The Abdus Salam International Centre for Theoretical Physics

2472-19

Advanced Workshop on Nonlinear Photonics, Disorder and Wave Turbulence

15 - 19 July 2013

Wave Turbulence in Astrophysics & Geophysics

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The Abdus Salam
International Centre for Theoretical Physics
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Advanced Workshop on
« Nonlinear Photonics, Disorder and Wave Turbulence »

Trieste, 15-19 July 2013

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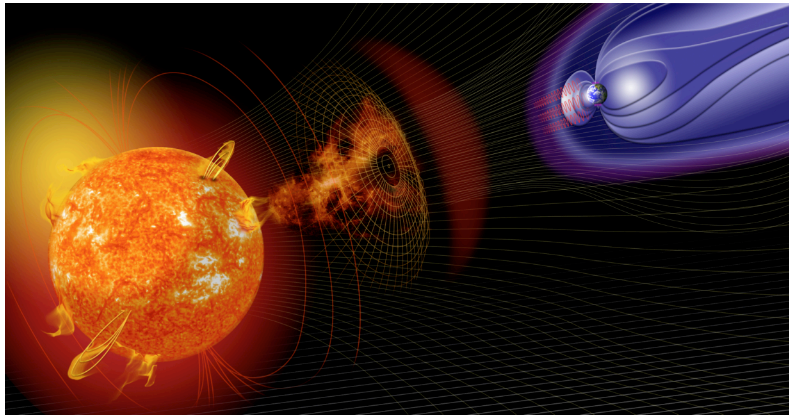
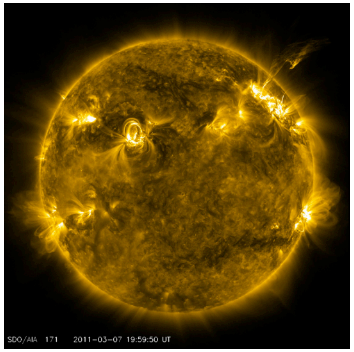
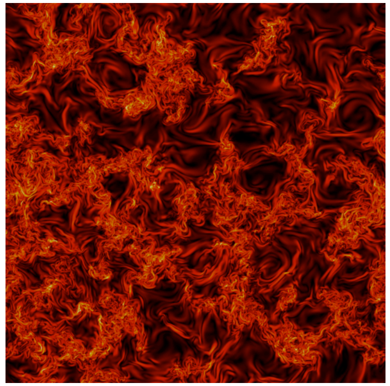
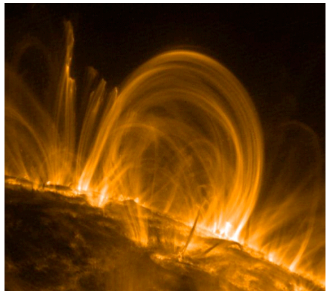
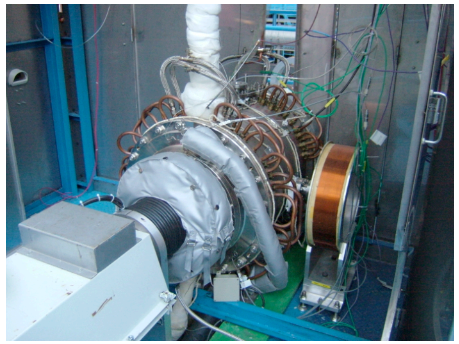








Wave Turbulence in Astrophysics & Geophysics



Sébastien **GALTIER**

IAS Orsay

UNIVERSITÉ PARIS SUD
Comprendre le monde, construire l'avenir®

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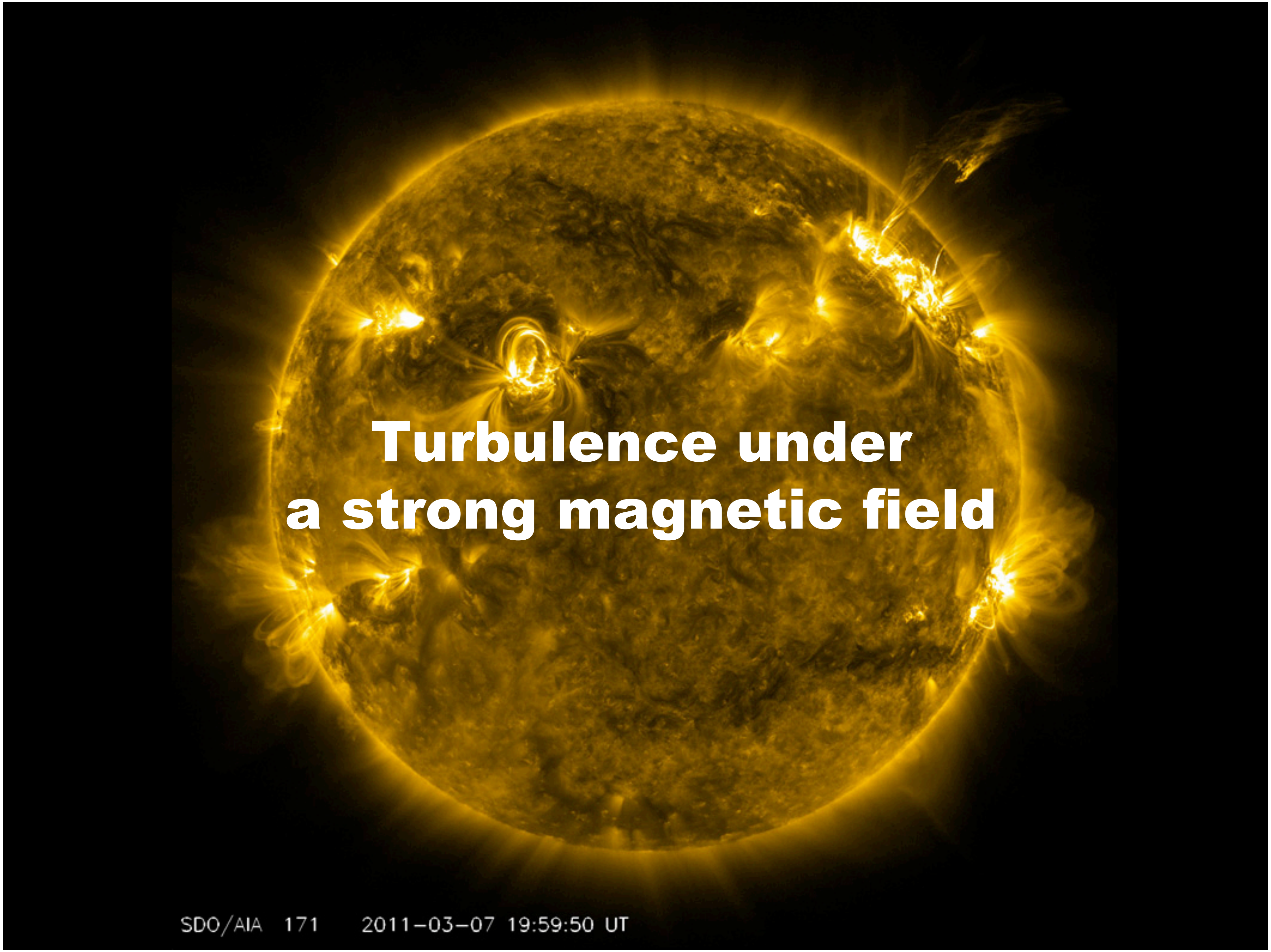
Turbulence under a strong magnetic field



MagnetoHydroDynamics (MHD)

H. Alfvén

$$\begin{cases} \dfrac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V} = -\nabla P_* + \mathbf{B}\cdot\nabla\mathbf{B} + \nu\nabla^2\mathbf{V} \\ \dfrac{\partial \mathbf{B}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{V} + \eta\nabla^2\mathbf{B} \\ \nabla\cdot\mathbf{V} = 0 \qquad \nabla\cdot\mathbf{B} = 0 \end{cases}$$

$$\mathbf{B} \to \sqrt{\mu_0 n m_\mathrm{i}}\,\mathbf{B}$$

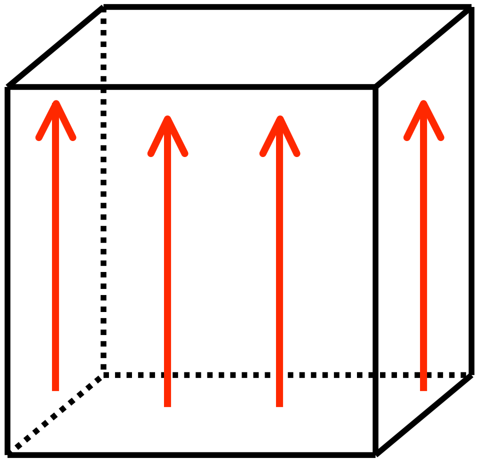
The three inviscid ($\nu = \eta = 0$) quadratic invariants

$$E = \frac{1}{2}\int (\mathbf{V}^2 + \mathbf{B}^2)\, d\mathscr{V}$$: **total energy** (direct cascade)

$$H_\mathrm{c} = (1/2)\int \mathbf{V}\cdot\mathbf{B}\, d\mathscr{V}$$: **cross-helicity** (direct cascade)

$$H_\mathrm{m} = \frac{1}{2}\int \mathbf{A}\cdot\mathbf{B}\, d\mathscr{V}$$: **magnetic helicity** (inverse cascade)

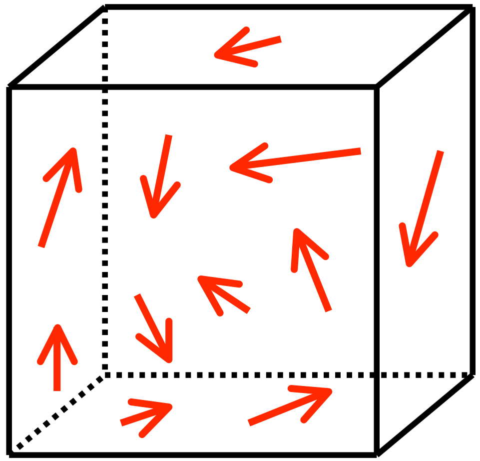
Regimes of turbulence



$\mathbf{B} = \mathbf{B}_0 + \varepsilon \mathbf{b}$

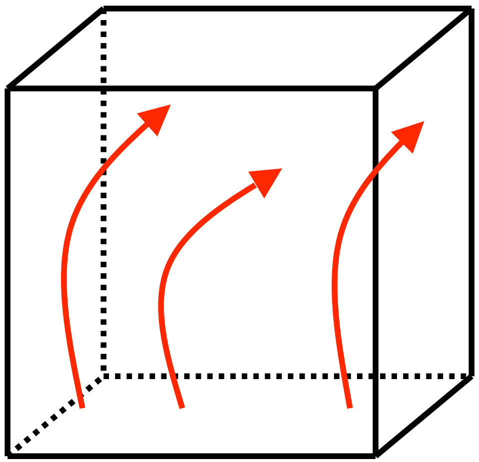
Wave (weak) turbulence
($\varepsilon \ll 1$)

