2472-10

Advanced Workshop on Nonlinear Photonics, Disorder and Wave Turbulence

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Review of optical turbulence in focusing and defocusing media

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Evgenii A. Kuznetsov
Gregory Falkovich
Pavel Lushnikov
Alexander Korotkevich

for explaining the foundations of the field

References cited in this talk are incomplete and subjective.

Nonlinear Schrödinger equation

$$i\psi_t + \nabla^2\psi \pm |\psi|^2\psi = 0$$

describes the evolution of a temporal envelope of a spectrally narrow wave packet, independent of the origin of the waves and the nature of the nonlinearity

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Benney & Newell (1967) — general settings
Zakharov (1968) — deep water waves
Hasegawa & Tappert (1973) — optical fibers

Why universal?

Linear wave:

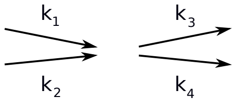
$$\frac{\partial a}{\partial t} + v\frac{\partial a}{\partial x} = 0$$

$$\frac{\partial a_k}{\partial t} + i\omega a_k = 0$$

$$\frac{\partial a_k}{\partial t} = -i\frac{\partial H_2}{\partial a_k^*}$$

$$H_2 = \int \omega_k |a_k|^2 d\mathbf{k}$$

Nonlinearity:



$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$$

$$\mathbf{k} = \mathbf{k}_0 + \mathbf{q}_k, \quad q_k \ll k_0$$

$$H_4 = \ldots$$

$$H = H_2 + H_4 = H_2 + \int T_{1234}\, a_1 a_2 a_3^* a_4^*\, \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4$$

Rewrite $\frac{\partial a_k}{\partial t} + i\omega a_k = -i\frac{\partial H_4}{\partial a_k^*}$ for the envelope, $a_k(t) = e^{-i\omega_0 t}\psi(q,t)$,

$$\frac{\partial \psi_q}{\partial t} - i\omega_0\psi_q + i\omega(q)\psi_q = -iT\int \psi_1^*\psi_2\psi_3\, \delta(\mathbf{q} + \mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) d\mathbf{q}_1 d\mathbf{q}_2 d\mathbf{q}_3$$

Why universal?

$$i\frac{\partial\psi_q}{\partial t}+\omega_0\psi_q-\omega(q)\psi_q=T\int\psi_1^*\psi_2\psi_3\,\delta(\mathbf{q}+\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)d\mathbf{q}_1d\mathbf{q}_2d\mathbf{q}_3$$

Assume $\omega=\omega(k)$ and expand for small $\mathbf{q}$

$$\omega(q)=\omega_0+q_i\left(\frac{\partial\omega}{\partial k_i}\right)_0+\frac{1}{2}q_iq_j\left(\frac{\partial^2\omega}{\partial k_i\partial k_j}\right)_0=\omega_0+vq_\parallel+\frac{1}{2}\left(\omega''q_\parallel^2+\frac{v}{k_0}q_\perp^2\right)$$

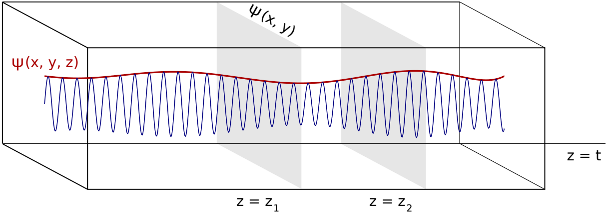
Back to r-space ($\mathbf{k}_0\parallel\hat{\mathbf{z}}$):

$$\underbrace{i\left(\frac{\partial\psi}{\partial t}+v\frac{\partial\psi}{\partial z}\right)}_{\frac{\partial\psi}{\partial t}\text{ in moving frame}}+\underbrace{\frac{\omega''}{2}\frac{\partial^2\psi}{\partial z^2}}_{\text{dispersion}}+\underbrace{\frac{v}{2k_0}\nabla_\perp^2\psi}_{\text{diffraction}}=\underbrace{T|\psi|^2\psi}_{\text{nonlinearity}}$$

Rescale ψ and spatial coordinates:

$$\boxed{i\psi_t+\nabla^2\psi\pm|\psi|^2\psi=0}$$

Why optical?



$$\boxed{\frac{1}{c^2}\left(\epsilon E\right)_{tt} - \nabla^2 E = 0}$$

Stationary envelope: $E = \frac{1}{2}\psi(x,y,z)e^{ikz-i\omega t}$, with $\omega = \frac{kc}{\sqrt{\epsilon_0}}$.

