The Abdus Salam International Centre for Theoretical Physics

2472-4

Advanced Workshop on Nonlinear Photonics, Disorder and Wave Turbulence

15 - 19 July 2013

Anderson Localization in Multilayered Metamaterials

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Outline

- Localization
- Metamaterials
- Model and methods
- General properties
- Long wave anomaly
- Conclusion

Collaboration: A. Asatryan, L. Botten, M. Byrne, V. Freilikher, Yu. Kivshar,

R. McPhedran, I. Shadrivov (We)

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1D Localization

Lyapunov exponent

currentless field $$e(z) = e^{+}\mathrm{e}^{ikz} + e^{-}\mathrm{e}^{-ikz}$$

$$\begin{pmatrix} e^{+} \\ \\ e^{-} \end{pmatrix} = e^{\xi} \begin{pmatrix} e^{i\theta} \\ \\ e^{-i\theta} \end{pmatrix} \qquad \xi(0) = 1$$

$$\gamma(L) \equiv \frac{1}{l_\xi(L)} = \frac{\xi(L)}{L} \qquad \overline{\gamma}(L) \equiv \frac{1}{\overline{l}_\xi(L)} = \frac{\langle \xi(L) \rangle}{L}$$

Lyapunov exponent $$\gamma = \lim_{L\to\infty} \overline{\gamma}(L) = \lim_{L\to\infty} \gamma(L)$$

Localization length $$l_\xi = \gamma^{-1} = \lim_{l\to\infty} l_\xi(L) = \lim_{L\to\infty} \overline{l}_\xi(L)$$

Transmission

transmission coefficient $T(L)$ **transmittivity** $\mathcal{T}(L) = |T(L)|^2$

$$\frac{1}{l_T(L))} = -\frac{\ln|T(L)|}{L} = -\frac{\mathrm{Re}\ln T(L)}{L}$$

$l_T(L)$ **- transmission length of a finite sample on a realization**

$$\frac{1}{\bar{l}_T(L)} = \left\langle \frac{1}{l_T(L)} \right\rangle$$

$\bar{l}_T(L)$ **- transmission length of a finite sample**

localization length $l_T \equiv \lim_{L\to\infty} l_T(L) = \lim_{L\to\infty} \bar{l}_T(L) = l_\xi$

$L \gg \bar{l}_T(L) \to$ **localized regime** $|T_L| \ll 1$ $|R_L| \approx 1$

$L \ll \bar{l}_T(L) \to$ **ballistic regime** $|T_L| \approx 1$ $|R_L| \ll 1$

continuous system $|T_L|^2 = \dfrac{4}{e^{2\xi_c(L)} + e^{2\xi_s(L)} + 2}$

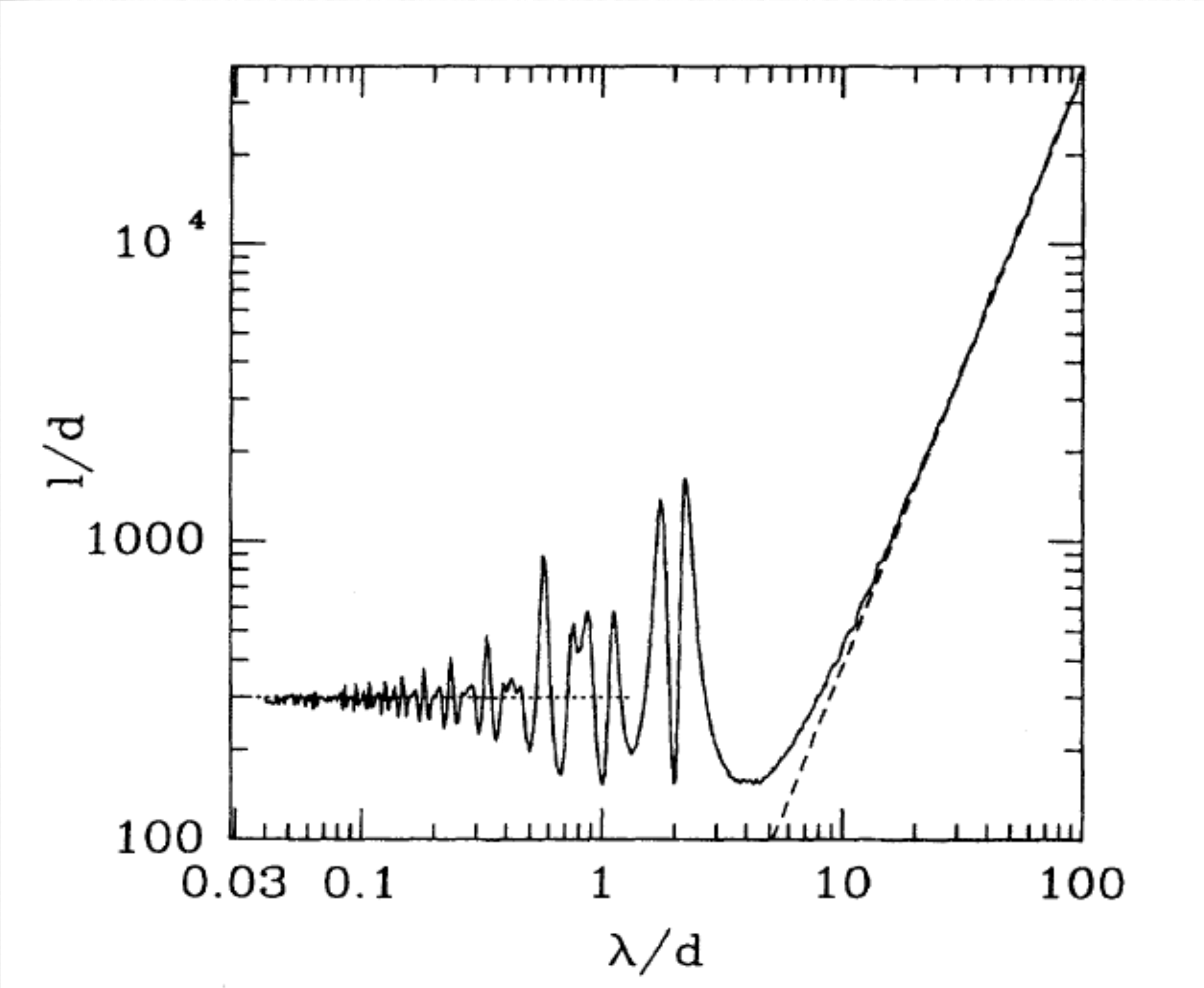
Multilayered random stack

$$l_T \to \text{const} \quad \lambda \to 0$$

Balluni & Willemsen 1985

$$l_T \propto \lambda^2 \quad \lambda \to \infty$$

Ping Sheng et. al. 1986



$$N = 10^4$$

de Sterke & McPhedran 1993

Metamaterials

Left -handed materials Veselago 1968

$$\varepsilon(\omega) < 0, \ \mu(\omega) < 0 \ \Rightarrow \ \mathbf{k} \cdot (\mathbf{E} \times \mathbf{H}) < \mathbf{0}$$

$$\underset{\longrightarrow}{\mathbf{E} \times \mathbf{H}} \qquad \underset{\longleftarrow}{\mathbf{k}} \qquad \mathbf{v}_g \cdot \mathbf{v}_p < 0$$

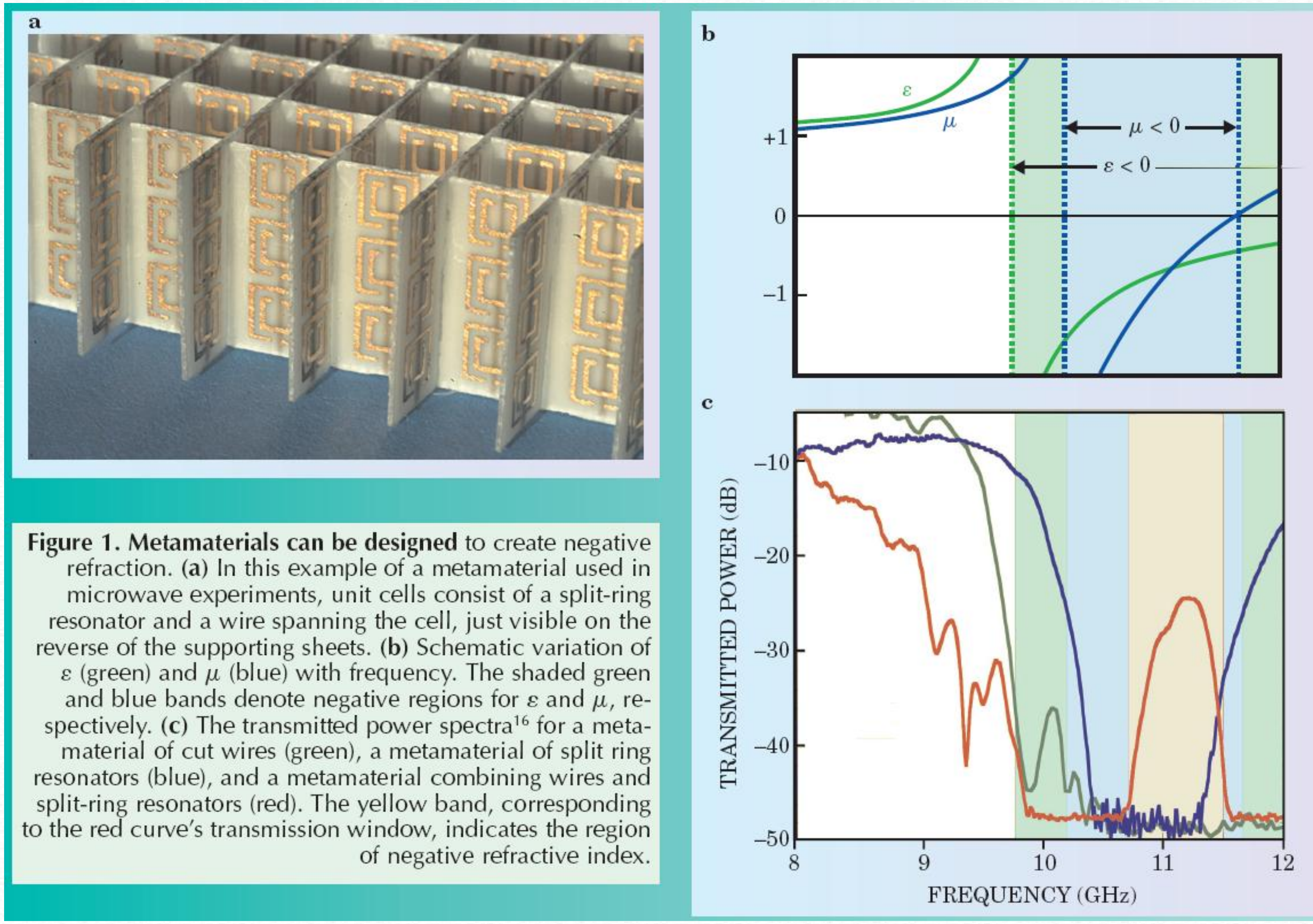


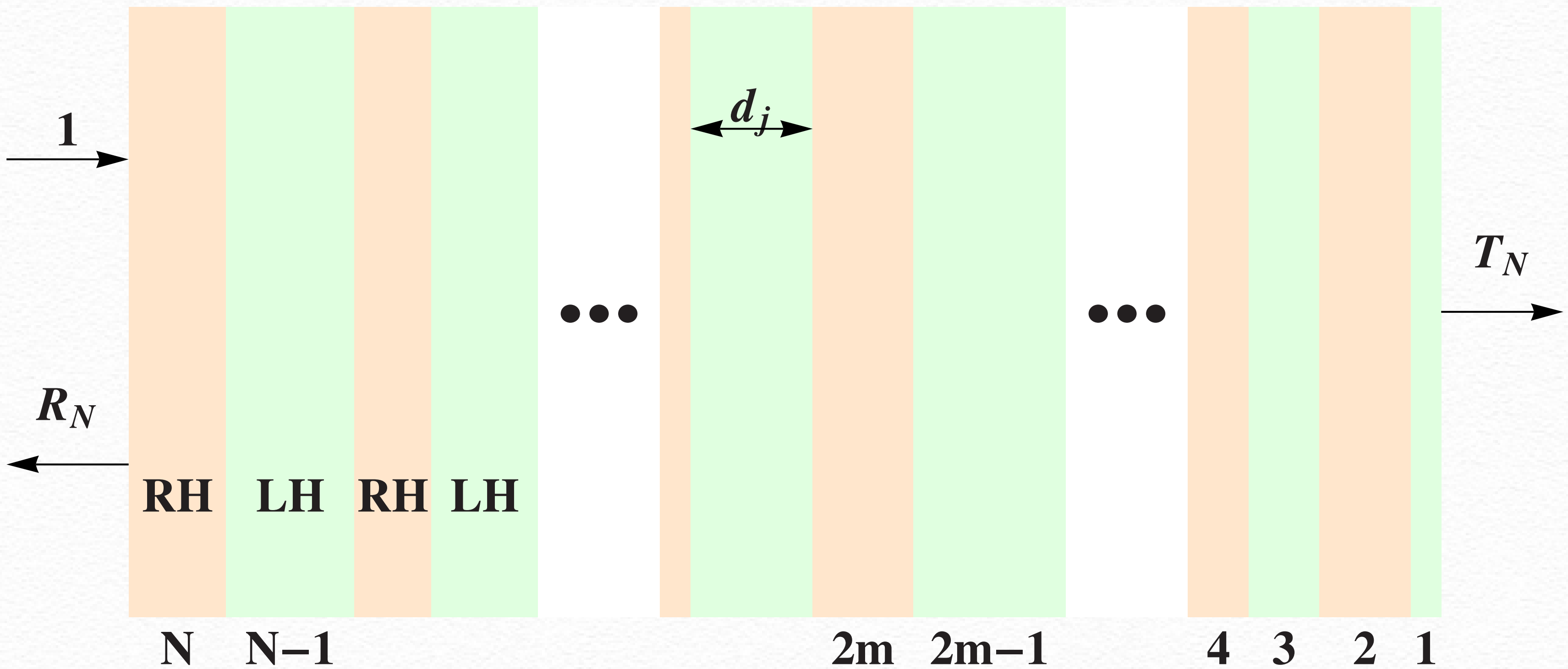
Figure 1. Metamaterials can be designed to create negative refraction. **(a)** In this example of a metamaterial used in microwave experiments, unit cells consist of a split-ring resonator and a wire spanning the cell, just visible on the reverse of the supporting sheets. **(b)** Schematic variation of ε (green) and μ (blue) with frequency. The shaded green and blue bands denote negative regions for ε and μ, respectively. **(c)** The transmitted power spectra[16] for a metamaterial of cut wires (green), a metamaterial of split ring resonators (blue), and a metamaterial combining wires and split-ring resonators (red). The yellow band, corresponding to the red curve's transmission window, indicates the region of negative refractive index.