MHD is non invariant under Galilean transformation



$\mathbf{B} = \mathbf{b}$

Isotropic (strong) turbulence



$\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$

Anisotropic (strong) turbulence

Alfvén wavepackets

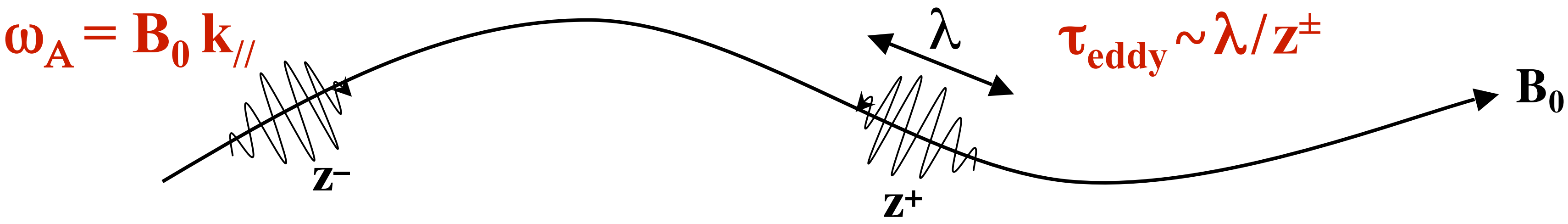
The main difference between neutral fluids and MHD is the presence of **Alfvén waves** [Alfvén, Nature, 1942]

$$\begin{cases} \partial_t \mathbf{z}^{\pm} \mp \mathbf{B}_0 \cdot \nabla \mathbf{z}^{\pm} = -\mathbf{z}^{\mp} \cdot \nabla \mathbf{z}^{\pm} - \nabla P_* \\ \nabla \cdot \mathbf{z}^{\pm} = 0 \end{cases}$$

$\mathbf{z}^{\pm} = \mathbf{u} \pm \mathbf{b}$ are the Elsässer fields

Wavepackets interact nonlinearly on a crossing time : $\tau_A \sim \lambda / B_0$

[Iroshnikov, Soviet Astron., 1964; Kraichnan, Phys. Fluids, 1965]



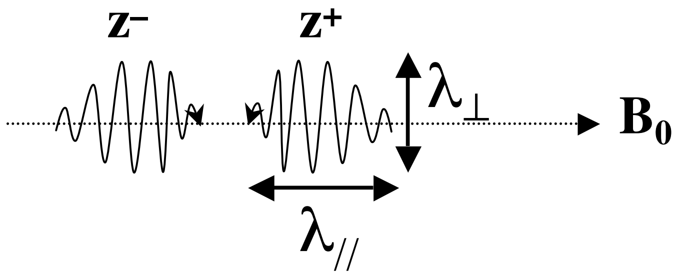
→ Stochastic **collisions** of wavepackets

Weak Alfvén wave turbulence

Incompressible MHD equations:

$$\partial_t \mathbf{z}^{\pm} \mp \mathbf{B_0} \cdot \nabla \mathbf{z}^{\pm} = -\mathbf{z}^{\mp} \cdot \nabla \mathbf{z}^{\pm} - \nabla P_*$$

$\mathbf{B_0} >> \mathbf{z}^{\pm}$



$$\begin{cases} \tau_A \sim \lambda_{//}/B_0 \\ \tau_{eddy} \sim \lambda_{\perp}/z^{\pm} \end{cases}$$

1 collision: $z_\ell(t+\tau_A) \sim z_\ell(t) + \tau_A \dfrac{\partial z_\ell}{\partial t} \sim z_\ell(t) + \tau_A \dfrac{z_\ell^2}{\ell_\perp}$

$\Delta_1 z_\ell \sim \tau_A z_\ell^2/\ell_\perp$ (distorsion of the wavepacket)

N stochastic collisions: $\displaystyle\sum_{i=1}^{N} \Delta_i z_\ell \sim \tau_A \frac{z_\ell^2}{\ell_\perp}\sqrt{\frac{t}{\tau_A}}$

Cumulative distorsion of order 1 defines $\boldsymbol{\tau_{tr}}$: $z_\ell \sim \tau_A \dfrac{z_\ell^2}{\ell_\perp}\sqrt{\dfrac{\tau_{tr}}{\tau_A}}$

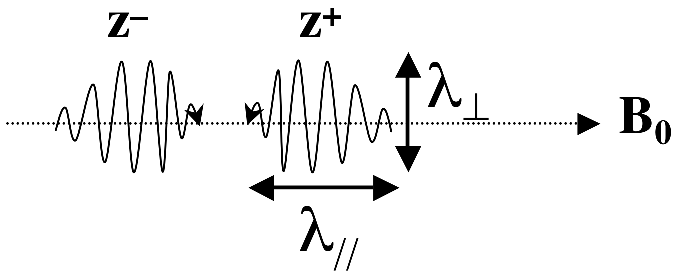
$$\tau_{tr} \sim \frac{1}{\tau_A}\frac{\ell_\perp^2}{z_\ell^2} \sim \frac{\tau_{eddy}^2}{\tau_A}$$

Hence the **spectrum**: $E(k_\perp, k_{||}) \sim \sqrt{\varepsilon B_0}\, k_\perp^{-2} k_{||}^{-1/2}$ With: $\varepsilon \sim z_\ell^2/\tau_{tr}$

[Galtier et al., J. Plasma Phys., 2000]

Weak Alfvén wave turbulence

$$\omega_A(k) = B_0\, k_{//}$$



Three-wave interactions :

$$\begin{cases} \omega_A(k) = \omega_A(p) - \omega_A(q) \\ \mathbf{k} = \mathbf{p} + \mathbf{q} \end{cases} \quad \Rightarrow \quad \begin{cases} k_{//} = p_{//} - q_{//} \\ \mathbf{k} = \mathbf{p} + \mathbf{q} \end{cases}$$

$$\longrightarrow \quad \boxed{q_{//} = 0}$$

No transfer along the uniform magnetic field !

Weak Alfvén wave turbulence

[Galtier et al., J. Plasma Phys., 2000]

- **Asymptotic** theory ($B_0 \to +\infty$; $\tau_A << \tau_{nl}$) :

($k_\perp >> k_{//}$)

$$\frac{\partial E^\pm(k_\perp, k_\parallel)}{\partial t} = \frac{\pi \varepsilon^2}{B_0} \iint_\Delta \cos^2\phi \sin\theta \, \frac{k_\perp}{q_\perp} E^\mp(q_\perp, 0) \left[k_\perp E^\pm(p_\perp, k_\parallel) - p_\perp E^\pm(k_\perp, k_\parallel) \right] dp_\perp dq_\perp$$

→ no transfer along $\mathbf{B_0}$, hence : $E^\pm(k_\perp, k_\parallel) = E^\pm(k_\perp) f_\pm(k_\parallel)$

→ **exact** stationary solutions:

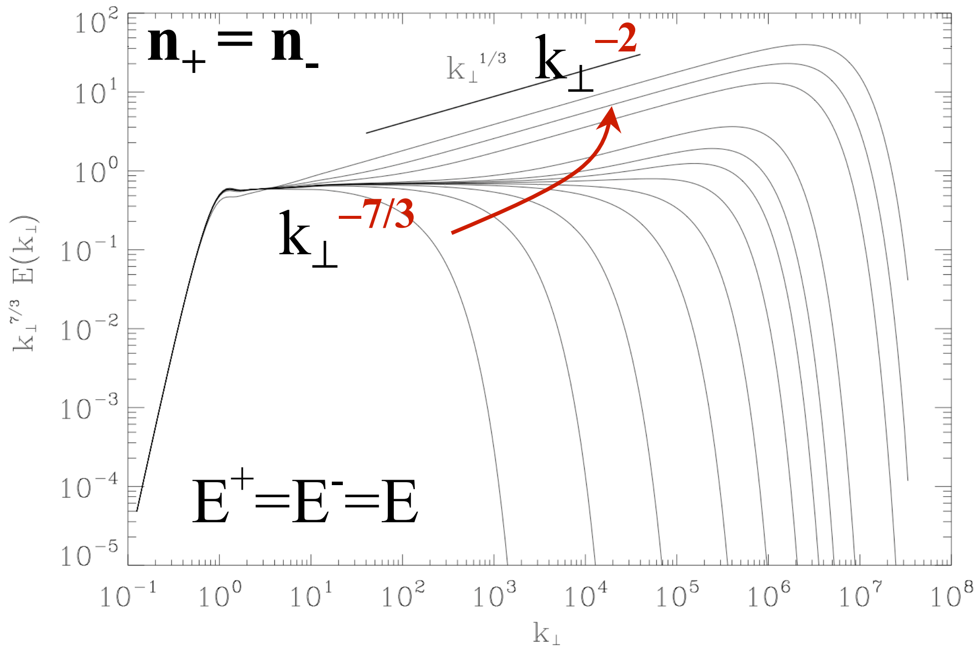
$E^\pm(k_\perp) \sim k_\perp^{n_\pm}$

$\mathbf{n_+ + n_- = -4}$

→ **anomalous** scaling in $k_\perp^{-7/3}$

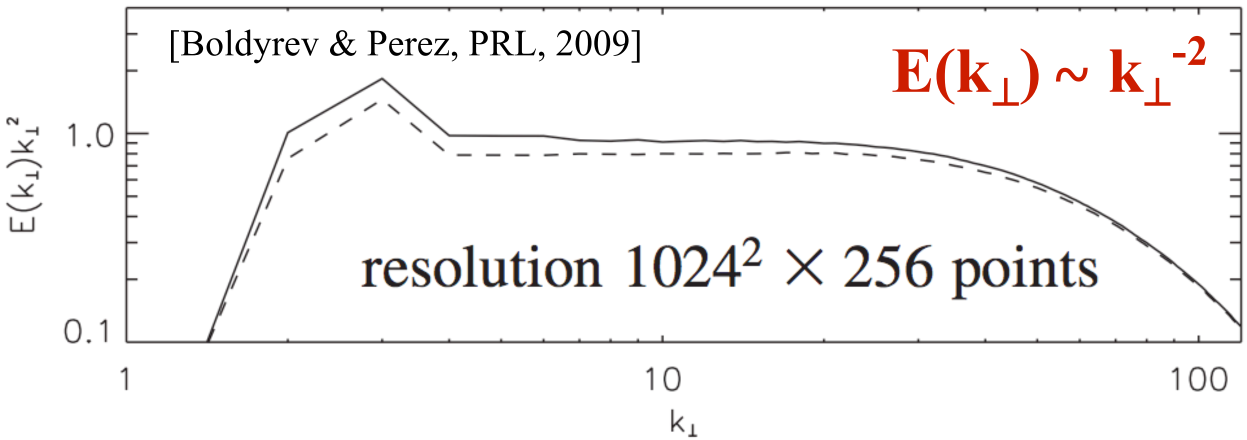
Generic behavior !