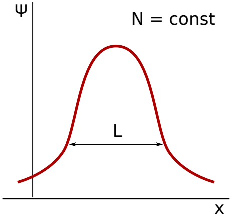
Kerr nonlinearity: $\epsilon = \epsilon_0 + \epsilon_2|E|^2 = \epsilon_0 + \epsilon_2|\psi|^2$.

$$\frac{1}{c^2}(i\omega)^2(\epsilon_0 + \epsilon_2|\psi|^2)\psi - \left[\nabla^2\psi + 2ik\psi_z - k^2\psi\right] = 0$$

Neglecting $\frac{\partial^2\psi}{\partial z^2}$ and using $kx \to x$, $\frac{1}{2}kz \to z$, and $\psi\left|\frac{\epsilon_2}{k\epsilon_0}\right|^{\frac{1}{2}} \to \psi$,

$$\boxed{i\psi_z + \nabla_\perp^2\psi + T|\psi|^2\psi = 0,} \qquad \text{with} \quad T = \pm 1$$

Collapses in focusing NSE



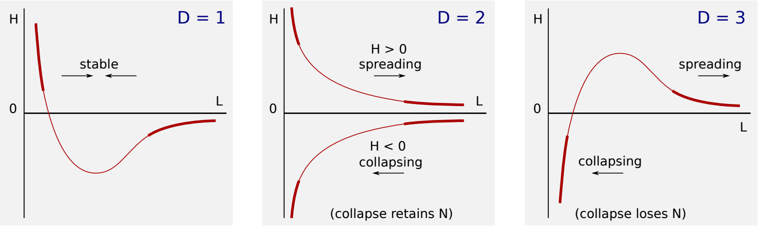
$$i\psi_t + \nabla^2\psi + |\psi|^2\psi = 0$$

Integrals of motion

$$N = \int |\psi|^2 \, d^D r$$
$$\mathcal{H} = \int \left(|\nabla\psi|^2 - \frac{1}{2}|\psi|^4 \right) \, d^D r$$

Within the packet

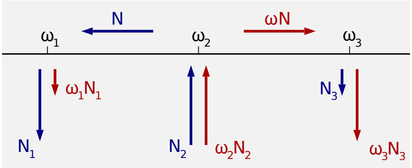
$$|\psi|^2 \sim N/L^D$$
$$\mathcal{H} \sim NL^{-2} - N^2 L^{-D}$$



Zakharov & Kuznetsov (1986)

Cascades of turbulence

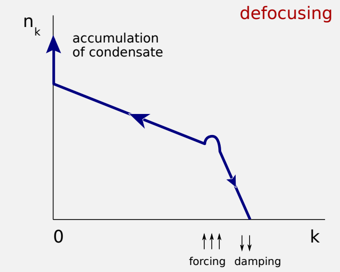
$$\mathcal{H} = \int \omega_k |a_k|^2 d\mathbf{k}$$
$$N = \int |a_k|^2 dk$$

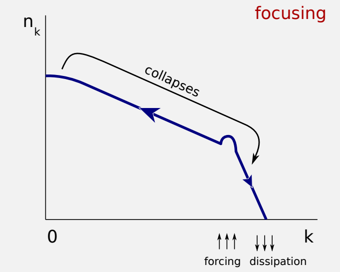


$$N_1 + N_3 = N_2$$
$$\omega_1 N_1 + \omega_3 N_3 = \omega_2 N_2$$

$$N_1 = N_2 \frac{\omega_3 - \omega_2}{\omega_3 - \omega_1} \approx N_2$$
$$N_3 = N_2 \frac{\omega_2 - \omega_1}{\omega_3 - \omega_1} \ll N_2$$

$$\omega_1 N_1 \ll \omega_2 N_2$$
$$\omega_3 N_3 \approx \omega_2 N_2$$





Dyachenko, Newell, Pushkarev, & Zakharov (1992)

Modulational instability

$$\boxed{i\psi_t = -\tfrac{1}{2}\omega''\nabla^2\psi + T|\psi|^2\psi}$$

Exact solution (condensate):

$$\Psi = \sqrt{N_0}e^{-iTN_0t}$$

For small perturbation $\psi := \Psi + \psi$,

$$i\psi_t = -\tfrac{1}{2}\omega''\nabla^2\psi + 2TN_0\psi + T\Psi^2\psi^* + O(|\psi|^2).$$

In k-space, using $(\psi^*)_k = \psi^*_{-k}$,

$$\begin{aligned} i\tfrac{d}{dt}\psi_k &= \left(\tfrac{1}{2}\omega'' k^2 + 2TN_0\right)\psi_k + T\Psi^2\psi^*_{-k}, \\ -i\tfrac{d}{dt}\psi^*_{-k} &= \left(\tfrac{1}{2}\omega'' k^2 + 2TN_0\right)\psi^*_{-k} + T\Psi^2\psi_k. \end{aligned}$$

Modulational instability

Looking for the solution in the form

$$\psi_k = \alpha e^{-i(TN_0+\Omega_k)t} \qquad \text{and} \qquad \psi^*_{-k} = \beta e^{i(TN_0-\Omega_k)t},$$

rewrite the system as

$$\begin{pmatrix} \frac{1}{2}\omega'' k^2 + TN_0 - \Omega_k & T\Psi^2 \\ T\Psi^{*2} & \frac{1}{2}\omega'' k^2 + TN_0 + \Omega_k \end{pmatrix} \begin{pmatrix} \alpha\, e^{-iTN_0 t} \\ \beta\, e^{iTN_0 t} \end{pmatrix} = 0$$

Bogoliubov dispersion relation:

$$\boxed{\Omega_k^2 = \omega'' TN_0 k^2 + \tfrac{1}{4}\omega''^{\,2} k^4}$$

Instability: $\omega'' T < 0$ (focusing nonlinearity).