Expectations

Anderson localization originates with interference of multiply scattered waves. In the case of comparatively long waves (where wave length is of order or larger than the layer thickness), the opposite signs of the phase velocity and group velocity in left-handed material result in well pronounced partial or complete cancellation of the phase accumulation in multilayered mixed stacks containing both normal and left-handed layers. This cancellation suppresses the interference and the localization itself and increases the stack length needed for manifestation of sharp transmission resonances.

Model and methods

The simplest model



$$< d_j >= 1$$

$$< L >= N$$

$$d_j = 1 + \delta_j^{(d)} \qquad \mu_j = \pm 1 \qquad \varepsilon_j = \pm(1 + \delta_j^{(\varepsilon)})^2$$

$$\nu_j = \pm(1 + \delta_j^{(\varepsilon)}) \qquad Z_j = \sqrt{\mu_j/\varepsilon_j} = 1/(1 + \delta_j^{(\varepsilon)})$$

$$a = d, \varepsilon \qquad \delta_j^{(a)} \sim U[-Q_a, Q_a] \qquad Q_a \ll 1$$

weak disorder

Methods

$l_T(L), \quad \bar{l}_T(L) \quad l_T \quad -$ **weak scattering approximation (WSA)** $|r_J| \ll 1$

$$T_j = \frac{T_{j-1} t_j}{1 - R_{j-1} r_j} \qquad R_j = r_j + \frac{R_{j-1} t_j^2}{1 - R_{j-1} r_j}$$

$$\ln T_j = \ln T_{j-1} + \ln t_j + R_{j-1} r_j \qquad R_j = r_j + R_{j-1} t_j^2$$

Methods T-matrix approach

$l_T(L), \quad \bar{l}_T(L) \quad l_T \quad -$ **weak scattering approximation (WSA)** $|r_n| \ll 1$

$$T_j = \frac{T_{j-1}t_j}{1 - R_{j-1}r_j} \qquad R_j = r_j + \frac{R_{j-1}t_j^2}{1 - R_{j-1}r_j}$$

$$\ln T_j = \ln T_{j-1} + \ln t_j + R_{j-1}r_j \qquad R_j = r_j + R_{j-1}t_j^2$$

$$l_\xi = \gamma^{-1} \qquad \gamma = \lim_{n\to\infty} \frac{\xi_n}{n} \qquad n = 2j$$

$$\left.\begin{aligned} \xi_n - \xi_{n-1} &= \Phi(\theta_{n-1}) \\ \theta_n - \theta_{n-1} &= \Theta\left(\theta_{n-1}\right) \end{aligned}\right\}$$

perturbation theory

$\gamma = \lim_{n\to\infty} \left(\xi_n - \xi_{n-1}\right) = \langle \Phi\left(\theta\right) \rangle_{\mathrm{st}}$ **Fokker-Plank equation for** $\rho\left(\theta\right)$

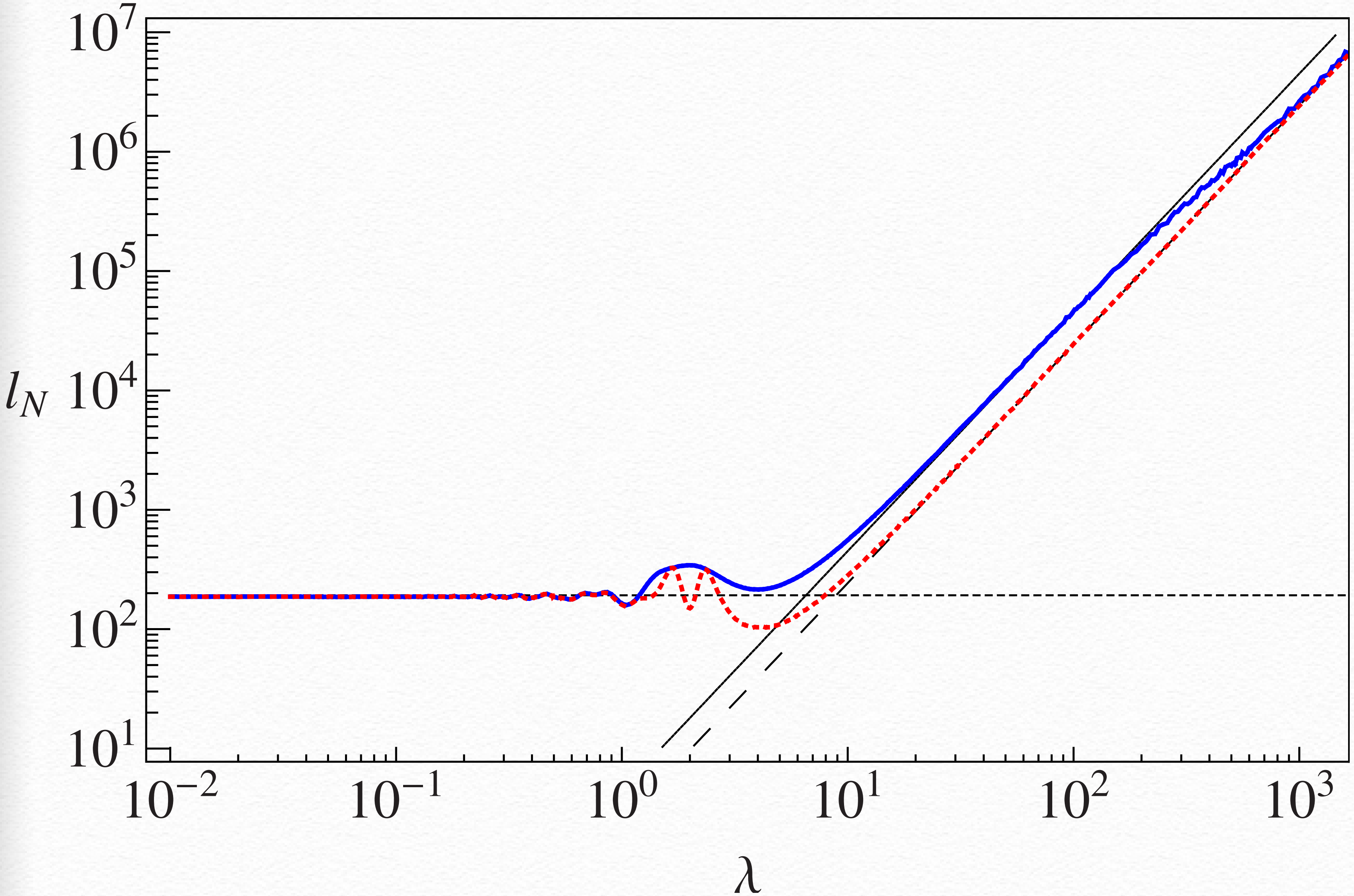
General properties

We *PRB* 81, 075124 (2010)

$$Q_\varepsilon = 0.25, \quad Q_d = 0.2$$

Suppression of localization

$$N = 10^5$$



$$l_T = \frac{3\lambda^2}{2\pi^2 Q_\varepsilon^2} \frac{3Q_\varepsilon^2 + 2Q_d^2}{Q_\varepsilon^2 + 2Q_d^2}$$

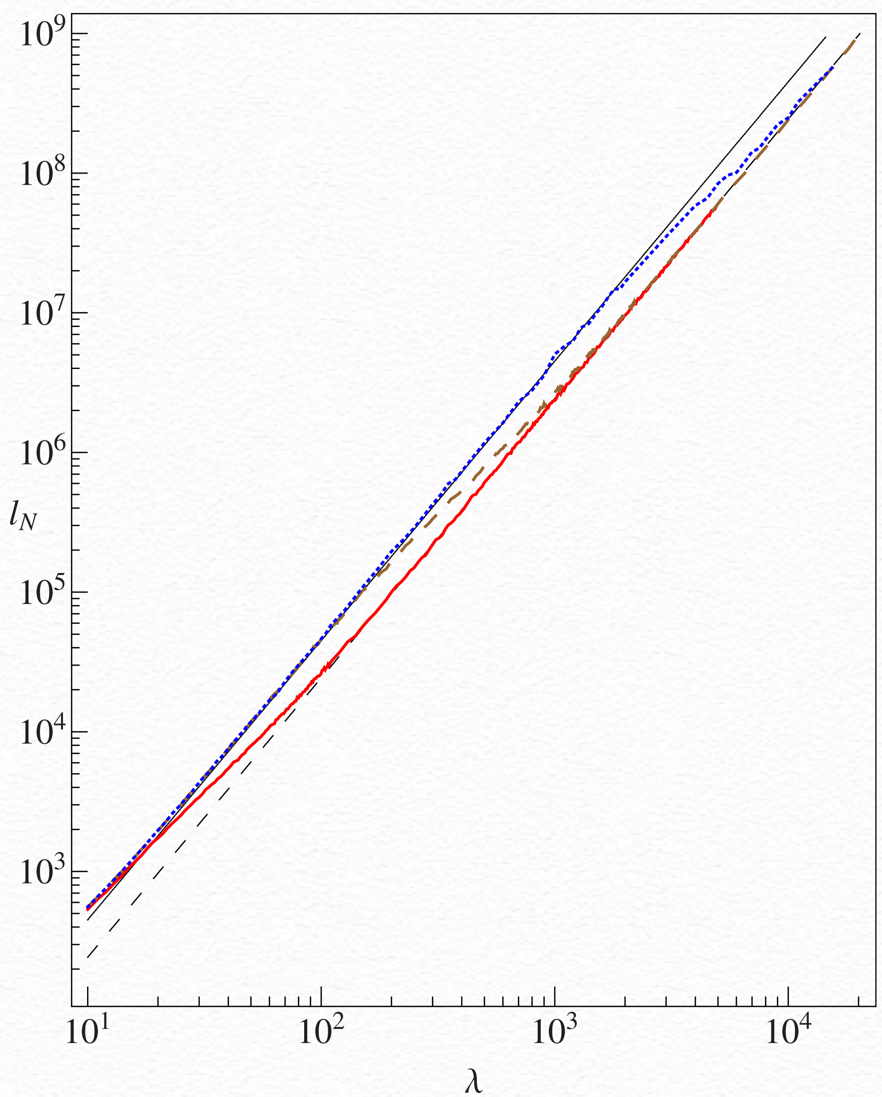
$$l_T = \frac{3\lambda^2}{2\pi^2 Q_\varepsilon^2}$$

blue solid curve - $\bar{l}_T(L)$ mixed alternating stack

red dashed curve - $\bar{l}_T(L)$ normal stack

} direct simulation and WSA

Long wave behavior: localization-ballistic crossover



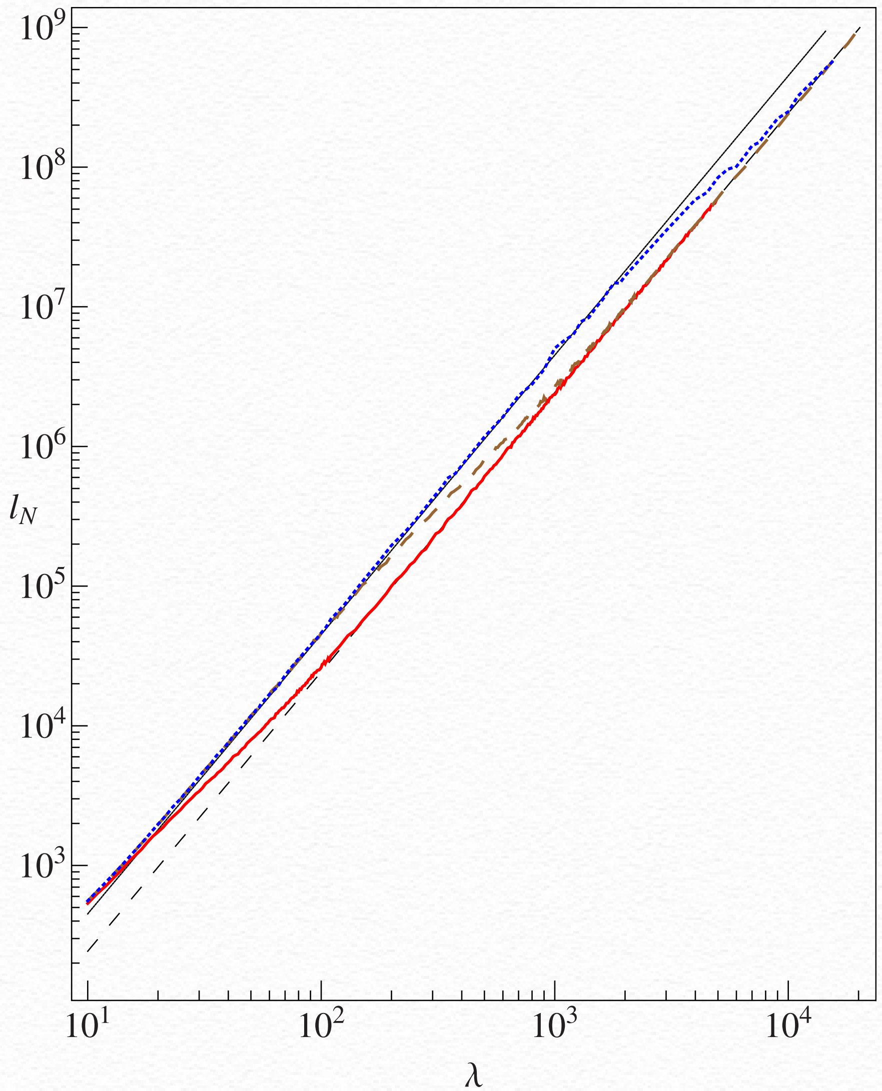
$$Q_\varepsilon = 0.25, \quad Q_d = 0.2$$

