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Advanced Workshop on Nonlinear Photonics, Disorder and Wave Turbulence

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Optical wave turbulence

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Optical wave turbulence

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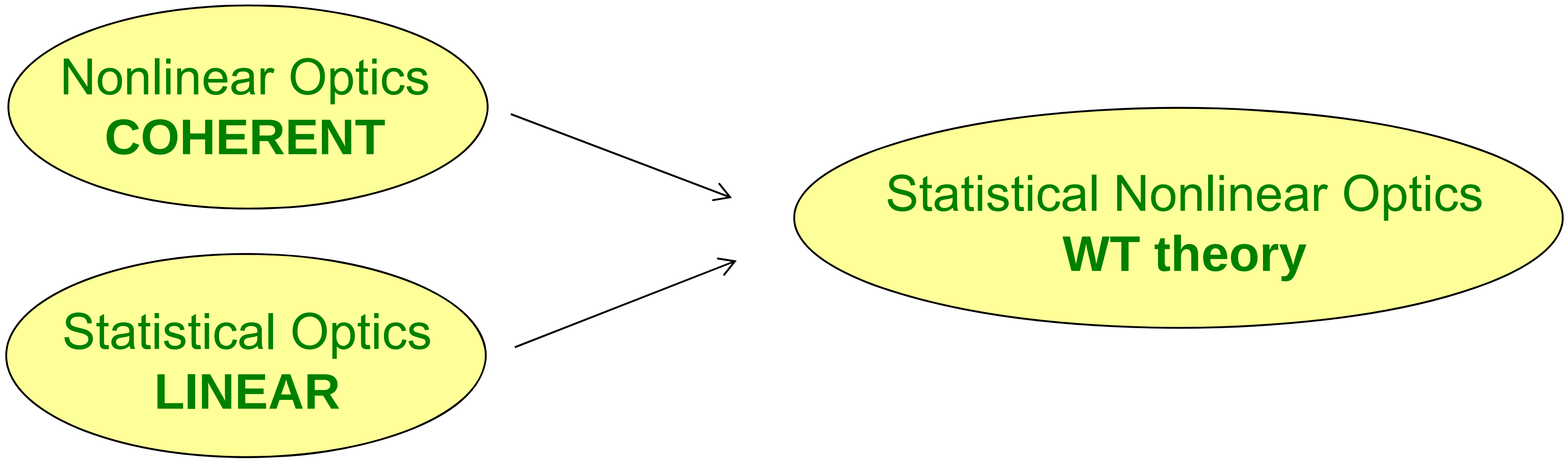
Theory

S. Rica (Un. Ibanez, Chile)
J. Garnier (ENS, Paris)

Experiments & Simulations

B. Kibler, S. Pitois, G. Millot (ICB)
P. Suret, S. Randoux (PhLam, Lille)
C. Michel, P. Aschieri (LPMC, Nice)
B. Barviau (CORIA, Rennes)
R. Kaiser (INLN, Nice)



Advanced Workshop on Nonlinear Photonics, Disorder and Wave Turbulence
Trieste – July 2013

Statistical nonlinear optics
$\longrightarrow$ kinetic approach: WT

1.- Homogeneous statistics & instantaneous response:

WT kinetic eqn

Thermalization / Anomalous thermalization -- Condensation

2.- Inhomogeneous statistics (& nonlocal response):

Vlasov kinetic eqn

Effective potential

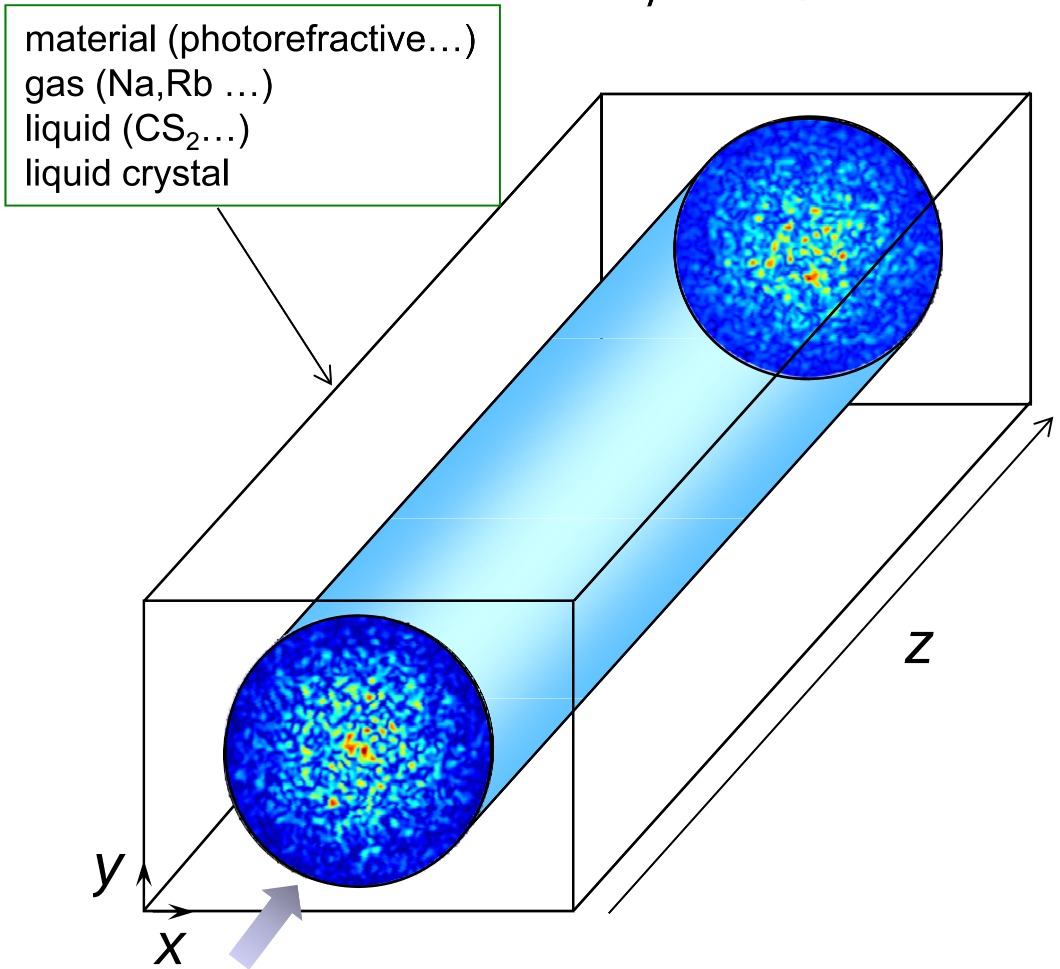
3.- Noninstantaneous response:

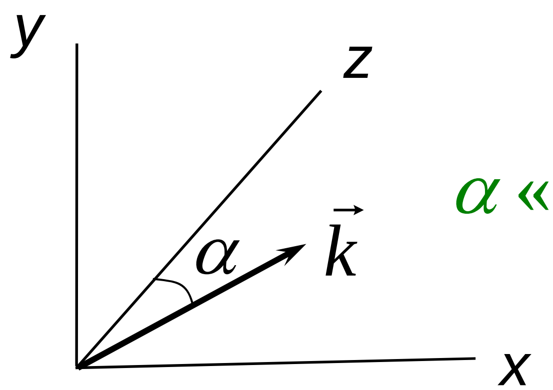
Weak Langmuir turbulence kinetic eqn

Causality condition

‘Spatial experiment’

ψ : amplitude of the optical field





$\alpha \ll 1$

$$\begin{cases} k = \sqrt{k_x^2 + k_y^2 + k_z^2} \\ \Delta k = k - k_z \approx (k_x^2 + k_y^2)/(2k_z) \\ \tilde{\psi}(\Delta k, k_x, k_y) \rightarrow \psi(z, x, y) \\ i\partial_z \psi(z, \vec{r}) = -\dfrac{1}{2k_z} \nabla^2 \psi(z, \vec{r}) \end{cases}$$

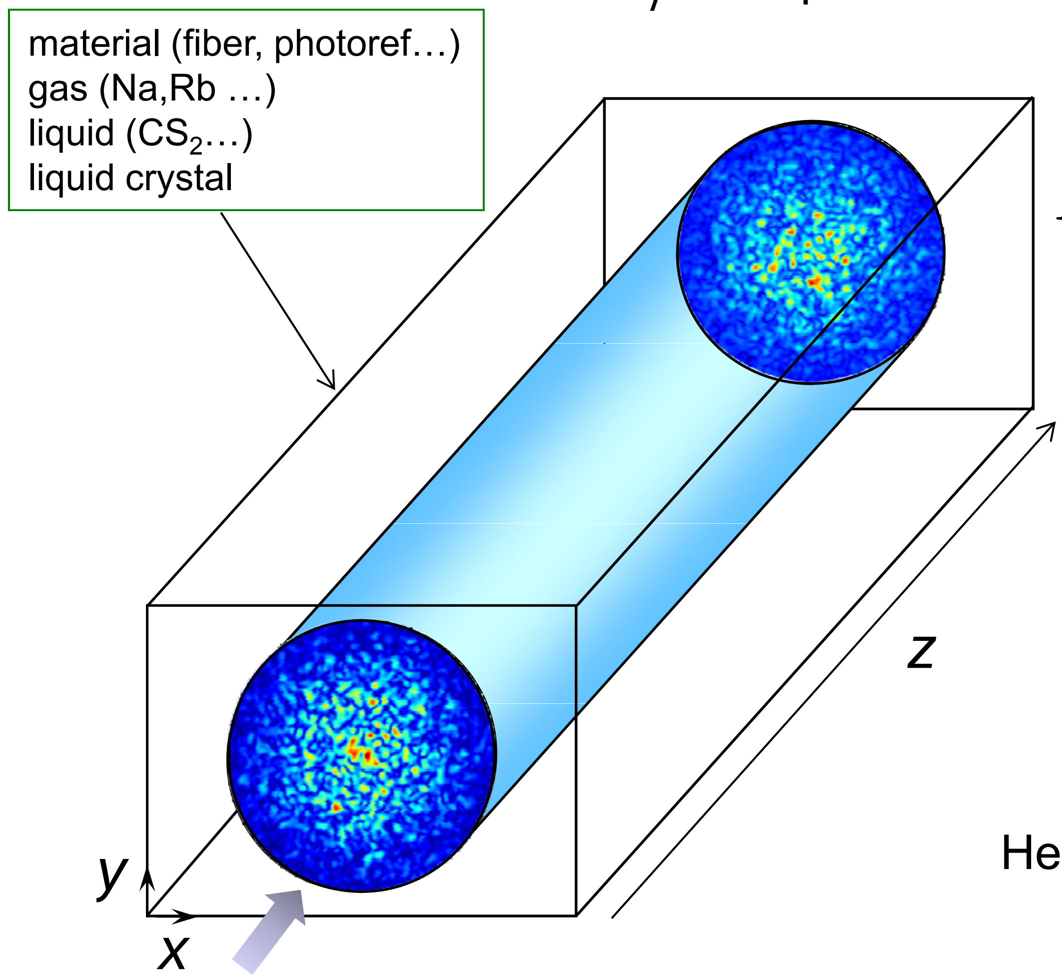
$$\begin{cases} n(I) = n_0 + n_{nl} I \\ k = n_0 \omega / c + n_{nl} I \omega / c \\ i\partial_z \psi = -\dfrac{1}{2k_z} \nabla^2 \psi + \dfrac{k n_{nl}}{n_0} |\psi|^2 \psi \end{cases}$$

weak NL

R.W. Boyd, *Nonlinear Optics* (2008)
J.V. Moloney, A.C. Newell, *Nonlinear Optics* (2004)

‘Spatial experiment’

ψ : amplitude of the optical field



$$\begin{cases} i\partial_z\psi = -\beta\,\nabla_\perp^2\psi + g\ f(|\psi|^2)\,\psi \\ \nabla_\perp^2 = \partial_{xx} + \partial_{yy} \\ f(|\psi|^2) = |\psi|^2 \end{cases} \qquad \left(f(|\psi|^2) = \frac{|\psi|^2}{1+|\psi|^2} \right)$$

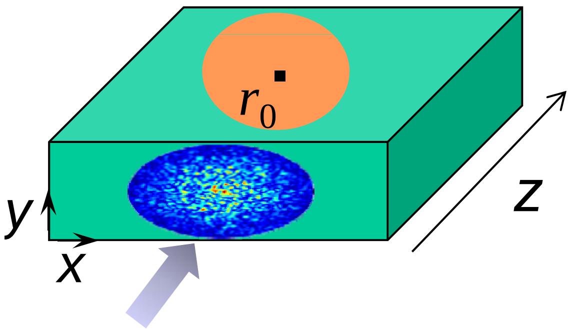
$$L_{lin} = \frac{\lambda_c^2}{\beta},\ L_{nl} = \frac{1}{g|\psi|^2}$$

$\downarrow$

Healing length: $\Lambda = \sqrt{\dfrac{\beta}{g|\psi|^2}}$

$\longrightarrow$ Typical size of a soliton, vortex, MI period

Nonlocal spatial response



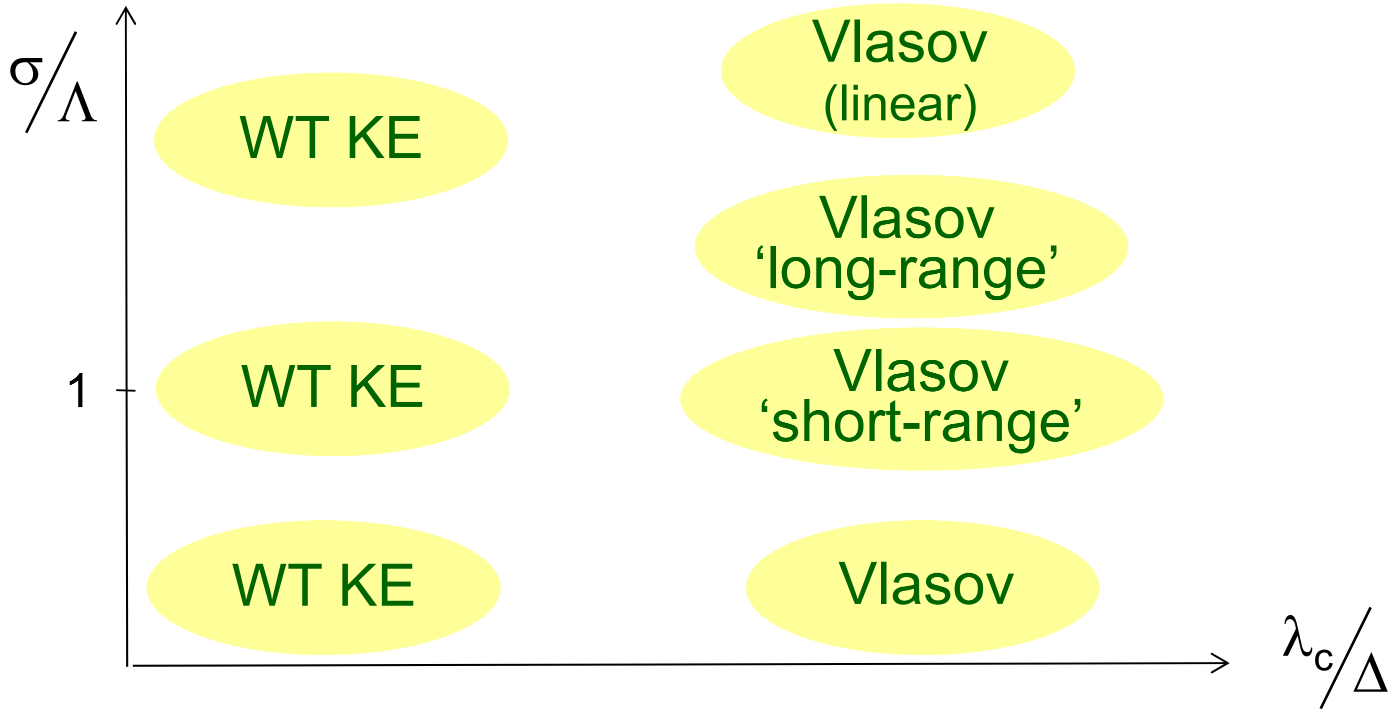
$$i\partial_z\psi = -\beta\,\nabla_\perp^2\psi + g\,\psi\int U(r')\,|\psi|^2(r-r')\,dr'$$

$U(x,y)$ nonlocal response (real, even): σ

$\Lambda = \sqrt{\beta/(g|\psi|^2)}$: healing length

Δ : scale of inhomogeneous statistics

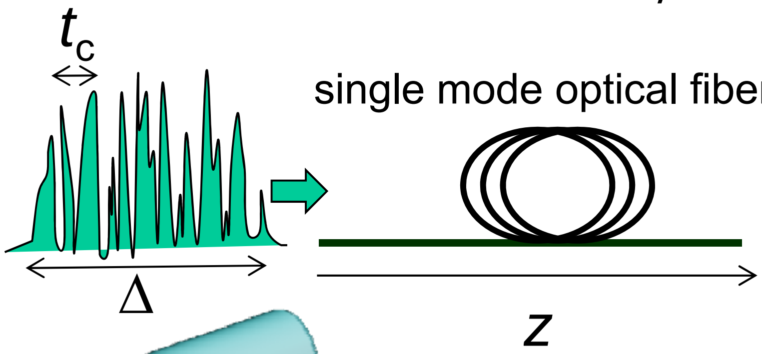
λ_c : correlation length

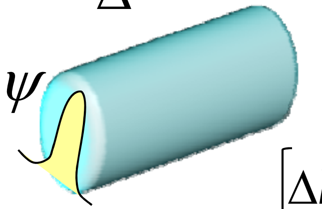


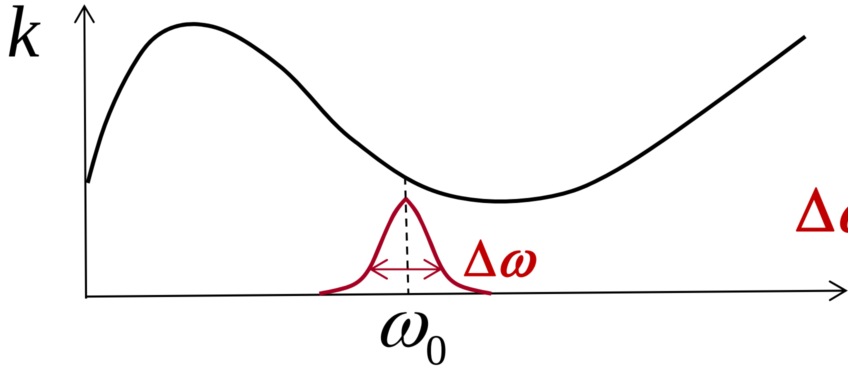
‘Temporal experiment’