Bogoliubov (1947)

Why turbulence?

- Wide energy spectra; cascades
- Statistical description
- High probability of extreme events (intermittency)
- Coherent structures — condensate or collapses
- Steady (with damping/forcing) or decaying

The rest of this talk is restricted to steady NSE turbulence in 2D.

Defocusing nonlinear Schrödinger equation

$$\boxed{i\psi_t + \nabla^2\psi - |\psi|^2\psi = i\hat{f}\psi}$$

Condensate

$$\Psi = \sqrt{N_0}\exp(-iN_0 t)$$

Notation:

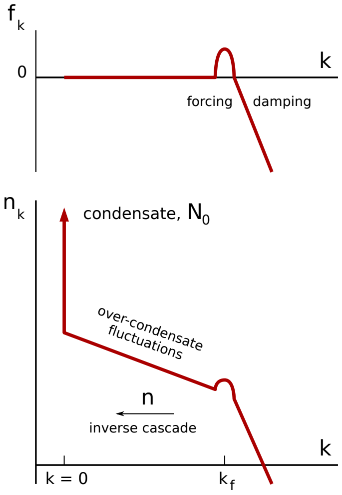
$$N = \overline{|\psi|^2}$$
$$N_0 = |\overline{\psi}|^2$$
$$n = N - N_0 = \int |\psi_k|^2 d^2k$$

We consider large condensate

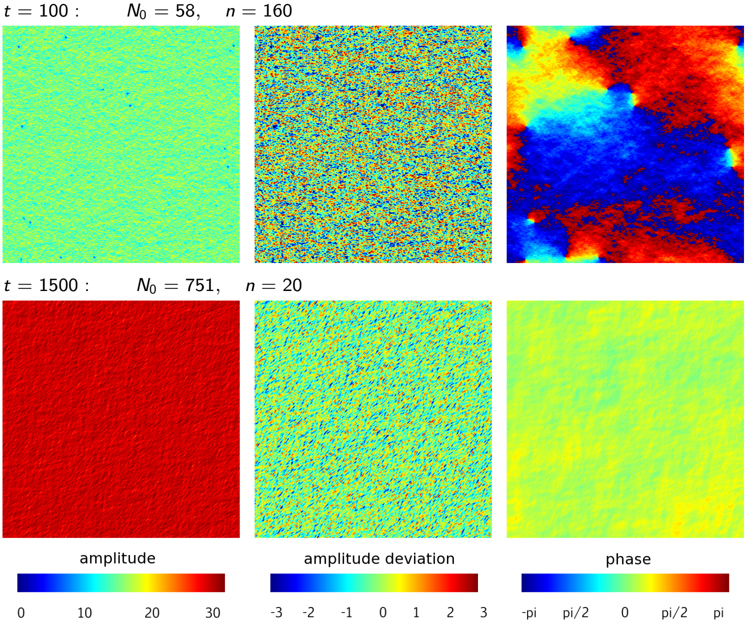
$$N_0 \gg n$$

Statistically quasi-steady

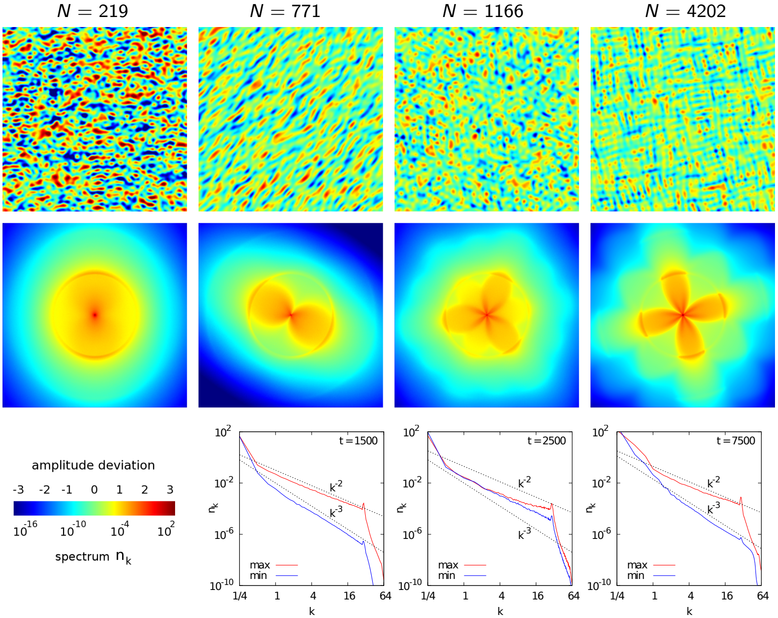
$$t \sim 10^4 \gg \frac{1}{\omega} \sim 10^{-3}$$