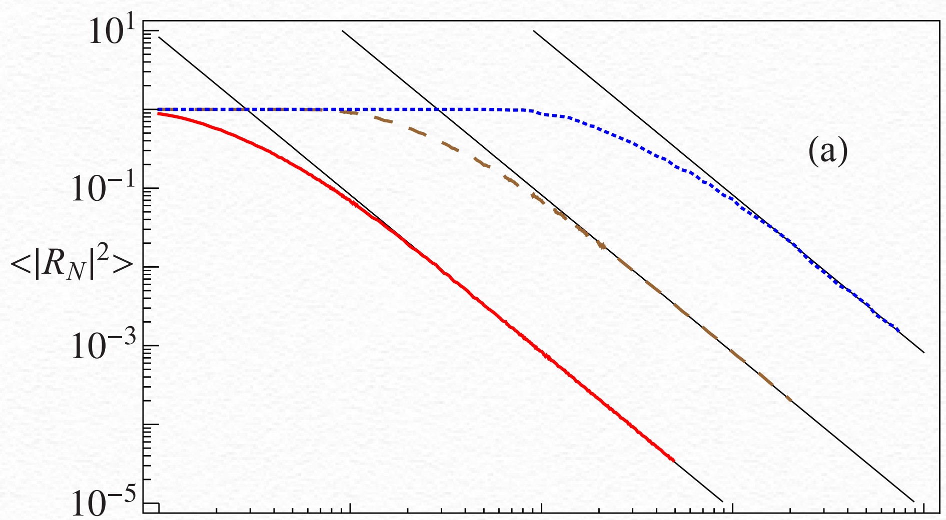
red solid line - $N = 10^3$

brown dashed line - $N = 10^5$

blue dotted line - $N = 10^7$

Long wave behavior: localization-ballistic crossover





$$Q_\varepsilon = 0.25, \quad Q_d = 0.2$$

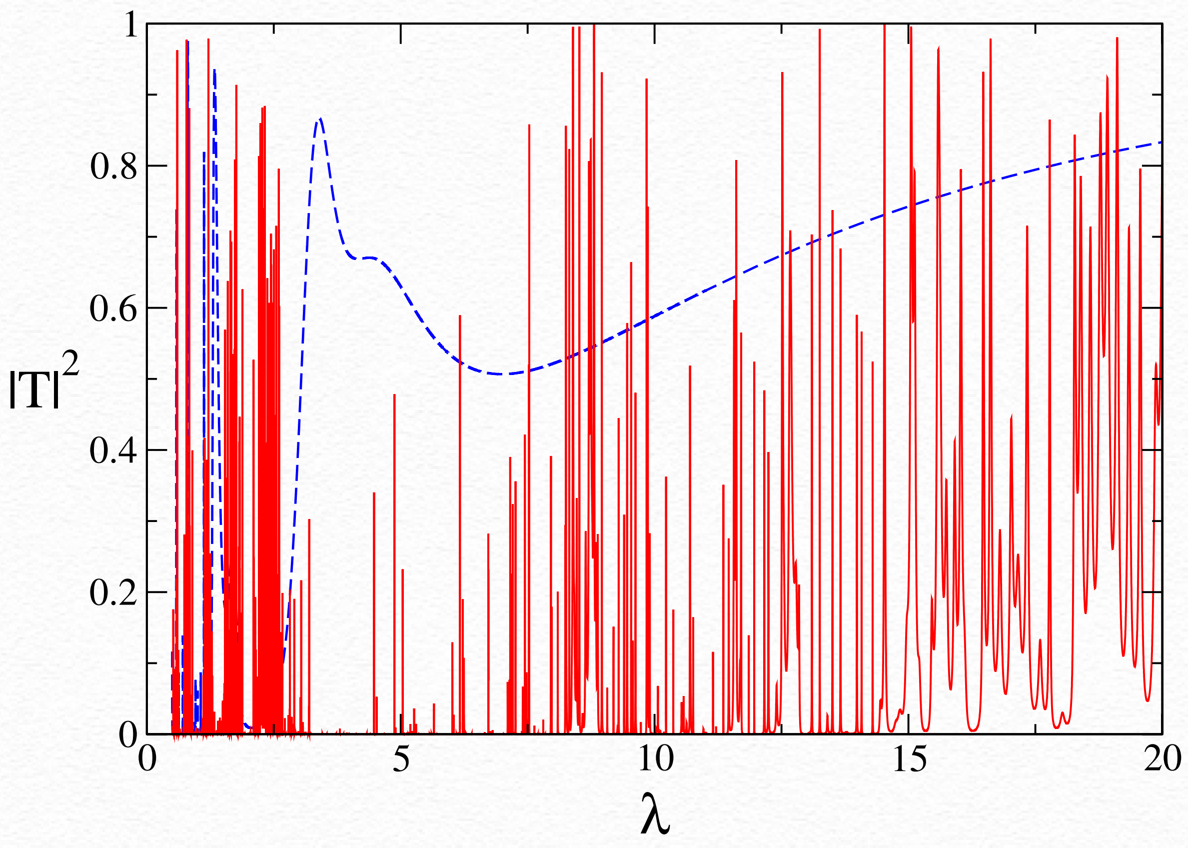
red solid line - $N = 10^3$

brown dashed line - $N = 10^5$

blue dotted line - $N = 10^7$

Transmission resonances

$$Q_\varepsilon = 0.25, \quad Q_d = 0$$

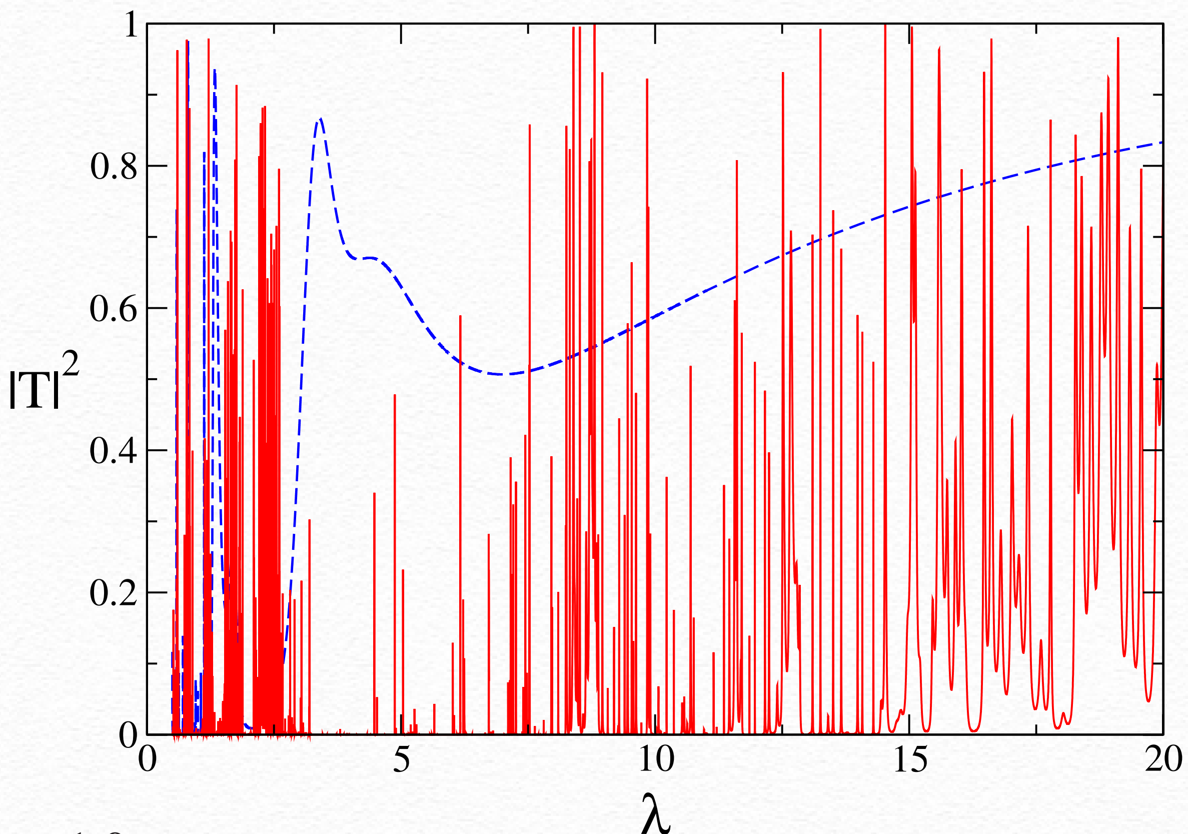


red solid line - normal stack $N = 10^3$

blue dotted line -mixed stack $N = 10^3$

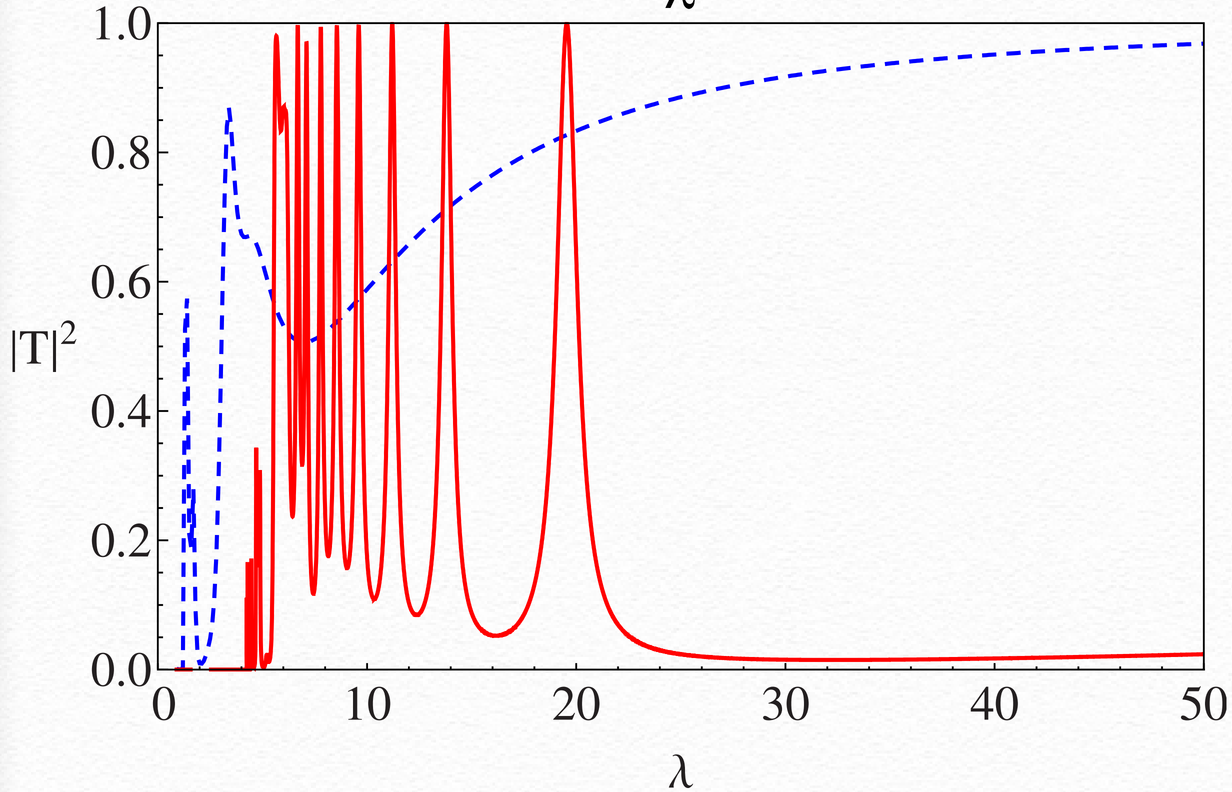
Transmission resonances

$$Q_\varepsilon = 0.25, \quad Q_d = 0$$



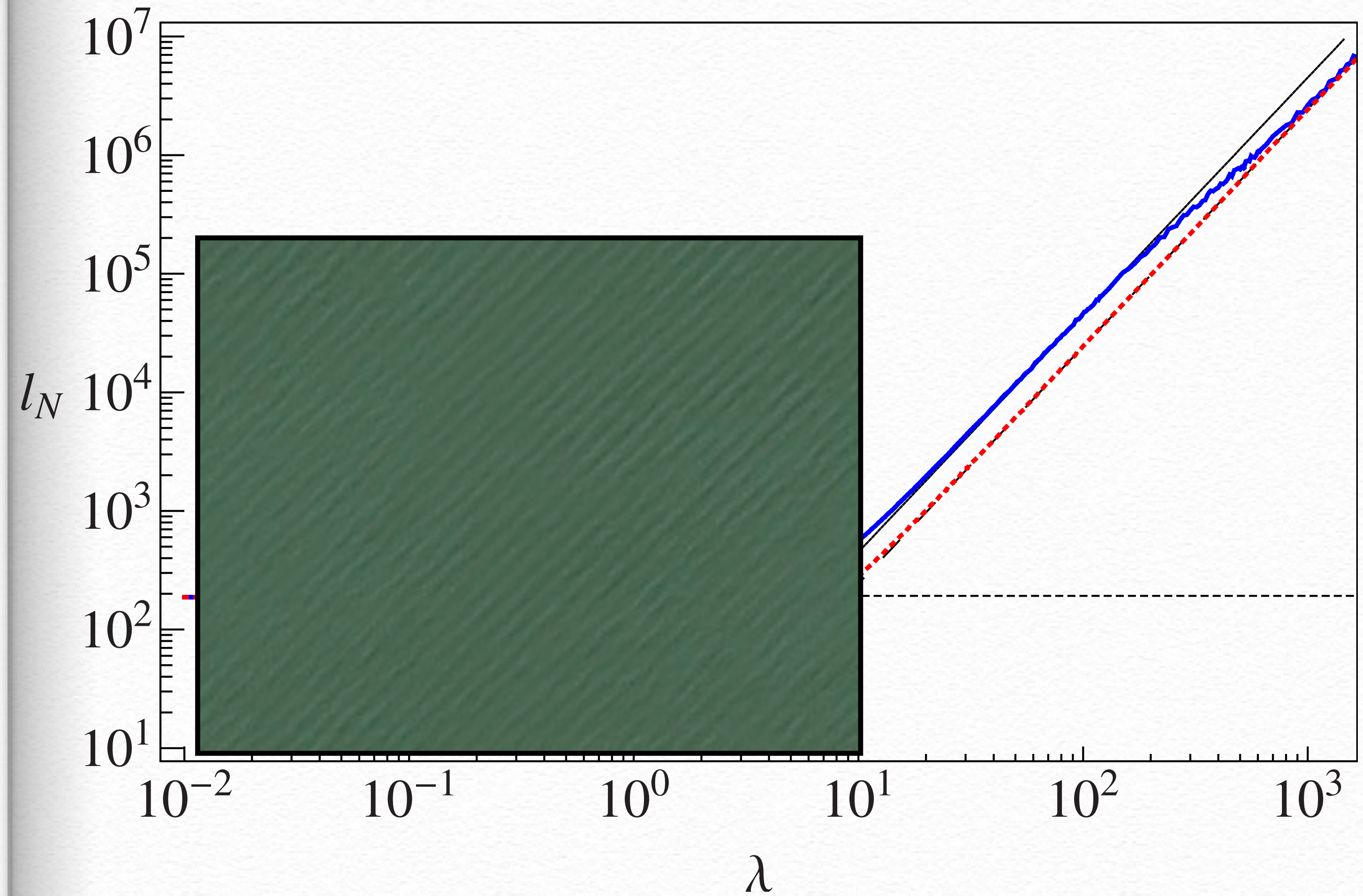
red solid line - normal stack $N = 10^3$

blue dotted line -mixed stack $N = 10^3$



red solid line -mixed stack $N = 10^5$

Transmission length and Lyapunov exponent



long wave region

$$\lambda \geq 1$$

$$l_T = \frac{3\lambda^2}{2\pi^2 Q_\varepsilon^2 d^2} \frac{3Q_\varepsilon^2 + 2Q_d^2}{Q_\varepsilon^2 + 2Q_d^2}$$

$$l_\xi = \frac{3\lambda^2}{2\pi^2 Q_\varepsilon^2 d^2}$$