[Connaughton et al., Phys. D, 2003]



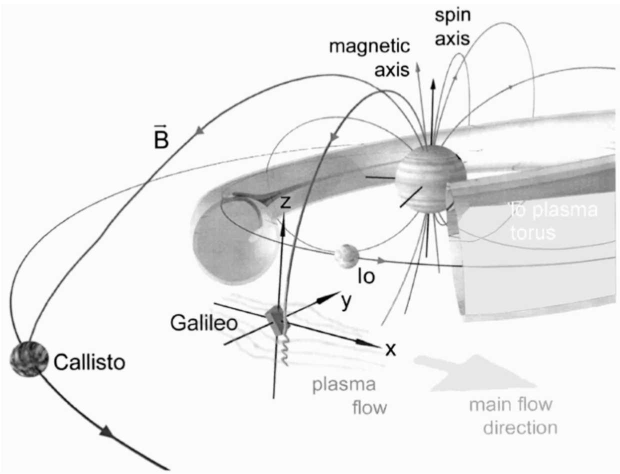
Weak Alfvén wave turbulence

- **First** signature with direct numerical simulations:



$B_0 \sim 5b$

- **Indirect** signature in the Jupiter's magnetosphere (Galileo/NASA)



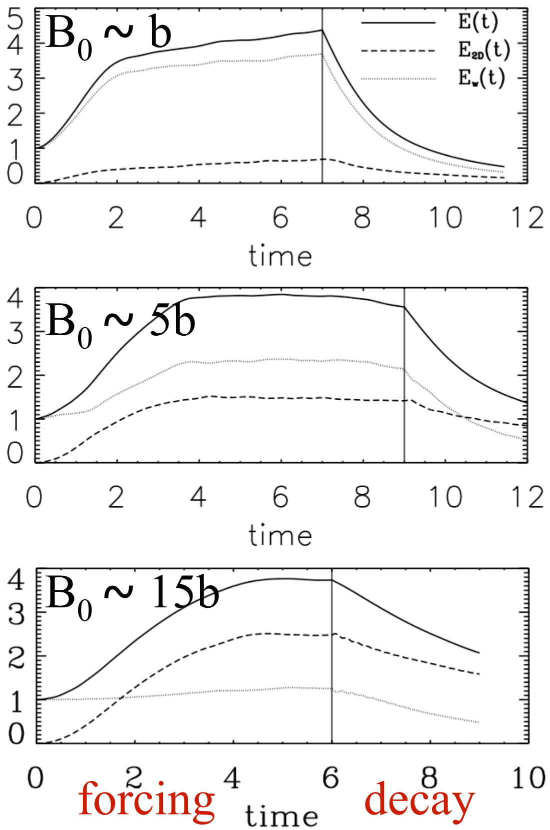
[Saur et al., A&A, 2002]

Weak Alfvén wave turbulence

- Role of the **slow mode** ($q_{//}$=0 is *not* weak turbulence):

$$\frac{\partial E^{\pm}(k_\perp, k_\|)}{\partial t} = \frac{\pi \varepsilon^2}{B_0} \iint_\Delta \cos^2\phi \sin\theta \, \frac{k_\perp}{q_\perp} \underbrace{E^{\mp}(q_\perp, 0)} \left[k_\perp E^{\pm}(p_\perp, k_\|) - p_\perp E^{\pm}(k_\perp, k_\|)\right] dp_\perp dq_\perp$$

512^3 [Bigot et al., PRE, 2011]



dynamics may be given by **strong** turbulence

$n_{2D} + n_W = -4$ always valid

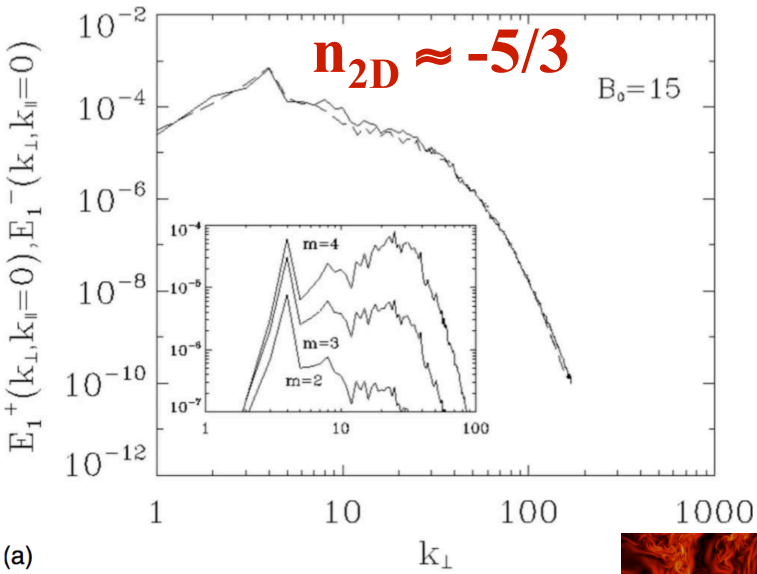
$E^{\pm}(k_\perp, k_{//}=0) < E^{\pm}(k_\perp, k_{//}>0)$: **$n_{2D} = n_W = -2$**

$E^{\pm}(k_\perp, k_{//}=0) > E^{\pm}(k_\perp, k_{//}>0)$: **$n_{2D} = -5/3$**, **$n_W = -7/3$**

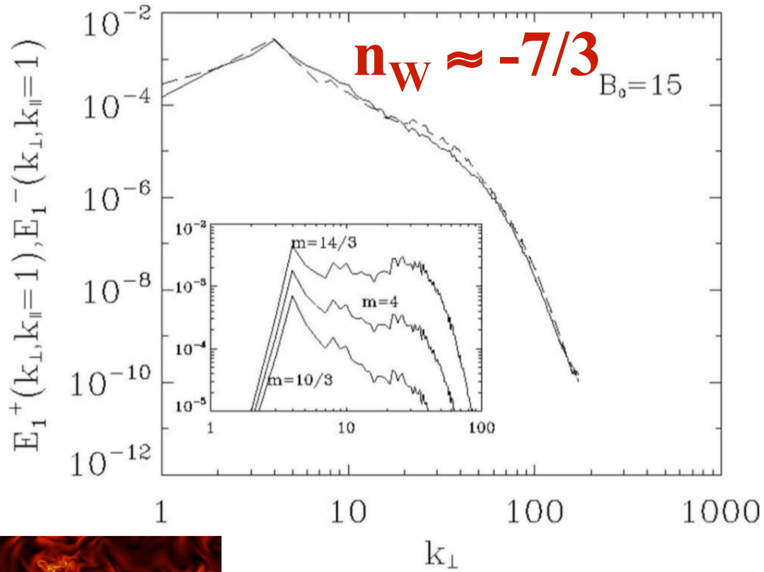
~ Kolmogorov scaling (with intermittency)

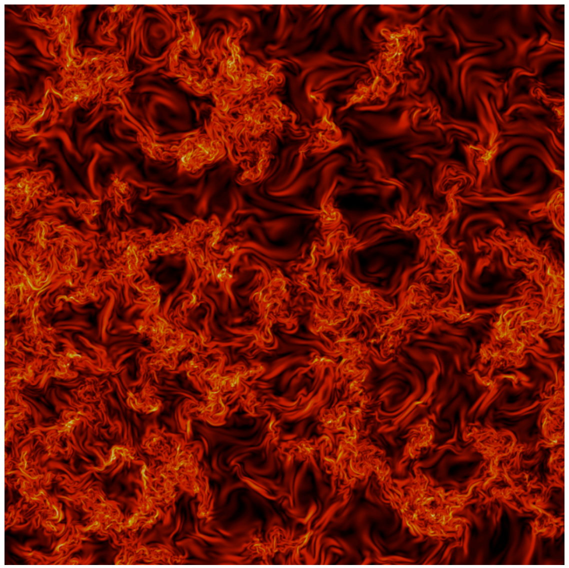
Weak Alfvén wave turbulence

[Bigot et al., PRE, 2008]



(a)

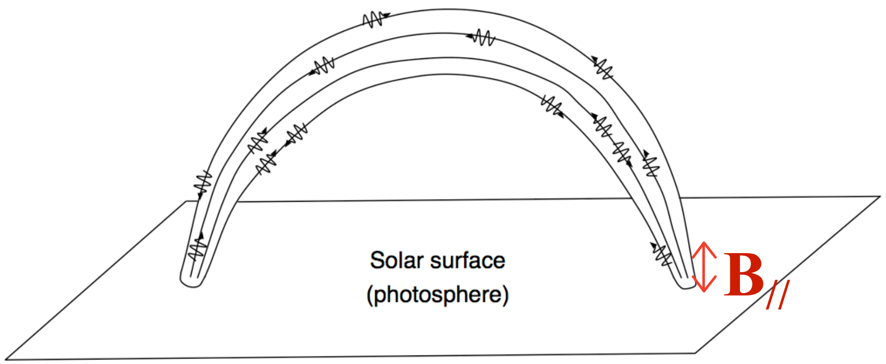


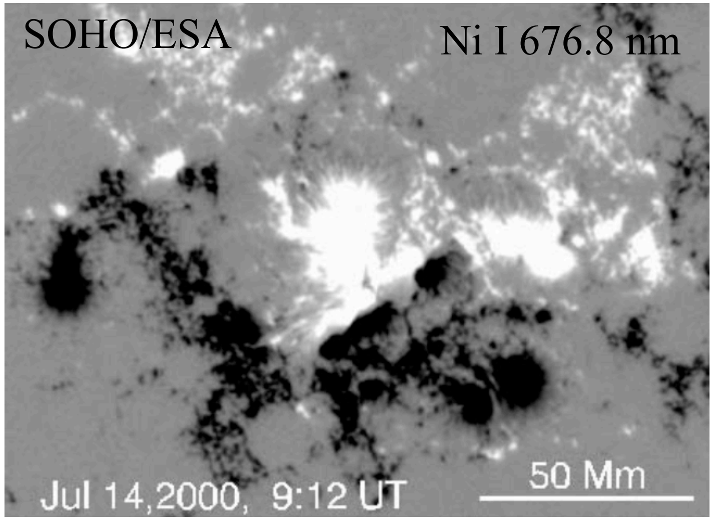


Electric current

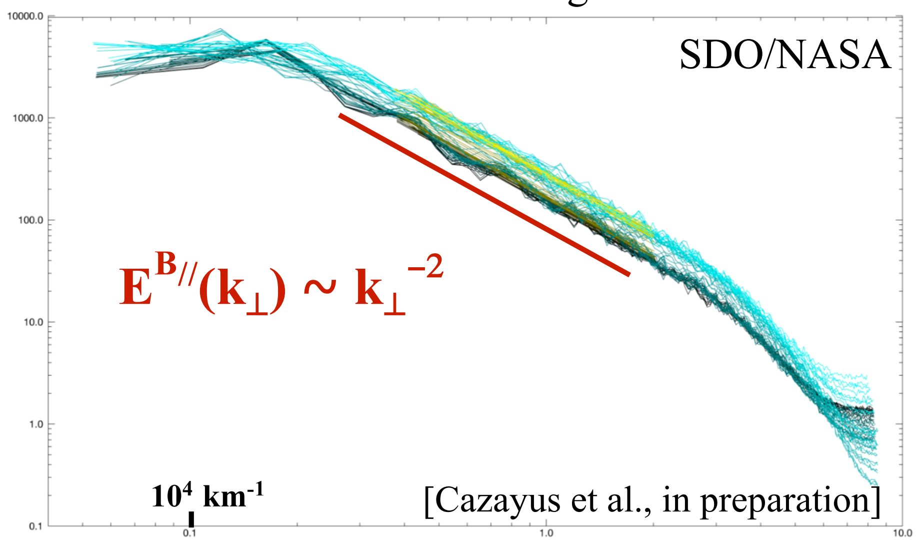
[Meyrand et al., 2013]