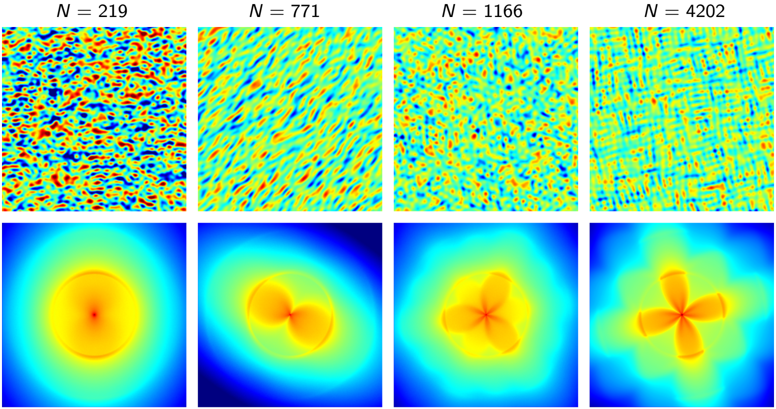
Onset of condensate

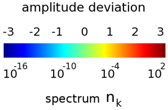


Phase transitions: breakdown of symmetries



Phase transitions: breakdown of symmetries

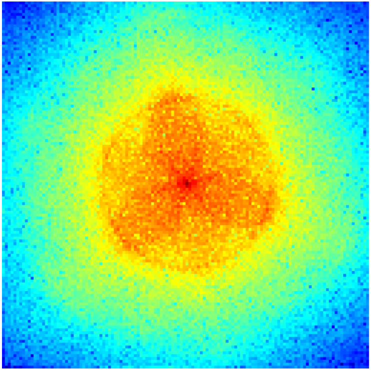




- Higher condensate ⇒ more ordered system
- Long-range orientational, short-range positional order
- What happens at even larger N?

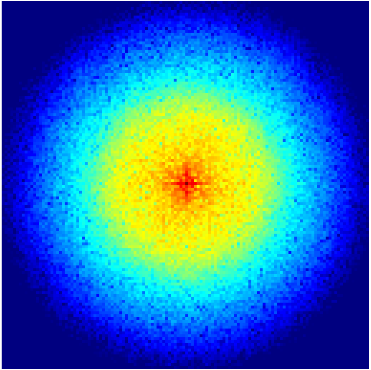
Vladimirova, Derevyanko, & Falkovich (2012)

Effect of forcing



Instability-driven force

$$i\psi_t + \nabla^2\psi - |\psi|^2\psi = i\hat{f}\psi$$



Random force

$$i\psi_t + \nabla^2\psi - |\psi|^2\psi = i\hat{F}$$

Small perturbations

Compare quadratic and cubic terms in Hamiltonian

$$\begin{aligned}
\langle \mathcal{H}_2 \rangle &= \Omega_k n = N_0^{1/2} k n \\
\langle \mathcal{H}_3 \rangle &= \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} V_{123} \langle \psi_{k_1} \psi_{k_2} \psi_{k_3}^* \rangle \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3) \\
&\simeq \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} |V_{123}|^2 n_1 n_2 \, \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3) \delta(\Omega_1 + \Omega_2 - \Omega_3) \\
&\simeq \frac{|V|^2 n^2 c}{k^3} \frac{k}{c} \simeq \frac{n^2 k}{N_0^{1/2}}
\end{aligned}$$

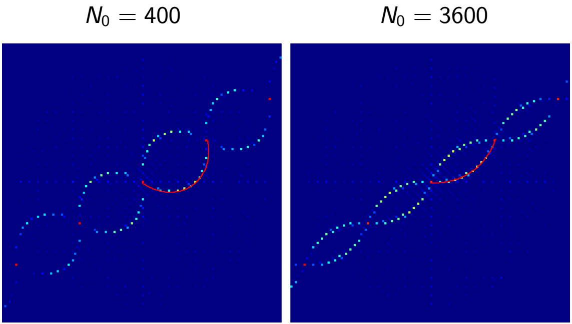
Effective nonlinearity parameter is small,

$$\frac{\mathcal{H}_3}{\mathcal{H}_2} \simeq \frac{n}{N_0}.$$

But: weak turbulence assumes random phases.

Angle of interaction: $k/c \sim k/\sqrt{N_0}$, where $c = \sqrt{2N_0}$.

Angle of interaction



Arch grows in k-space from the condensate to a preset mode, $\mathbf{k}_0$.
Arch equation:

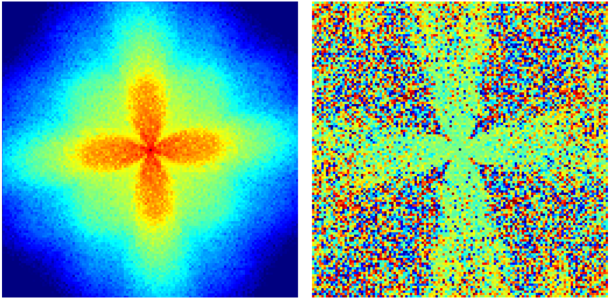
$$\begin{aligned} \omega(k_0) &= \omega(k) + \omega(|\mathbf{k}_0 - \mathbf{k}|) \\ \omega^2(k) &= 2N_0 k^2 + k^4 \end{aligned}$$

Angle of interaction:

$$\phi_{max} \approx \frac{k}{\sqrt{3N_0/2}} \sim \frac{k}{c}$$

Phase coherence

n_k $\qquad$ $\theta_k = 2\phi_0 - \phi_k - \phi_{-k}$



$$\boxed{2\phi_0 - \phi_k - \phi_{-k} = \pi}$$

Three-wave model

Consider condensate interacting with two waves

$$\psi_{\pm k} = \sqrt{n}\exp(\pm ikx + iN_0 t + i\phi_{\pm k})$$

with $\theta = 2\phi_0 - \phi_k - \phi_{-k}$.