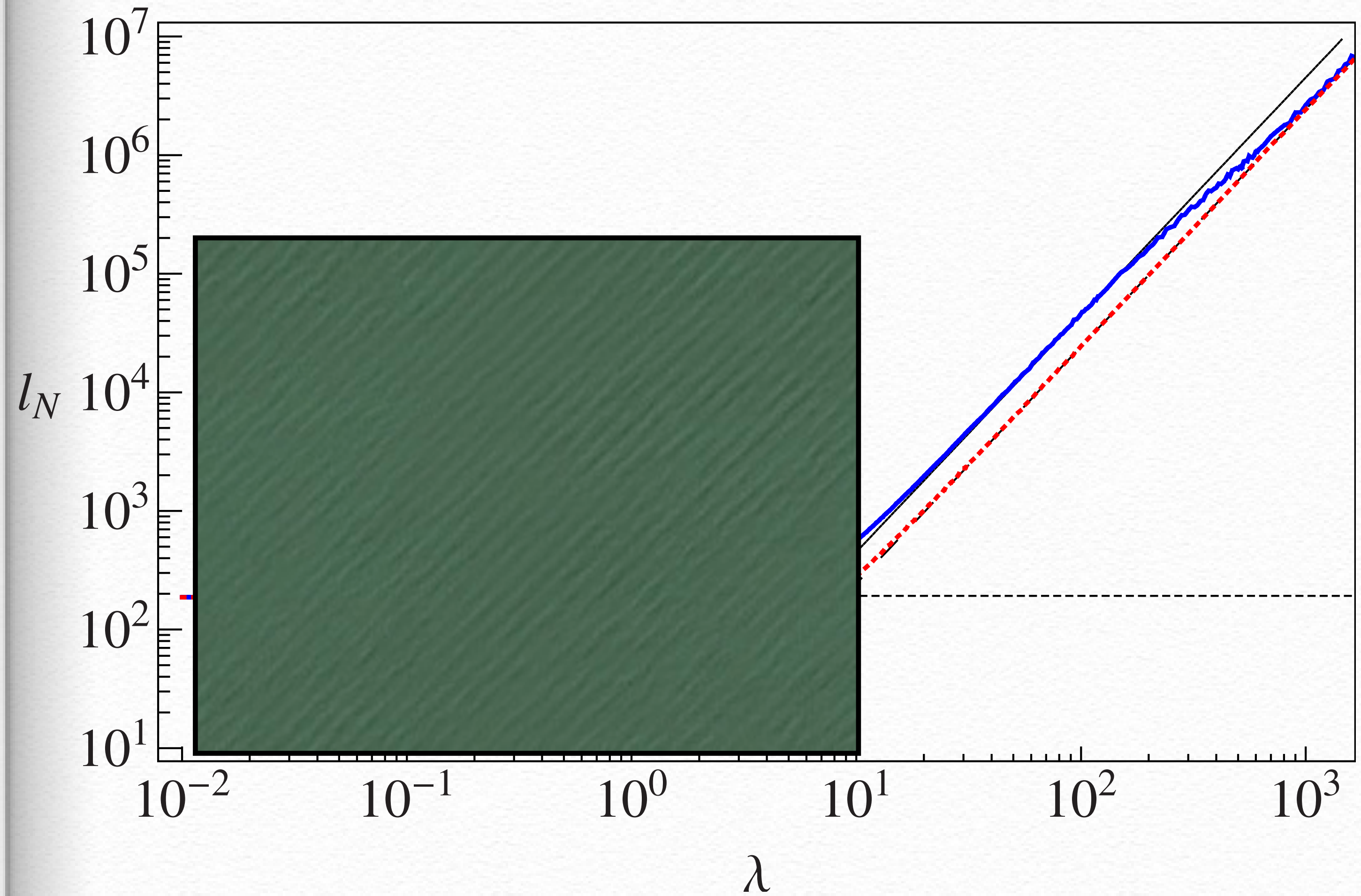
blue solid curve - $\bar{l}_T(L)$ **mixed alternating stack**

red dashed curve - $\begin{cases} \bar{l}_T(L) = l_\xi & \textbf{normal stack} \\ l_\xi & \textbf{mixed alternating stack} \end{cases}$

direct simulation and analytic WSA and PT

mixed alternating stack $l_\xi \neq l_T$ **!**

Transmission length and Lyapunov exponent



long wave region

$$\lambda \geq 1$$

$$l_T = \frac{3\lambda^2}{2\pi^2 Q_\varepsilon^2 d^2} \frac{3Q_\varepsilon^2 + 2Q_d^2}{Q_\varepsilon^2 + 2Q_d^2}$$

$$l_\xi = \frac{3\lambda^2}{2\pi^2 Q_\varepsilon^2 d^2}$$

blue solid curve - $\bar{l}_T(L)$ **mixed alternating stack**

red dashed curve - $\left\{ \begin{array}{ll} \bar{l}_T(L) = l_\xi & \textbf{normal stack} \\ l_\xi & \textbf{mixed alternating stack} \end{array} \right.$

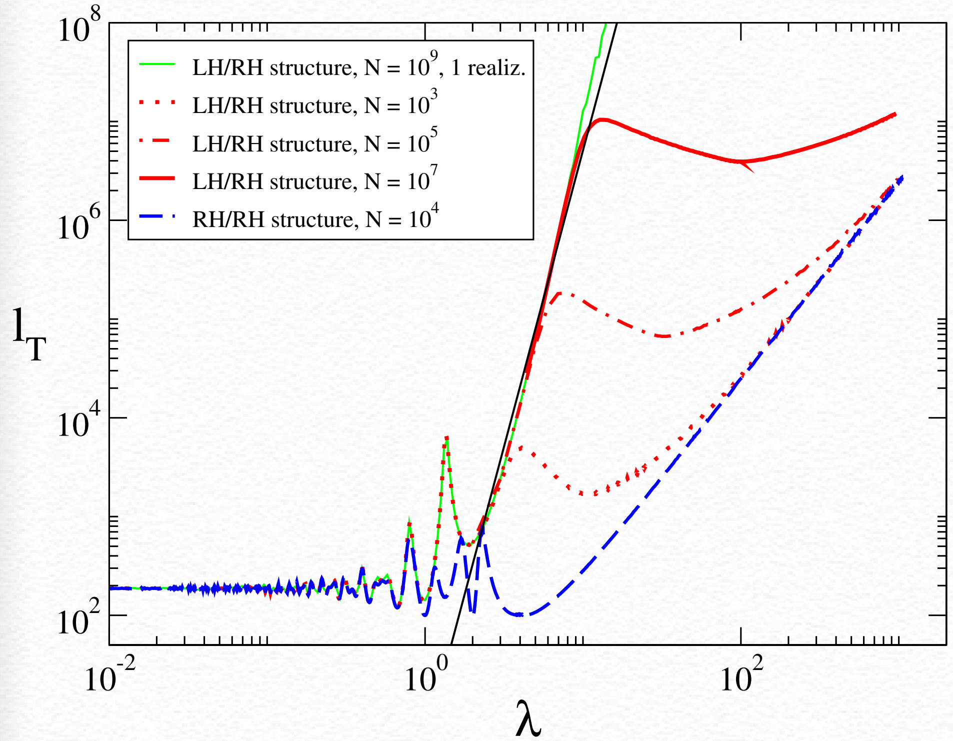
direct simulation and analytic WSA and PT

mixed alternating stack $l_\xi \neq l_T$ **!**

$$\mathcal{T}_N \equiv |T_N|^2 = \frac{4}{e^{2\xi_c(N)} + e^{2\xi_s(N)} + 2e^{\xi_c(N)+\xi_s(N)} \sin\left(\theta_c(N) - \theta_s(N)\right)}$$