Observation of weak MHD turbulence

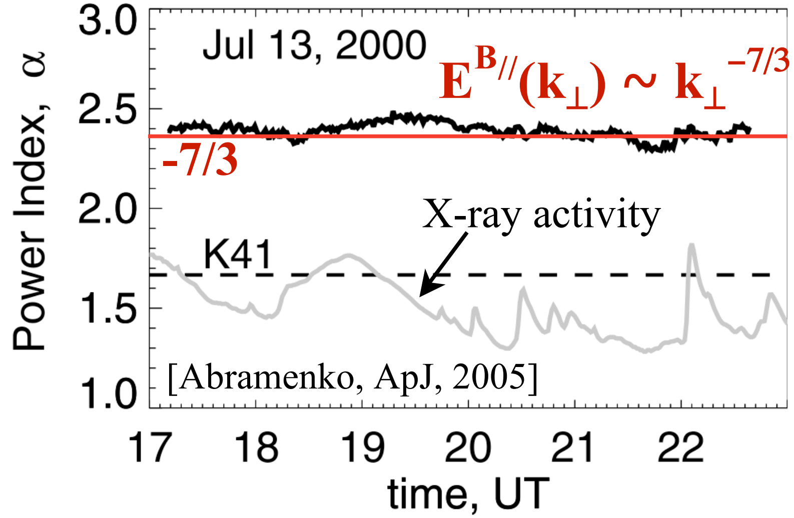




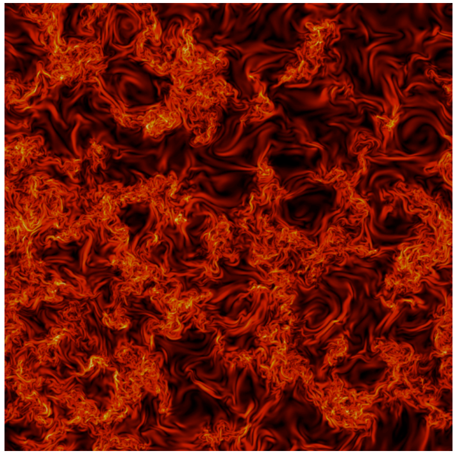
Moderate active region

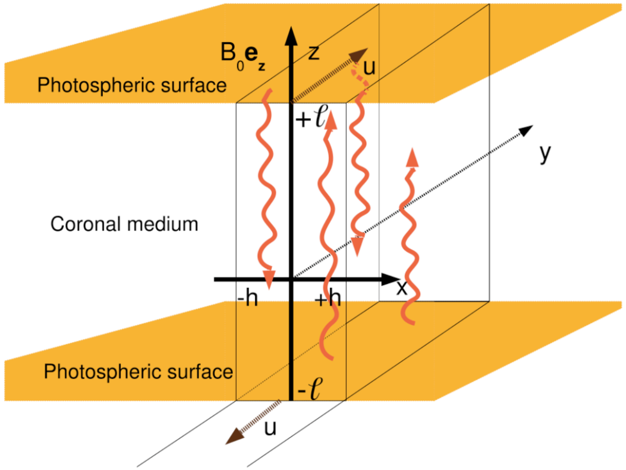


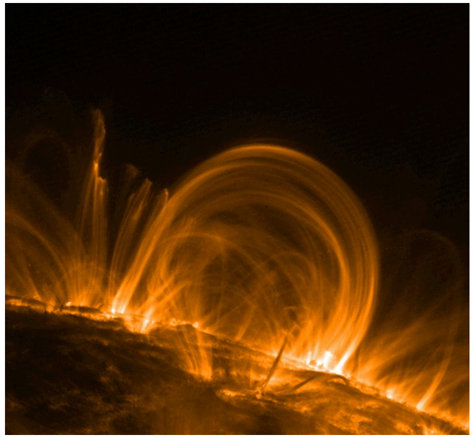
Strongly active region

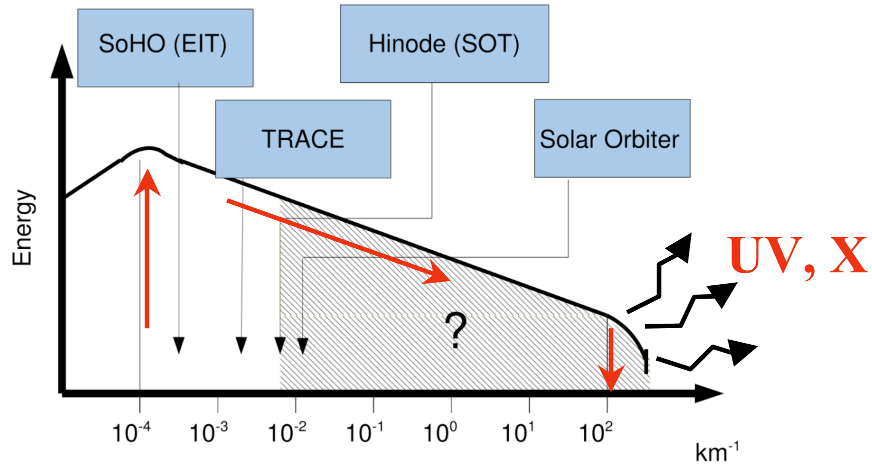


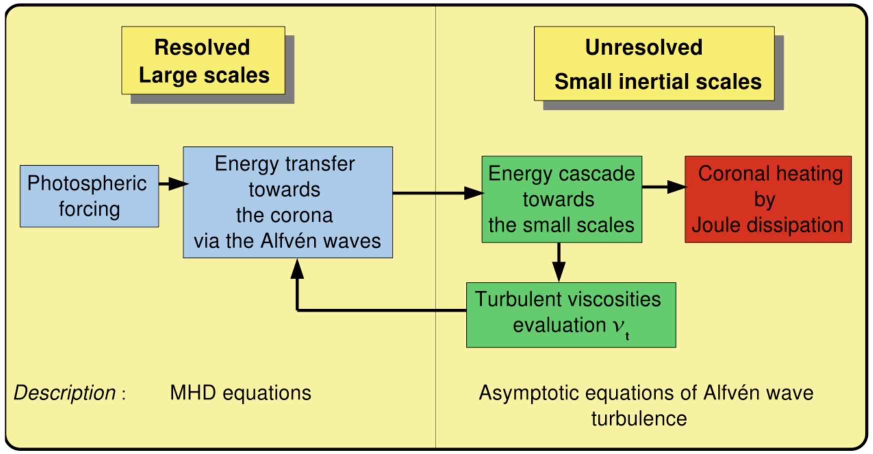
Application to solar coronal heating









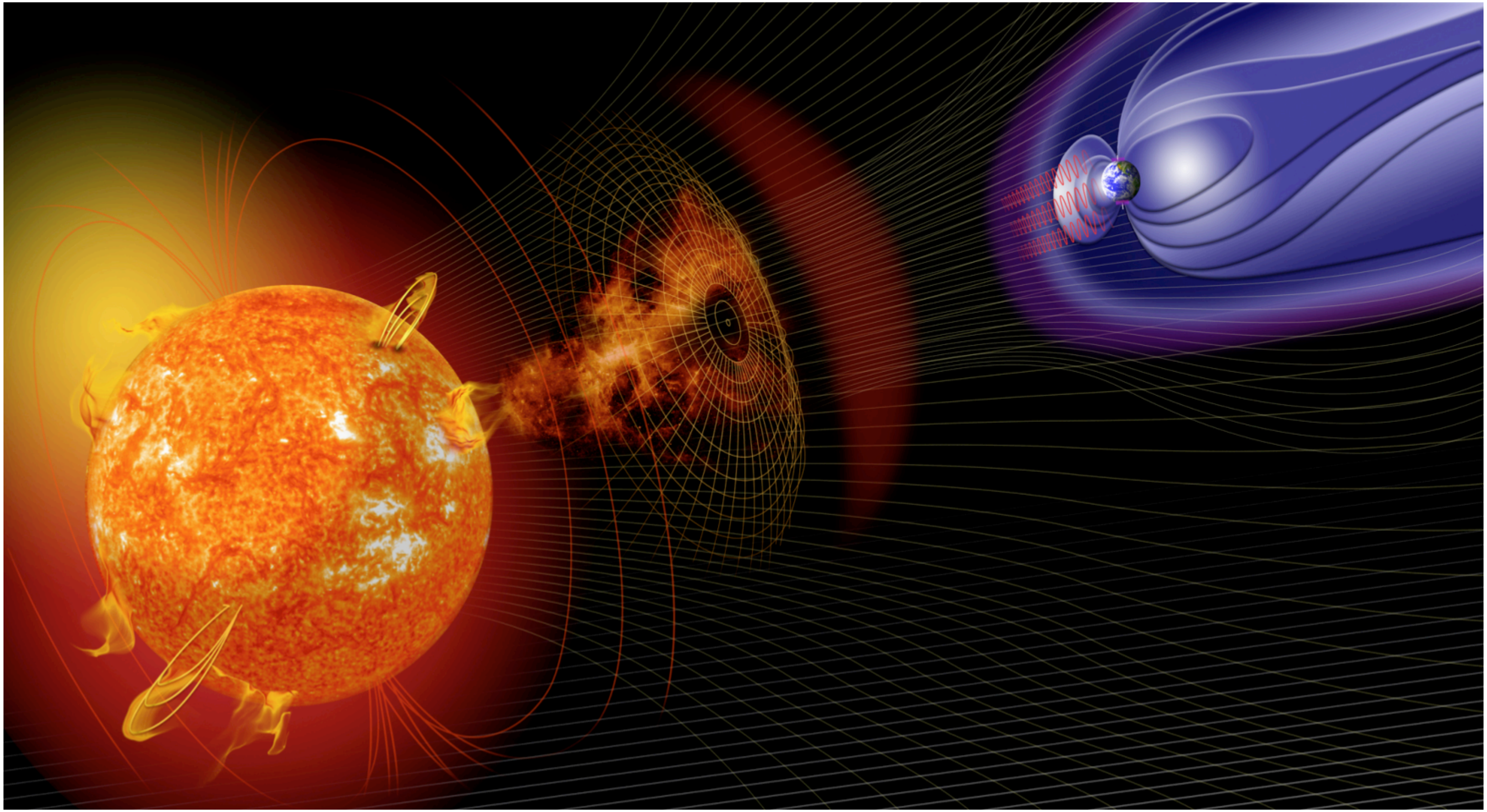


[Bigot et al., A&A, 2008]

Predictions:

$$
\begin{cases}
u_\perp = 50\,\mathcal{H}^{1/2}\,\mathcal{U}^{1/3}\,\mathcal{B}_0^{2/3}\,\mathcal{M}^{-1/6}\,\mathcal{L}^{-1/2}\ \mathrm{km\,s^{-1}} \\
|\mathcal{F}_z| = 1249\,\mathcal{H}^{2/3}\,\mathcal{U}^{4/3}\,\mathcal{B}_0^{5/3}\,\mathcal{M}^{1/6}\ \mathrm{J\,m^{-2}\,s^{-1}}
\end{cases}
$$