Hamiltonian:

$$H = 2k^2 n + \frac{1}{2}N^2 + 2n(N-2n)(1+cos\theta) + n^2$$

Equations of motion:

$$\begin{aligned} \dot{n} &= 2n(N-2n)\sin\theta \\ \dot{\theta} &= 2k^2 + 2(N-3n) + 2(N-4n)\cos\theta \end{aligned}$$

Stability points:

$$\begin{array}{llcl} \theta = \pi, & n = -\frac{1}{2}k^2 & \Rightarrow & \text{unphysical} \\ \theta = 0, & n = (4N+k^2)/14 & \Rightarrow & \text{too high } n \end{array}$$

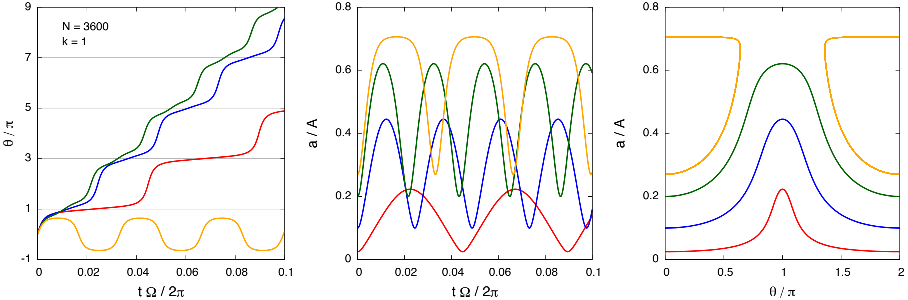
Falkovich (2011), Miller, Vladimirova & Falkovich (2013)

Predictions of three-wave model

$$\begin{aligned}\dot{n} &= 2n(N-2n)\sin\theta \\ \dot{\theta} &= 2k^2 + 2(N-3n) + 2(N-4n)\cos\theta\end{aligned}$$

For $n \ll N$:

- the system spends most of its time around $\theta = \pi$ state
- the frequency of oscillations $2\Omega \approx 2\sqrt{2Nk^2 + k^4}$
- the amplitude $a \equiv \sqrt{n(t)}$ exhibits complicated cusped shape



Individual modes in turbulence

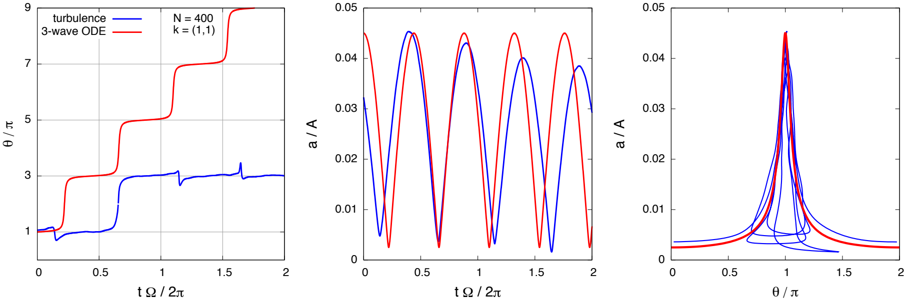
In turbulence, $n \ll N$ condition is well satisfied.

As predicted:

- the system spends most of its time around $\theta = \pi$ state
- the frequency of oscillations approaches $2\Omega = 2\sqrt{2Nk^2 + k^4}$
- the amplitude $a \equiv \sqrt{n(t)}$ exhibits complicated cusped shape

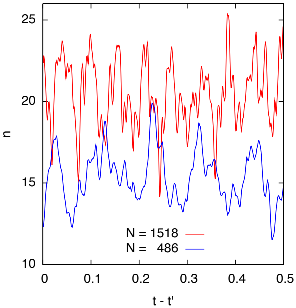
However:

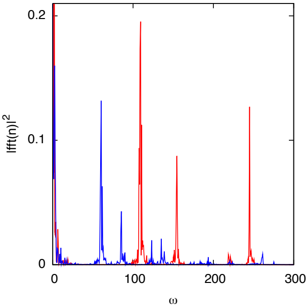
The 3-wave model cannot grasp closed trajectories with $\theta \approx \pi$.