Long wave anomaly $Q_d = 0$ $Q_\varepsilon = 0.25$

We ***PRL*** **99, 193902 (2007);** ***PRB*** **81, 075124 (2010)**



LH/RH structure, N = 10^9, $l_T(N)$

LH/RH structure, N = 10^3

LH/RH structure, N = 10^5

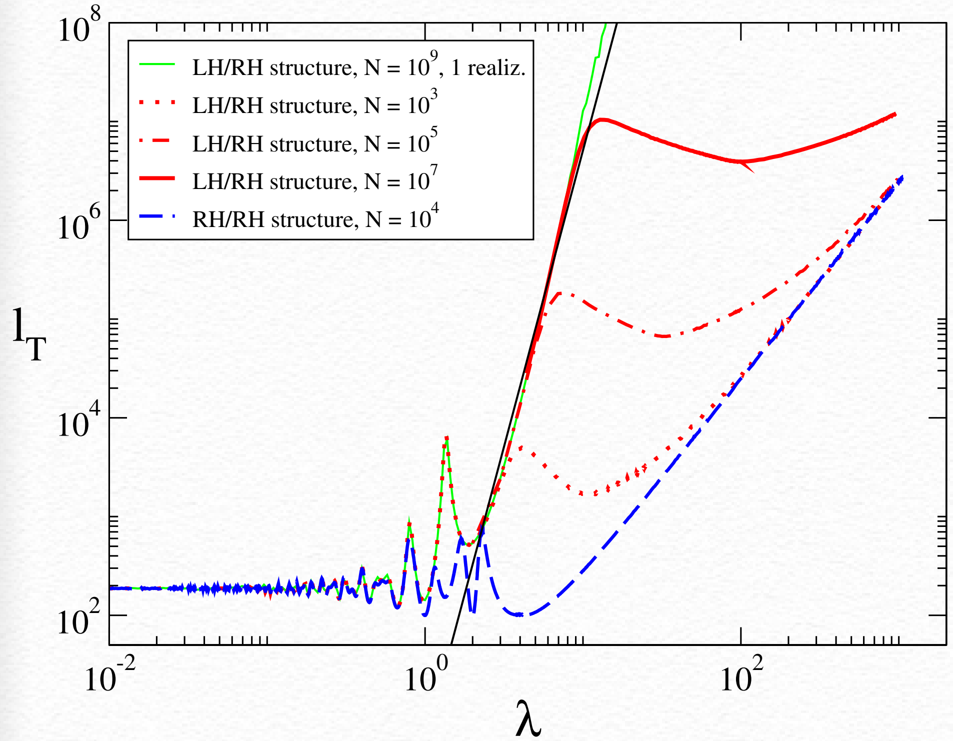
LH/RH structure, N = 10^7

RH/RH structure, N = 10^4

$\bar{l}_T(N)$

Long wave anomaly $\quad Q_d = 0 \quad Q_\varepsilon = 0.25$

We ***PRL*** **99, 193902 (2007);** ***PRB*** **81, 075124 (2010)**



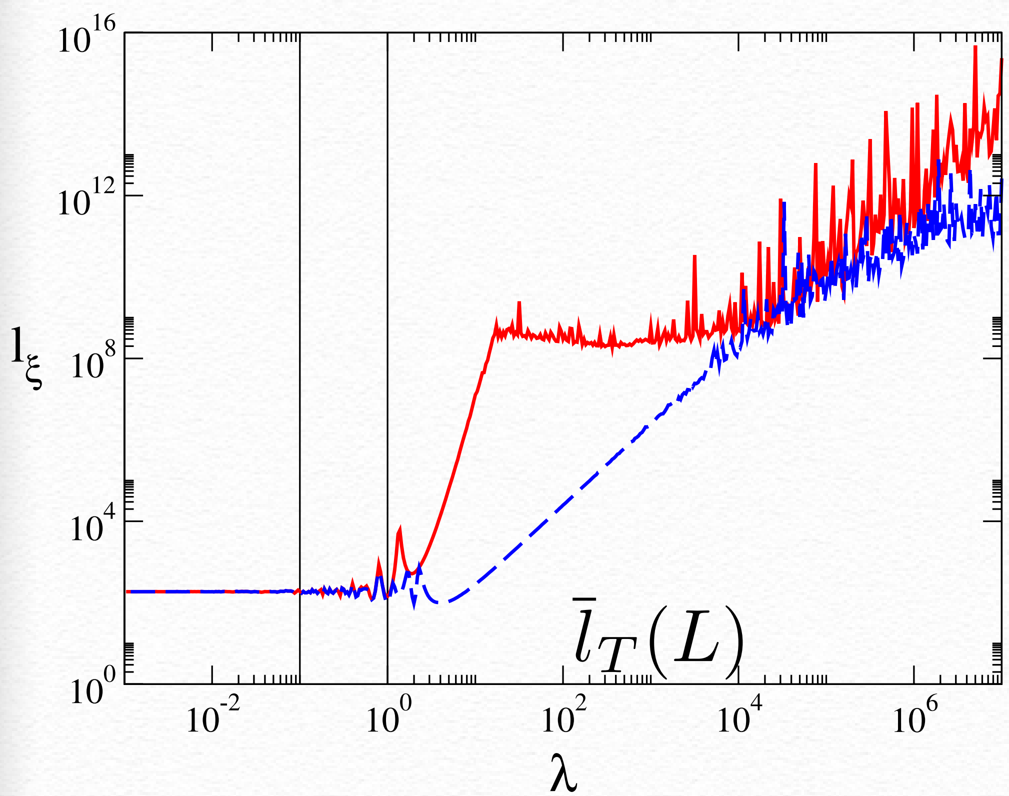
LH/RH structure, N = 10^9, $\quad l_T(N)$

LH/RH structure, N = 10^3

LH/RH structure, N = 10^5

LH/RH structure, N = 10^7 $\quad \bar{l}_T(N)$

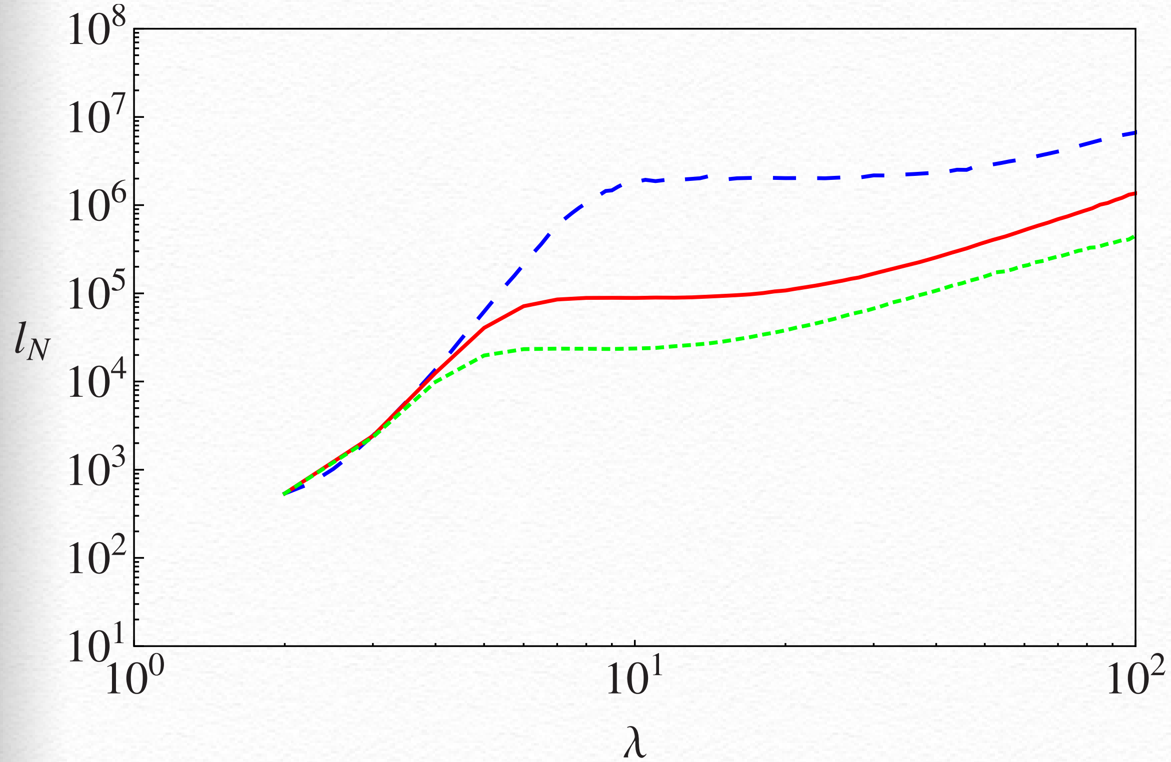
RH/RH structure, N = 10^4



$l_\xi(N)$ {
red solid curve - mixed stack
blue dashed curve - normal stack

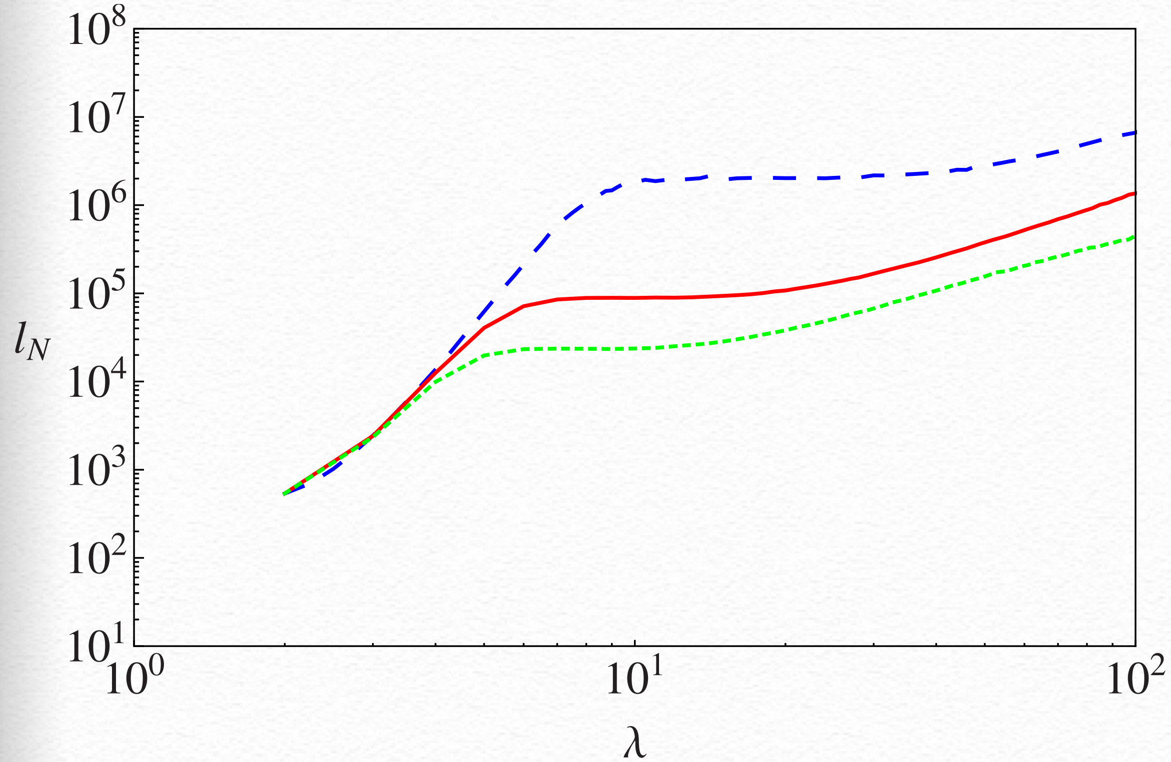
$N = 10^9$ **direct simulation**

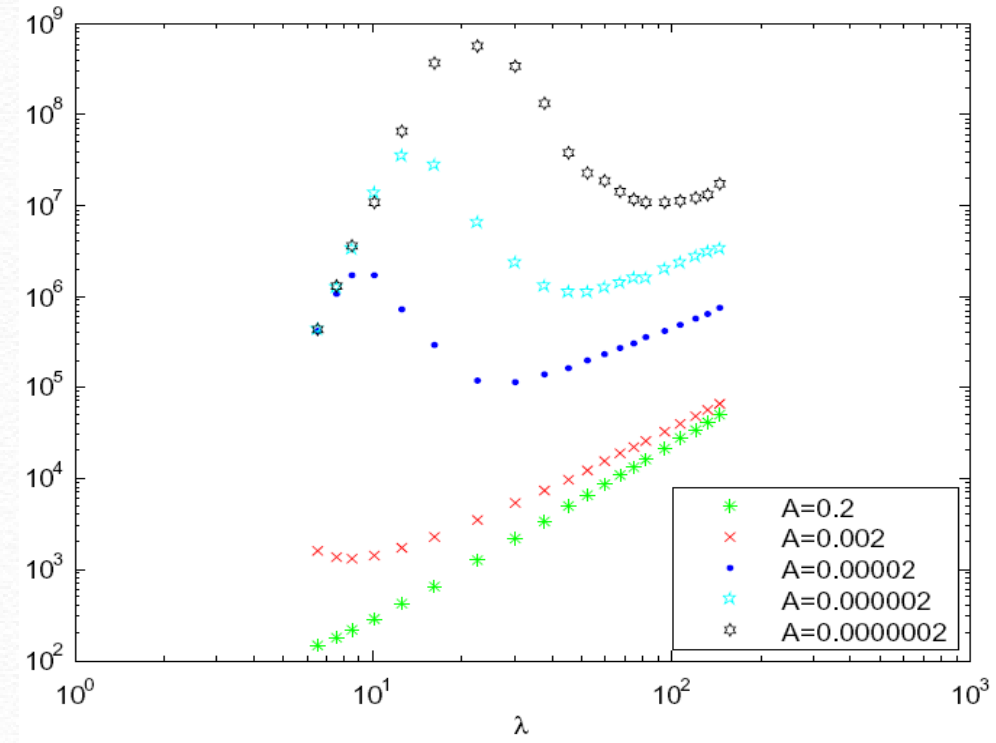
$Q_\varepsilon = 0.25,\ N = 10^9$



$$Q_d = \begin{cases} 0.01 & \text{dotted green line} \\ 0.05 & \text{red solid line} \\ 0.001 & \text{blue dashed line} \end{cases}$$