Solar wind turbulence

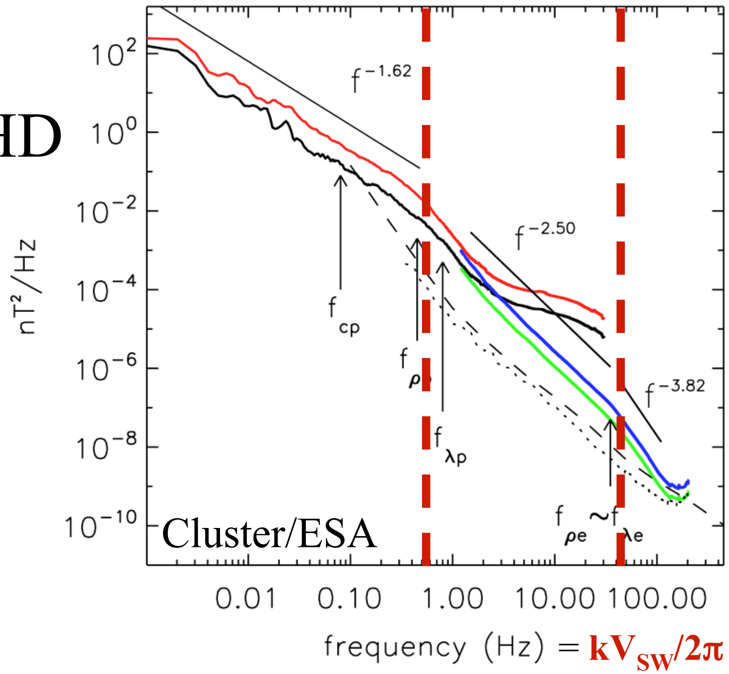


Solar wind turbulence

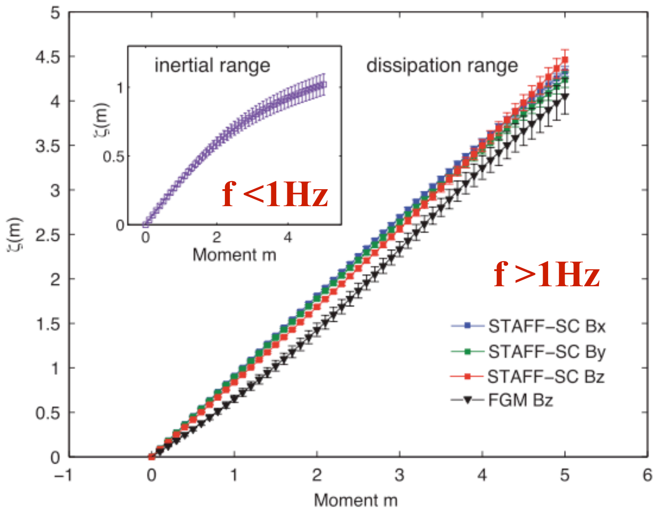
- Physics at low frequency (f<1Hz) governed by MHD
  → **waves** and **anisotropy** are observed

- Physics at high frequency (f>1Hz) ≠ MHD
  → spectra between **$k^{-2.5}$** and **$k^{-8/3}$**
  → **absence** of intermittency

Magnetic spectrum



[Sahraoui et al., PRL, 2009]



[Kiyani et al., PRL, 2009]

Solar wind turbulence at f >1Hz

$$\begin{cases} (\partial_t + \mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P_* + \mathbf{b}\cdot\nabla\mathbf{b} + \nu\Delta\mathbf{v} \\ (\partial_t + \mathbf{v}\cdot\nabla)\mathbf{b} = \mathbf{b}\cdot\nabla\mathbf{v} - \boxed{d_I\nabla\times[(\nabla\times\mathbf{b})\times\mathbf{b}]} + \eta\Delta\mathbf{b} \end{cases}$$

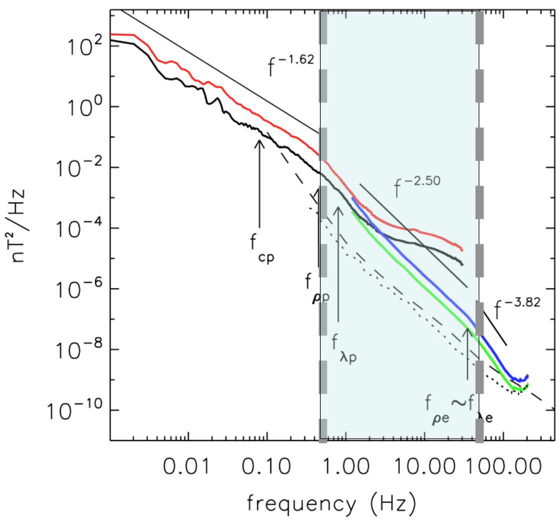
$$d_I = c/\omega_{pi}$$

Kolmogorov's law for **Hall** MHD : [Galtier, PRE, 2008]

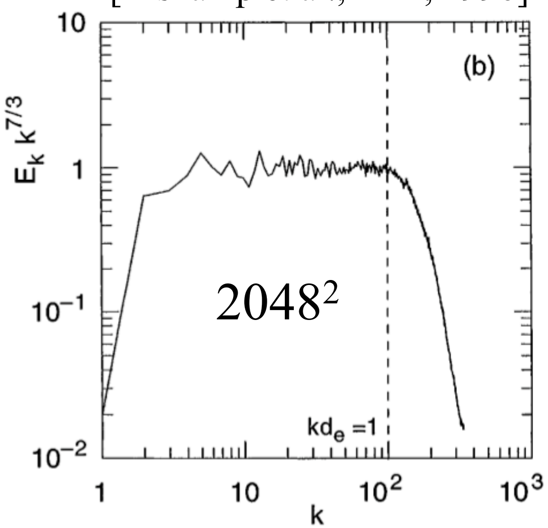
$$-\frac{4}{3}\varepsilon^T r = \langle[(\delta\mathbf{v})^2 + (\delta\mathbf{b})^2]\delta v_r\rangle - 2\langle[\delta\mathbf{v}\cdot\delta\mathbf{b}]\delta b_r\rangle \quad \} \; E \sim \mathbf{k}^{-5/3}$$

$$+4d_I\langle[(\mathbf{J}\times\mathbf{b})\times\delta\mathbf{b}]_r\rangle \quad \} \; E^b \sim \mathbf{k}^{-7/3}$$

[Sahraoui et al., PRL, 2009]



[Biskamp et al., PRL, 1996]



Whistler wave turbulence

[Galtier & Bhattacharjee, Phys. Plasmas, 2003; Galtier, J. Plasma Phys., 2006]

- Expansion in ε: **B** (**x**,t) = B_o **e**$_{//}$ + **εb**(**x**,t) with **0 < ε << 1**
- Complex **helicity** decomposition (whistler waves): **$\omega_w = d_i B_0 k_{//} k$**
- Leads to the wave amplitude equation: (three-wave interactions)

$$\partial_t a_\Lambda^s = \frac{\epsilon}{4d_\mathrm{i}} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} \xi_\Lambda^{s\,2} \frac{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p}}{\xi_\Lambda^s - \xi_\Lambda^{-s}} \underbrace{M_{\;-k\,p\,q}^{\;\Lambda \Lambda_p \Lambda_q}_{\;s\, s_p\, s_q}}_{} a_{\Lambda_p}^{s_p} a_{\Lambda_q}^{s_q} e^{-i\Omega_{pq,k} t} \delta_{pq,k} \, d\mathbf{p} \, d\mathbf{q},$$

matrix of 9 indices (but symmetric)