$$\begin{cases} z \leftrightarrow t \\ k \leftrightarrow \omega \end{cases}$$

ψ : amplitude of the optical field







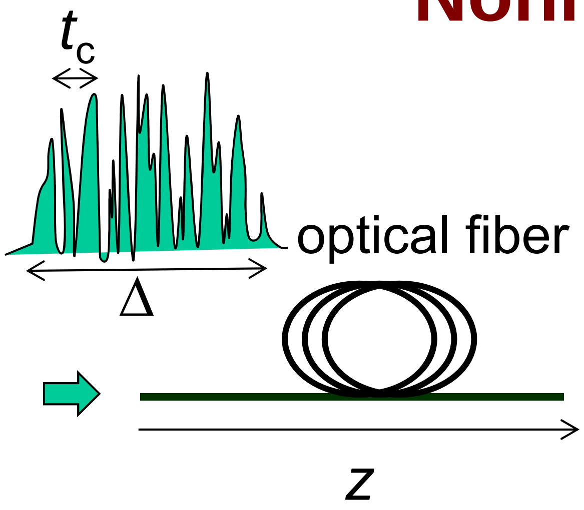
$\Delta\omega \ll \omega_0$

$$\begin{cases} \Delta k = k(\omega) - k(\omega_0) \approx (\omega - \omega_0) k'(\omega_0) + \frac{1}{2}(\omega-\omega_0)^2 k''(\omega_0) + \frac{1}{6}(\omega-\omega_0)^3 k'''(\omega_0) + \ldots \\ \tilde{\psi}(\Delta k, \omega - \omega_0) \rightarrow \psi(z,t) \\ i\partial_z \psi(z,t) = ik'(\omega_0)\partial_t \psi(z,t) + \frac{1}{2} k''(\omega_0)\partial_{tt}\psi(z,t) + \frac{1}{6} k'''(\omega_0)\partial_{ttt}\psi(z,t) + \ldots \end{cases}$$

$$\begin{cases} n(I) = n_0 + n_{nl} I \\ k = n_0 \omega_0 / c + n_{nl} I \omega_0 / c \\ \tau = t - k'(\omega_0) z \\ i\partial_z \psi = \frac{n_{nl} k}{n_0} |\psi|^2 \psi + \frac{1}{2} k''(\omega_0)\, \partial_{\tau\tau}\psi + \frac{1}{6} k'''(\omega_0)\, \partial_{\tau\tau\tau}\psi + \ldots \end{cases}$$

weak NL

Noninstantaneous response

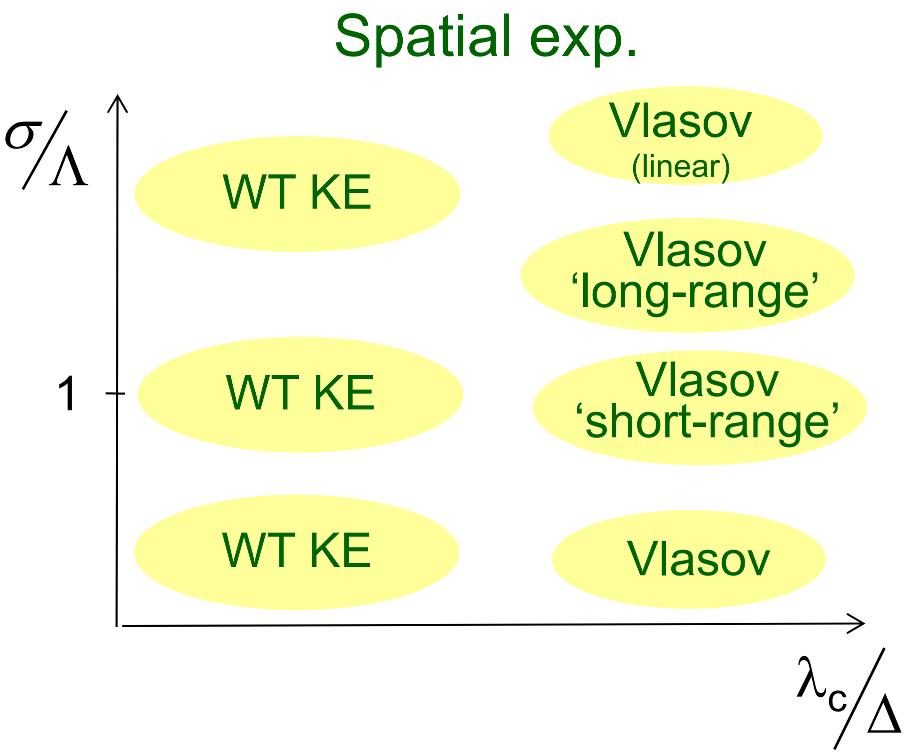


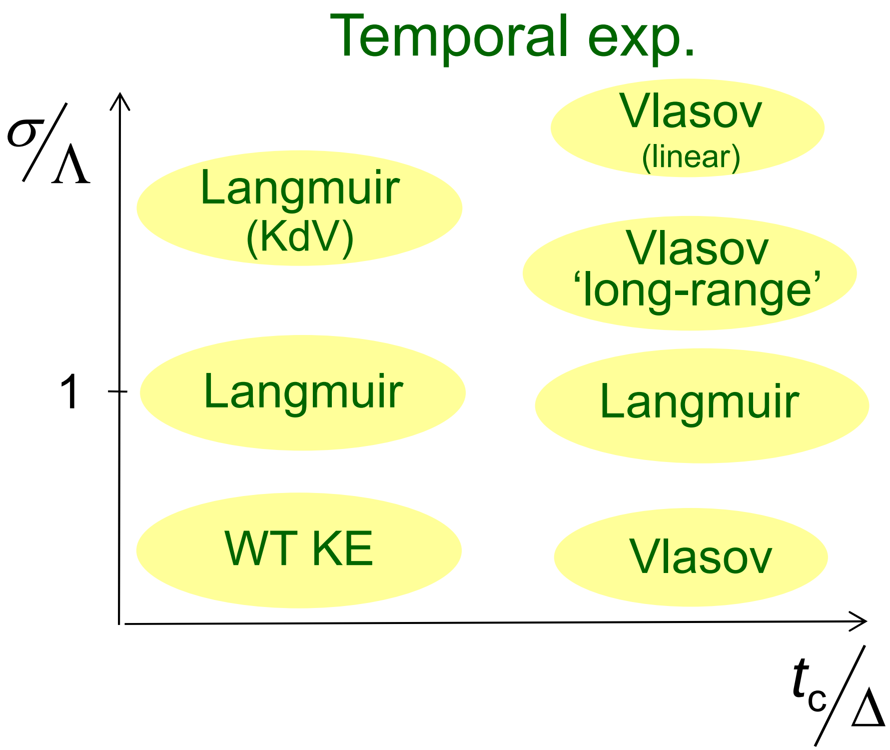
$$i\partial_z\psi = -\beta\,\partial_{tt}\psi + g\,\psi\int U(t')\,|\psi|^2(t-t')\,dt'$$

- $U(t)$ is causal: $U(t)=0$ for $t<0$
- σ : response time

- $\Lambda = \sqrt{\beta/(g|\psi|^2)}$: healing time
- Δ : scale of inhomogeneous statistics
- t_c : correlation time

Optical wave turbulence





Review: Garnier, Lisak, Picozzi, JOSA B (2012)

Optical wave turbulence

1.- <u>Homogeneous statistics & instantaneous response:</u>

WT kinetic eqn

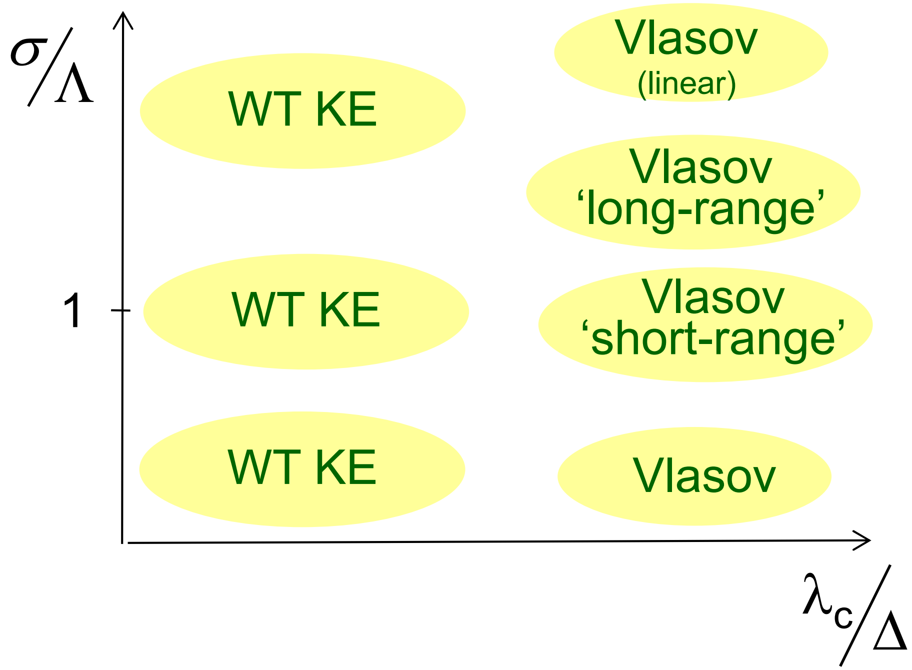
a.- Wave condensation in a trap

b.- Thermalization / Anomalous thermalization

2.- <u>Inhomogeneous statistics & nonlocal response:</u>

Vlasov kinetic eqn

Effective potential



3.- <u>Noninstantaneous response:</u>

Weak Langmuir turbulence kinetic eqn

Causality condition

Summary of basic results on wave condensation

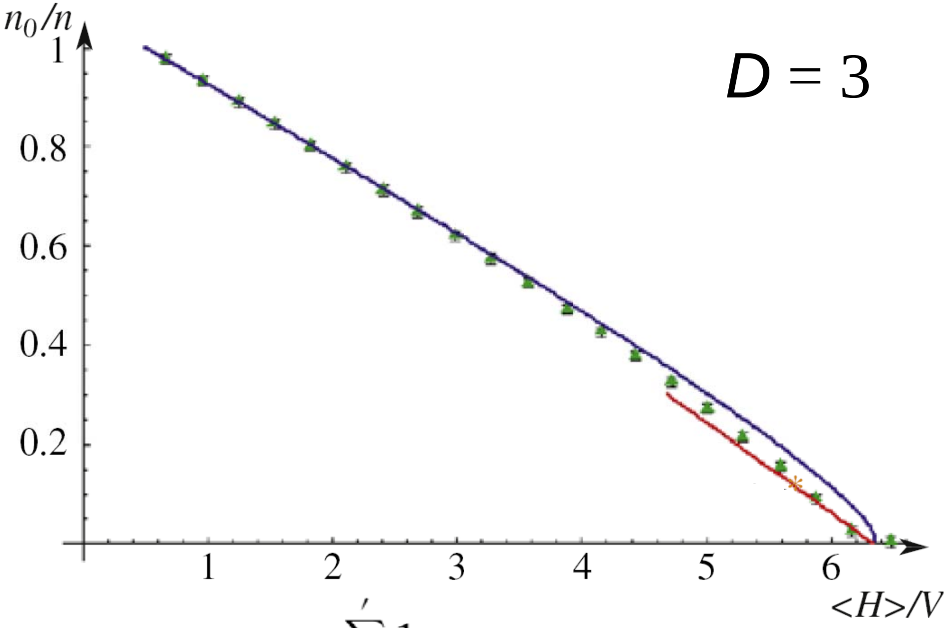
⟶ **role of a frequency cut-off k_c: $E_c \sim k_c^2$**

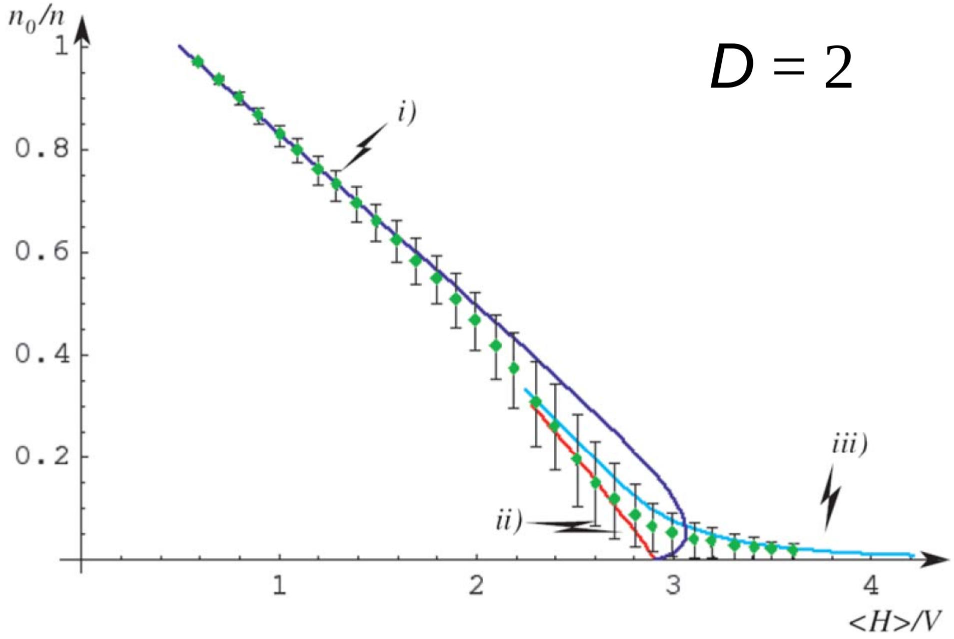
↓

Wave condensation with a trap *V*(*r*)

⟶ Wave condensation in the thermodynamic limit, $D = 2$

Wave condensation at equilibrium: theory/numerics





$$\frac{\langle H\rangle}{V} = (n - n_0)\frac{\sum_{\mathbf{k}}' 1}{\sum' \frac{1}{k^2}} + a\left(n^2 - \frac{1}{2}n_0^2\right)$$

$$\langle H\rangle / V = a\frac{n^2}{2} + \frac{a}{2}(n - n_0)^2 + (n - n_0)\frac{\sum_{\mathbf{k}}' 1}{\sum_{\mathbf{k}}' \frac{k^2+an_0}{k^4+2an_0k^2}}$$

$$\begin{cases} \dfrac{\langle H\rangle(\mu)}{V} = (n - n_0)\dfrac{\sum_{\mathbf{k}}' \frac{\alpha k^2}{\alpha k^2 - \mu}}{\sum_{\mathbf{k}}' \frac{1}{\alpha k^2 - \mu}} + g\left(n^2 - \dfrac{1}{2}n_0^2\right) \\ \dfrac{n_0(\mu)}{n} = \dfrac{1}{-\mu}\dfrac{1}{\sum_{\mathbf{k}} \frac{1}{\alpha k^2 - \mu}} \end{cases}$$

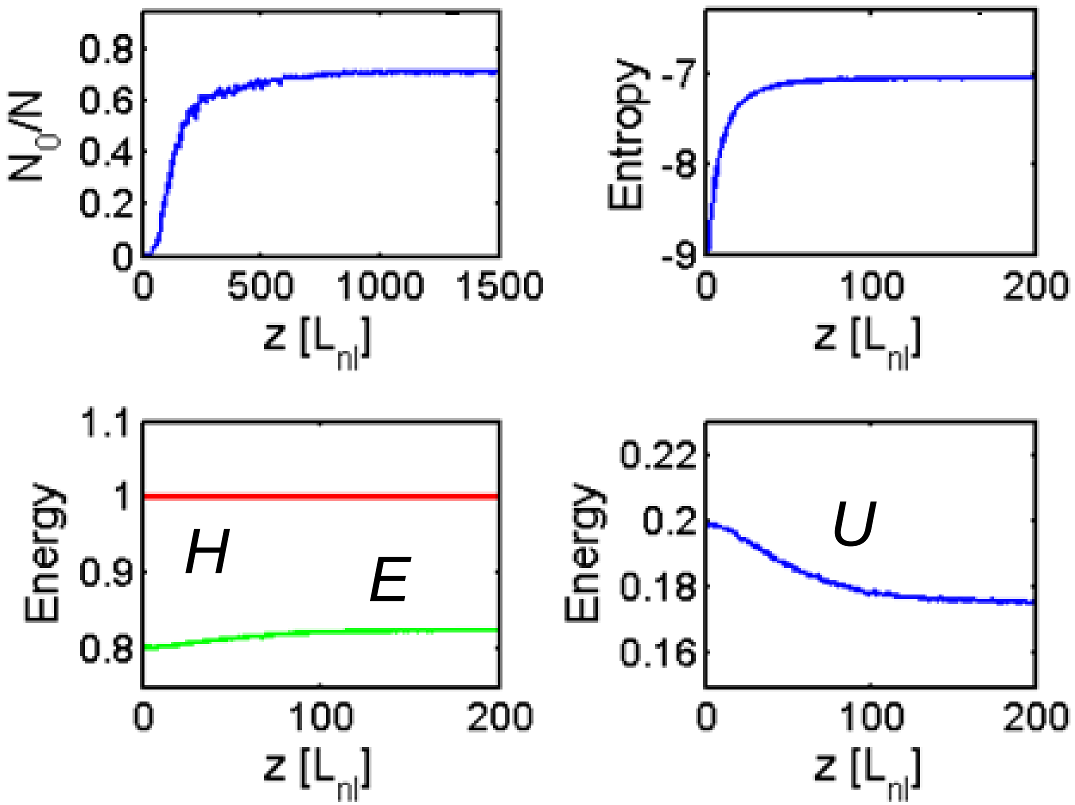
Small Cond. Amp.: WT

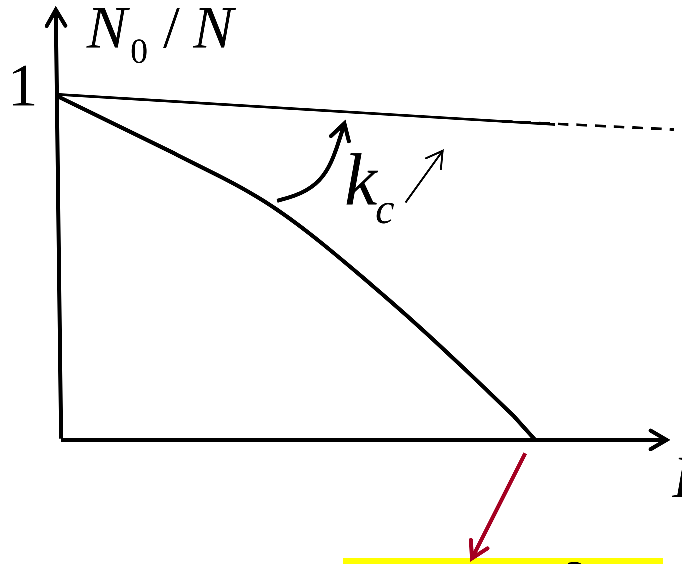
High Cond. Amp.: Bogoliubov

Beyond Thermodyn. Limit

Nazarenko, Zakharov, Physica D (2005)
Connaughton, Josserand, Picozzi, Pomeau, Rica PRL (2005)
Düring, Picozzi, Rica, Physica D - 2009

An increase of disorder (entropy) requires the generation of the coherent plane-wave





$$E_c = Nk_c^2/3$$

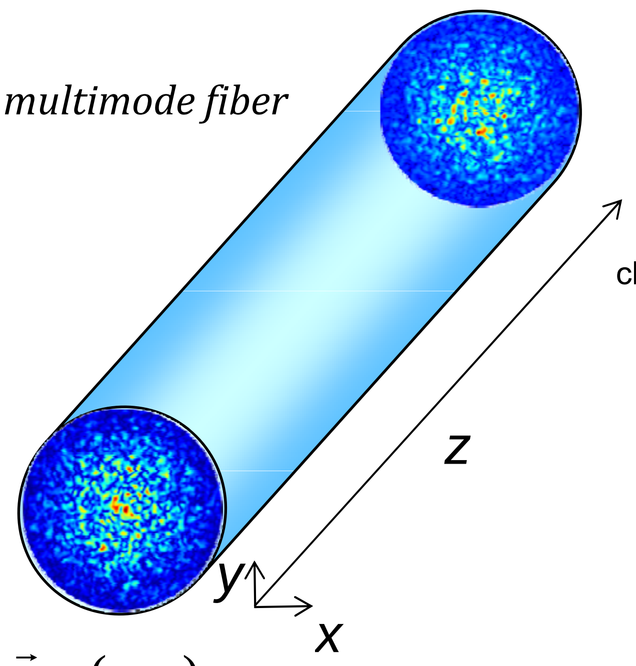
Dyachenko, Newell, Pushkarev, Zakharov, Physica D (1992)
Pomeau, Physica D (1992)

$$\begin{cases} E = \int |\nabla \psi|^2 \, dx \\ U = \int \frac{1}{2} |\psi|^4 \, dx \end{cases} \qquad \begin{cases} H = E + U = const \\ N = \int |\psi|^2 \, dx = const \end{cases}$$