$Q_\varepsilon = 0.25, \;\; N = 10^9$



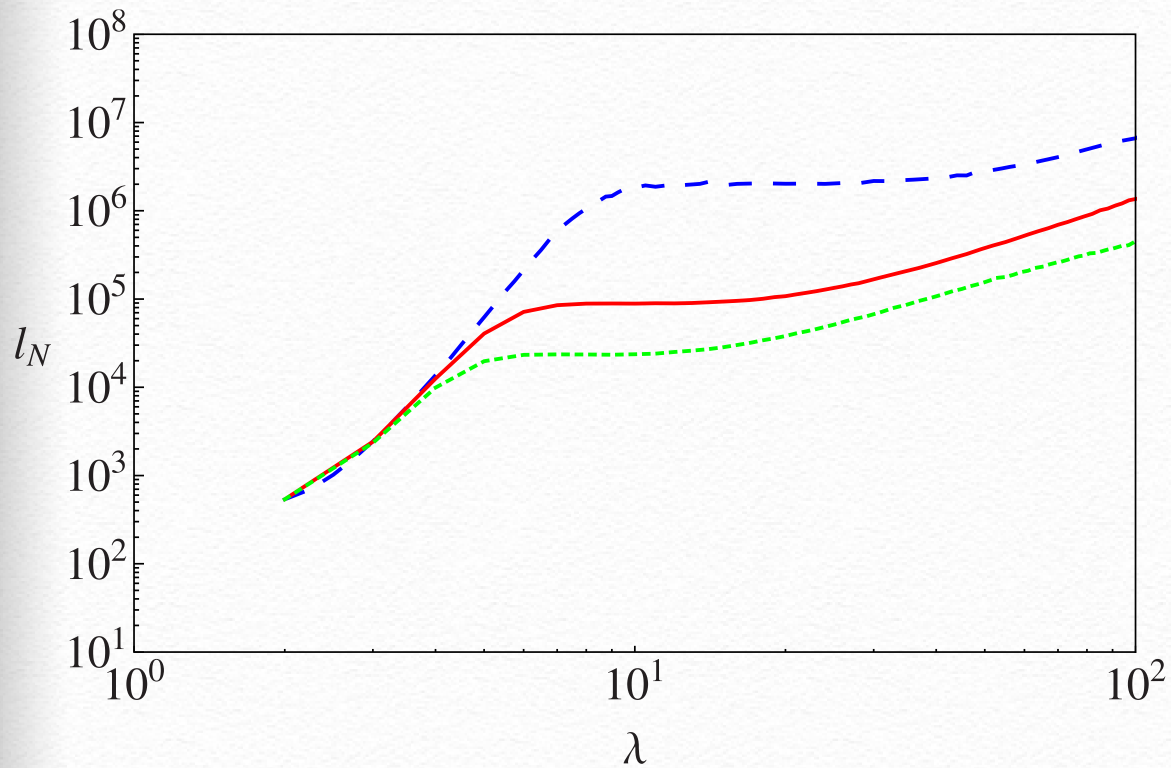


$$Q_d = \begin{cases} 0.01 & \text{dotted green line} \\ 0.05 & \text{red solid line} \\ 0.001 & \text{blue dashed line} \end{cases}$$

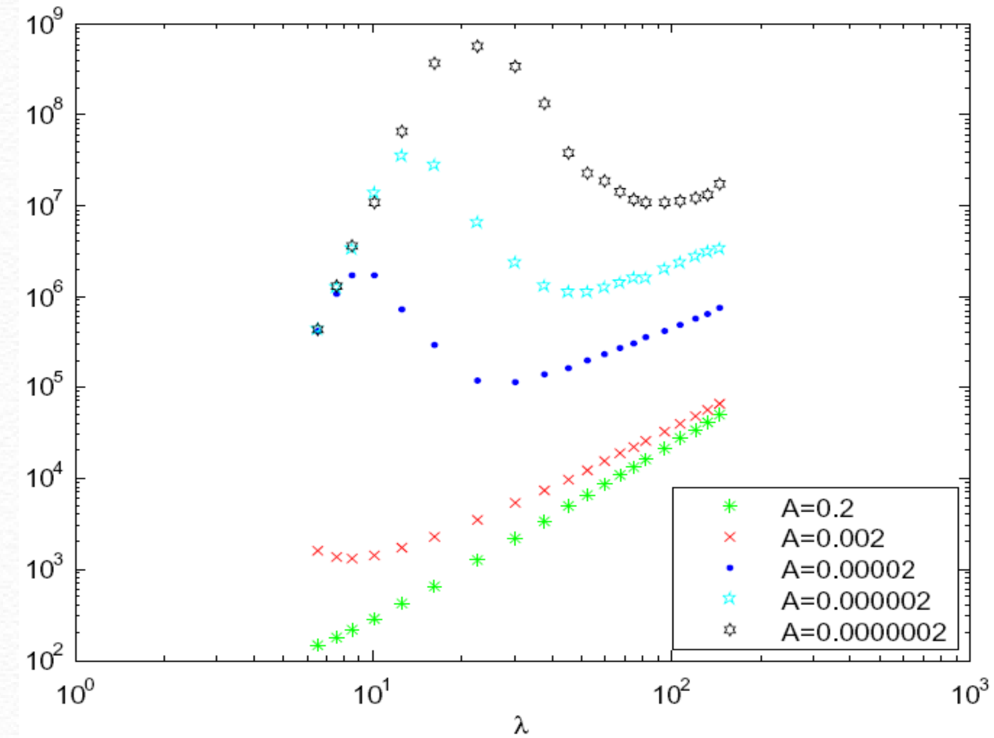
A=0.2
A=0.002
A=0.00002
A=0.000002
A=0.0000002

$$A = \frac{|d_L - d_R|}{d_L + d_R}$$

$Q_{\varepsilon} = 0.25, \;\; N = 10^9$

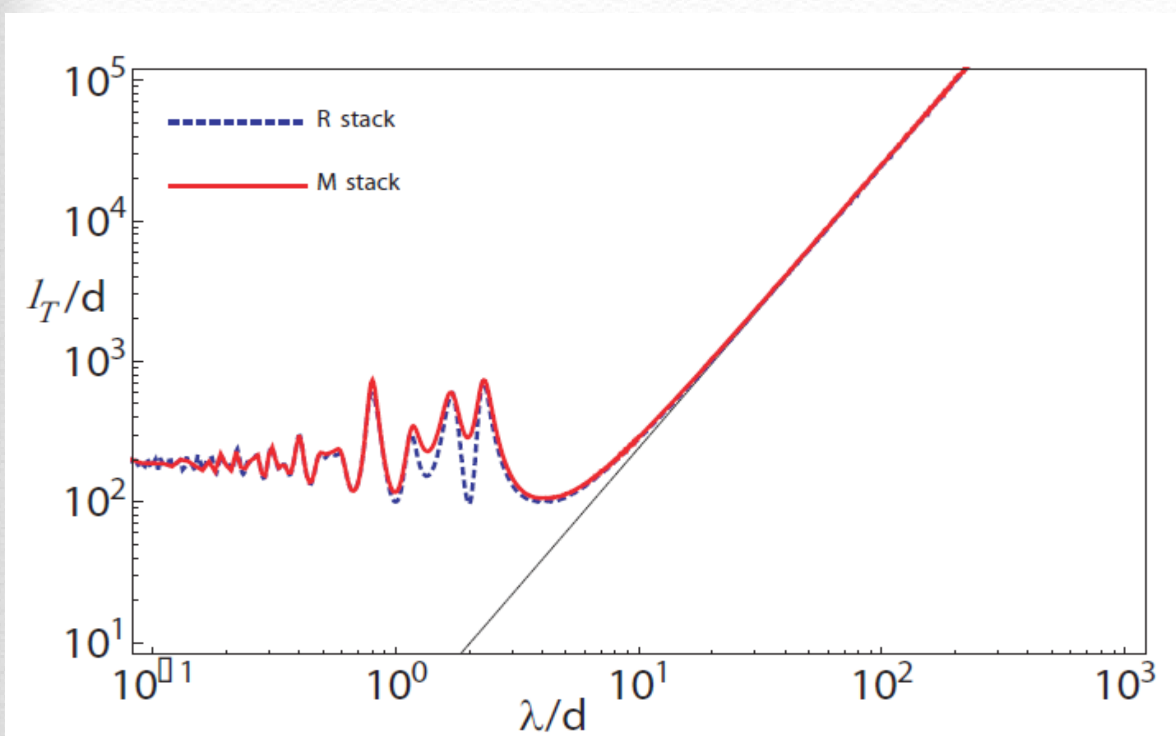


$$Q_d = \begin{cases} 0.01 & \text{dotted green line} \\ 0.05 & \text{red solid line} \\ 0.001 & \text{blue dashed line} \end{cases}$$

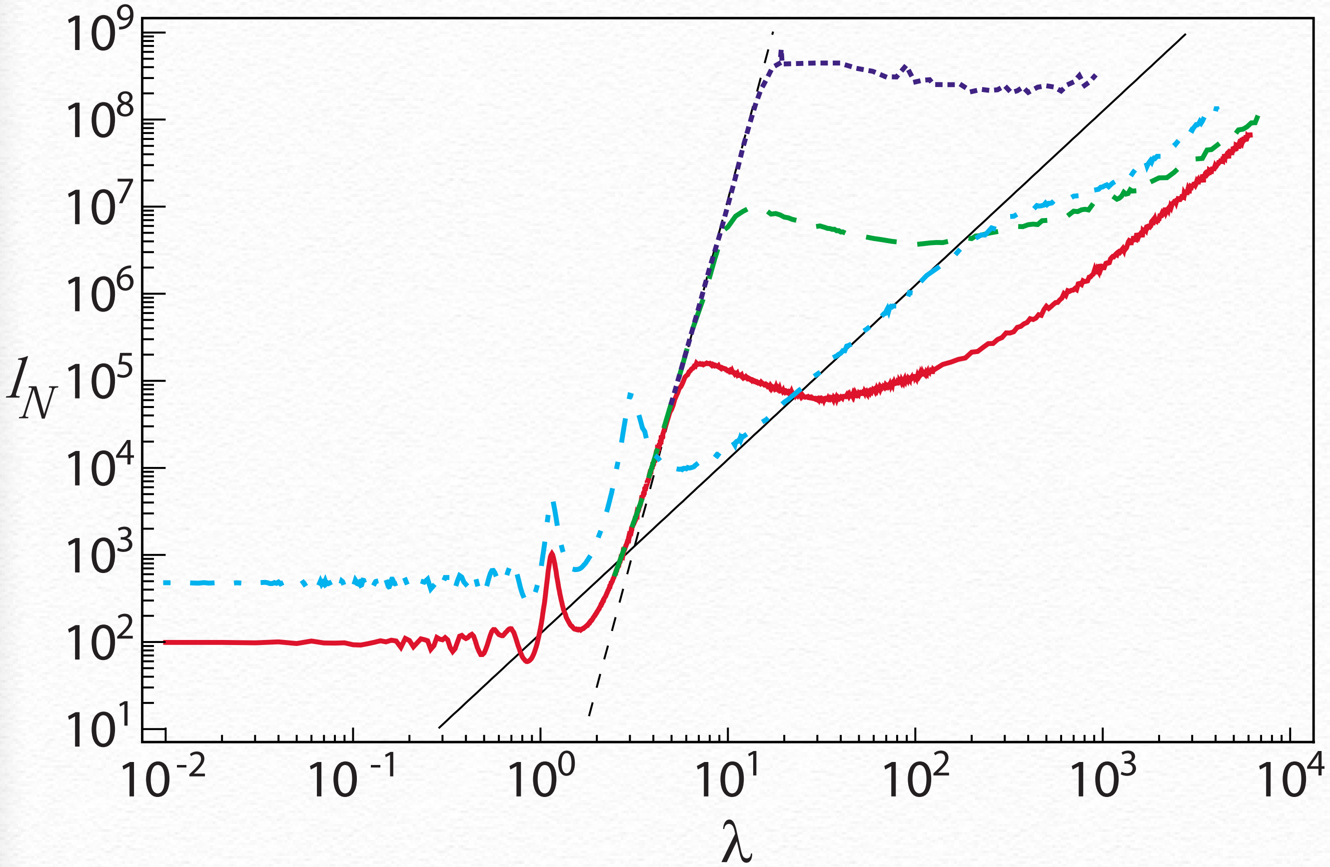


A=0.2
A=0.002
A=0.00002
A=0.000002
A=0.0000002

$$A = \frac{|d_L - d_R|}{d_L + d_R}$$



random arrangement of the layers; numbers of normal ad left-handed layers coincide