⋮

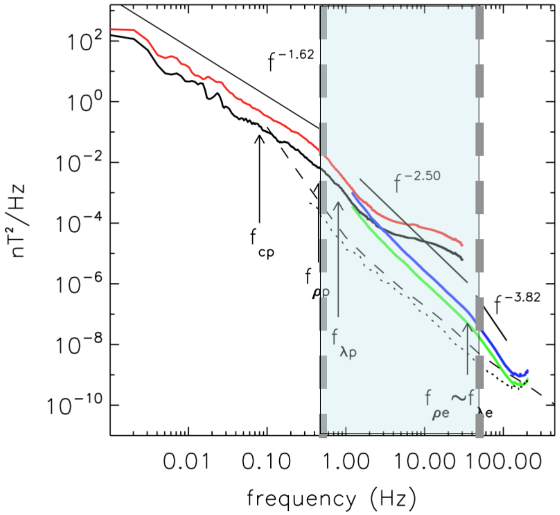
- **Kinetic equations:**

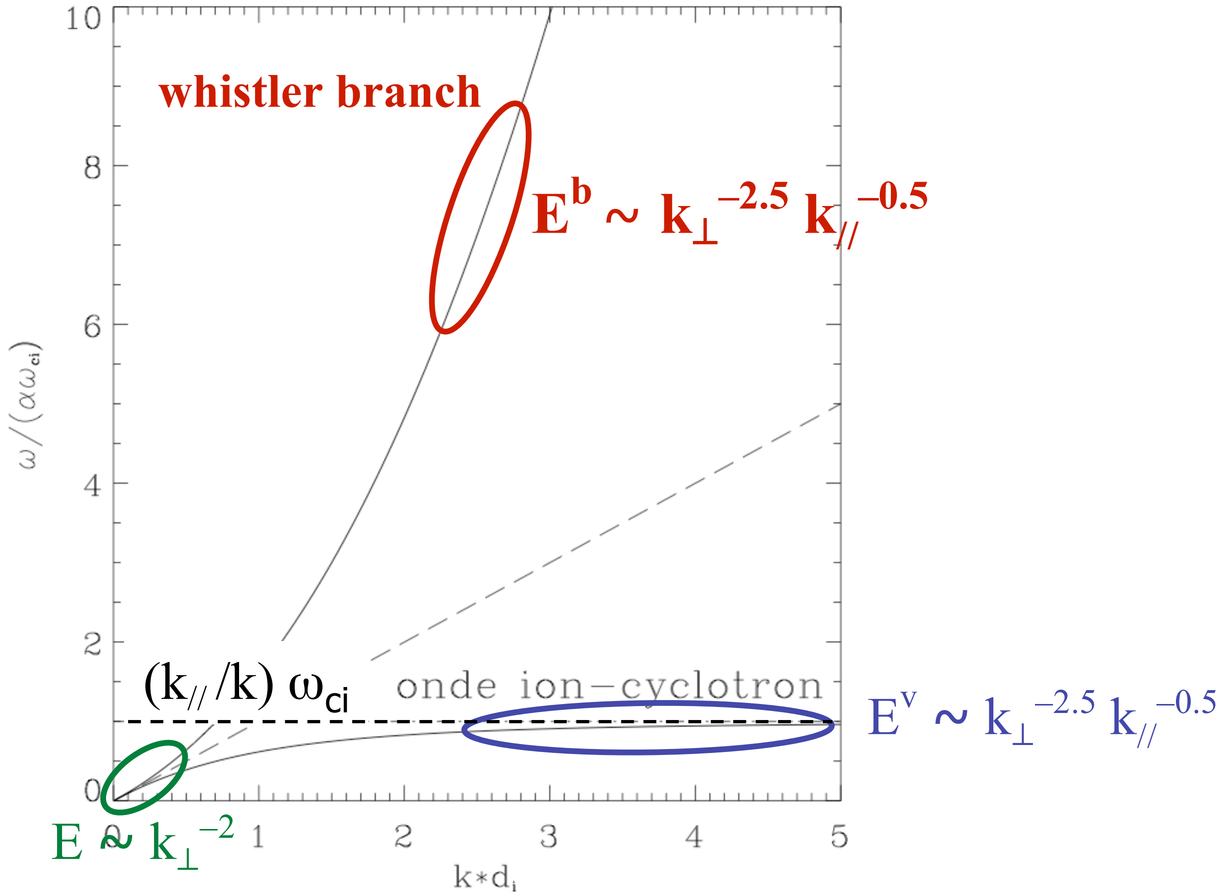
$$\partial_t q_\Lambda^s(\mathbf{k}) = \frac{\pi \epsilon^2}{4 d_\mathrm{i}^2 B_0^2} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} \left(\frac{\sin \psi_k}{k} \right)^2 (\Lambda k + \Lambda_p p + \Lambda_q q)^2$$
$$\times (1 - \xi_\Lambda^{-s\,2} \xi_{\Lambda_p}^{-s_p\,2} \xi_{\Lambda_q}^{-s_q\,2})^2 \left(\frac{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p}}{k_\parallel} \right)^2 \left(\frac{\omega_\Lambda^s}{1 + \xi_\Lambda^{-s\,2}} \right)$$
$$\times \left[\left(\frac{\omega_\Lambda^s}{1 + \xi_\Lambda^{-s\,2}} \right) \frac{1}{q_\Lambda^s(\mathbf{k})} - \left(\frac{\omega_{\Lambda_p}^{s_p}}{1 + \xi_{\Lambda_p}^{-s_p\,2}} \right) \frac{1}{q_{\Lambda_p}^{s_p}(\mathbf{p})} - \left(\frac{\omega_{\Lambda_q}^{s_q}}{1 + \xi_{\Lambda_q}^{-s_q\,2}} \right) \frac{1}{q_{\Lambda_q}^{s_q}(\mathbf{q})} \right]$$
$$\times q_\Lambda^s(\mathbf{k}) q_{\Lambda_p}^{s_p}(\mathbf{p}) q_{\Lambda_q}^{s_q}(\mathbf{q}) \delta(\Omega_{k,pq}) \delta_{k,pq} \, d\mathbf{p} \, d\mathbf{q}. \qquad (3.23)$$

Whistler wave turbulence

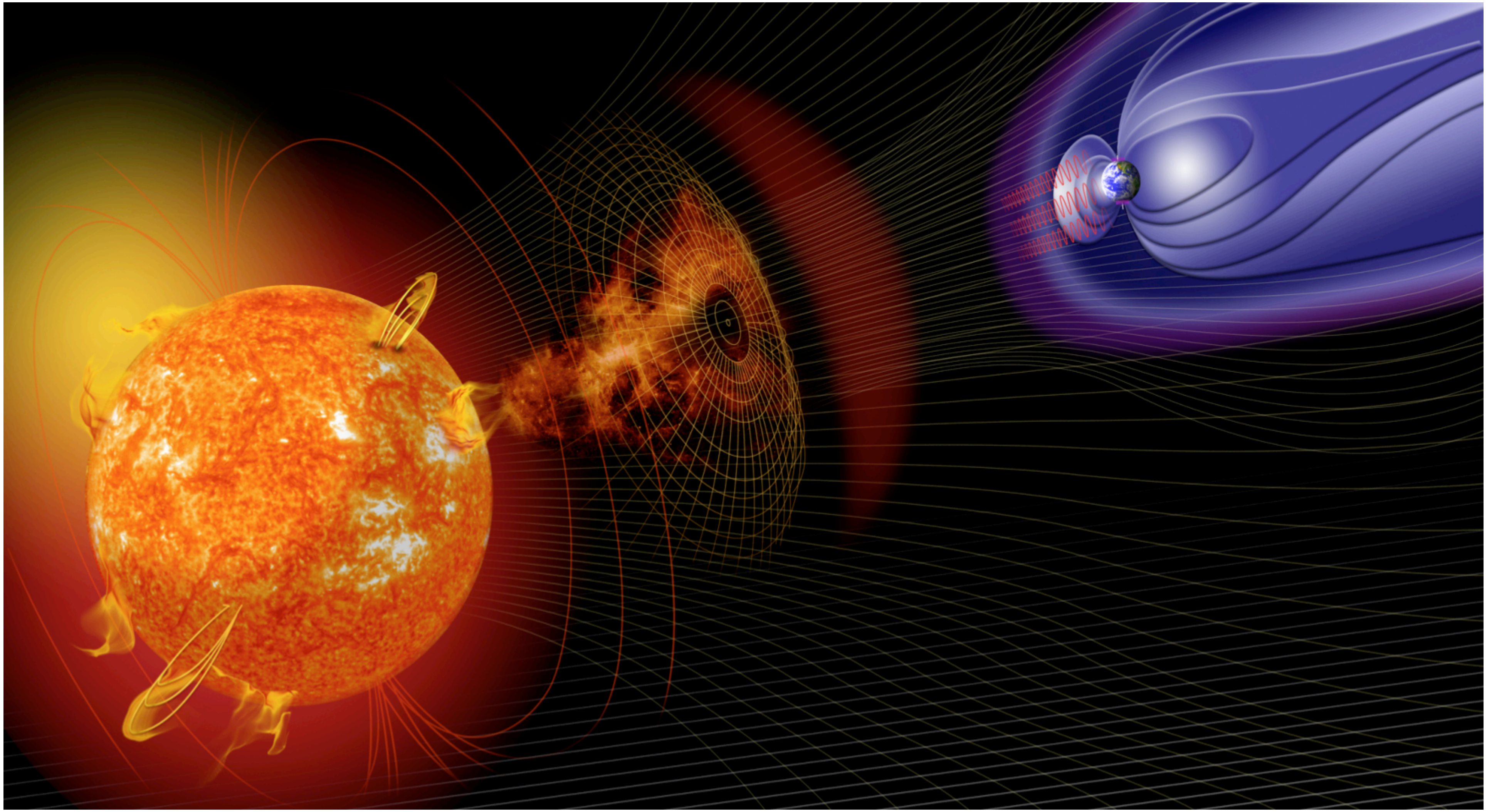
[Galtier, J. Plasma Phys., 2006]

$(k_\perp >> k_{//})$

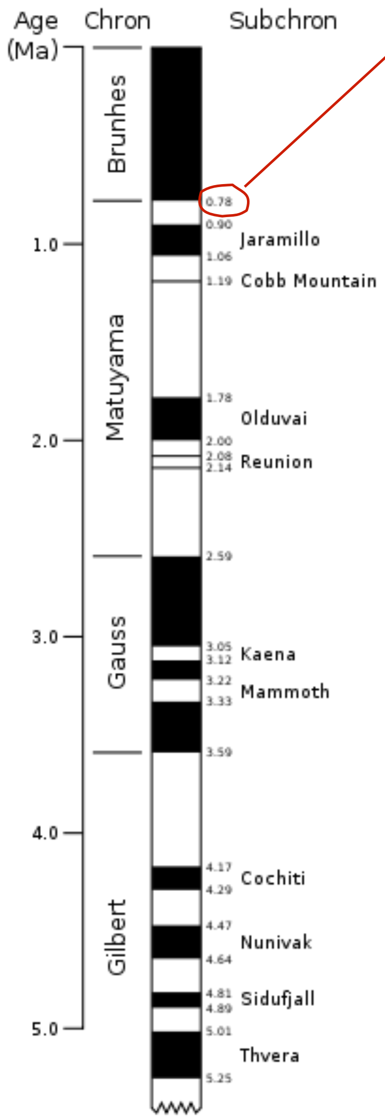




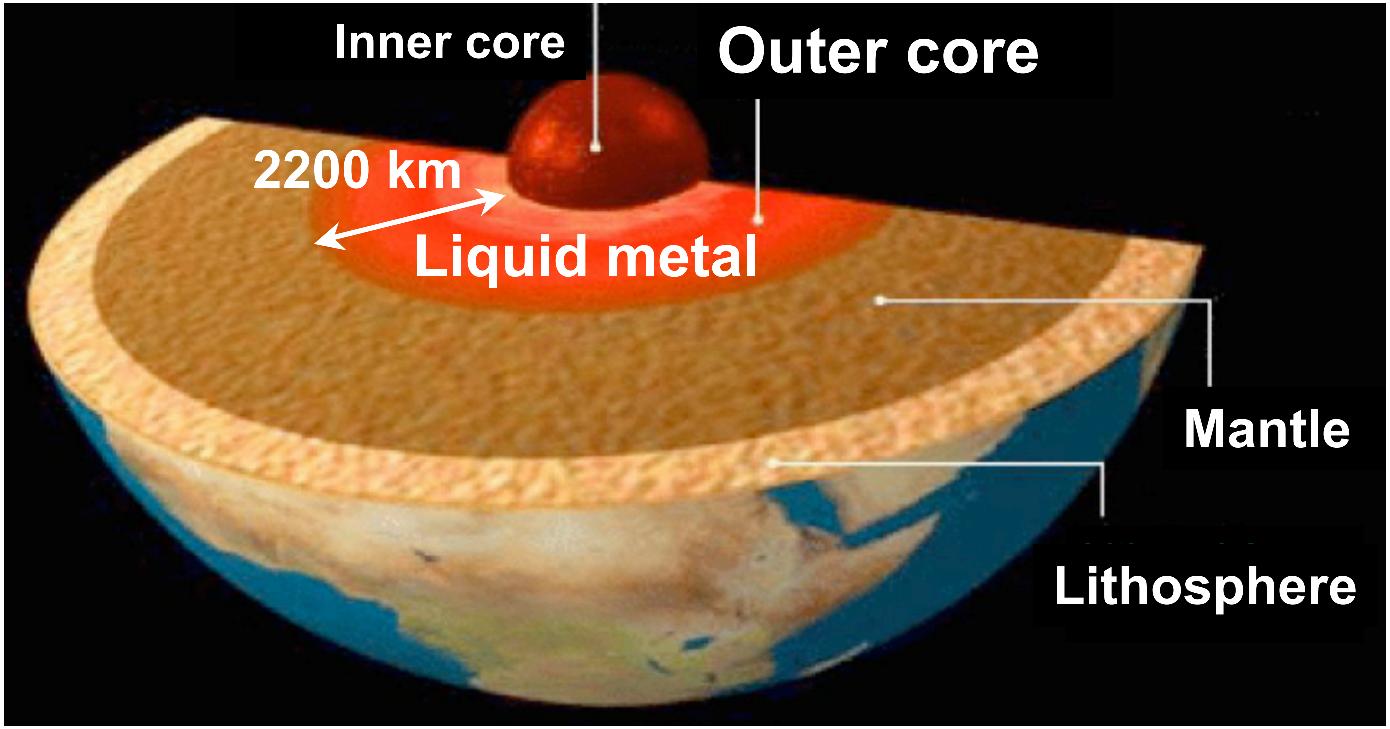
Wave turbulence in geophysics



Origin of Earth's magnetic field ?



