family

- Using the scaling transformations $r \to a^{-1/2} r(t/a)$ remaining N it is possible to see that soliton realizes minimum of $H(a)$ $\chi > 0$. For $\chi < 0$ (defocussing) solitons have maxima.
- By means of exact integral inequalities it is possible to get the following estimation

$$H \geq -\frac{\chi^2 N^3}{8\sqrt{3}} \left[1 - \left(\frac{N}{N_{cr}} \right)^2 \right]$$

whence one can see the boundedness of H if $N \leq N_{cr}$.

- This proves stability of the soliton family with $\chi > 0$. This is stability in the Lyapunov sense, i.e. stability not only with respect to small perturbations but also against finite ones

Collapse of solitons

- At $\chi = 0$ in this case this model can be considered as the critical NLS (critical in the same sense as 2D cubic NLS).
- Another reason for collapse existence in the GNLS is connected with unboundedness of the Hamiltonian from below (only for defocussing cubic nonlinearity!).

$$H\left(a\right) = \left(\frac{1}{a} - \frac{1}{2a^2}\right) \frac{\chi}{2} \int |\psi_s|^4 \, dt$$

where a is the scaling parameter.

- For $\beta = 0$ the virial theorem is valid:

$$R_{zz} = 8\left(H - \frac{\chi}{4} \int |\psi|^4 dt\right), \; R = \int t^2 |\psi|^2 dt,$$

At $\chi < 0$ collapse takes place for $H \leq H_s$.

Collapse of solitons

- Numerics at $\chi < 0$ show self-similar behavior for collapsing soliton. Near the collapse point the term $\sim \chi$ can be neglected. In this case GNLS admits self-similar solutions,

$$r(z,t) = (z_0 - z)^{-1/4} f\left(\frac{t}{(z_0 - z)^{1/2}}\right).$$

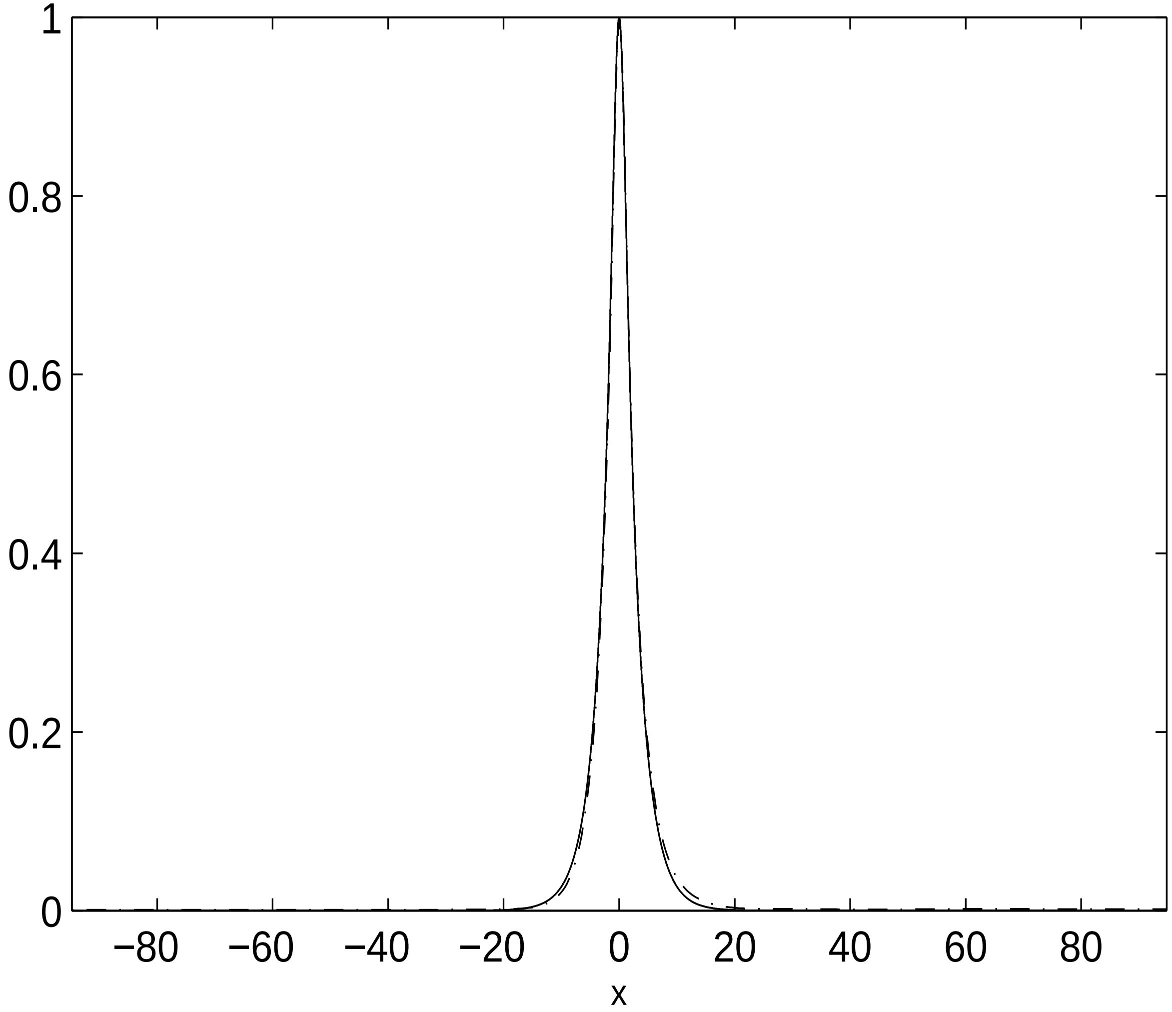
- To verify the asymptotic behavior, we normalized ψ by $\max|\psi| \equiv M$:

$$\psi(x,t) = M\overline{\psi}(\xi,\tau),\ \xi = M^2(x - x_{max}),\ \ \tau = \text{log}\, M.$$

New more convenient variables do not require the determination of the collapsing time t_0.