WT in a trap *V(r)*

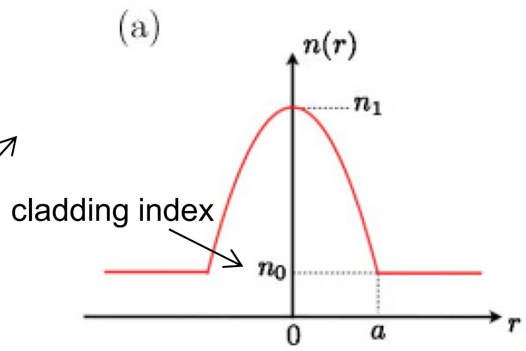
$$i\partial_z \psi = -\alpha \nabla^2 \psi + V(\mathbf{r})\psi + \gamma |\psi|^2 \psi$$

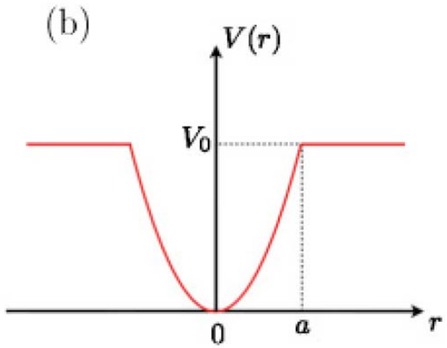
(D = 2)



$\vec{r} = (x, y)$

$$V(\mathbf{r}) = k_0 \left(\frac{n_1^2 - n^2(\mathbf{r})}{2n_0} \right)$$





$$N = \int |\psi|^2 d\mathbf{r}$$

$$H = \underbrace{\int \alpha |\nabla \psi|^2 d\mathbf{r} + \int V(\mathbf{r}) |\psi|^2 d\mathbf{r}}_{\text{Linear, } E} + \underbrace{\frac{\gamma}{2} \int |\psi|^4 d\mathbf{r}}_{\text{Nonlinear, } U}$$

WT in a trap *V(r)*

Basic considerations

$$i\partial_z\psi = -\alpha\nabla^2\psi + V(\mathbf{r})\psi + \gamma|\psi|^2\psi$$

$$\psi(\mathbf{r},z) = \sum_m c_m(z)u_m(\mathbf{r})\exp(-i\beta_m z)$$

$\{m\}$ labels the two numbers (m_x, m_y)

[parabolic trap: $\beta_m \simeq \beta_0(m_x + m_y + 1)$]

$$\int u_m(\mathbf{r})\, u_n^*(\boldsymbol{r})\, d\boldsymbol{r} = \delta^K_{n,m}$$

$$\beta_m u_m(\boldsymbol{r}) = -\alpha\nabla^2 u_m(\boldsymbol{r}) + V(\boldsymbol{r})u_m(\boldsymbol{r})$$

$$\langle c_m(z)c_n^*(z)\rangle = n_m(z)\delta^K_{n,m}$$

$$N = \sum_m n_m(z)$$

$$E(z) = \sum_m \mathcal{E}_m(z) = \sum_m n_m(z)\,\beta_m = \int \alpha|\nabla\psi|^2\, d\boldsymbol{r} + \int V(r)|\psi|^2\, d\boldsymbol{r}$$

Number of particles in the mode *m*:

$$n_m(z) = \langle\left|\int \psi(\mathbf{r},z)u_m^*(\mathbf{r})d\mathbf{r}\right|^2\rangle = \langle|c_m(z)|^2\rangle$$

Condensation: $n_0 \gg n_m$

WT in a trap *V(r)*

$$i\partial_z\psi = -\alpha\nabla^2\psi + V(\mathbf{r})\psi + \gamma|\psi|^2\psi \qquad \psi(\mathbf{r},z) = \sum_m c_m(z)u_m(\mathbf{r})\exp(-i\beta_m z)$$

$$i\partial_z a_m = \beta_m a_m + \gamma \sum_{p,q,s} W_{mpqs} a_p a_q^* a_s \qquad a_m(z) = c_m(z)\exp(-i\beta_m z)$$

$$W_{mpqs} = \int u_m^*(\mathbf{r})u_p(\mathbf{r})u_q^*(\mathbf{r})u_s(\mathbf{r})d\mathbf{r}$$

$$H = \sum_m \beta_m |a_m|^2 + \frac{\gamma}{4}\sum_{m,p,q,s}(W_{mpqs}a_m^* a_p a_q^* a_s + W_{mpqs}^* a_m a_p^* a_q a_s^*)$$

$$N = \sum\nolimits_m |a_m|^2$$

Continuous limit: $V_0/\beta_0 \gg 1$

$$\partial_z \tilde{n}_\kappa = \frac{4\pi\gamma^2}{\beta_0^6}\iiint d\kappa_1 d\kappa_2 d\kappa_3 \delta(\tilde{\beta}_{\kappa_1} + \tilde{\beta}_{\kappa_3} - \tilde{\beta}_{\kappa_2} - \tilde{\beta}_\kappa)$$
$$\times \left|\widetilde{W}_{\kappa\kappa_1\kappa_2\kappa_3}\right|^2 \tilde{n}_\kappa \tilde{n}_{\kappa_1}\tilde{n}_{\kappa_2}\tilde{n}_{\kappa_3}(\tilde{n}_\kappa^{-1} + \tilde{n}_{\kappa_2}^{-1} - \tilde{n}_{\kappa_1}^{-1} - \tilde{n}_{\kappa_3}^{-1})$$
$$+ \frac{8\pi\gamma^2}{\beta_0^2}\int d\kappa_1 \delta(\tilde{\beta}_{\kappa_1} - \tilde{\beta}_\kappa)\left|\widetilde{U}_{\kappa\kappa_1}(\tilde{n})\right|^2(\tilde{n}_{\kappa_1} - \tilde{n}_\kappa)$$

$$\widetilde{U}_{\kappa\kappa_1}(\tilde{n}) = \frac{1}{\beta_0^2}\int d\kappa'\, \widetilde{W}_{\kappa\kappa_1\kappa'\kappa'}\tilde{n}_{\kappa'}$$

$$\tilde{\beta}_\kappa = \beta_{[k/\beta_0]}$$
$$\tilde{n}_\kappa(z) = n_{[k/\beta_0]}(z)$$
$$\widetilde{W}_{\kappa\kappa_1\kappa_2\kappa_3} = W_{[k/\beta_0][k_1/\beta_0][k_2/\beta_0][k_3/\beta_0]}$$
$$\kappa = \beta_0(m_x, m_y)$$

Aschieri, Garnier, Michel, Doya, Picozzi PRA (2011)

WT in a trap *V(r)*

$$\partial_z \tilde{n}_\kappa = \frac{4\pi\gamma^2}{\beta_0^6}\iiint d\kappa_1 d\kappa_2 d\kappa_3 \delta(\tilde{\beta}_{\kappa_1}+\tilde{\beta}_{\kappa_3}-\tilde{\beta}_{\kappa_2}-\tilde{\beta}_\kappa)$$
$$\times \left|\tilde{W}_{\kappa\kappa_1\kappa_2\kappa_3}\right|^2 \tilde{n}_\kappa \tilde{n}_{\kappa_1}\tilde{n}_{\kappa_2}\tilde{n}_{\kappa_3}(\tilde{n}_\kappa^{-1}+\tilde{n}_{\kappa_2}^{-1}-\tilde{n}_{\kappa_1}^{-1}-\tilde{n}_{\kappa_3}^{-1})$$
$$+\frac{8\pi\gamma^2}{\beta_0^2}\int d\kappa_1 \delta(\tilde{\beta}_{\kappa_1}-\tilde{\beta}_\kappa)\left|\tilde{U}_{\kappa\kappa_1}(\tilde{n})\right|^2(\tilde{n}_{\kappa_1}-\tilde{n}_\kappa)$$

$$\tilde{U}_{\kappa\kappa_1}(\tilde{n}) = \frac{1}{\beta_0^2}\int d\kappa' \, \tilde{W}_{\kappa\kappa_1\kappa'\kappa'}\tilde{n}_{\kappa'}$$

[parabolic trap: $\tilde{\beta}_\kappa = \kappa_x + \kappa_y + \beta_0$]

$$W_{mpqs} = \int u_m^*(\mathbf{r})u_p(\mathbf{r})u_q^*(\mathbf{r})u_s(\mathbf{r})d\mathbf{r}$$
$$\tilde{W}_{\kappa\kappa_1\kappa_2\kappa_3} = W_{[\kappa/\beta_0][\kappa_1/\beta_0][\kappa_2/\beta_0][\kappa_3/\beta_0]}$$

Plane-wave expansion & periodic boundary conditions:

$$u_{m_x,m_y}(\boldsymbol{r}) = \frac{1}{L}\exp[2i\pi(m_x x + m_y y)/L]$$

$$|\tilde{W}_{\kappa\kappa_1\kappa_2\kappa_3}|^2 = \tfrac{(2\pi)^2}{L^6}\delta(\boldsymbol{k}_1+\boldsymbol{k}_3-\boldsymbol{k}_2-\boldsymbol{k})$$

$$\partial_z \tilde{n}_{\boldsymbol{k}}(z) = \frac{4\pi\gamma^2}{(2\pi)^2}\iiint d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \delta\left(\alpha\left(k_1^2+k_3^2-k_2^2-k^2\right)\right)\delta(\boldsymbol{k}_1+\boldsymbol{k}_3-\boldsymbol{k}_2-\boldsymbol{k})\mathcal{N}(\tilde{\mathbf{n}})$$

$$\mathcal{N}(\tilde{\mathbf{n}}) = \tilde{n}_{\boldsymbol{k}}\tilde{n}_{\boldsymbol{k}_1}\tilde{n}_{\boldsymbol{k}_2}\tilde{n}_{\boldsymbol{k}_3}\left(\tilde{n}_{\boldsymbol{k}}^{-1}+\tilde{n}_{\boldsymbol{k}_2}^{-1}-\tilde{n}_{\boldsymbol{k}_1}^{-1}-\tilde{n}_{\boldsymbol{k}_3}^{-1}\right)$$

$$\mathbf{P} = \int \boldsymbol{k}\,\tilde{n}_{\boldsymbol{k}}\,d\boldsymbol{k}$$

WT in a trap *V(r)*

$$\partial_z \tilde{n}_\kappa = \frac{4\pi\gamma^2}{\beta_0^6} \iiint d\kappa_1 d\kappa_2 d\kappa_3 \delta(\tilde{\beta}_{\kappa_1} + \tilde{\beta}_{\kappa_3} - \tilde{\beta}_{\kappa_2} - \tilde{\beta}_\kappa)$$
$$\times |\tilde{W}_{\kappa\kappa_1\kappa_2\kappa_3}|^2 \tilde{n}_\kappa \tilde{n}_{\kappa_1} \tilde{n}_{\kappa_2} \tilde{n}_{\kappa_3} (\tilde{n}_\kappa^{-1} + \tilde{n}_{\kappa_2}^{-1} - \tilde{n}_{\kappa_1}^{-1} - \tilde{n}_{\kappa_3}^{-1})$$
$$+ \frac{8\pi\gamma^2}{\beta_0^2} \int d\kappa_1 \delta(\tilde{\beta}_{\kappa_1} - \tilde{\beta}_\kappa) |\tilde{U}_{\kappa\kappa_1}(\tilde{n})|^2 (\tilde{n}_{\kappa_1} - \tilde{n}_\kappa)$$

$$\tilde{U}_{\kappa\kappa_1}(\tilde{n}) = \frac{1}{\beta_0^2} \int d\kappa' \, \tilde{W}_{\kappa\kappa_1\kappa'\kappa'} \tilde{n}_{\kappa'}$$

$$N = \beta_0^{-2} \int d\boldsymbol{\kappa} \tilde{n}_{\boldsymbol{\kappa}}$$
$$E = \beta_0^{-2} \int d\boldsymbol{\kappa} \tilde{\beta}_{\boldsymbol{\kappa}} \tilde{n}_{\boldsymbol{\kappa}}$$
$$\mathcal{S}(z) = \beta_0^{-2} \int d\boldsymbol{\kappa} \ln(\tilde{n}_{\boldsymbol{\kappa}})$$
: *H*-theorem

$$\boxed{\tilde{n}_\kappa^{eq} = \frac{T}{\tilde{\beta}_\kappa - \mu}}$$

$$\tilde{\varepsilon}_\kappa = \tilde{\beta}_\kappa \tilde{n}_\kappa^{eq} \sim T$$

Energy equipartition

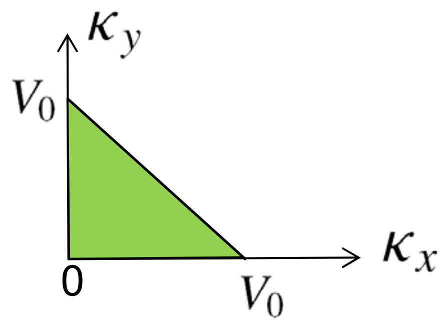
Wave condensation in the thermodynamic limit *D*=2 thanks to a parabolic trap *V(r)=qr²*

parabolic trap: $\beta_m \simeq \beta_0(m_x + m_y + 1)$ $\quad \tilde{n}_\kappa^{eq} = \dfrac{T}{\tilde{\beta}_\kappa - \mu}$

$\beta_0 \leq \tilde{\beta}(\boldsymbol{\kappa}) \leq V_0 \quad \beta_0 \ll V_0$

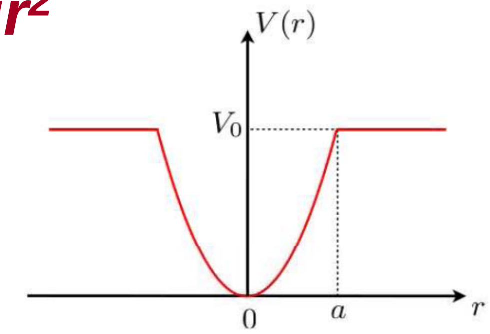
$$N = (T/\beta_0^2) \int_0^{V_0} d\kappa_x \int_0^{V_0-\kappa_x} (\kappa_x + \kappa_y + \beta_0 - \mu)^{-1} \, d\kappa_y$$

$$\boxed{N = \frac{T}{\beta_0^2}\left[V_0 - \tilde{\mu}\ln\left(\frac{-\tilde{\mu}}{V_0 - \tilde{\mu}}\right)\right]} \qquad \tilde{\mu} = \mu - \beta_0$$



Wave condensation in the thermodynamic limit *D*=2 thanks to a parabolic trap *V(r)=qr²*

$$\tilde{n}_\kappa^{eq} = \frac{T}{\tilde{\beta}_\kappa - \mu}$$



$$N = (T/\beta_0^2) \int_0^{V_0} d\kappa_x \int_0^{V_0-\kappa_x} (\kappa_x + \kappa_y + \beta_0 - \mu)^{-1} \, d\kappa_y$$

$$\boxed{N = \frac{T}{\beta_0^2}\left[V_0 - \tilde{\mu} \ln\left(\frac{-\tilde{\mu}}{V_0 - \tilde{\mu}}\right)\right]} \qquad \tilde{\mu} = \mu - \beta_0$$

$\tilde{\mu} \to 0$ for a non-vanishing critical temperature $\boxed{T_c = 4\alpha N q / V_0}$ $\quad V(r) = qr^2$

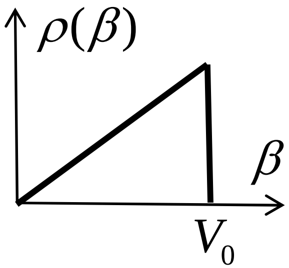
Thermod. limit $\begin{cases} N \to \infty \\ q \to 0 \end{cases}$ $\qquad Nq = \text{const}$

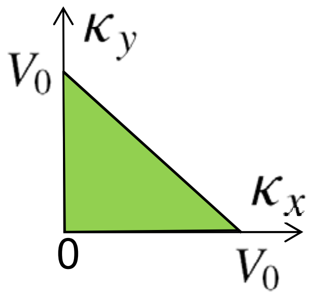
We also need a truncated potential: $V_0 < \infty$

Density of states:

$$\rho(\beta) = \frac{1}{\beta_0^2} \iint_{\mathcal{D}} d^2\boldsymbol{\kappa} \, \delta(\beta - \kappa_x - \kappa_x - \beta_0) \qquad \iint_{\mathcal{D}} d^2\boldsymbol{\kappa} = \int_0^{V_0} d\kappa_x \int_0^{V_0-\kappa_x} d\kappa_y$$

$$\begin{cases} \rho(\beta) = \beta/\beta_0^2 \text{ for } \beta \leq V_0 \\ \rho(\beta) = 0 \text{ for } \beta > V_0 \end{cases}$$





$$N = \int_0^{V_0} d\beta \, \rho(\beta) \, n_\beta^{eq} = T \int_0^{V_0} d\beta \, \rho(\beta)/\beta$$

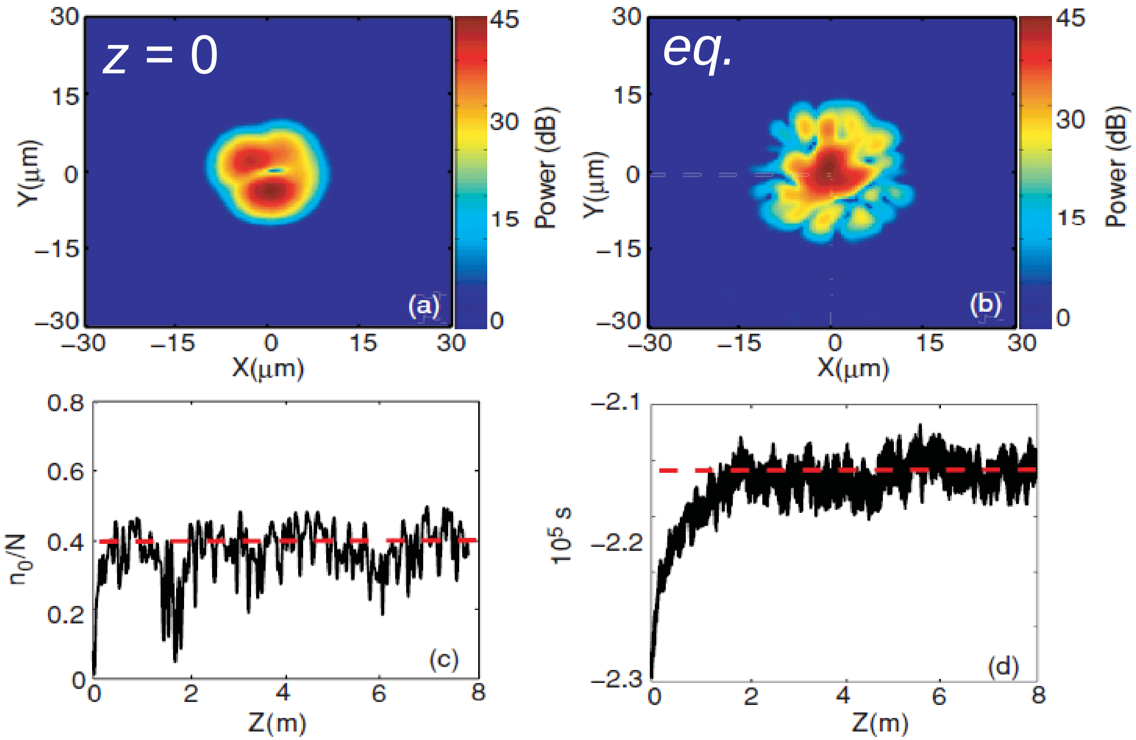
$$\left(V(r) = 0 \to \rho(\beta) = const\right)$$