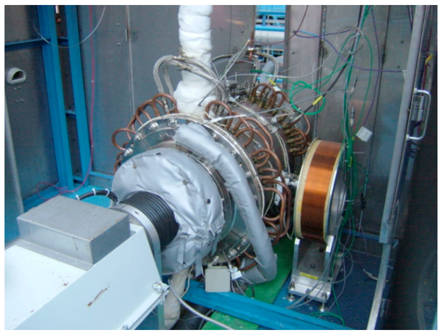
780 000 years ago => very slow time-scale !

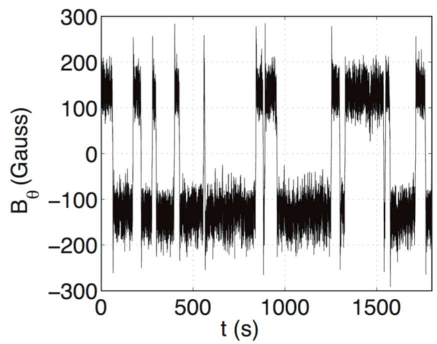


VKS experiment :

(Cadarache)

[Monchaux et al., PRL, 2007]





Incompressible rotating MHD

Coriolis force — Laplace force

$$\frac{\partial \mathbf{u}}{\partial t} + \underbrace{2\mathbf{\Omega_0} \times \mathbf{u}}_{\text{Coriolis force}} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P_* + \underbrace{\mathbf{b_0} \cdot \nabla \mathbf{b} + \mathbf{b} \cdot \nabla \mathbf{b}}_{\text{Laplace force}} + \nu \nabla^2 \mathbf{u} \,,$$

$$\frac{\partial \mathbf{b}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{b} = \mathbf{b_0} \cdot \nabla \mathbf{u} + \mathbf{b} \cdot \nabla \mathbf{u} + \eta \nabla^2 \mathbf{b} \,,$$

$$\mathbf{\Omega_0} = \Omega_0 \hat{\mathbf{e}}_{||} \,, \qquad \mathbf{b_0} = b_0 \hat{\mathbf{e}}_{||}$$

$d \equiv \dfrac{b_0}{\Omega_0}$: magneto-inertial length

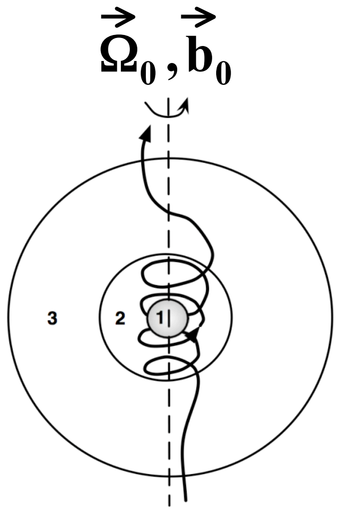
Two inviscid invariants:

$$E = \frac{1}{2} \int (\mathbf{u}^2 + \mathbf{b}^2) \, d\mathcal{V} \,,$$

$$H = \frac{1}{2} \int \left(\mathbf{u} \cdot \mathbf{b} - \frac{\mathbf{a} \cdot \mathbf{b}}{d} \right) d\mathcal{V}$$

$$\begin{cases} \dfrac{\partial E}{\partial t} = -\displaystyle\int (\nu \, \mathbf{w}^2 + \eta \, \mathbf{j}^2) \, d\mathcal{V} \,, \\ \dfrac{\partial H^c}{\partial t} = \mathbf{\Omega_0} \cdot \displaystyle\int (\mathbf{b} \times \mathbf{u}) \, d\mathcal{V} - (\nu + \eta) \displaystyle\int (\mathbf{j} \cdot \mathbf{w}) \, d\mathcal{V} \,, \\ \dfrac{\partial H^m}{\partial t} = \mathbf{b_0} \cdot \displaystyle\int (\mathbf{b} \times \mathbf{u}) \, d\mathcal{V} - 2\eta \displaystyle\int (\mathbf{j} \cdot \mathbf{b}) \, d\mathcal{V} \,, \end{cases}$$

Geophysical conditions



$U_0 = 0.5$ mm/s

$L_0 = 10^3$ km

$\Omega_0 = 7\ 10^{-5}\ s^{-1}$

$\nu = 10^{-6}\ m^2/s$

$\eta = 1\ m^2/s$

=>

$R_0 \sim \mathbf{10^{-5}}$: Rossby number

$R_e \sim \mathbf{10^8}$: Reynolds number

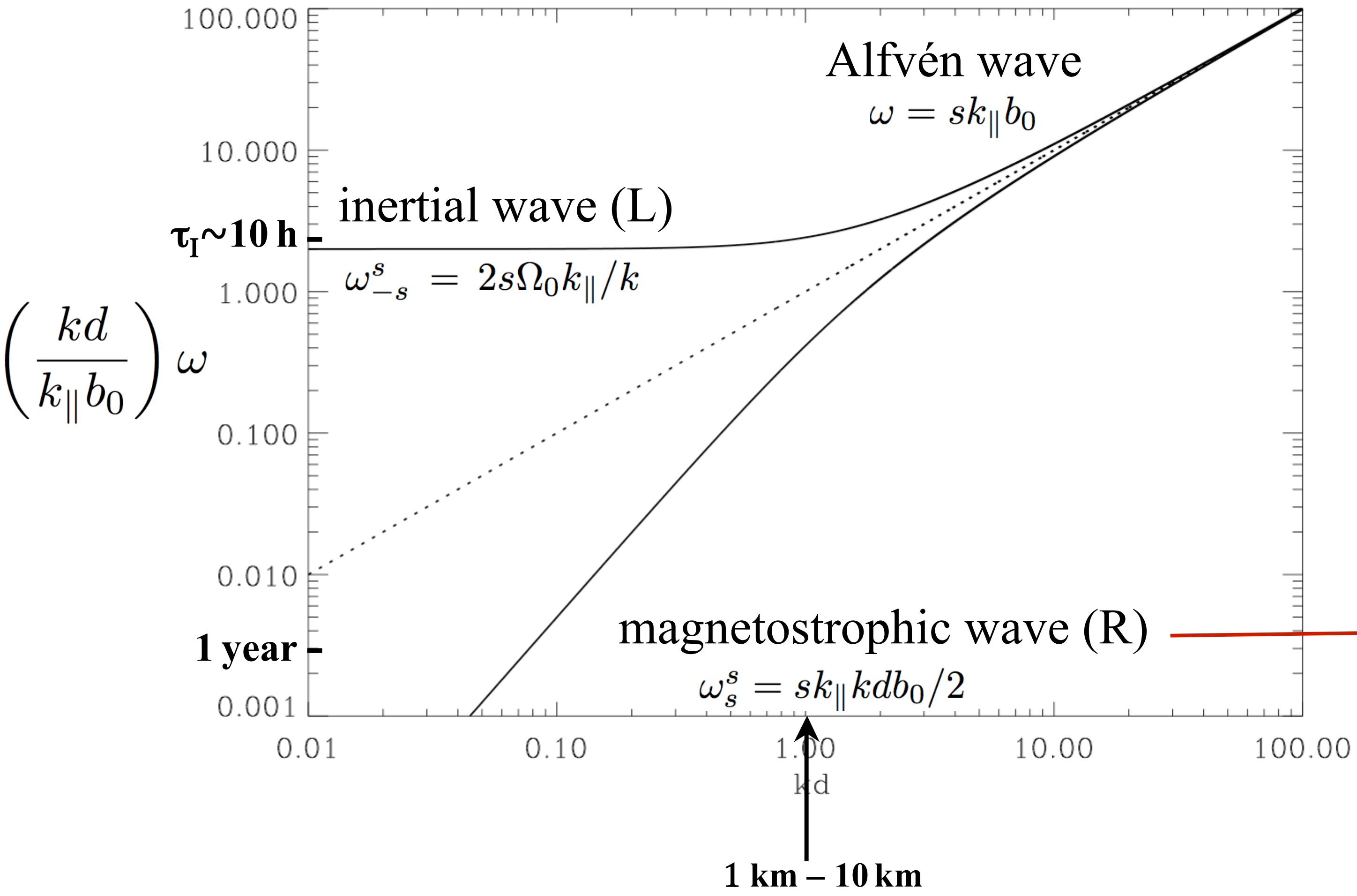
$R_m \sim \mathbf{500}$: Magnetic Reynolds number

$d = b_0/\Omega_0 \sim 100\,m - 1\,km$

Rossby number = small parameter for weak turbulence

Dispersion relation in the outer core

$$\omega \equiv \omega^s_\Lambda = \frac{s k_\| \Omega_0}{k}\left(-s\Lambda + \sqrt{1 + k^2 d^2}\right)$$