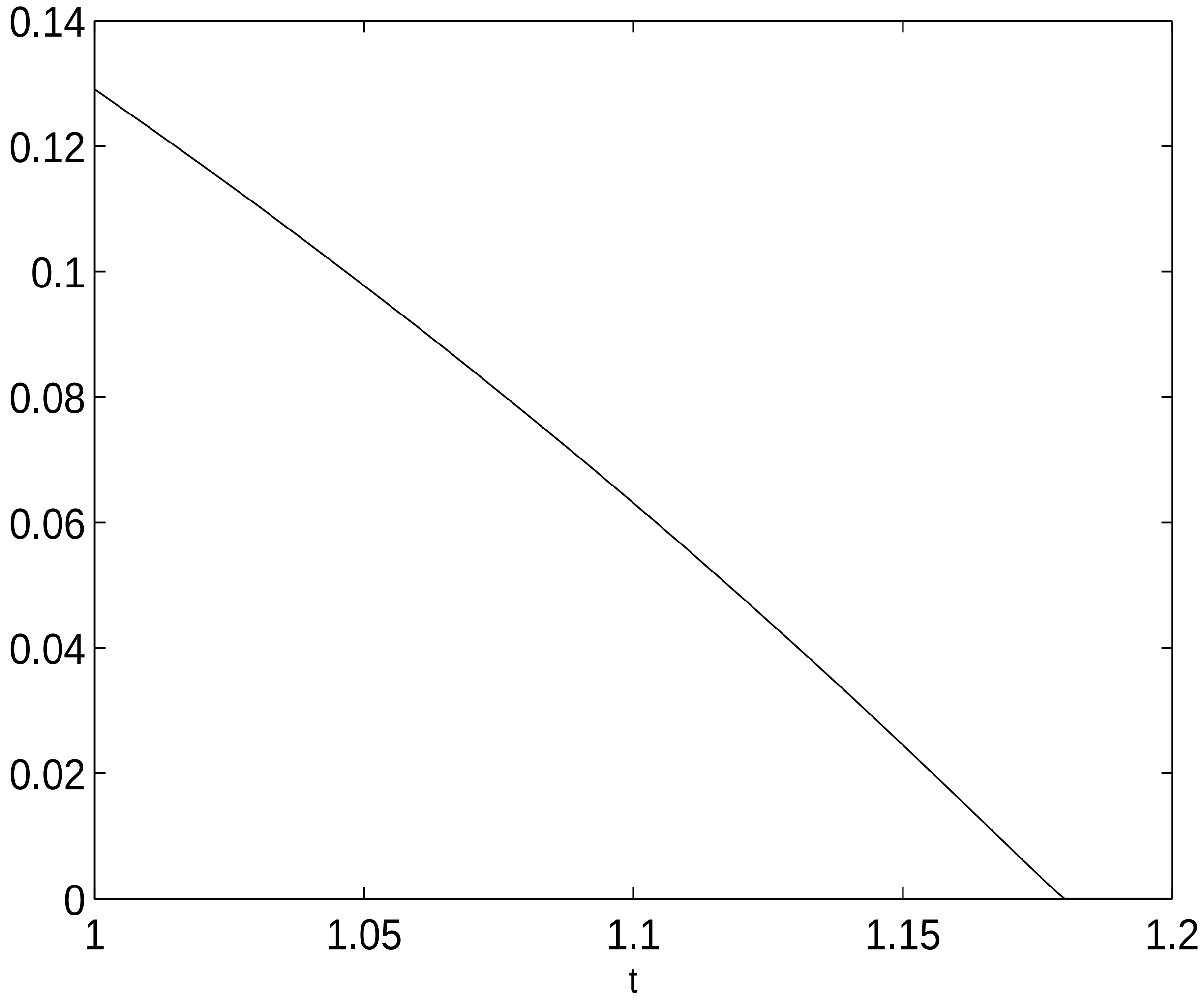
Collapse of solitons: numerical results

Initial (solid line) and final (dashed line) at $t = 2.7192$ dependences $|\overline{\psi}|$ on ξ.



Collapse of solitons: numerical results

Dependence of $1/\max|\psi|^4$ on time.



Conclusions

- We analyzed the stability of solitons. In particular, we have found a region of wave intensity, $N \leq N_{cr}$, where solitons stable in accordance with the Lyapunov theorem. Their stability is based on the boundedness of H from below.
- For solitons with $\chi < 0$ (defocusing Kerr nonlinearity) we get instability.
- Nonlinear stage of this instability leads to wave collapse. The form of the central part of the collapsing pulse in self-similar variables approaches the soliton shape. Asymptotically collapse tends to the critical one for NLS systems.

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