Aschieri, Garnier, Michel, Doya, Picozzi PRA (2011)

Wave condensation in a parabolic trap $V(r) = qr^2$: Numerics

Condensation:

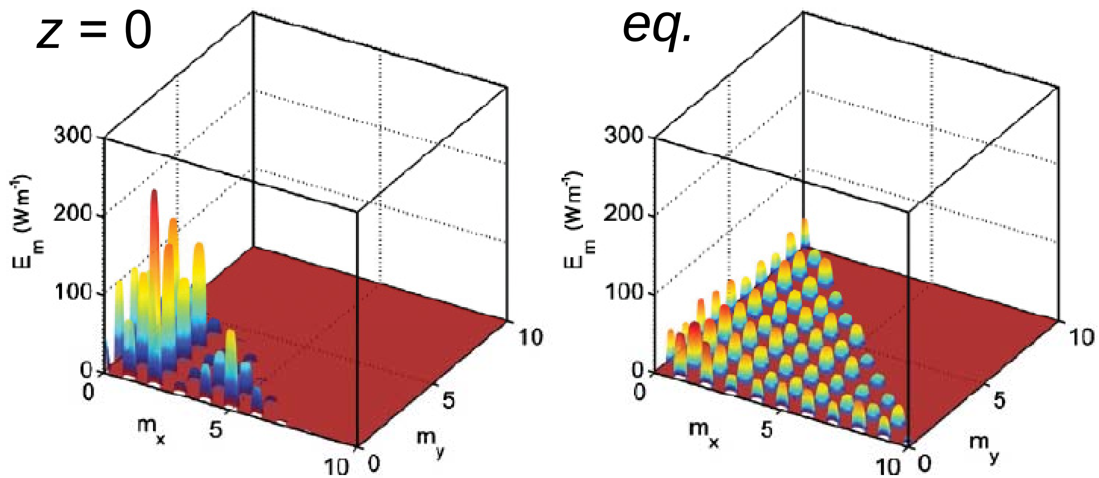
Log-scale



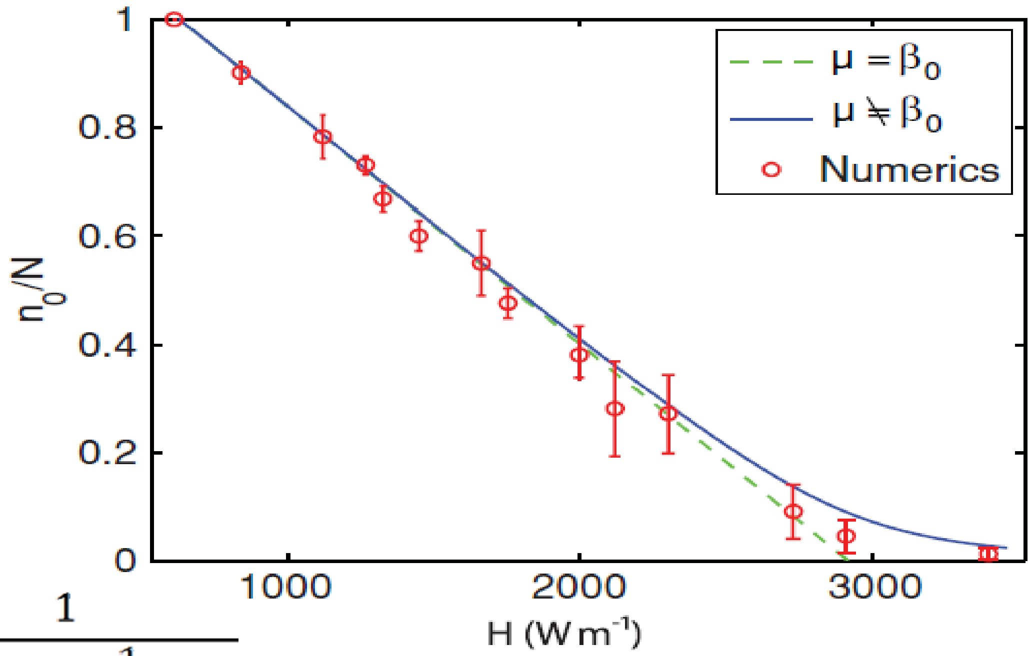
Energy equipartition:

$$n_m(z) = \langle \left| \int \psi(\mathbf{r},z) u_m^*(\mathbf{r}) d\mathbf{r} \right|^2 \rangle = \langle |c_m(z)|^2 \rangle$$

$$\beta_m \simeq \beta_0 (m_x + m_y + 1)$$



Wave condensation in a parabolic trap $V(r)=qr^2$: Numerics



$$\begin{cases} \dfrac{n_0}{N}(\mu) = \dfrac{1}{-\mu \sum_m \dfrac{1}{\beta_m - \beta_0 - \mu}} \\ \langle H\rangle(\mu) = N \dfrac{\sum_m \dfrac{\beta_m}{\beta_m - \beta_0 - \mu}}{\sum_m \dfrac{1}{\beta_m - \beta_0 - \mu}} + \langle U\rangle(\mu) \\ \langle U\rangle(\tilde{\mu}) = \gamma \left[\dfrac{\rho}{2} n_0^2 - 2 n_0^2 \tilde{\mu} \displaystyle\int |u_0|^2(r) \sum_m{}' \dfrac{|u_m(\boldsymbol{r})|^2}{\beta_m - \beta_0 - \tilde{\mu}} d\boldsymbol{r} + n_0^2 \tilde{\mu}^2 \int \Big(\sum_m{}' \dfrac{|u_m(\boldsymbol{r})|^2}{\beta_m - \beta_0 - \tilde{\mu}} \Big)^2 d\boldsymbol{r} \right] \end{cases}$$

→ Experiment in progress in Lille
S. Randoux & P. Suret

↓ V(r) = 0

$$\langle U\rangle = \gamma (n^2 - \tfrac{1}{2} n_0^2)$$

Aschieri, Garnier, Michel, Doya, Picozzi PRA (2011)

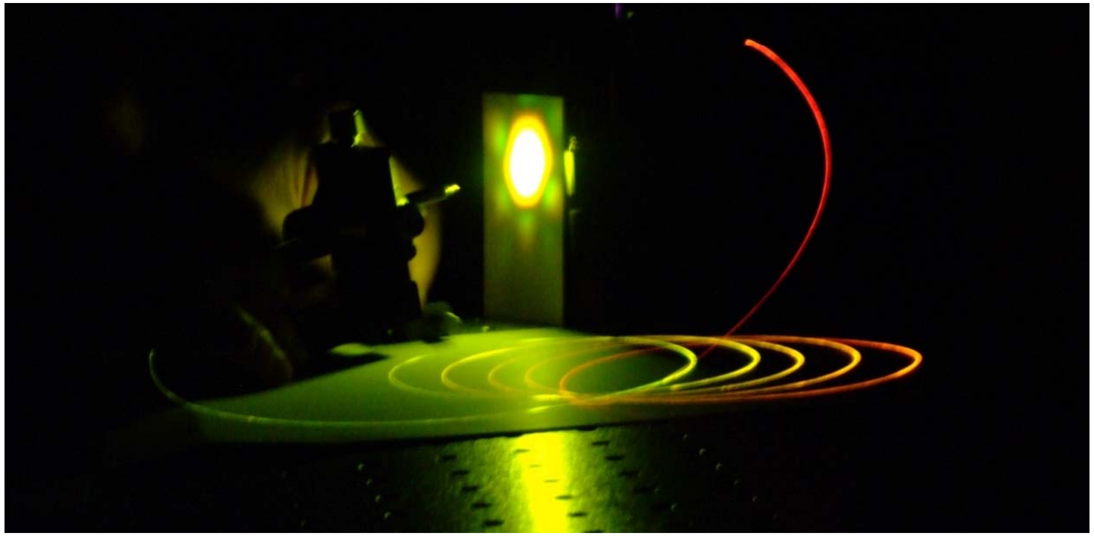
WT KE & thermalization

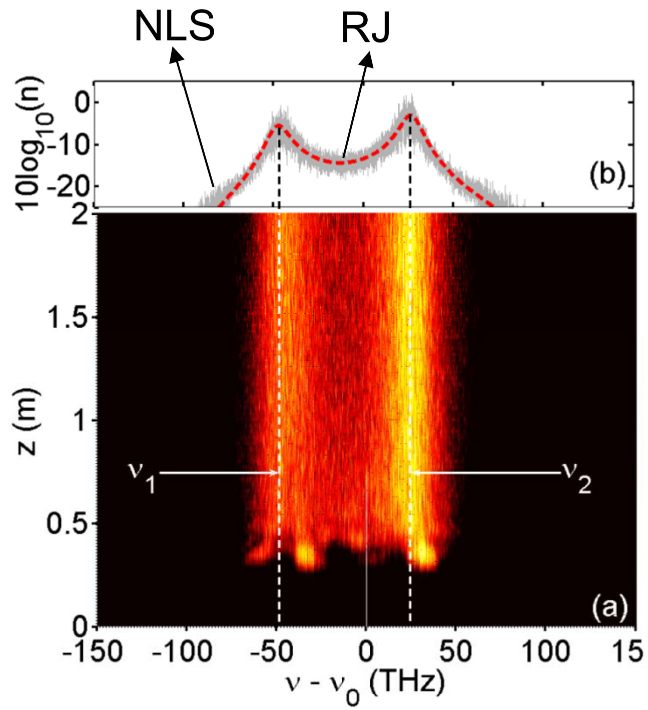
- 'Trapping' of incoherent wave-packets Pitois, Lagrange, Jauslin, Picozzi, PRL (2006)

- Spontaneous polarization of unpolarized light Picozzi, Opt. Exp. (2008)

- Condensation induced by coherence absorption Picozzi, Rica, EPL (2008)

- *Supercontinuum generation*

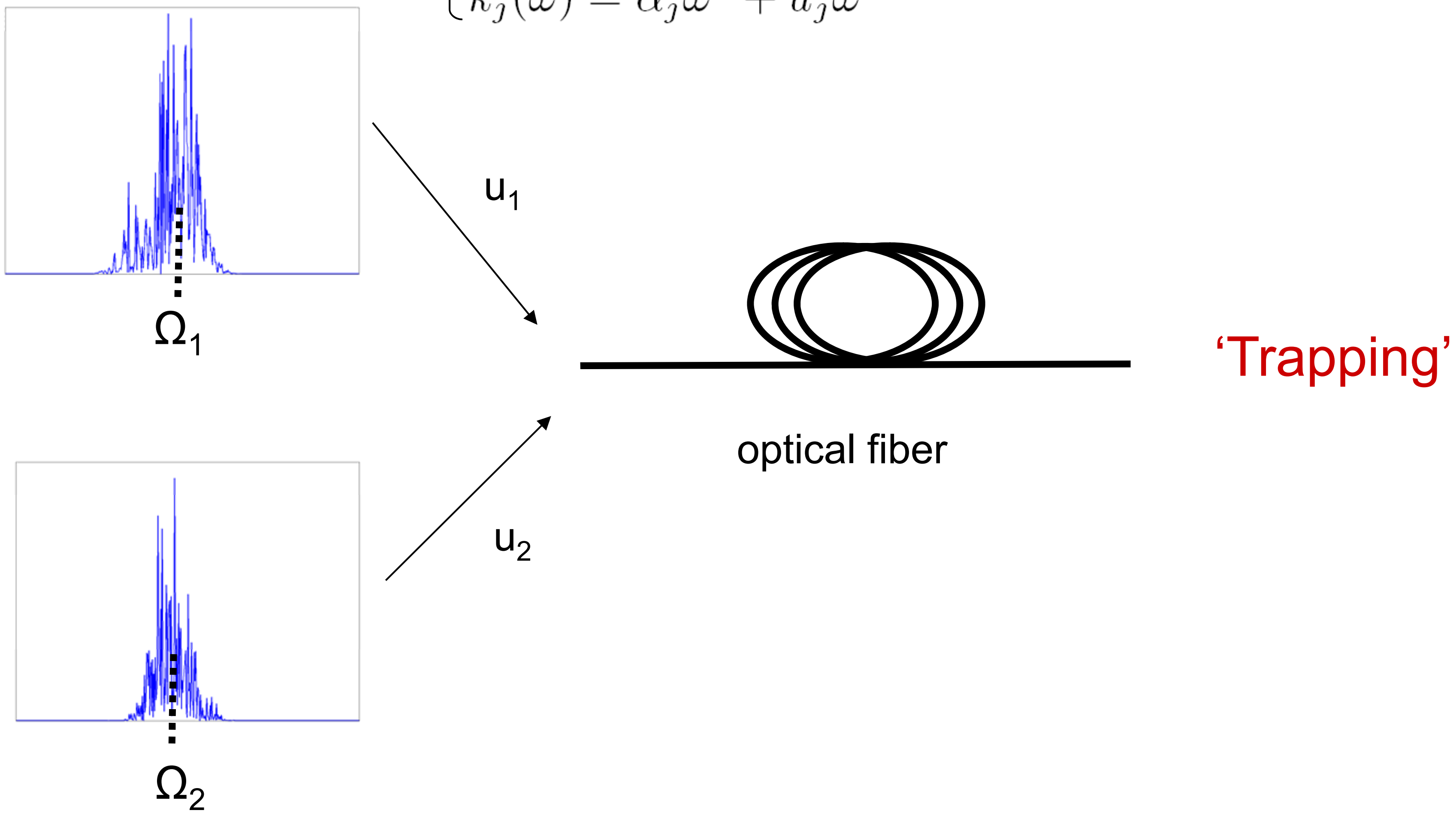




Barviau, Kibler, Coen, Picozzi OL (2008)
Barviau, Kibler, Kudlinski, Mussot, Millot, Picozzi OE (2009)
Barviau, Kibler, Picozzi PRA (2009)
Barviau, Garnier, Xu, Kibler, Millot, Picozzi, PRA (2013)

Thermalization $\longrightarrow$ 'Trapping' of incoherent wave-packets

$$\begin{cases} i(\partial_z + u_j \partial_t)\psi_j = -\alpha_j \partial_{tt}\psi_j + \gamma_j \big(|\psi_j|^2 + \kappa \sum_{i \neq j} |\psi_i|^2\big)\psi_j \\ k_j(\omega) = \alpha_j \omega^2 + u_j \omega \end{cases}$$

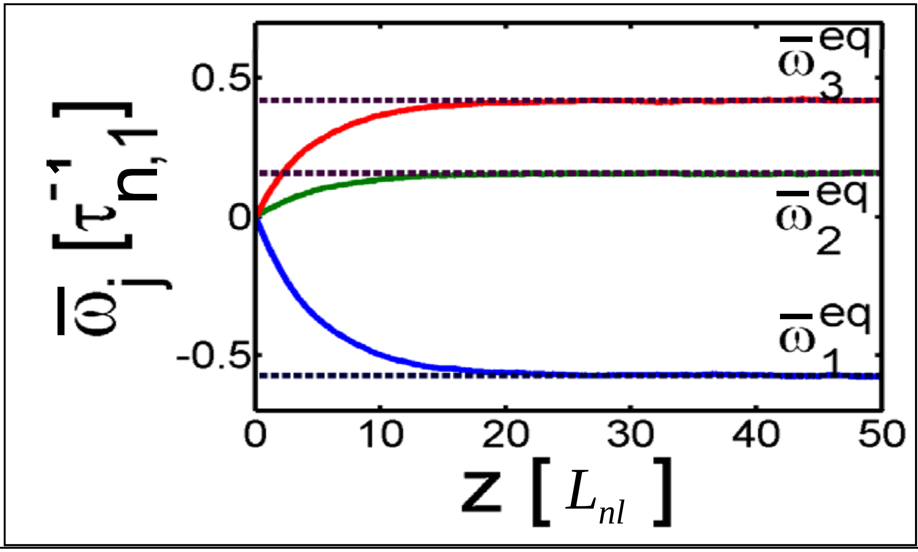


Pitois, Lagrange, Jauslin, Picozzi PRL (2006)

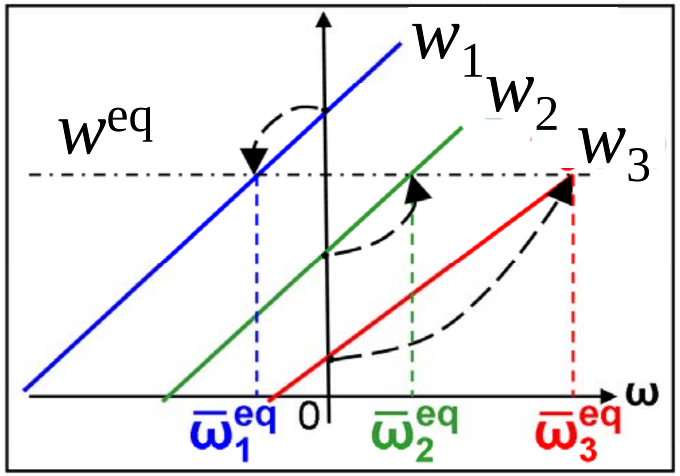
Trapping of incoherent wave-packets

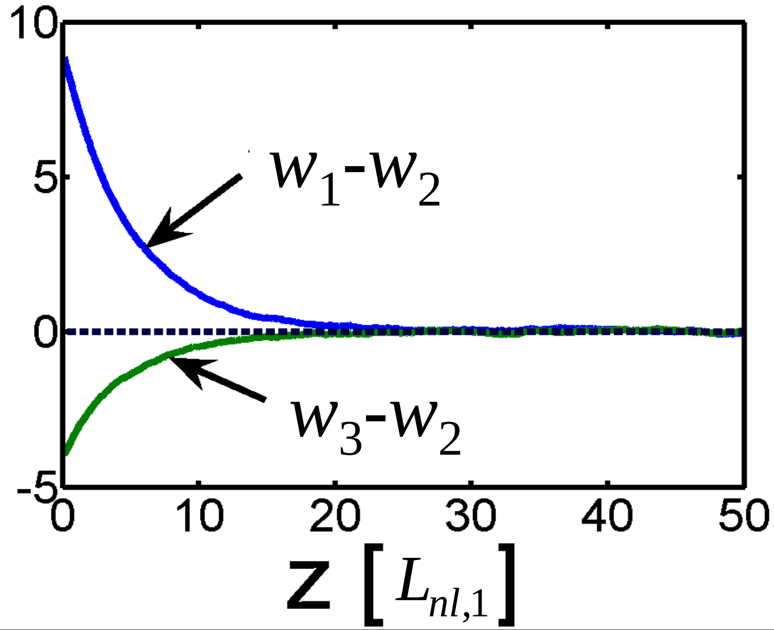
$$\overline{\omega}_j(z) = \int \omega n_j \, d\omega \Big/ \int n_j \, d\omega$$

$(M = 3)$



$$w_j(\omega) = \partial k_j / \partial \omega = u_j + 2\alpha_j \omega$$





Thermalization ⟶ 'trapping'

$$\begin{cases} n_j^{eq}(\omega) = \dfrac{T}{\alpha_j \omega^2 + (\lambda + u_j)\omega - \mu_j} \\ k_j(\omega) = \alpha_j \omega^2 + u_j \omega \end{cases}$$