We ***PRB*** **82 205124 (2010)**

oblique incidence $\theta = \pi/6$

$p-$polarization

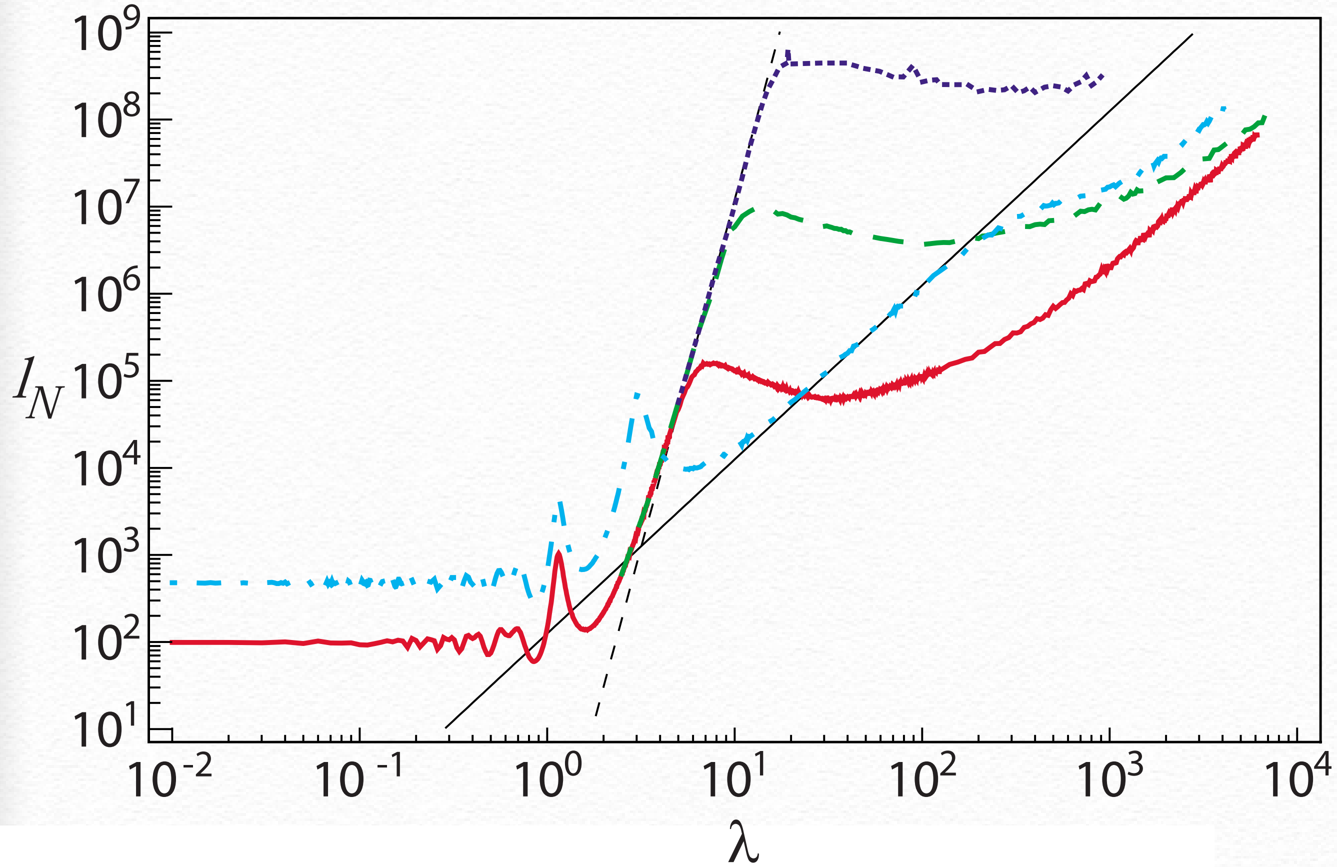
red solid curve $N = 10^5$

green dashed curve $N = 10^7$

blue dotted curve $N = 10^9$

$s-$polarization

cyan dashed-dotted curve $N = 10^6$



We *PRB* 82 205124 (2010)

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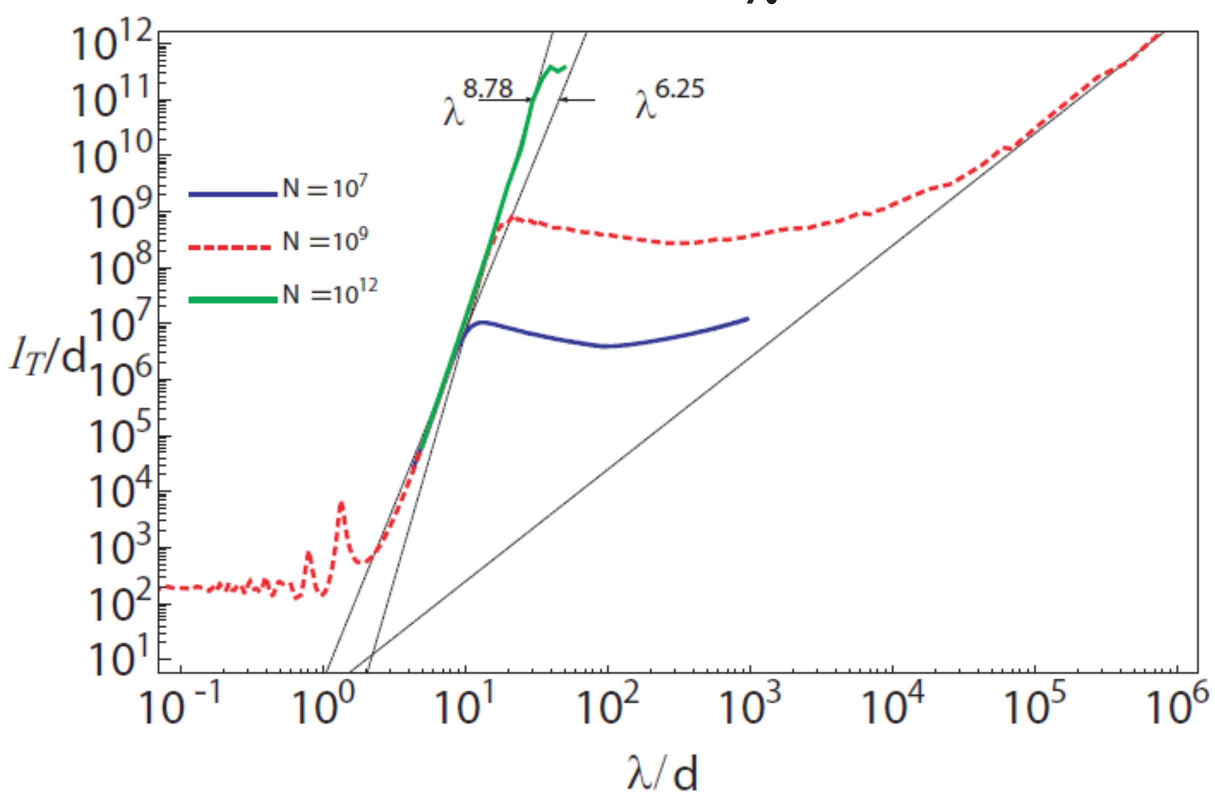


$$N = 10^7,\ 10^9,\ 10^{12}$$

$$l_T \propto \lambda^{\kappa}$$

$$6.25 \le \kappa \le 8.78$$

dispersive model R.A. Shelby, D.R. Smith, and S. Shultz, Science 292,77 (2001)

$$Q_d = 0 \qquad d = 0.003\,\mathrm{m}$$

left handed layer

$$\varepsilon(f) = 1 - \frac{f_{ep}^2 - f_e^2}{f^2 - f_e^2 + i\gamma f} \qquad \mu(f) = 1 - \frac{f_{mp}^2 - f_m^2}{f^2 - f_m^2 + i\gamma f}$$

$$f_e = \bar{f}_e(1+\delta_e), \quad f_m = \bar{f}_m(1+\delta_m) \qquad 2\pi f = \omega$$

$$\gamma = 0 \qquad f_e < f < f_{mp} \quad \Rightarrow \quad \varepsilon(f),\ \mu(f)\ < 0$$

normal layer $\varepsilon_n(f) = -\varepsilon^*(f) \qquad \mu_n(f) = -\mu^*(f)$

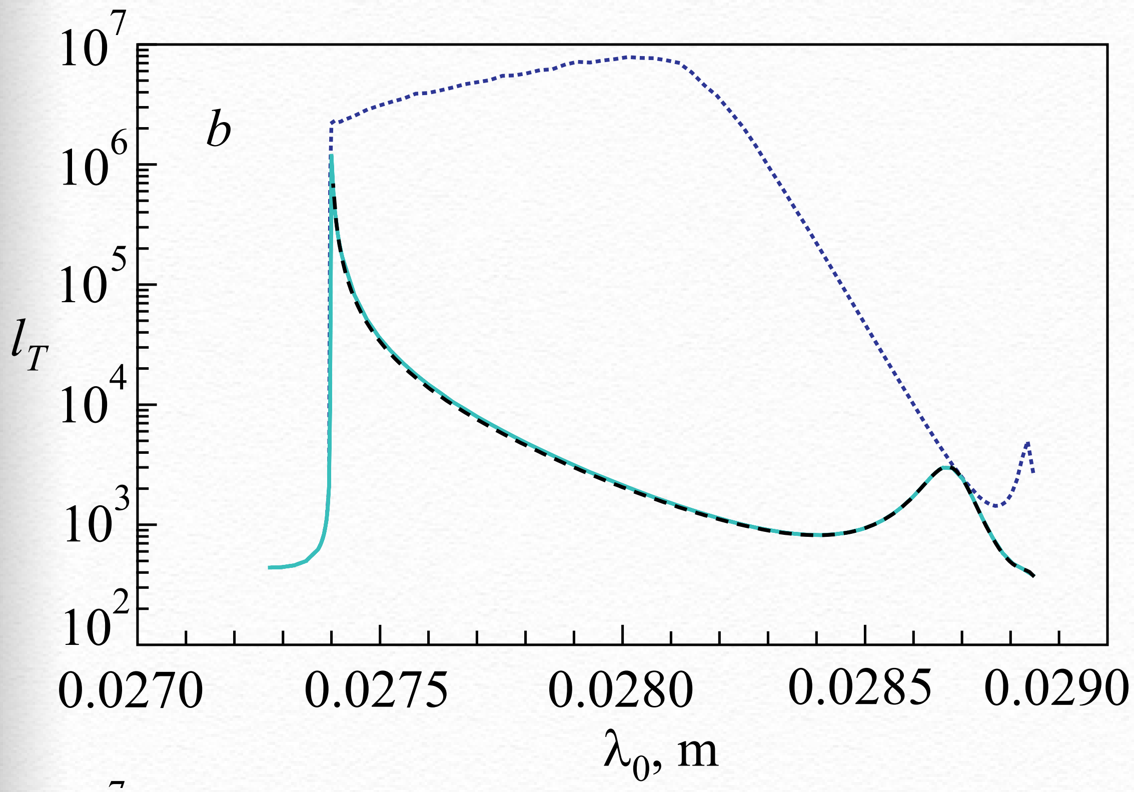
$$f_{mp} = 10.95\ \mathrm{GHz} \qquad \bar{f}_m = 10.05\ \mathrm{GHz}$$

$$f_{ep} = 12.8\ \mathrm{GHz} \qquad \bar{f}_e = 10.3\ \mathrm{GHz}$$

$$\gamma = 10\mathrm{MHz} \qquad Q_{e,m} < 5 \times\ 10^{-3}$$

$$10.3\ \mathrm{GHz} \leq f \leq 10.95\ \mathrm{GHz} \qquad \mathrm{Re}\ \mu(f),\ \mathrm{Re}\ \varepsilon(f) < 0$$

We **PRB 85, 045122 (2012)**



$l_T(N)$ versus free space wavelength λ_0

blue dotted curve

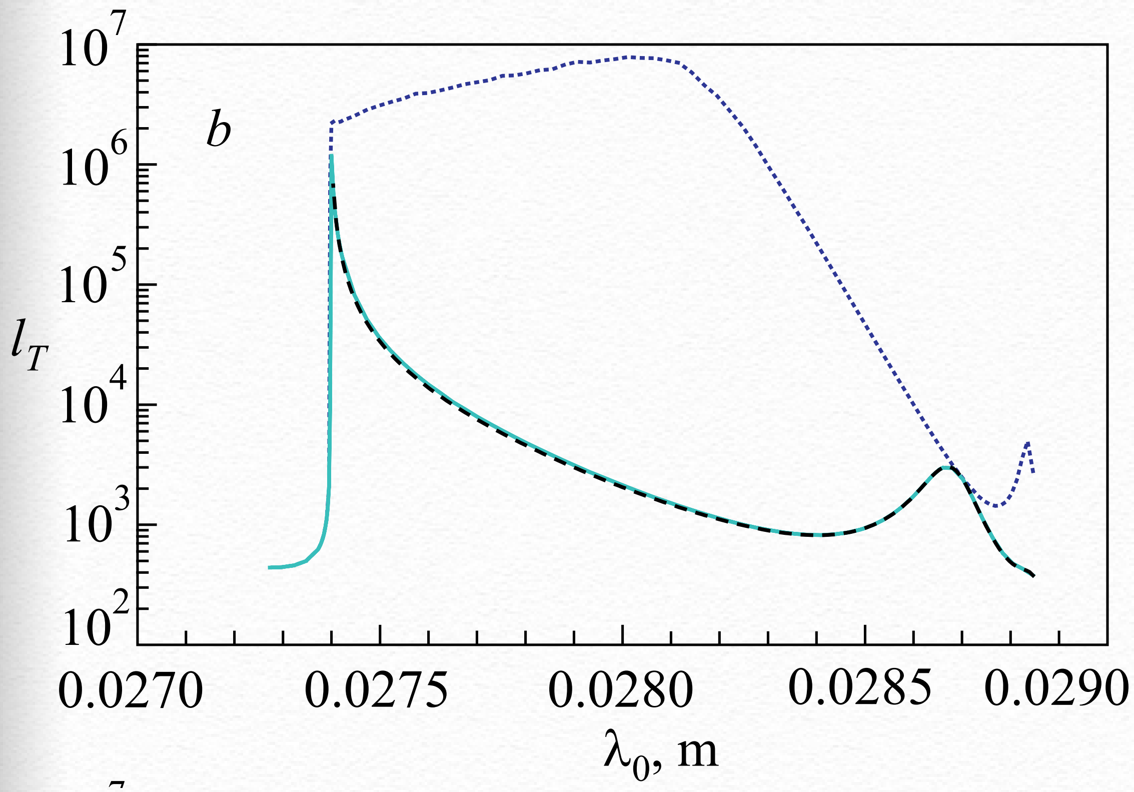
$$Q_\varepsilon = 0.005 \quad Q_\mu = 0 \quad N = 10^7$$