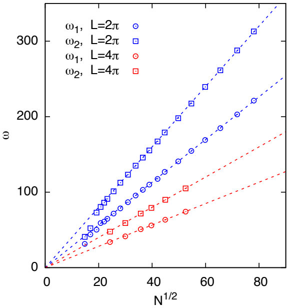


Collective oscillations

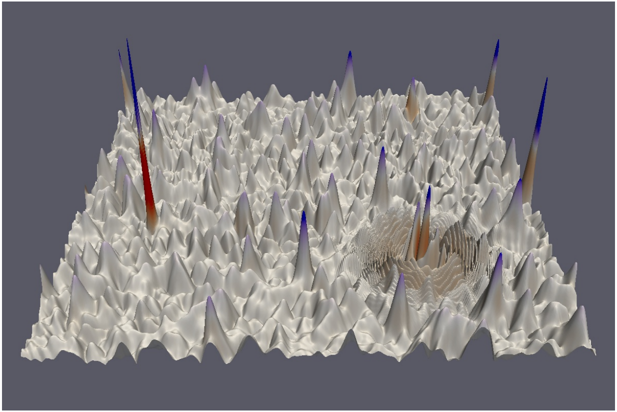
- The system periodically oscillates around a steady state.
- Turbulence and condensate exchange a small fraction of waves.
- The condensate imposes the phase coherence between the pairs of counter-propagating waves (anomalous correlation).
- Collective oscillations are not of a predator-prey type; they are due to phase coherence and anomalous correlations.







Focusing nonlinear Schrödinger equation



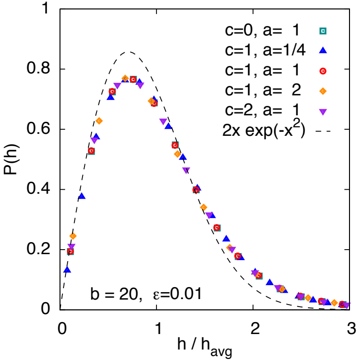
$$i\psi_t + \nabla^2\psi + |\psi|^2\psi = 0$$

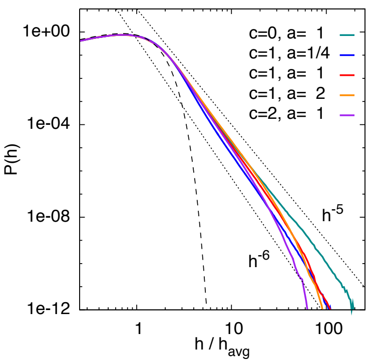
$$i\psi_t + (1 - i\epsilon a)\nabla^2\psi + (1 + i\epsilon c)|\psi|^2\psi = i\epsilon b\psi$$

Can we explain the statistics of a turbulent field with statistics and properties of individual collapses?

Distribution of $|\psi|$ in the field

Notation: $h \equiv |\psi|$, $h_{avg} = \langle|\psi|\rangle \propto \sqrt{N}$, $N = \int |\psi|^2 d^2 r$.





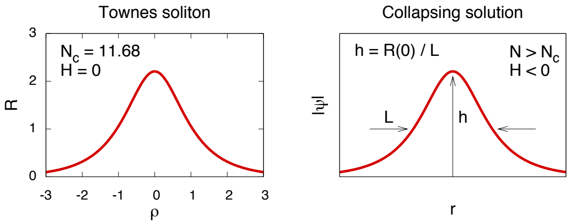
Turbulent background:

- is well described by h_{avg}
- what determines h_{avg}?

Collapse contribution:

- power-law (?) for $h \gg h_{avg}$
- depend on a, c but not b

Catastrophic collapse



Townes soliton, $\psi = \frac{1}{L} e^{i\omega t} R(\rho)$, with $\rho = \frac{r}{L}$, $\omega = \frac{1}{L^2}$

$$i\psi_t + \nabla^2\psi + |\psi|^2\psi = 0 \qquad \Rightarrow \qquad \boxed{R'' + \frac{1}{\rho}R' = R - R^3.}$$

Unstable: if $N > N_c$, the soluton blows up in finite time, t_c.

Collapsing solution is self-similar and rescales to Townes soliton.

Lens transform maps $t_c \to \infty$.

What is the functional form of $L(t)$?

Townes (1964), Talanov (1970)

Lens transform

In slow variables, $\rho = \frac{r}{L}$ and $\tau = \int L^{-2}(t')dt'$, look for solution

$$\psi(r,t) = \frac{1}{L} V(\rho,\tau) e^{i\tau + i\gamma(\tau)\rho^2/4}$$

$V(\rho,\tau)$ is real when $\gamma = LL_t$. Using $\beta \equiv -L^3 L_{tt}$ we get,

$$\underbrace{iV_\tau}_{\text{neglect}} + \nabla^2 V + V^3 - V + \frac{1}{4}\beta\rho^2 V = 0.$$

β is the measure of excess of N over critical, $\nu(\beta)$ is the loss rate.