Lagrange multiplier: *P* = const

$$\overline{\omega}_j(z) = \int \omega n_j \, d\omega \Big/ \int n_j \, d\omega$$

$$\bar{\omega}_j^{eq} = -(\lambda + u_j)/2\alpha_j$$

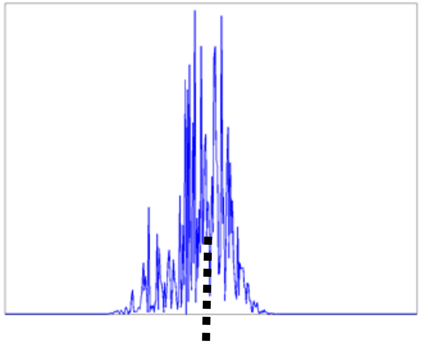
⇓

$$w_j(\bar{\omega}_j^{eq}) = u_j + 2\alpha_j \bar{\omega}_j^{eq} = -\lambda$$

$$\langle w^{eq} \rangle = -\lambda = \frac{2P + \sum_i N_i u_i/\alpha_i}{\sum_i N_i/\alpha_i}$$

$$N_j = \int n_j(z,\omega) d\omega$$

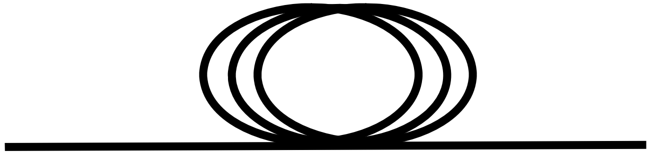
Thermalization → Trapping of incoherent wave-packets



Ω_1

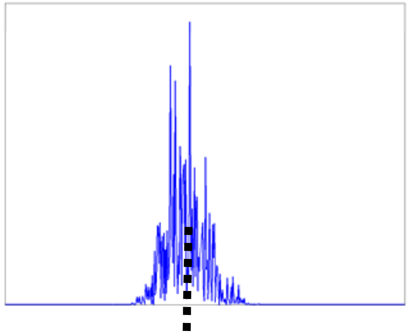
u_1

$$\begin{cases} i(\partial_z + u_1^{-1}\partial_t)A_1 = -\alpha_1 \partial_{tt} A_1 + \gamma(|A_1|^2 + \kappa |A_2|^2)A_1 \\ i(\partial_z + u_2^{-1}\partial_t)A_2 = -\alpha_2 \partial_{tt} A_2 + \gamma(|A_2|^2 + \kappa |A_1|^2)A_2 \end{cases}$$



‘trapping’

optical fiber



Ω_2

u_2

$$\langle w^{eq} \rangle = -\lambda = \frac{2P + \sum_i N_i u_i/\alpha_i}{\sum_i N_i/\alpha_i}$$

Lagrange multiplier: P = const

L. Landau and E. Lifchitz, *Statistical Physics*, Sec. 10:

« An isolated system at equilibrium can only exhibit a uniform motion of translation as a whole, while any macroscopic internal motion is not possible. »

Pitois, Lagrange, Jauslin, Picozzi, PRL (2006)

Thermalization does not necessarily occurs…

→ The Fermi-Pasta-Ulam Problem: A Status Report,
Ed. by G. Gallavotti, Lecture Notes in Physics (Springer, New York, 2007)

1.- Truncated RJ

$$i\partial_z \psi = -\partial_t^2 \psi + i\alpha\, \partial_t^3 \psi + \beta\, \partial_t^4 \psi + |\psi|^2 \psi$$

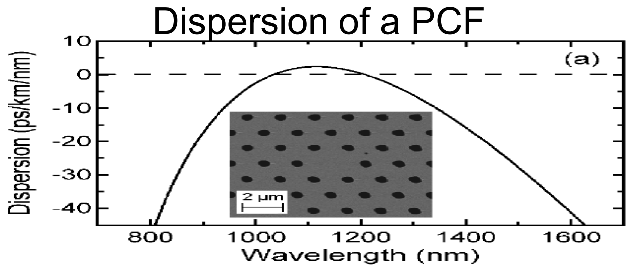
2.- Anomalous thermalization

$$i\partial_z \psi = -\partial_t^2 \psi + i\alpha\, \partial_t^3 \psi + |\psi|^2 \psi$$

3.- Integrable 1D-NLS

$$i\partial_z \psi = -\partial_t^2 \psi + |\psi|^2 \psi$$

1.- Truncated Rayleigh-Jeans



$$i\partial_z\psi = -\partial_t^2\psi + i\alpha\partial_t^3\psi + \beta\partial_t^4\psi + |\psi|^2\psi$$

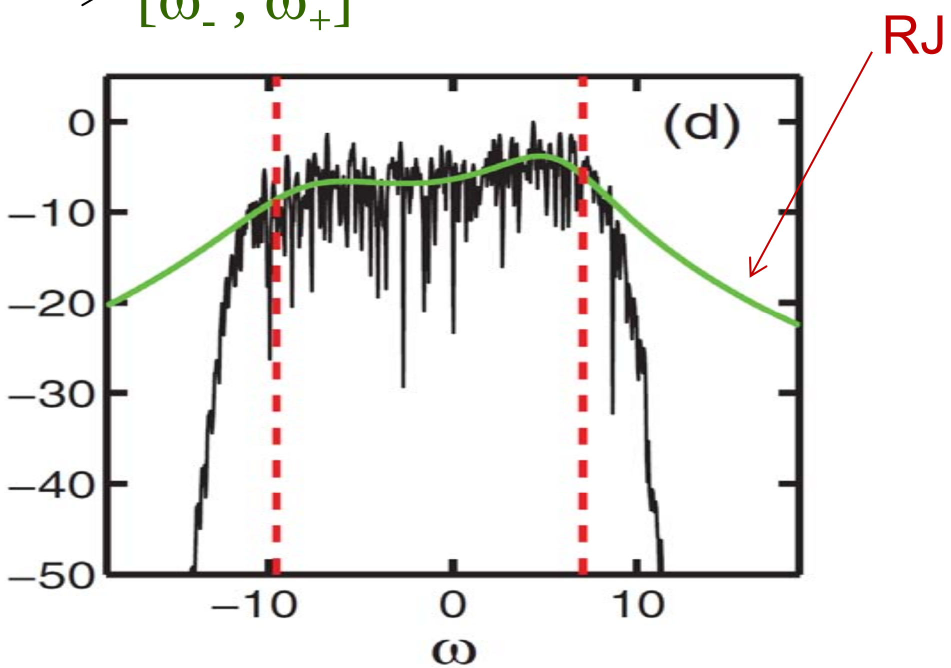
$$\partial_z n(z,\omega) = \frac{1}{\pi}\int d\omega_1\, d\omega_2\, d\omega_3\; \mathcal{N}_{\omega_1\omega_2\omega_3}(\mathbf{n})\; \delta(\omega+\omega_1-\omega_2-\omega_3)\,\delta[k(\omega)+k(\omega_1)-k(\omega_2)-k(\omega_3)]$$

$$\mathcal{N}_{\omega_1\omega_2\omega_3}(\mathbf{n}) = n(\omega)\,n(\omega_1)\,n(\omega_2)\,n(\omega_3)\;\left[n^{-1}(\omega)+n^{-1}(\omega_1)-n^{-1}(\omega_2)-n^{-1}(\omega_3)\right]$$

$$\partial_z n(z,\omega) = \frac{1}{\pi}\int \frac{\tilde{\mathcal{N}}_{\tilde\Omega_2\tilde\Omega_3}(\mathbf{n})\quad \delta\left[(\tilde\Omega_2/a_2)^2+(\tilde\Omega_3/a_3)^2-\rho\right]}{|\tilde\Omega_2+\tilde\Omega_3-r_\omega|\,|\tilde\Omega_2-\tilde\Omega_3-r_\omega|}\; d\tilde\Omega_2 d\tilde\Omega_3$$

$$\begin{cases} \Omega_2 = \frac{1}{\sqrt{2}}(\omega_2+\omega_3) \\ \Omega_3 = \frac{1}{\sqrt{2}}(\omega_2-\omega_3) \\ \tilde\Omega_2 = \Omega_2 + q/(7\beta) \\ \tilde\Omega_3 = \Omega_3 \\ r_\omega = 6\sqrt{2}\omega/7 + 3\sqrt{2}\alpha/(14\beta) \end{cases}$$

$$\begin{cases} \rho = 1 - \frac{3}{28}\left(8\beta\omega^2 + 4\alpha\omega - \frac{3\alpha^2}{\beta}\right) \geq 0 \\ \omega_\pm = -\frac{\alpha}{4\beta} \pm \frac{\sqrt{21}}{12\beta}\sqrt{3\alpha^2+8\beta} \end{cases} \longrightarrow [\omega_-\,,\,\omega_+]$$



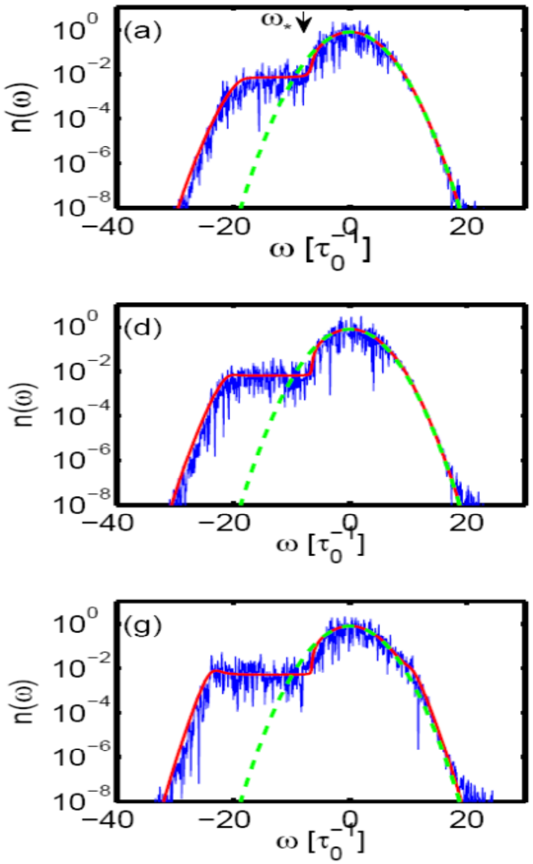
Barviau, Garnier, Xu, Millot, Picozzi, PRA (2013)

2.- Anomalous thermalization

$$i\partial_z A = -\partial_t^2 A - i\alpha\partial_t^3 A + |A|^2 A$$

$$\begin{cases} J_\omega = n(z,\omega) - n(z, q-\omega) \\ q = 2/3\alpha \end{cases}$$

$$\partial_z n(\omega,z) = \frac{1}{3\pi|\alpha|}\int \frac{n_\omega(n_\omega - J_\omega)n_{\omega_1}(n_{\omega_1} - J_{\omega_1})}{|\omega-\omega_1|\,|\omega+\omega_1-q|}\left(\frac{1}{n_\omega} + \frac{1}{n_\omega - J_\omega} - \frac{1}{n_{\omega_1}} - \frac{1}{n_{\omega_1} - J_{\omega_1}}\right) d\omega_1$$



z = 30 L_{nl}

z = 500 L_{nl}

z = 2000 L_{nl}

Kinetic equation

Initial condition

Numerics: NLS

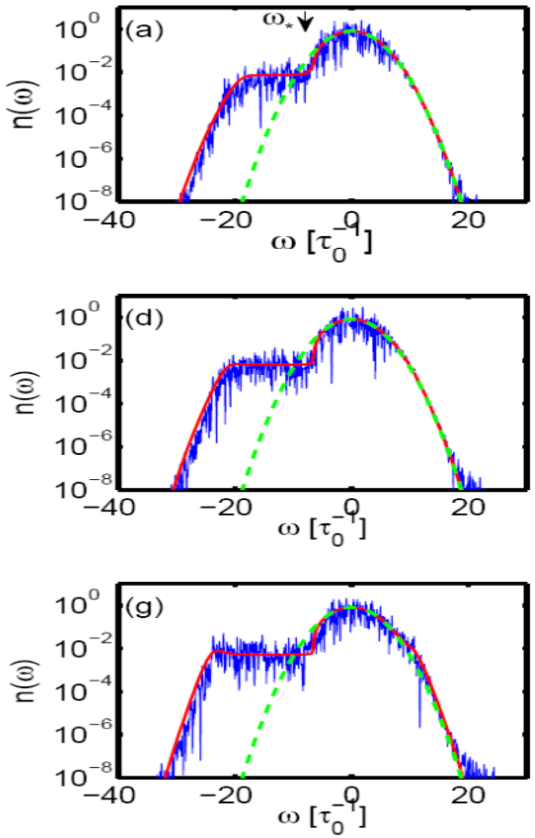
2.- Anomalous thermalization

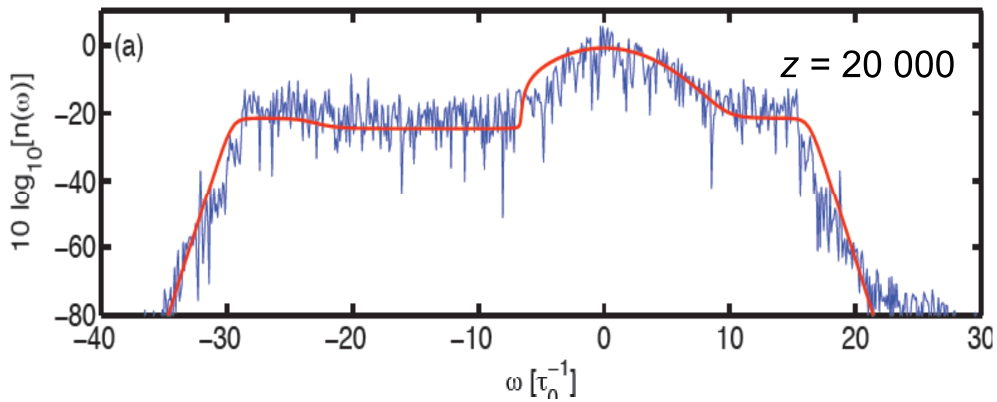
$$J_\omega = n(z,\omega) - n(z,q-\omega)$$

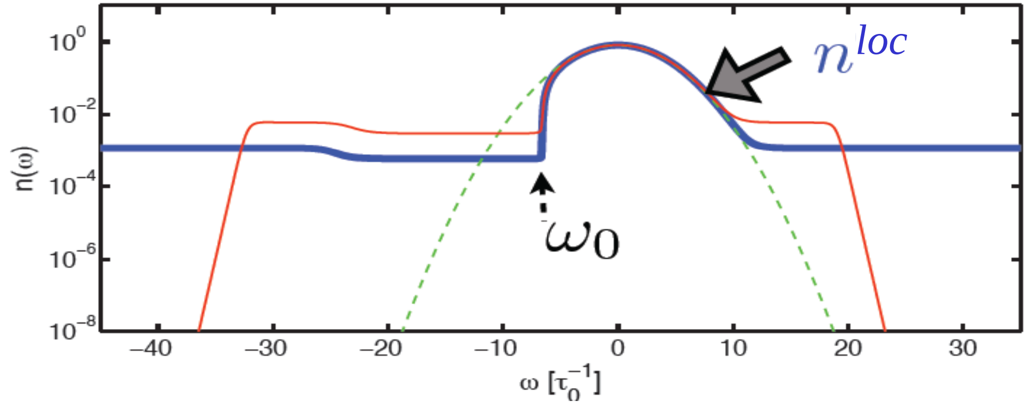
$$\partial_z n(\omega,z) = \frac{1}{3\pi|\alpha|}\int \frac{n_\omega(n_\omega - J_\omega)n_{\omega_1}(n_{\omega_1} - J_{\omega_1})}{|\omega-\omega_1|\,|\omega+\omega_1-q|}\left(\frac{1}{n_\omega} + \frac{1}{n_\omega - J_\omega} - \frac{1}{n_{\omega_1}} - \frac{1}{n_{\omega_1} - J_{\omega_1}}\right)\mathrm{d}\omega_1$$

$$\partial_z \mathcal{S} \geq 0 \qquad \mathcal{S}(z)/T_0 = \frac{1}{2\pi}\int \log[n_\omega(z)]\,\mathrm{d}\omega \qquad N/T_0 = \frac{1}{2\pi}\int n^{\mathrm{loc}}(\omega)\,\mathrm{d}\omega$$

$$\boxed{n^{loc}(\omega) = J_\omega/2 + [1 + \sqrt{1+\lambda^2 J_\omega^2/4}]/\lambda}$$



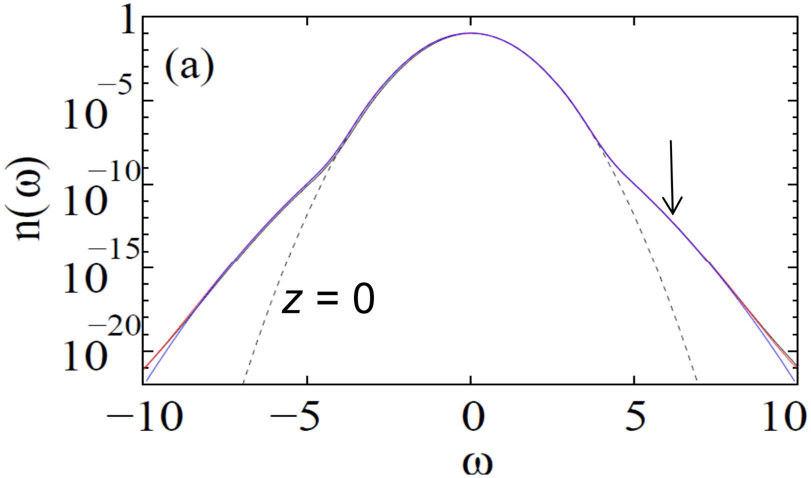


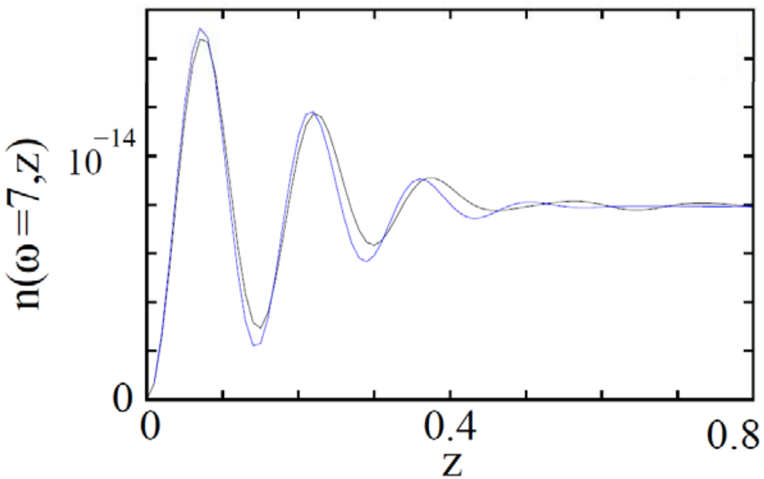


Suret, Randoux, Jauslin, Picozzi, OL (2010)
Michel, Suret, Randoux, Jauslin, Picozzi, PRL (2010)