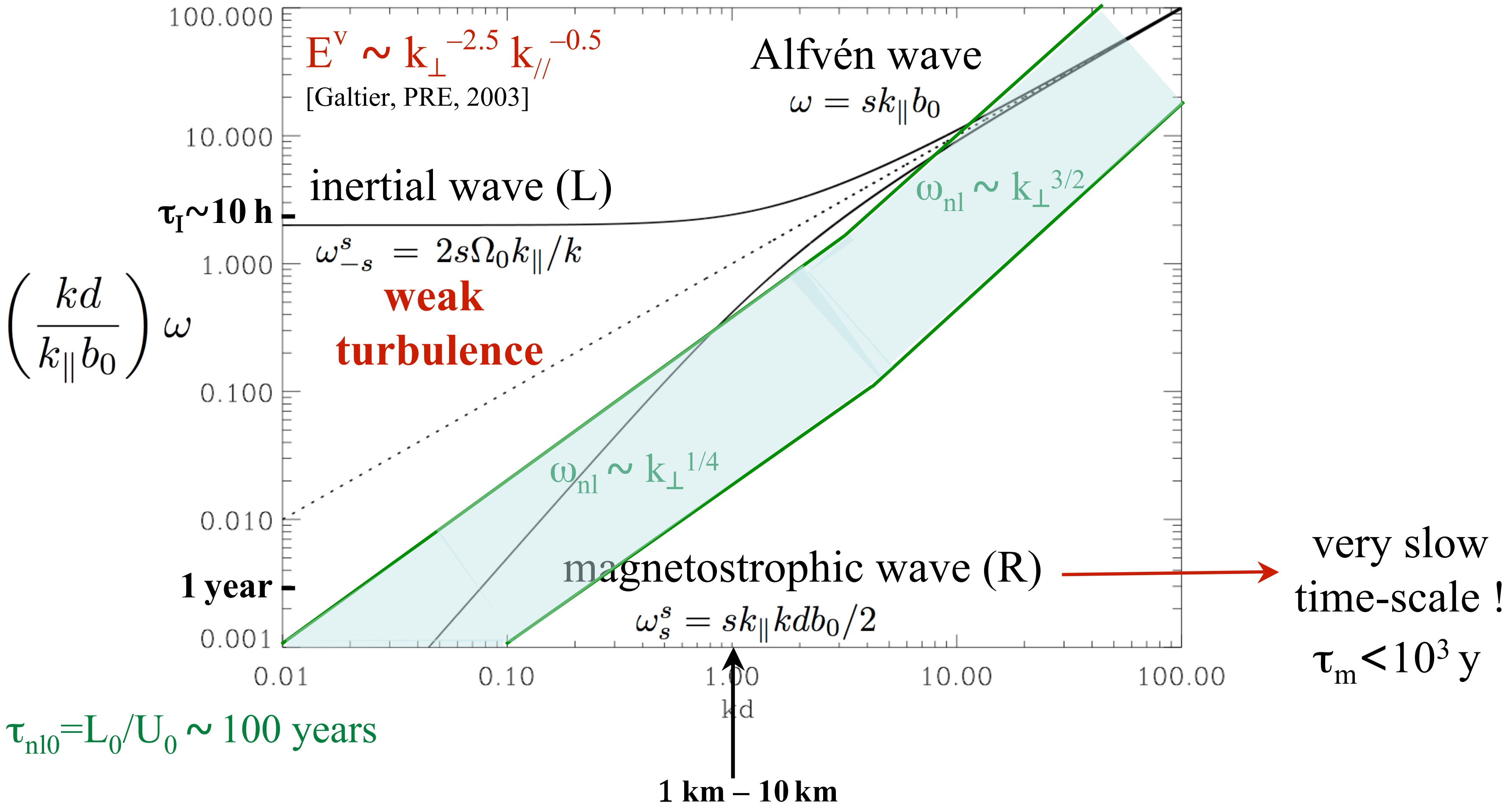


very slow time-scale !

$\tau_m < 10^3$ y

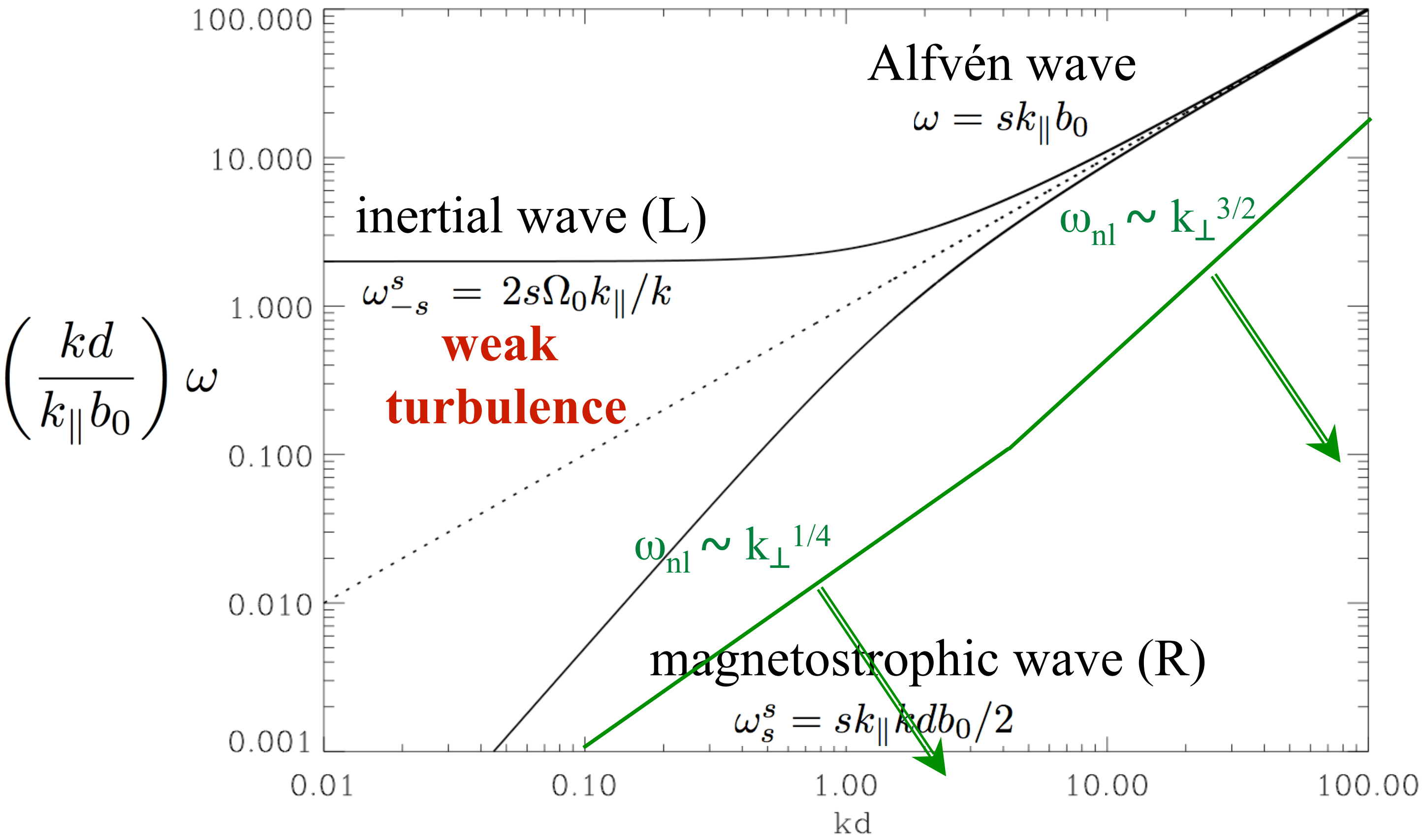
Dispersion relation in the outer core

$$\omega \equiv \omega^s_\Lambda = \frac{s k_\parallel \Omega_0}{k}\left(-s\Lambda + \sqrt{1+k^2d^2}\right)$$



For exoplanets (>1000) ?

$$\omega \equiv \omega_\Lambda^s = \frac{sk_\parallel \Omega_0}{k}\left(-s\Lambda + \sqrt{1+k^2d^2}\right)$$



Weak turbulence theory

[Galtier, in preparation]

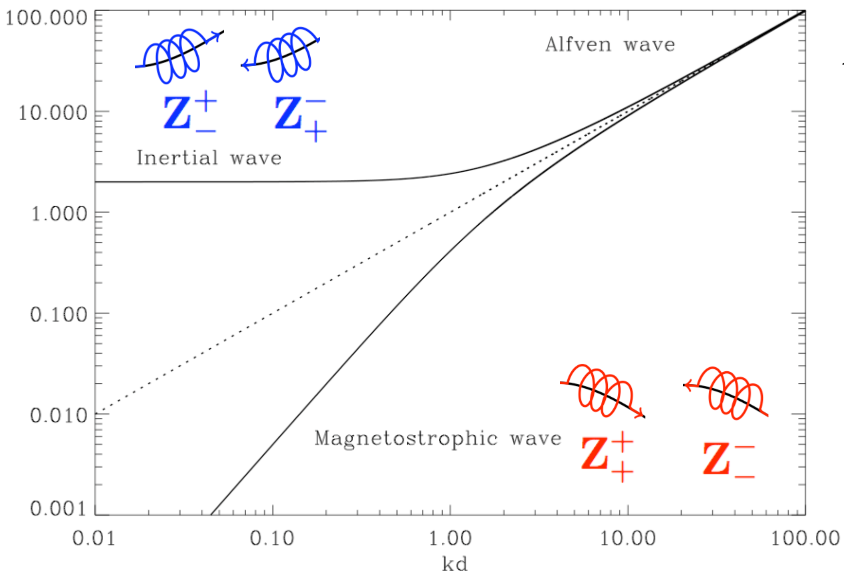
$$\partial_t q^s_\Lambda(\mathbf{k}) = \qquad (3.28)$$

$$\frac{\pi\,\epsilon^2 d^4}{64\,b_0^2} \int \sum_{\substack{\Lambda_p,\Lambda_q \\ s_p,s_q}} \left(\frac{\sin\psi_k}{k}\right)^2 k^2p^2q^2(\Lambda k+\Lambda_p p+\Lambda_q q)^2 \xi^{s\,2}_\Lambda \xi^{s_p\,2}_{\Lambda_p} \xi^{s_q\,2}_{\Lambda_q} \left(\frac{\xi^{-s_q}_{\Lambda_q}-\xi^{-s_p}_{\Lambda_p}}{k_\parallel}\right)^2$$

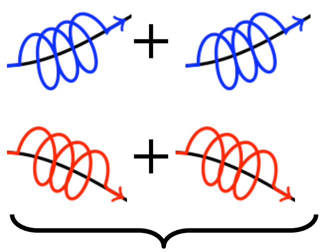
$$\left(2+\xi^{-s\,2}_\Lambda \xi^{-s_p\,2}_{\Lambda_p} \xi^{-s_q\,2}_{\Lambda_q} - \xi^{-s\,2}_\Lambda - \xi^{-s_p\,2}_{\Lambda_p} - \xi^{-s_q\,2}_{\Lambda_q}\right)^2 \left(\frac{\omega^s_\Lambda}{1+\xi^{-s\,2}_\Lambda}\right) q^s_\Lambda(\mathbf{k})\, q^{s_p}_{\Lambda_p}(\mathbf{p})\, q^{s_q}_{\Lambda_q}(\mathbf{q})$$

$$\left[\frac{\omega^s_\Lambda}{(1+\xi^{-s\,2}_\Lambda)q^s_\Lambda(\mathbf{k})} - \frac{\omega^{s_p}_{\Lambda_p}}{(1+\xi^{-s_p\,2}_{\Lambda_p})q^{s_p}_{\Lambda_p}(\mathbf{p})} - \frac{\omega^{s_q}_{\Lambda_q}}{(1+\xi^{-s_q\,2}_{\Lambda_q})q^{s_q}_{\Lambda_q}(\mathbf{q})}\right] \delta(\Omega_{k,pq})\,\delta_{k,pq}\,d\mathbf{p}\,d\mathbf{q}\,.$$

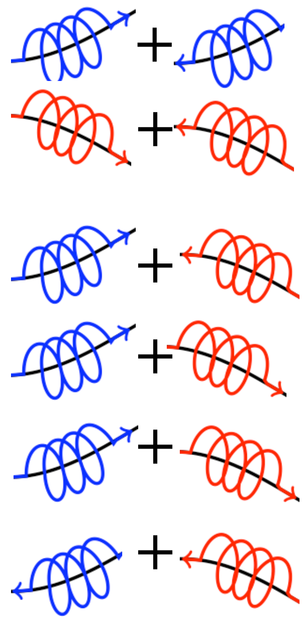
$$\xi^s_\Lambda \equiv \frac{-skd}{(-s\Lambda+\sqrt{1+k^2d^2})}$$



three-wave interactions:



weak transfer



Summary

- Waves and turbulence are ubiquitous in astrophysics

- Weak (Hall) MHD turbulence relevant for understanding:
  - solar/stellar coronal loops
  - Jupiter's and Saturn's magnetospheres
  - solar wind (f >1Hz)
  - geodynamo (Earth and exoplanets)

New book in wave turbulence !