Using β as a small parameter,

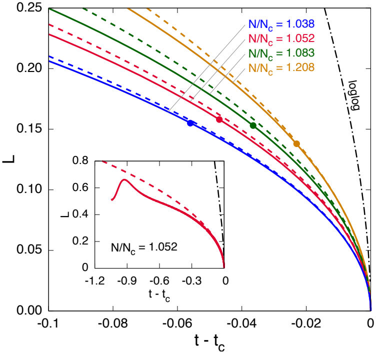
$$\begin{aligned} L_{\tau\tau} - 2L_\tau^2/L &= -\beta L && \text{small scales} \\ \beta_\tau &= -\nu(\beta) && \text{large scales} \end{aligned}$$

Loglog scaling at $t \to t_c$

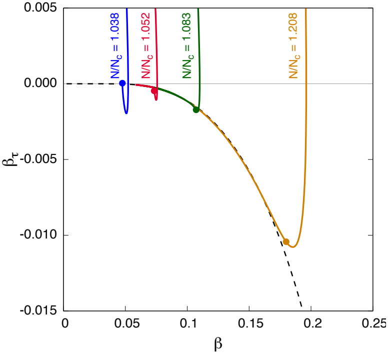
$$L \sim \left[\ln\ln\frac{1}{t_c - t}\right]^{-\frac{1}{2}} (t_c - t)^{\frac{1}{2}}$$

Talanov (1970), Zakharov (1972), Kuznetzov & Turitzin (1985)
Fraiman (1985), LeMesurier, Papanicolaou, Sulem & Sulem (1988)

Loglog and early evolution bridged!



$L(t)$ depends on N



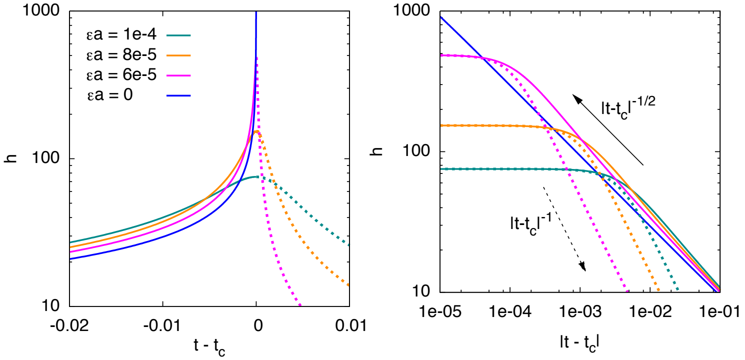
$\beta_\tau(\beta)$ is universal

Loglog law: $t \to t_c$.
Adiabatic approximation, $\beta = const$: short times or small β.

New approximation for $\nu(\beta)$ accounts for initial excess of N and bridges early evolution and loglog asymptotics.

Malkin (1993), Fibich (1996), Lushnikov, Dyachenko, Vladimirova (2013)

Collapses with stabilization

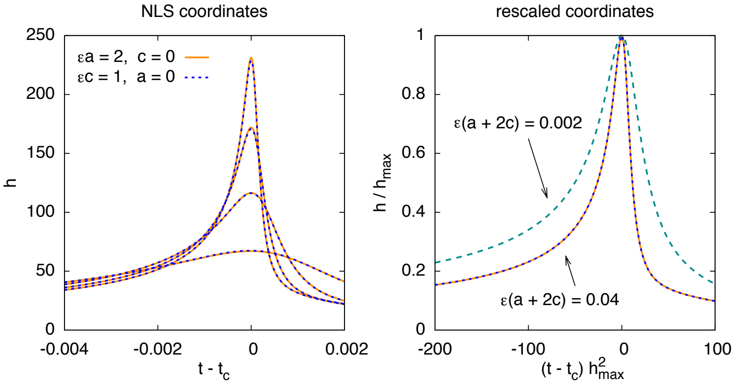


$$i\psi_t + (1 - i\epsilon a)\nabla^2\psi + (1 + i\epsilon c)|\psi|^2\psi = i\epsilon b\psi$$

Growth: $L \propto (t_c - t)^{\alpha}$, $\alpha \approx \frac{1}{2}$, rate depends on a and c.
Decay: $L \propto (t - t_c)^{-1}$, Talanov solution, unstable.

Talanov (1970)

Rescaled evolution: $(a + 2c)$ similarity



Growth and saturation of a collapse is controlled by $a + 2c$ combination.

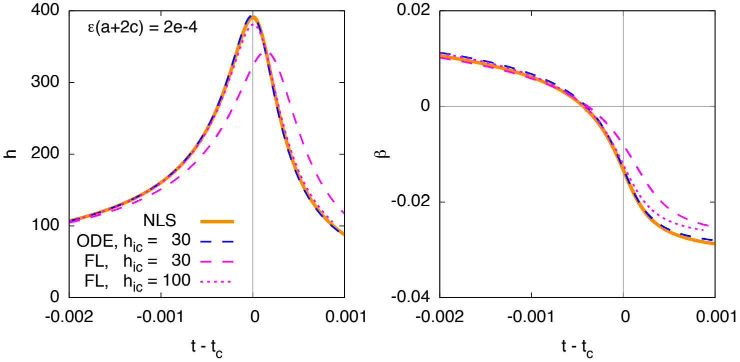
From direct integration of NSE,

$$N_t = -2\epsilon(aH_k - 2cH_p - bN), \quad \text{where} \quad H_k + H_p = H.$$

For critical collapse: $\psi = e^{i\omega t}u(r)$, $H_k = -H_p = \omega N_c$, $\omega = h^2$