3.- Integrable 1D – NLS

$$i\partial_z\psi = -\partial_{tt}\psi + |\psi|^2\psi$$





$$\begin{cases} \dfrac{\partial n_{\omega_1}(z)}{\partial z} = \dfrac{1}{\pi^2}\displaystyle\int_0^z dz' \int\int\int d\omega_{2-4}\, \mathcal{N}(z')\cos(\Delta k(z'-z))\,\delta(\omega_1+\omega_2-\omega_3-\omega_4) \\ \mathcal{N}(z) = n_{\omega_1}(z)n_{\omega_3}(z)n_{\omega_4}(z) + n_{\omega_2}(z)n_{\omega_3}(z)n_{\omega_4}(z) - n_{\omega_1}(z)n_{\omega_2}(z)n_{\omega_3}(z) - n_{\omega_1}(z)n_{\omega_2}(z)n_{\omega_4}(z) \end{cases}$$

$$\frac{\sin(\Delta k z)}{\Delta k} \underset{z\to\infty}{\to} \pi\delta(\Delta k) \longrightarrow \text{WT kinetic eq.} \longrightarrow \partial_z n_{\omega_1} = 0$$

Suret, Picozzi, Randoux,, Opt. Exp. (2011)

Optical wave turbulence

1.- <u>Homogeneous statistics & instantaneous response:</u>

WT kinetic eqn

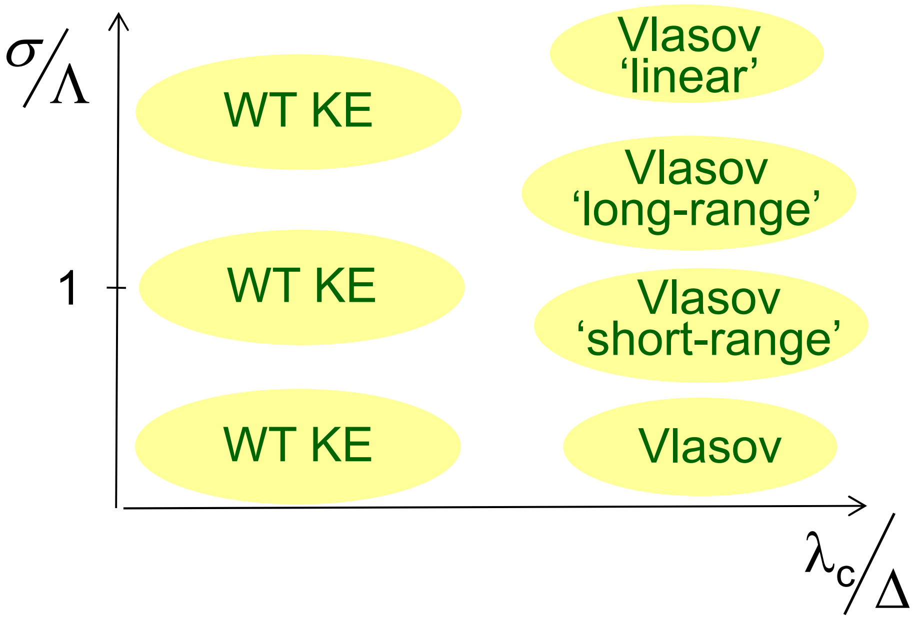
Thermalization / Anomalous thermalization

Wave condensation in a trap

2.- <u>Inhomogeneous statistics & nonlocal response:</u>

Vlasov kinetic eqn

Effective potential



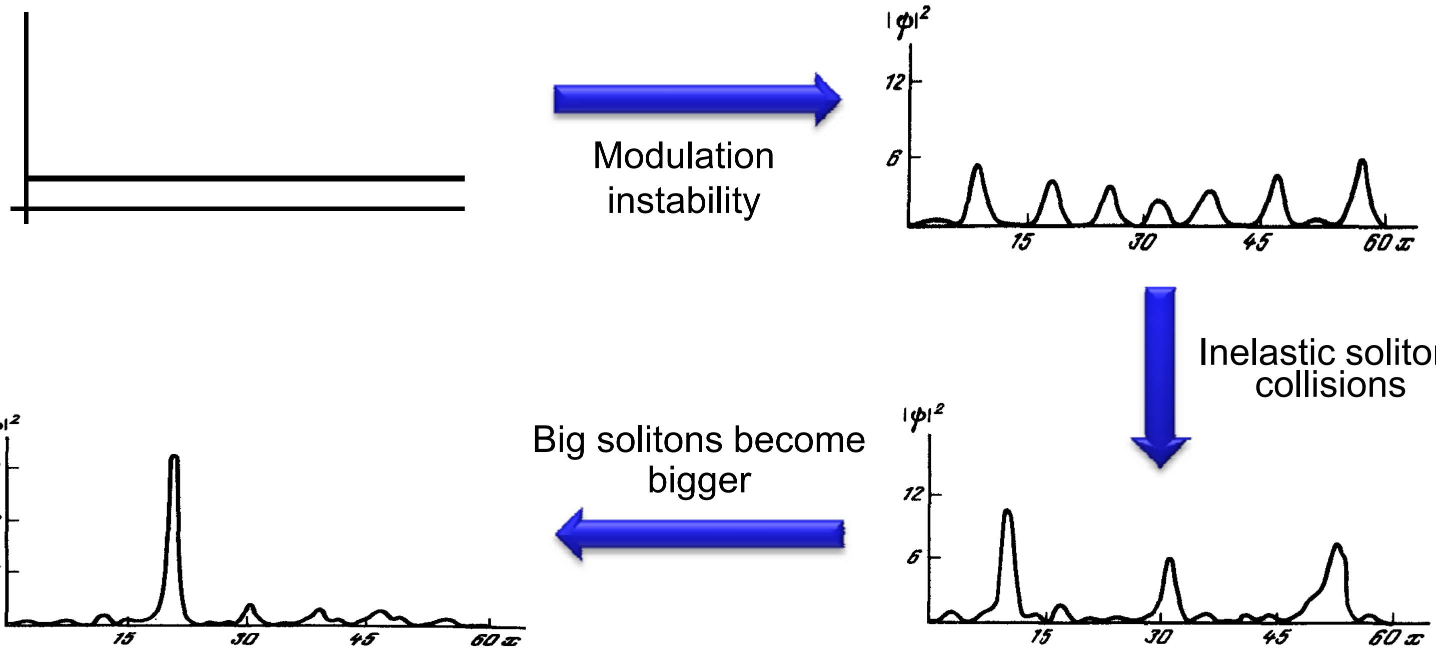
3.- <u>Noninstantaneous response:</u>

Weak Langmuir turbulence kinetic eqn

Causality condition

‘Soliton turbulence’: Focusing nonlinearity

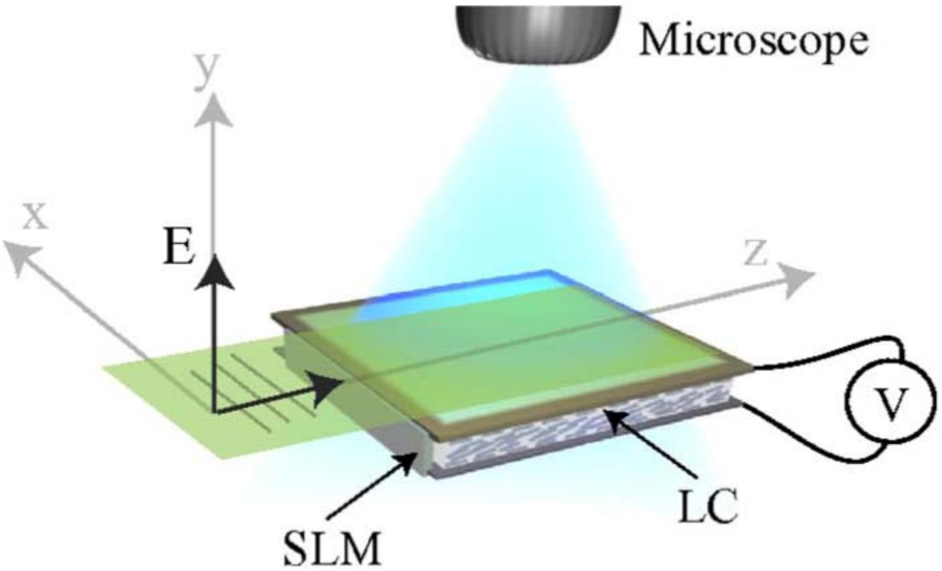
$$i\partial_z \psi = \partial_{xx}\psi + |\psi|\psi$$

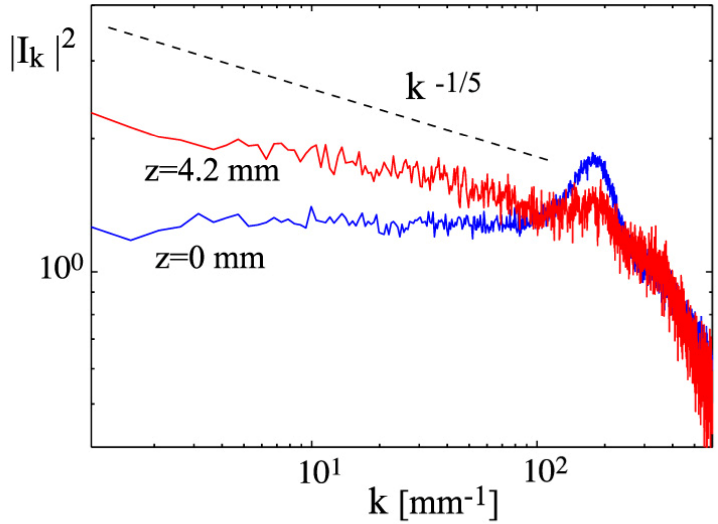


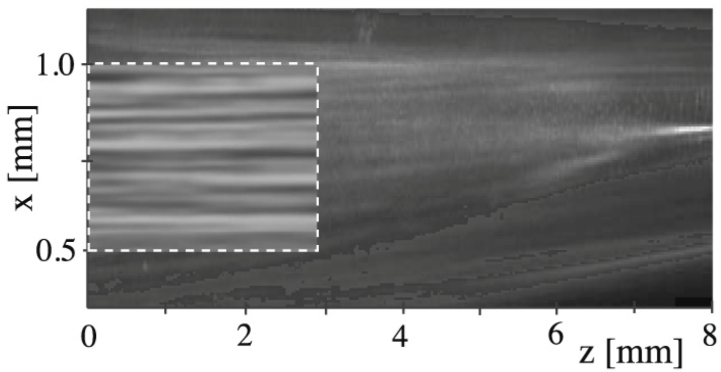
V.E. Zakharov, et al., Pis’ma Zh. Eksp. Teor. Fiz. 48 (1988) 79 [JETP Lett. 48 (1988) 83]

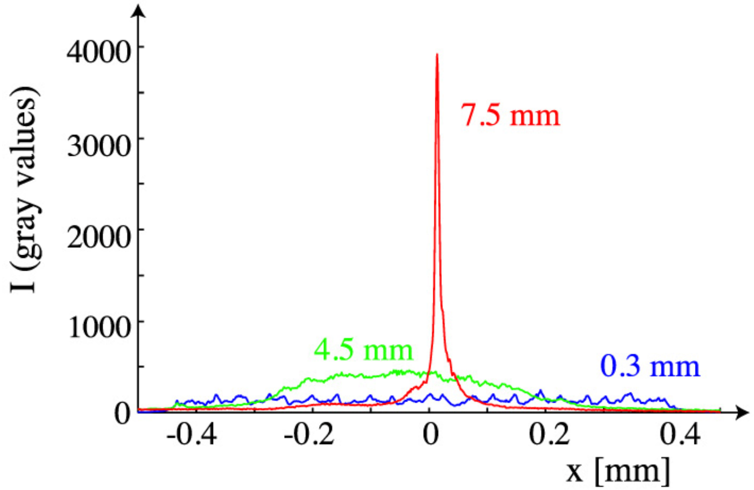
B. Rumpf, A.C. Newell, Phys. Rev. Lett. (2001)
B. Rumpf, A.C. Newell, Physica D (2003)
B. Rumpf PRE (2004)
R. Jordan, B. Turkington, C. Zirbel, Physica D (2000)
R. Jordan, C. Josserand, Phys. Rev. E (2000)

Focusing: Liquid crystal experiment









U. Bortolozzo, J. Laurie, S. Nazarenko, and S. Residori
J. Opt. Soc. Am. B, (2009)

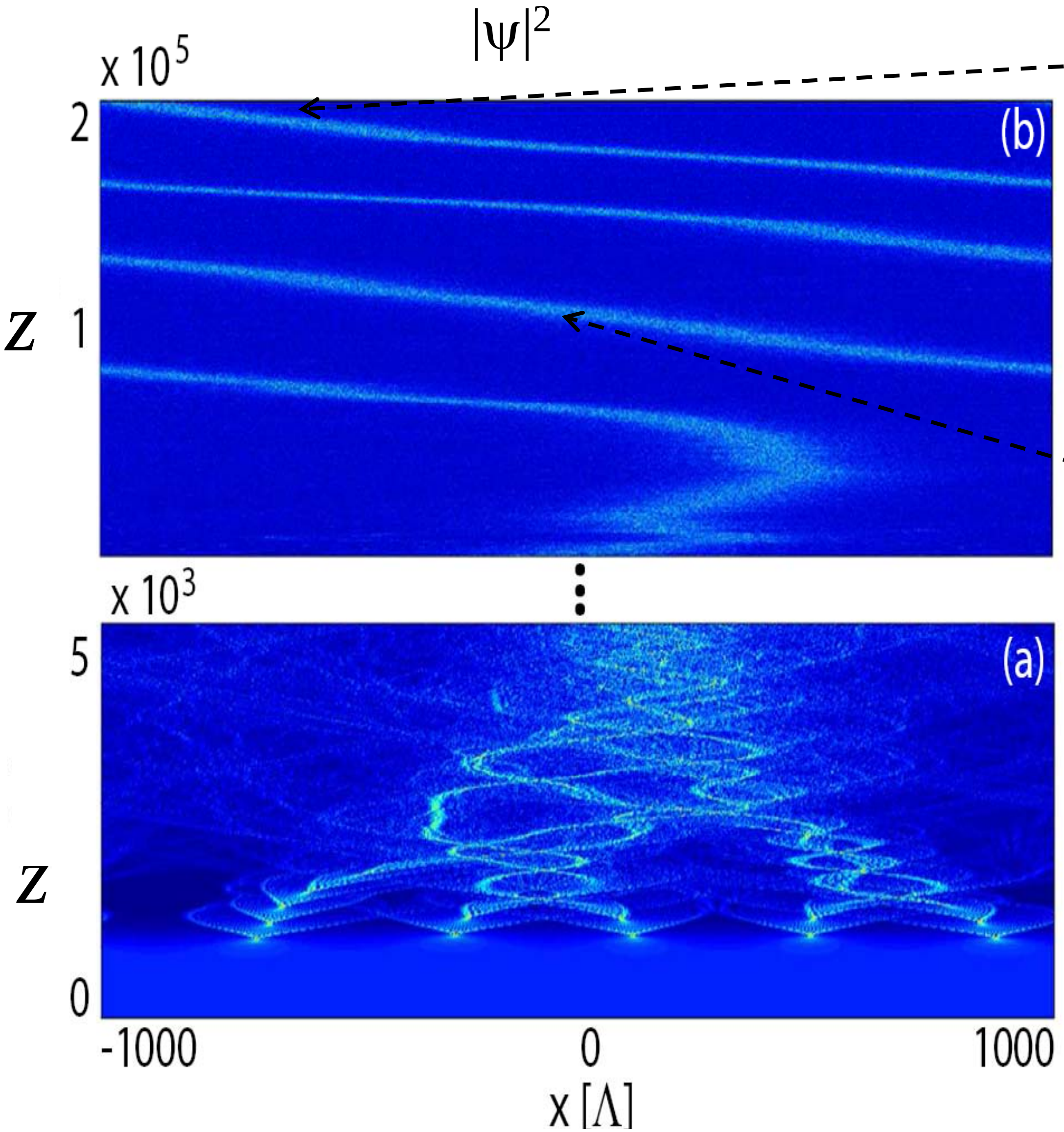
Highly nonlocal nonlinear regime

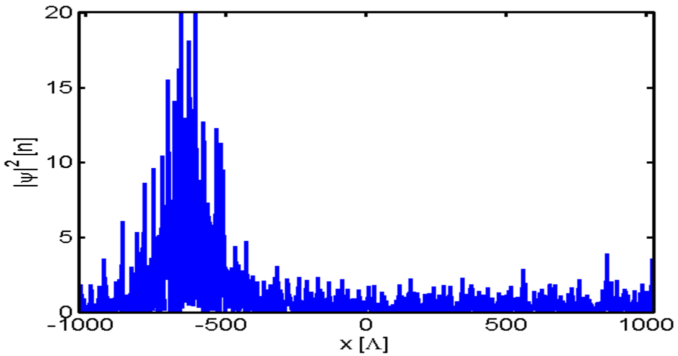
$\left(\Lambda = \sqrt{\beta / g|\psi|^2}\right)$

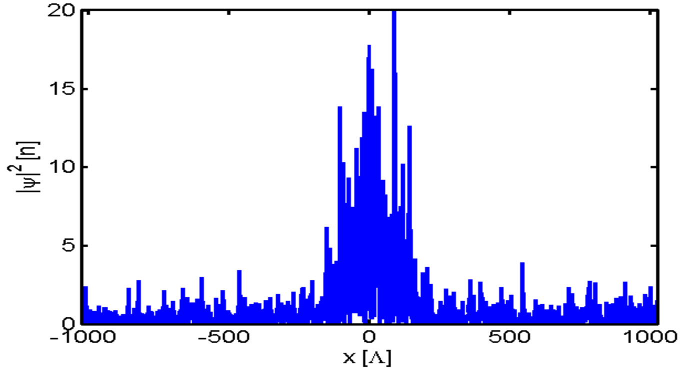
$$i\partial_z\psi + \beta\partial_{xx}\psi + \gamma\psi \int_{-\infty}^{+\infty} U(x-x')\ |\psi|^2(z,x')\ dx' = 0$$

$\sigma = 100\Lambda$

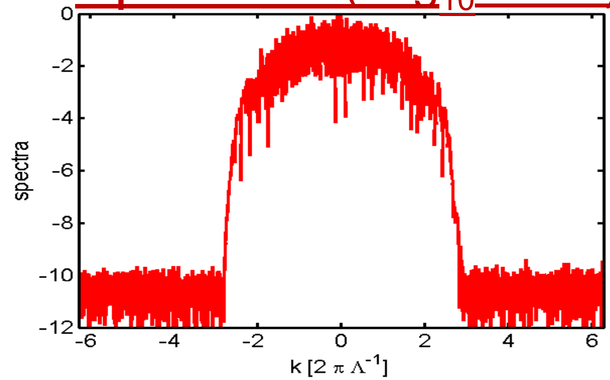
$U(x)$ is Gaussian







Spectrum (Log$_{10}$-scale):



Picozzi, Garnier, PRL (2011)

Why a coherent soliton is not generated?

$$i\partial_t\psi + \beta\partial_{xx}\psi + \gamma\psi\int_{-\infty}^{+\infty}U(x-x')|\psi|^2(t,x')dx' = 0 \qquad \rho = \mathcal{N}/L$$

width of the soliton: l
amplitude of the soliton: a

$$al \sim \sqrt{\beta/\gamma}$$

$l > \sigma$ otherwise the self-consitent potential would be smoothed-out by the nonlocal potential

$$a < \sqrt{\beta/\gamma}/\sigma \qquad \Lambda = \sqrt{\beta/(\gamma\rho)}$$

$$a/\sqrt{\rho} < \Lambda/\sigma$$

$$\sigma \gg \Lambda$$

$$a \ll \sqrt{\rho}$$

⟶ The small amplitude soliton would not be able to feel its self-induced potential