bottom curves $Q_\varepsilon = Q_\mu = 0.005, N = 10^7$

cyan solid curve - direct simulation

black dashed curve - WS approximation

We **PRB 85, 045122 (2012)**



$l_T(N)$ versus free space wavelength λ_0

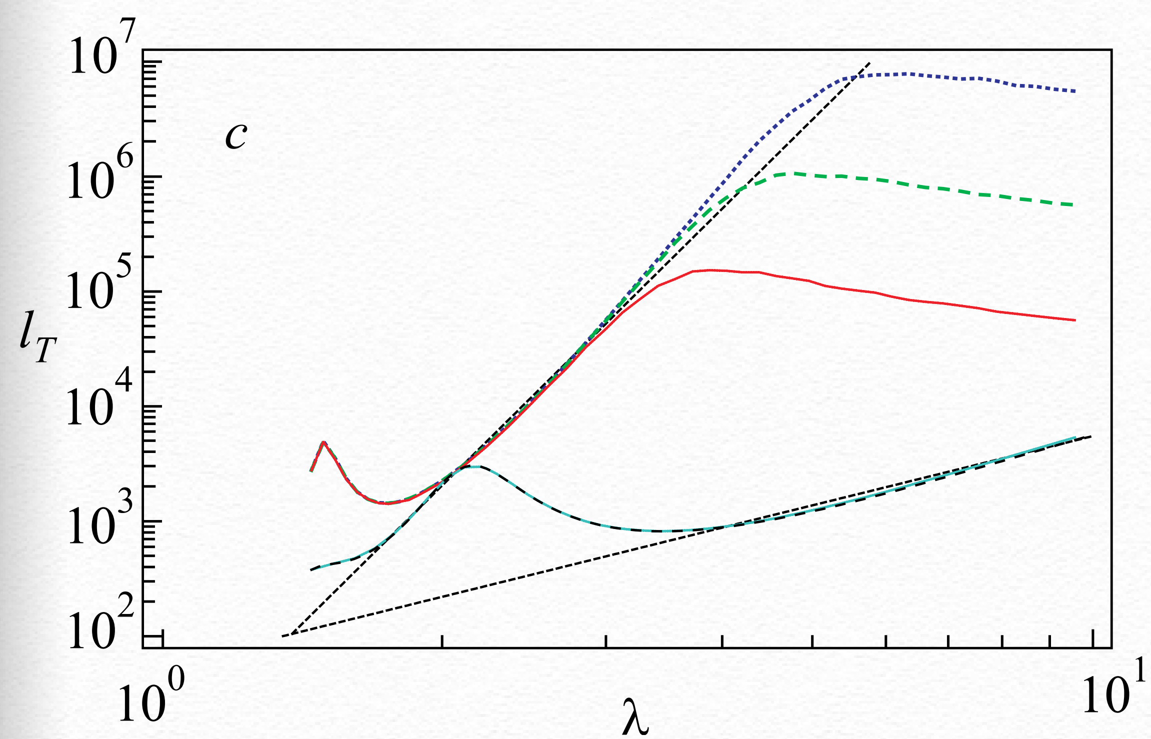
blue dotted curve

$Q_\varepsilon = 0.005 \quad Q_\mu = 0 \quad N = 10^7$

bottom curves $Q_\varepsilon = Q_\mu = 0.005, N = 10^7$

cyan solid curve - direct simulation

black dashed curve - WS approximation



$l_T(N)$ versus wavelength λ within the stack

three top curves $Q_\varepsilon = 0.05 \quad Q_\mu = 0$

red solid curve $N = 10^5$

green dashed curve $N = 10^6$

blue dotted curve $N = 10^7$

bottom curves $Q_\varepsilon = Q_\mu = 0.005, N = 10^7$

cyan solid curve - direct simulation

black dashed curve - WS approximation

analytical treatment of the λ^8 anomaly

E. Torres-Herrera, F. Israilev, N. Makarov, EL **98**, 27003 (2012)

currentless field $$e_n = \begin{pmatrix} P_n \\ Q_n \end{pmatrix} = e^{\xi_n} \begin{pmatrix} \cos\theta_n \\ \sin\theta_n \end{pmatrix}$$

$$\tan\theta_{n+1} = \frac{T_{21}\cos\theta_n + T_{22}\sin\theta_n}{T_{11}\cos\theta_n + T_{12}\sin\theta_n}$$

$$e_{n+1} = \hat{T}_{n+1} e_n$$

$$\xi_{n+1} = \xi_n + \Phi(\theta_n)$$

$$\Phi(\theta_n) = \frac{1}{2}\ln\left[(T_{11}\cos\theta_n + T_{12}\sin\theta_n)^2 + (T_{21}\cos\theta_n + T_{22}\sin\theta_n)^2\right]$$

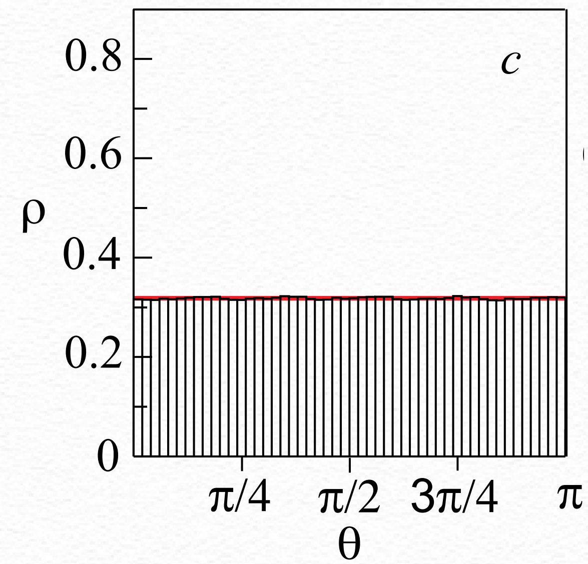
Fokker-Planck equation $\longrightarrow$ **stationary probability density** $\rho(\theta)$

$$l_\xi^{-1} \equiv \gamma = \langle \Phi(\theta) \rangle = \int \rho(\theta) d\theta$$

Q^2 accuracy

$H-$ stack :

$$\rho(\theta) = \frac{1}{\pi}$$



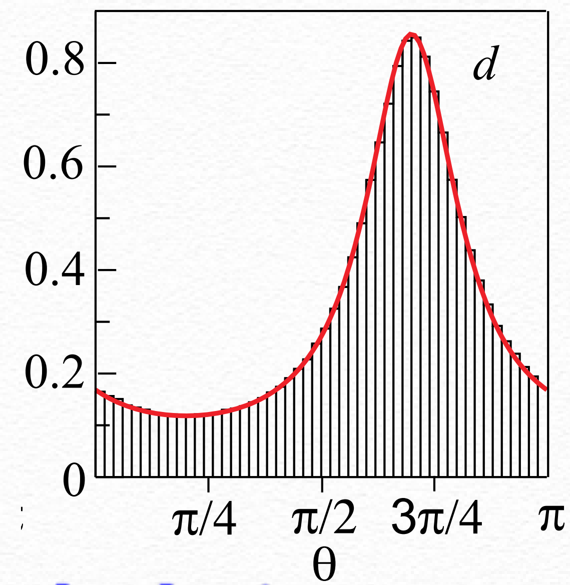
$$\gamma = \frac{2}{3}\left(\frac{\pi Q_\varepsilon}{\lambda}\right)^2$$

$$l_\xi \propto \lambda^2$$

$M-$ stack :

$$\rho(\theta) = \frac{\sqrt{\varphi^2 - \sin^2\varphi}}{\pi[\varphi + sin(2\theta - \varphi)]}$$

$$\varphi = \frac{\pi d}{\lambda}$$



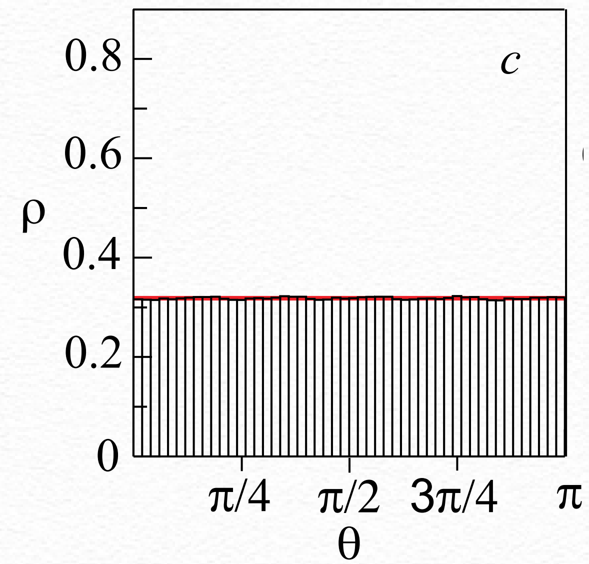
$$\gamma = 0$$

Q^4 accuracy : cumbersome calculations

Q^2 accuracy

$H-$ stack :

$$\rho(\theta) = \frac{1}{\pi}$$



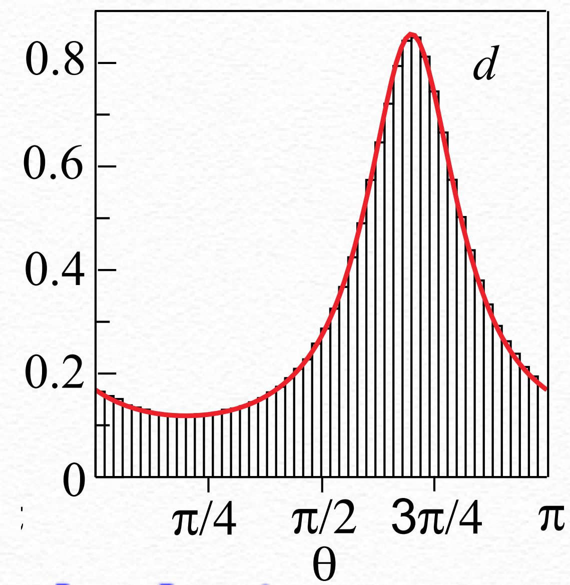
$$\gamma = \frac{2}{3}\left(\frac{\pi Q_\varepsilon}{\lambda}\right)^2$$

$$l_\xi \propto \lambda^2$$

$M-$ stack :

$$\rho(\theta) = \frac{\sqrt{\varphi^2 - \sin^2\varphi}}{\pi[\varphi + sin(2\theta - \varphi)]}$$

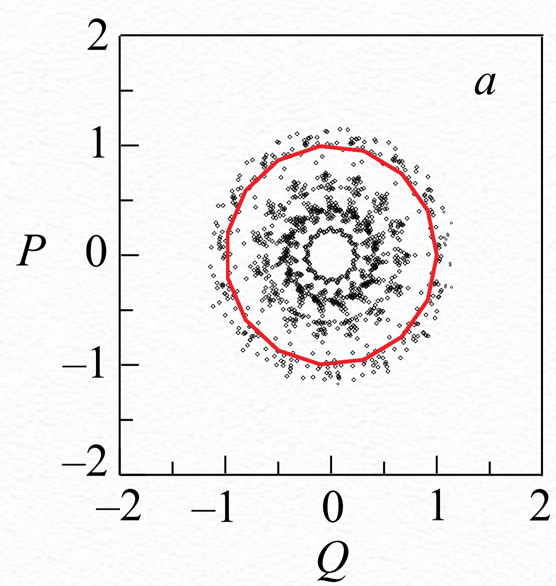
$$\varphi = \frac{\pi d}{\lambda}$$



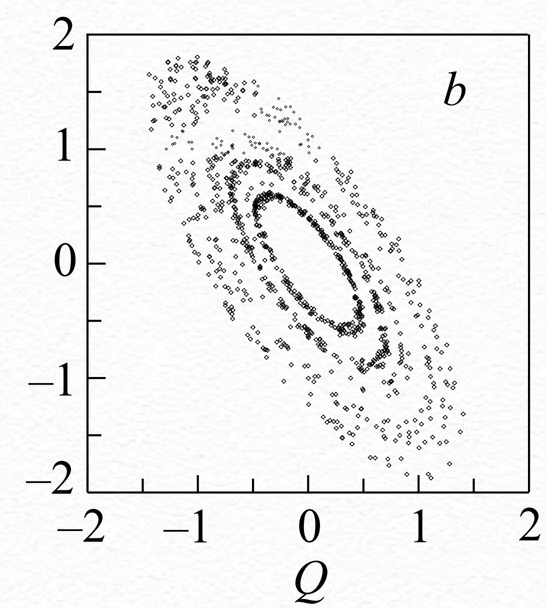
$$\gamma = 0$$

Q^4 accuracy : cumbersome calculations

$H-$ stack



$M-$ stack



$$\tilde{e}_n = \hat{R} e_n = e^{\Xi_n} \begin{pmatrix} \cos\Theta_n \\ \sin\Theta_n \end{pmatrix} \qquad \hat{R} = \begin{pmatrix} \sqrt{\eta}\cos\tau & \sqrt{\eta}\sin\tau \\ -\frac{\sin\tau}{\sqrt{\eta}} & \frac{\cos\tau}{\sqrt{\eta}} \end{pmatrix}$$

Fokker-Planck equation $\rightarrow \ \rho(\Theta) = \frac{1}{\pi} \quad \tau = \frac{\varphi}{2} \quad \eta^2 = \frac{\varphi + \sin\varphi}{\varphi - \sin\varphi}$

$$\gamma = \left(\frac{4}{15}\right)^3 \left(\frac{2Q_\varepsilon^2}{3}\right)^2 \left(\frac{2\pi}{\lambda}\right)^8 \propto \frac{Q_\varepsilon^4}{\lambda^8}$$

$$l_\xi \propto \lambda^8$$

Conclusions

Left-handed layers suppress localization of electromagnetic waves in multilayered stack. In special case of the layers with the same thickness and random only dielectric constant, such a suppression leads to an anomalously enhanced transmission in the long-wave region.

At the long wave localization region, the localization length of a mixed alternating stack determined by reciprocal Lyapunov exponent differs from that determined by transmission decrement.

Thank you for attention!