$$N_t = -2\epsilon(a + 2c)h^2 N_c$$

ODE model for collapse saturation



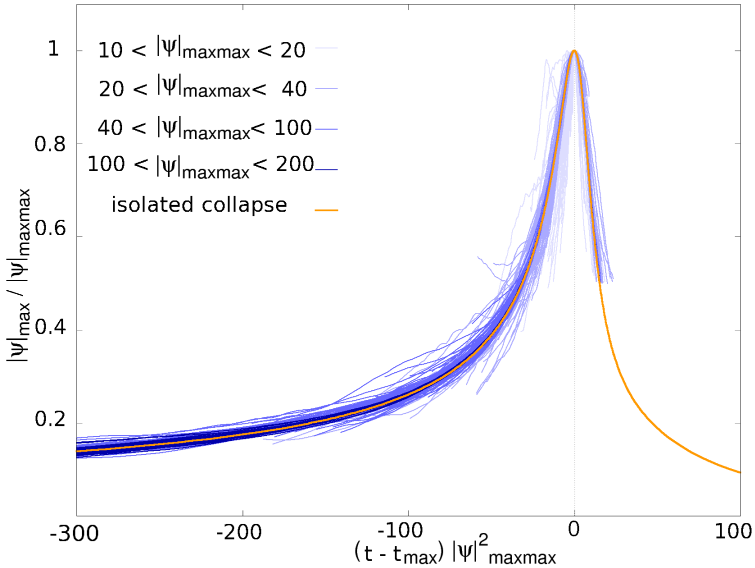
$$
\begin{aligned}
L^2\beta_t &= -k_1 - k_2\beta + k_3(LL_t)^2 - \nu(\beta) \\
L^3 L_{tt} &= -\beta
\end{aligned}
$$

$$k_1 \approx 6.76\,\epsilon(a+2c),\ k_2 \approx 2.01\,\epsilon a,\ k_3 \approx 0.01\,\epsilon a.$$

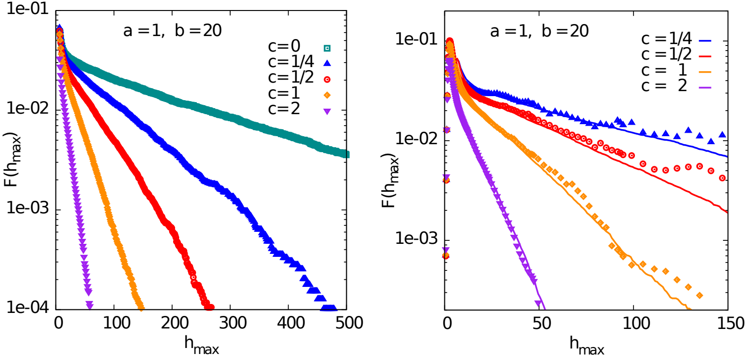
Data suggest: k_1 term describes first order effects well;
$k_2 \approx -37\epsilon(a+2c)$ instead of original $k_2 \approx 2\epsilon a$.

Fibich & Levy (1998)

Universality of collapses in turbulence



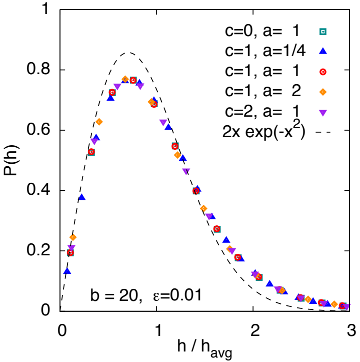
Connecting PDF of $|\psi|$ to frequency of collapses



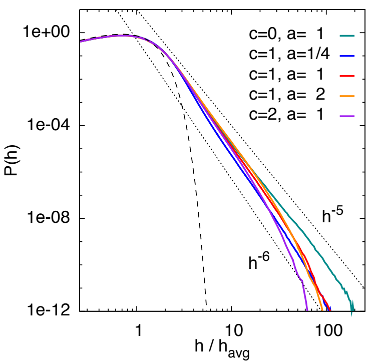
- Assume similarity among collapses, $h \sim (t_c - t)^{-\frac{1}{2}}$.
- Left: frequency of collapses with $h > h_{\text{max}}$ per unit area.
- Right: solid lines show $F(h_{\text{max}})$; points show $0.012\, h^5 P(h)$.
- Conclude that PDF in the field $P(h) \sim h^{-5} F_{\text{max}}(h)$.

Lushnikov & Vladimirova (2010)

Distribution of $|\psi|$ in the field



turbulent background



collapse contribution

A lot about focusing NSE turbulence not understood yet:

- universality of the backgound
- how collapes are seeded
- how breaking of collapses contribute to turbulence
- how saturation parameters affect the system

Conclusion

- The nonlinear Schrödinger equation is a universal model, describing a spectrally narrow wave packet.
- Focusing and defocusing nonlinearities exhibit different behavior.
- In 2D, focusing nonlinearity results in collapses.
- In 2D, defocusing nonlinearity results in condensate.
- Interaction of either collapses or condensate with background turbulence is nontrivial.
- The study of NSE turbulence has just begun...

References (books and reviews)

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- J. Yang, *Nonlinear Waves in Integrable and Nonintegrable Systems*, SIAM, 2010
- G. Falkovich *Fluid mechanics (a short course for physicists)*, Cambridge University Press, 2011