⟶ The formation of an IS leads to an increase of the kinetic energy

‘Short-range’ Vlasov eqn

$$i\partial_z \psi = -\beta \partial_{xx}\psi + \gamma\psi \int_{-\infty}^{+\infty} U(x')\ |\psi|^2(t, x-x')\ dx'$$

$$B(z,x,\xi) = \langle \psi(z, x+\xi/2)\psi^*(z, x-\xi/2)\rangle$$

(i) Weak nonlinearity

$$\begin{cases} i\partial_z B(x,\xi) = -2\beta\partial^2_{x\xi}B(x,\xi) + \gamma P(x,\xi) + \gamma Q(x,\xi) \\ P(x,\xi) \ = \ B(x,\xi)\displaystyle\int U(y)\big[N(x-y+\xi/2) - N(x-y-\xi/2)\big]dy \\ Q(x,\xi) \ = \ \displaystyle\int U(y)\big[B(x-y/2+\xi/2,y)B(x-y/2,\xi-y) - B(x-y/2,\xi+y)B(x-y/2-\xi/2,-y)\big]dy \end{cases}$$

$$N(z,x) \equiv B(z,x,0) = \left\langle |\psi(z,x)|^2 \right\rangle$$

$$n_k(x,z) = \int B(z,x,\xi)\ \exp(-ik\xi)\ d\xi$$

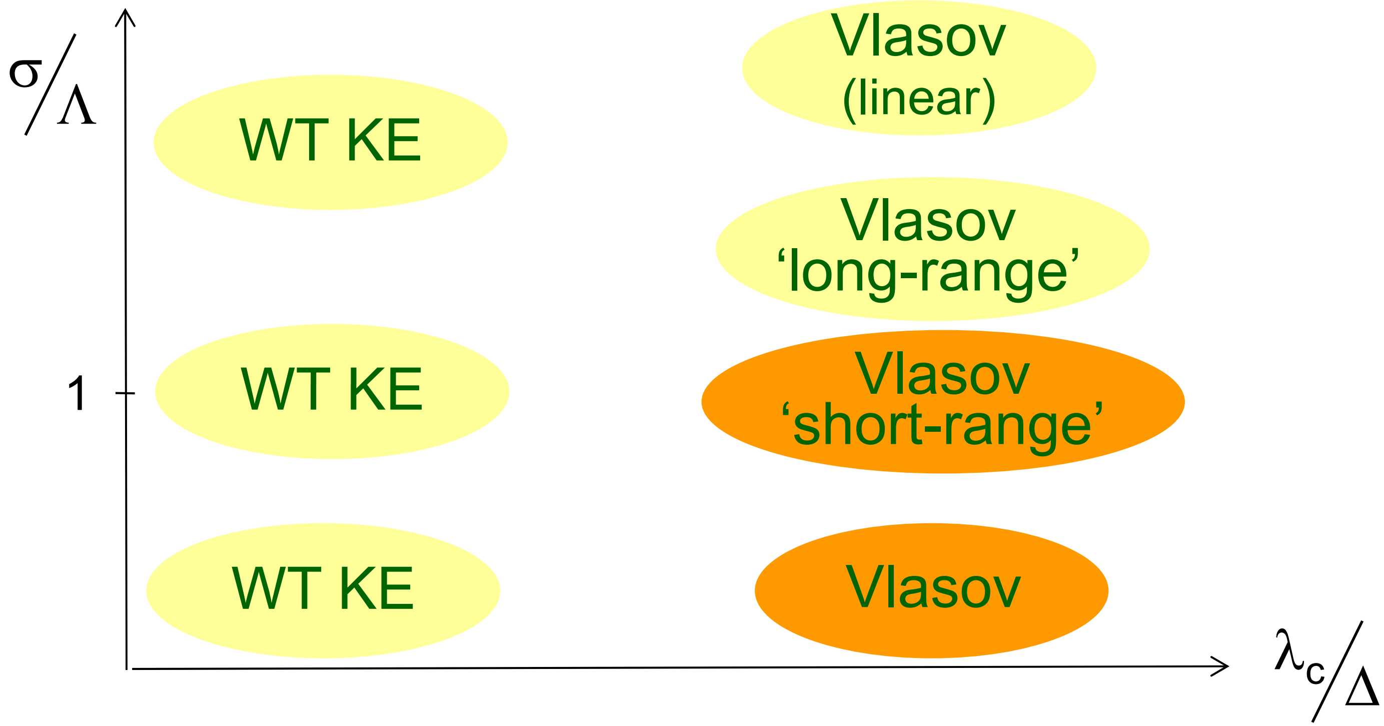
(ii) Quasi-homogeneous statistics: $\varepsilon = \lambda_c / \Delta \ll 1$

$$\boxed{B(x,\xi) = B^{(0)}(\varepsilon x,\xi) = \tfrac{1}{2\pi}\textstyle\int n_k^{(0)}(\varepsilon x)\exp(ik\xi)\,dk}$$

$$\begin{cases} \partial_z n_k(x,z) + \partial_k\tilde{\omega}_k(x,z)\,\partial_x n_k(x,z) - \partial_x\tilde{\omega}_k(x,z)\,\partial_k n_k(x,z) = 0 \\ \tilde{\omega}_k(x,z) = \omega(k) - V_k(x,z) \\ V_k(x,z) = \gamma N(x) + \gamma \displaystyle\int \tilde{U}_{k-k'}\, n_{k'}(x,z) dk' \end{cases}$$

$$N(x,z) = \frac{1}{2\pi}\int n_k(x,z)\,dk$$

‘Short-range’ Vlasov eqn



$$\begin{cases} \partial_z n_{\boldsymbol{k}}(\boldsymbol{x},z) + \partial_{\boldsymbol{k}}\tilde{\omega}_{\boldsymbol{k}}(\boldsymbol{x},z).\partial_{\boldsymbol{x}} n_{\boldsymbol{k}}(\boldsymbol{x},z) - \partial_{\boldsymbol{x}}\tilde{\omega}_{\boldsymbol{k}}(\boldsymbol{x},z).\partial_{\boldsymbol{k}} n_{\boldsymbol{k}}(\boldsymbol{x},z) = 0 \\ \tilde{\omega}_{\boldsymbol{k}}(\boldsymbol{x},z) = \omega(\boldsymbol{k}) + V_{\boldsymbol{k}}(\boldsymbol{x},z) : \textit{renormalization of the frequency} \\ V_{\boldsymbol{k}}(\boldsymbol{x},z) = \frac{\gamma}{(2\pi)^d}\int (1+\tilde{U}_{\boldsymbol{k}-\boldsymbol{k}'})\, n_{\boldsymbol{k}'}(\boldsymbol{x},z)\, d\boldsymbol{k}' \xrightarrow[U(\boldsymbol{x})=\delta(\boldsymbol{x})]{} \begin{cases} V(\boldsymbol{x},z) = \frac{2\gamma}{(2\pi)^d} N(\boldsymbol{x},z) \\ N(\boldsymbol{x},z) = \int n_{\boldsymbol{k}'}(\boldsymbol{x},z)\, d\boldsymbol{k}' \end{cases} \\ \mathcal{H}_{Vl} = \iint \omega(\boldsymbol{k})\, n_{\boldsymbol{k}}(\boldsymbol{x})\, d\boldsymbol{x} d\boldsymbol{k} + \frac{1}{2(2\pi)^{2d}} \iiint n_{\boldsymbol{k}_1}(\boldsymbol{x})\, \tilde{U}_{\boldsymbol{k}_1-\boldsymbol{k}_2}\, n_{\boldsymbol{k}_2}(\boldsymbol{x}) d\boldsymbol{x} d\boldsymbol{k}_1 d\boldsymbol{k}_2 \end{cases}$$

'Long-range' Vlasov eqn $(\sigma/\Lambda \gg 1)$

$$i\partial_z\psi = -\beta\partial_{xx}\psi - \gamma\psi \int_{-\infty}^{+\infty} U(x')\ |\psi|^2(t, x-x')\ dx'$$

$$B(z,x,\xi) = \langle \psi(z, x+\xi/2)\psi^*(z, x-\xi/2)\rangle$$

$$\begin{cases} i\partial_z B(x,\xi) = -2\beta\partial^2_{x\xi}B(x,\xi) + \gamma P(x,\xi) + \gamma Q(x,\xi) \\ P(x,\xi) \ = \ B(x,\xi)\displaystyle\int U(y)\big[N(x-y+\xi/2) - N(x-y-\xi/2)\big]dy \\ Q(x,\xi) \ = \ \displaystyle\int U(y)\big[B(x-y/2+\xi/2, y)B(x-y/2, \xi-y) - B(x-y/2, \xi+y)B(x-y/2-\xi/2, -y)\big]dy \end{cases}$$

$$N(z,x) \equiv B(z,x,0) = \big\langle|\psi(z,x)|^2\big\rangle$$

$$n_k(x,z) = \int B(z,x,\xi)\ \exp(-ik\xi)\, d\xi$$

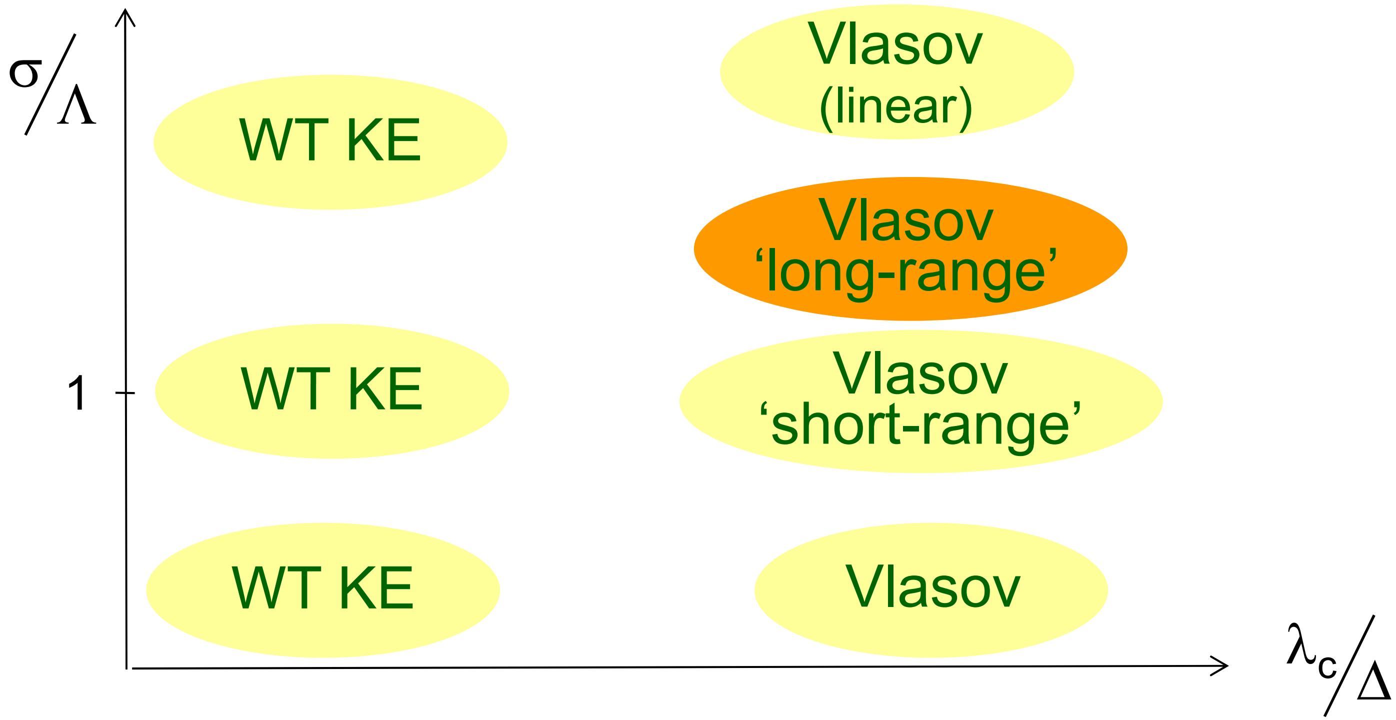
quasi-homogeneous statistics: $\varepsilon = \lambda_c / \Delta \ll 1$

$$\boxed{B(x,\xi) = B^{(0)}(\varepsilon x, \xi) = \tfrac{1}{2\pi}\int n_k^{(0)}(\varepsilon x)\exp(ik\xi)\, dk}$$

$$\boxed{U(x) \ = \ \varepsilon U^{(0)}(\varepsilon x)}$$

$$\begin{cases} \partial_z n_k(x,z) + \partial_k\tilde{\omega}_k(x,z)\,\partial_x n_k(x,z) - \partial_x\tilde{\omega}_k(x,z)\,\partial_k n_k(x,z) = 0 \\ \tilde{\omega}_k(x,z) = \beta k^2 - V(x,z) \\ V(x,z) = \gamma\displaystyle\int U(x-x')N(x',z)\, dx' \\ N(x,z) = \dfrac{1}{2\pi}\displaystyle\int n_k(x,z)\, dk \end{cases}$$

‘Long-range’ Vlasov eqn



$$\begin{cases} \partial_z n_k(x,z) + \partial_k \tilde{\omega}_k(x,z)\,\partial_x n_k(x,z) - \partial_x \tilde{\omega}_k(x,z)\,\partial_k n_k(x,z) = 0 \\ \tilde{\omega}_k(x,z) = \beta k^2 - V(x,z) \\ V(x,z) = \gamma \int U(x-x')N(x',z)\,dx' \\ N(x,z) = \frac{1}{2\pi}\int n_k(x,z)\,dk \end{cases}$$

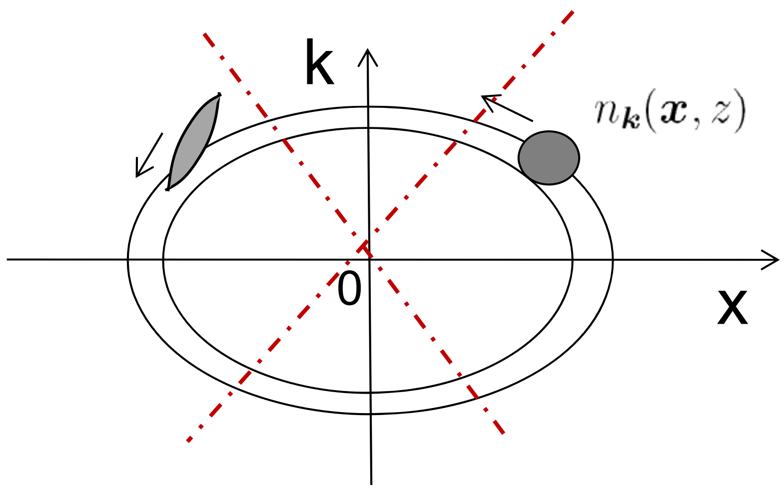
Vlasov eqn

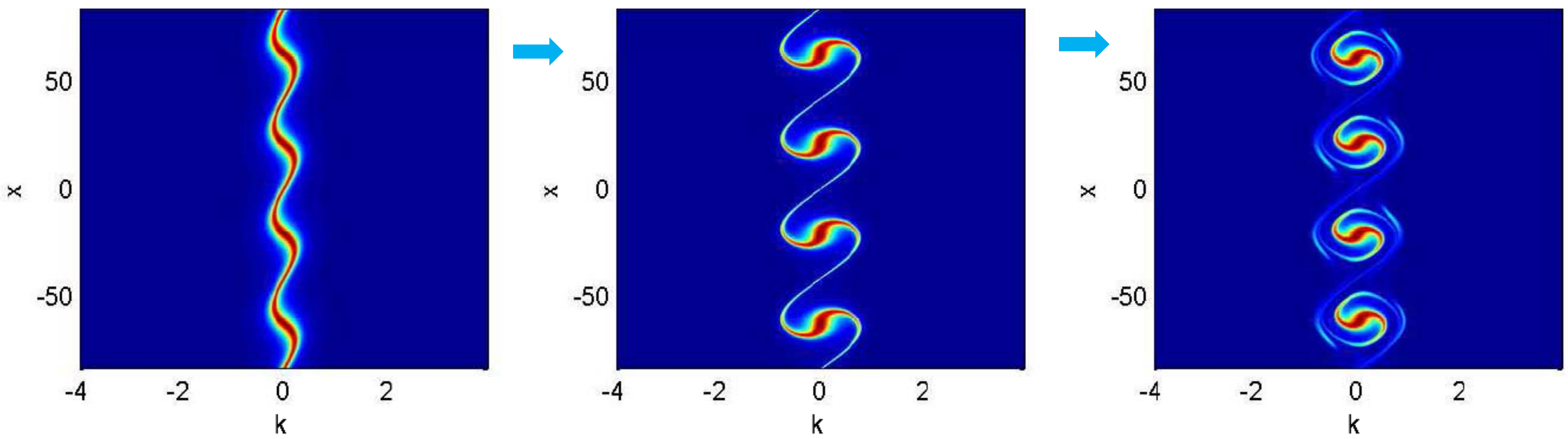
$$\partial_z n_{\boldsymbol{k}}(\boldsymbol{x}, z) + \partial_{\boldsymbol{k}} \tilde{\omega}_{\boldsymbol{k}}(\boldsymbol{x}, z).\partial_{\boldsymbol{x}} n_{\boldsymbol{k}}(\boldsymbol{x}, z) - \partial_{\boldsymbol{x}} \tilde{\omega}_{\boldsymbol{k}}(\boldsymbol{x}, z).\partial_{\boldsymbol{k}} n_{\boldsymbol{k}}(\boldsymbol{x}, z) = 0$$

$$\begin{cases} \dot{\boldsymbol{x}} = \partial_{\boldsymbol{k}} \tilde{\omega} \\ \dot{\boldsymbol{k}} = -\partial_{\boldsymbol{x}} \tilde{\omega} \end{cases} \qquad \begin{aligned} \tilde{\omega}_{\boldsymbol{k}}(\boldsymbol{x}, z) &= \omega(\boldsymbol{k}) + V(\boldsymbol{x}, z) \\ \omega(\boldsymbol{k}) &= \beta_s k^2 \end{aligned}$$

$$V(\boldsymbol{x}, z) = \gamma \int U(\boldsymbol{x} - \boldsymbol{x}')\, N(\boldsymbol{x}', z)\, d\boldsymbol{x}'$$

$$N(\boldsymbol{x}, z) = \frac{1}{(2\pi)^d} \int n_{\boldsymbol{k}}(\boldsymbol{x}, z)\, d\boldsymbol{k}$$





The particles trap themselves into a spiralling behavior ('phase-mixing')

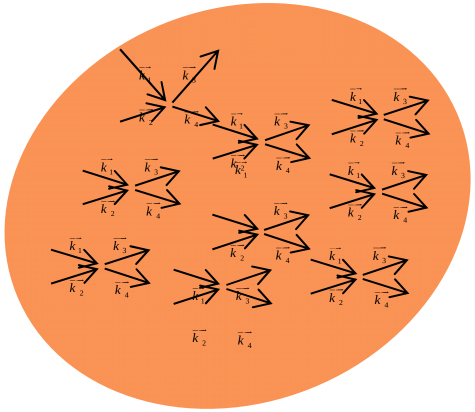
Vlasov eqn

$$\partial_z n_{\boldsymbol{k}}(\boldsymbol{x},z) + \partial_{\boldsymbol{k}}\tilde{\omega}_{\boldsymbol{k}}(\boldsymbol{x},z).\partial_{\boldsymbol{x}} n_{\boldsymbol{k}}(\boldsymbol{x},z) - \partial_{\boldsymbol{x}}\tilde{\omega}_{\boldsymbol{k}}(\boldsymbol{x},z).\partial_{\boldsymbol{k}} n_{\boldsymbol{k}}(\boldsymbol{x},z) = 0$$

$$\begin{cases} \dot{\boldsymbol{x}} = \partial_{\boldsymbol{k}}\tilde{\omega} \\ \dot{\boldsymbol{k}} = -\partial_{\boldsymbol{x}}\tilde{\omega} \end{cases} \qquad \begin{aligned} \tilde{\omega}_{\boldsymbol{k}}(\boldsymbol{x},z) &= \omega(\boldsymbol{k}) + V(\boldsymbol{x},z) \\ \omega(\boldsymbol{k}) &= \beta_s k^2 \end{aligned}$$

$$V(\boldsymbol{x},z) = \gamma \int U(\boldsymbol{x}-\boldsymbol{x}')\, N(\boldsymbol{x}',z)\, d\boldsymbol{x}'$$

$$N(\boldsymbol{x},z) = \frac{1}{(2\pi)^d}\int n_{\boldsymbol{k}}(\boldsymbol{x},z)\, d\boldsymbol{k}$$



- invariant under the transformation $(z,\boldsymbol{k}) \to (-z,-\boldsymbol{k})$

$$\begin{cases} \mathcal{N} = (2\pi)^{-d} \iint n_{\boldsymbol{k}}(\boldsymbol{x},z)\, d\boldsymbol{x} d\boldsymbol{k} \\ \mathcal{H}_{Vl} = \iint \omega(\boldsymbol{k})\, n_{\boldsymbol{k}}(\boldsymbol{x},z)\, d\boldsymbol{x} d\boldsymbol{k} + \frac{1}{2}\int V(\boldsymbol{x}) N(\boldsymbol{x})\, d\boldsymbol{x} \\ \mathcal{M} = \iint f[n]\, d\boldsymbol{x} d\boldsymbol{k} \end{cases}$$

- $f[n]$ is an arbitrary functional of $n_{\boldsymbol{k}}(\boldsymbol{x},z)$ $\big(f[n] = \log(n)\big)$

Soliton solution of the 'long-range' Vlasov eqn

$$y = x + vz \qquad \partial_k \tilde{\omega}_k(y)\, \partial_y n_k(y) - \partial_y \tilde{\omega}_k(y)\, \partial_k n_k(y) + v \partial_y n_k(y) = 0$$

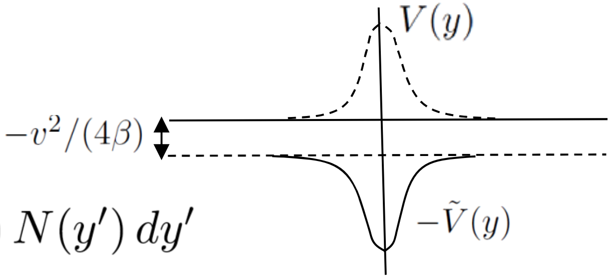
$$\left.\begin{matrix} h = \tilde{\omega} + kv \\ p = k + v/(2\beta) \end{matrix}\right\} \quad h(p,y) = \beta p^2 - \tilde{V}(y) \qquad \tilde{V}(y) = V(y) + v^2/(4\beta)$$

$$\partial_p h(p,y)\, \partial_y n_p(y) - \partial_y h(p,y)\, \partial_p n_p(y) = 0$$

Any function $n_p(y) = f(h + const)$ is solution of this equation

Bernstein, Green, Kruskal, PR '57
Hasegawa, Phys. Fluids '75
$\rightarrow U(\boldsymbol{x}) = \delta(\boldsymbol{x})$

$$h \le -v^2/(4\beta) \qquad \begin{cases} N(y) = \frac{1}{2\pi} \int_{-p_c}^{+p_c} n_p(y) dp \\ p_c = \sqrt{\tilde{V}/\beta - v^2/(4\beta^2)} \end{cases}$$



$$N = (2\pi\sqrt{\beta})^{-1} \int_{-\tilde{V}}^{-v^2/4\beta} (h + \tilde{V})^{-\frac{1}{2}}\, n^{st}(h)\, dh \qquad V(y) = \int U(y - y')\, N(y')\, dy'$$

$$\begin{cases} f(\lambda) = \int_0^\infty \exp(-\lambda t)\, F(t)\, dt = \mathcal{L}(F) \quad F_1 \star F_2 = \int_0^t F_1(t-y)\, F_2(y)\, dy \quad \mathcal{L}(F_1 \star F_2) = \mathcal{L}(F_1)\, \mathcal{L}(F_2) \\ N(x) = \mathrm{n}(2\pi\sigma_N^2)^{-1/2} \exp[-x^2/(2\sigma_N^2)] \quad U(x) = (2\pi\sigma^2)^{-1/2} \exp(-x^2/(2\sigma^2)) \\ n^{st}(h) \sim (-h - v^2/(4\beta))^{\frac{1}{\alpha} - \frac{1}{2}} \quad \alpha = (1 + (\sigma/\sigma_N)^2)^{-1} \xrightarrow{U(\boldsymbol{x}) = \delta(\boldsymbol{x})} n^{st}(h) \sim \sqrt{-h} \end{cases}$$

$$n_k^{st}(x + vz) = Q_\alpha \left[c_\alpha \gamma N^\alpha(x + vz) - \beta(k + v/(2\beta))^2\right]^{\frac{1}{\alpha} - \frac{1}{2}}$$

$$Q_\alpha = 2\pi\beta^{\frac{1}{2}} \Gamma(\alpha^{-1} + 1)/[\Gamma(\alpha^{-1} + 1/2)\Gamma(1/2)(c_\alpha\gamma)^{1/\alpha}]$$

$$c_\alpha = (2\pi)^{\frac{\alpha}{2} - \frac{1}{2}} \sigma_N^\alpha \mathrm{n}^{1-\alpha} (\sigma^2 + \sigma_N^2)^{-\frac{1}{2}}$$

Picozzi, Garnier PRL (2011)

Soliton solution of the 'long-range' Vlasov eqn

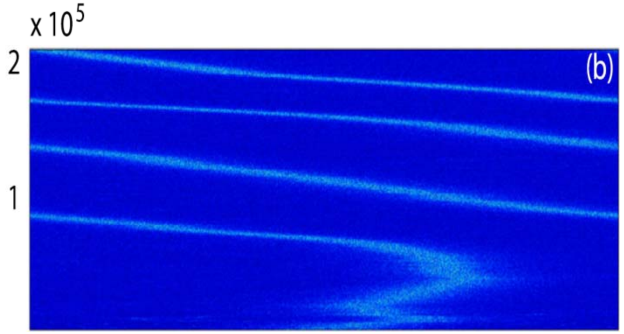
$$\partial_k \tilde{\omega}_k(y)\, \partial_y n_k(y) - \partial_y \tilde{\omega}_k(y)\, \partial_k n_k(y) + v \partial_y n_k(y) = 0$$

$$\begin{cases} N(x) = \mathrm{n}(2\pi\sigma_N^2)^{-1/2} \exp[-x^2/(2\sigma_N^2)] \\ U(x) = (2\pi\sigma^2)^{-1/2} \exp(-x^2/(2\sigma^2)) \end{cases}$$

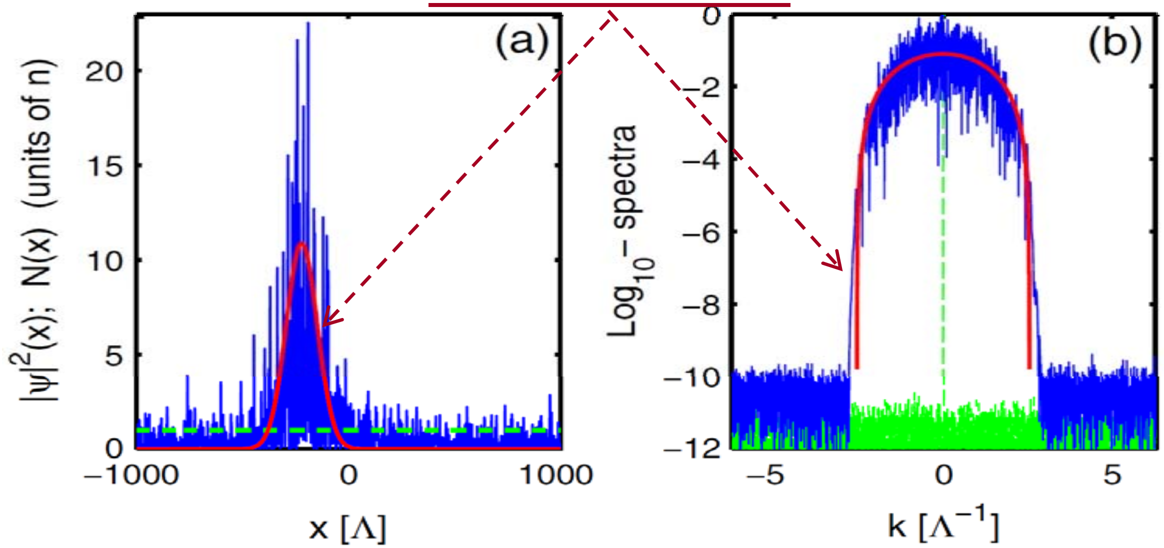
$$n_k^{st}(x+vz) = Q_\alpha \left[c_\alpha \gamma N^\alpha(x+vz) - \beta(k + v/(2\beta))^2\right]^{\frac{1}{\alpha}-\frac{1}{2}}$$

$$Q_\alpha = 2\pi\beta^{\frac{1}{2}}\Gamma(\alpha^{-1}+1)/[\Gamma(\alpha^{-1}+1/2)\Gamma(1/2)(c_\alpha\gamma)^{1/\alpha}]$$

$$\alpha = (1 + (\sigma/\sigma_N)^2)^{-1} \qquad c_\alpha = (2\pi)^{\frac{\alpha}{2}-\frac{1}{2}}\sigma_N^\alpha \mathrm{n}^{1-\alpha}(\sigma^2+\sigma_N^2)^{-\frac{1}{2}}$$



Soliton solution



$$\mathcal{U} = -\frac{\gamma}{2}\int\int U(x-x')\,|\psi(x)|^2\,|\psi(x')|^2\,dx\,dx'$$

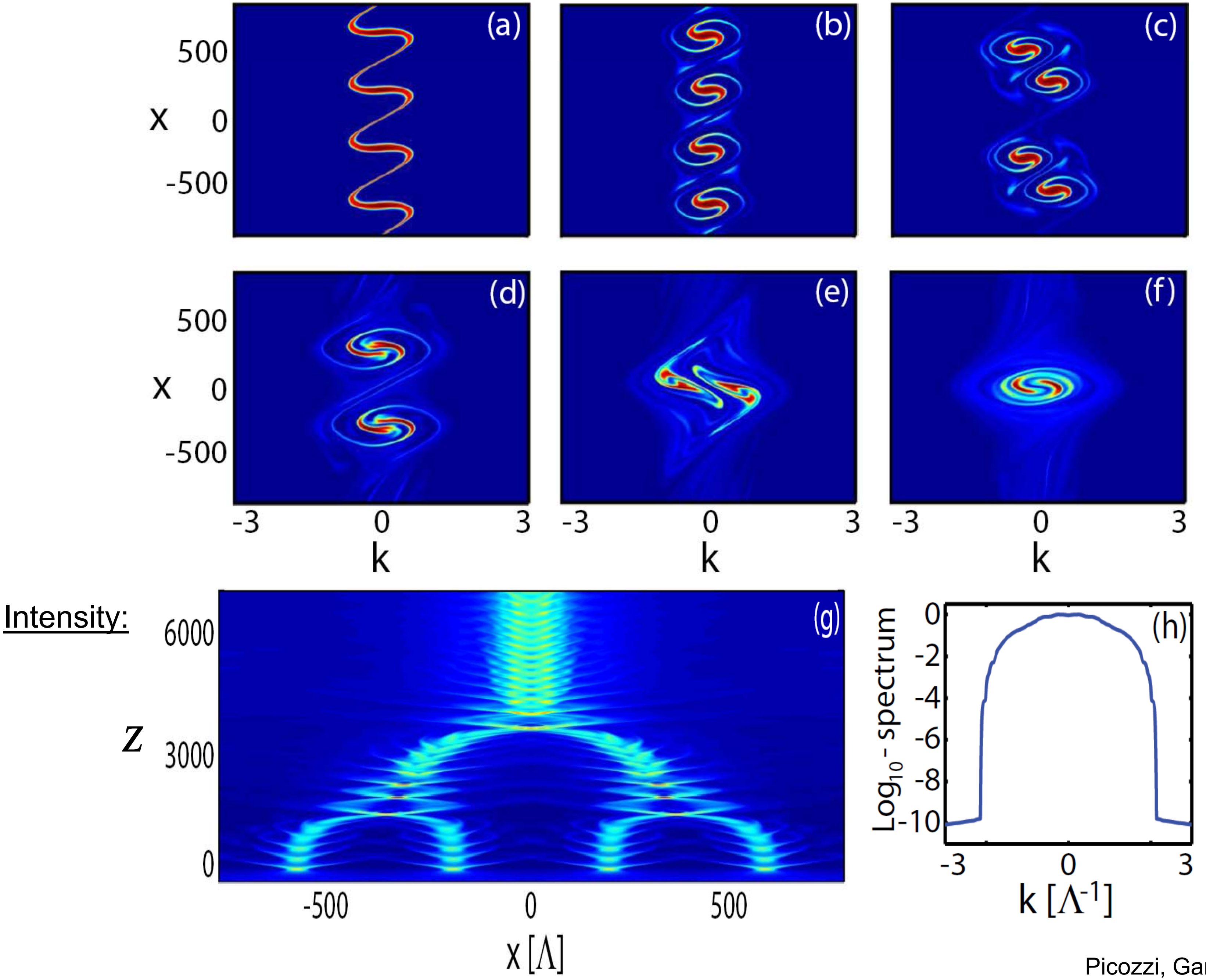
$$\mathcal{E} = \beta\int|\partial_x\psi|^2\,dx$$

$$\boxed{|\mathcal{U}/\mathcal{E}| \simeq 1.5}$$

$$\begin{cases} \boxed{\beta k^2 - V(x) \le 0} \\ k_c \sim \sqrt{V/\beta} \end{cases}$$

$$\left(V(x) = \gamma U * N = (\gamma\rho L/\sigma)\int U_0(\tilde{x}-\tilde{x}')N_0(\tilde{x}')d\tilde{x}' \sim \gamma\rho L/\sigma\right) \qquad k_c^{-1} \sim \Lambda\sqrt{\sigma/L}$$

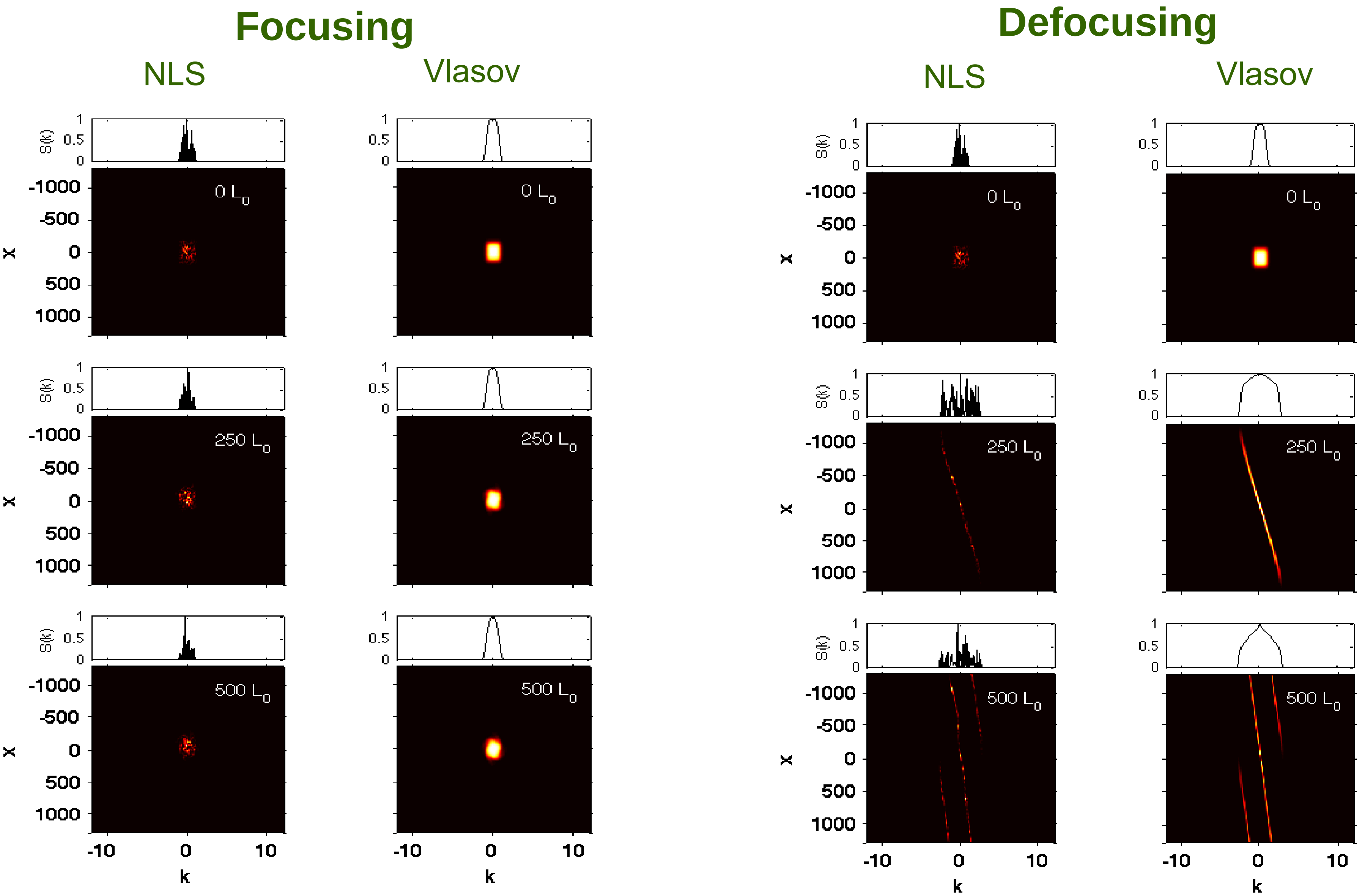
Simulation 'long-range' Vlasov eqn.



Picozzi, Garnier PRL (2011)

Good agreement NLS vs long-range-Vlasov

⟶ in the nonlinear regime: $H_{nl} \sim 10\, H_{lin}$



‘Long-range’ Vlasov eqn

This Vlasov eqn should provide an ‘exact’ statistical description of the random nonlinear wave

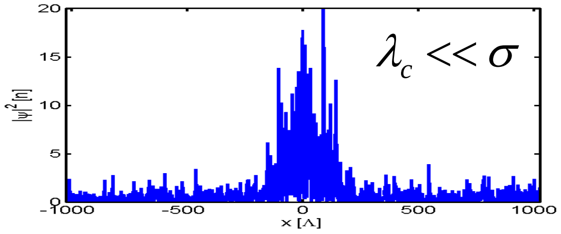
Highly nonlocal nonlinearity

(i) Weak NL: Closure

$$\int U(\boldsymbol{x}-\boldsymbol{x}')|\psi(\boldsymbol{x}',z)|^2 d\boldsymbol{x}' \simeq \int U(\boldsymbol{x}-\boldsymbol{x}')N(\boldsymbol{x}',z)d\boldsymbol{x}'$$

Gaussian statistics (CLT)

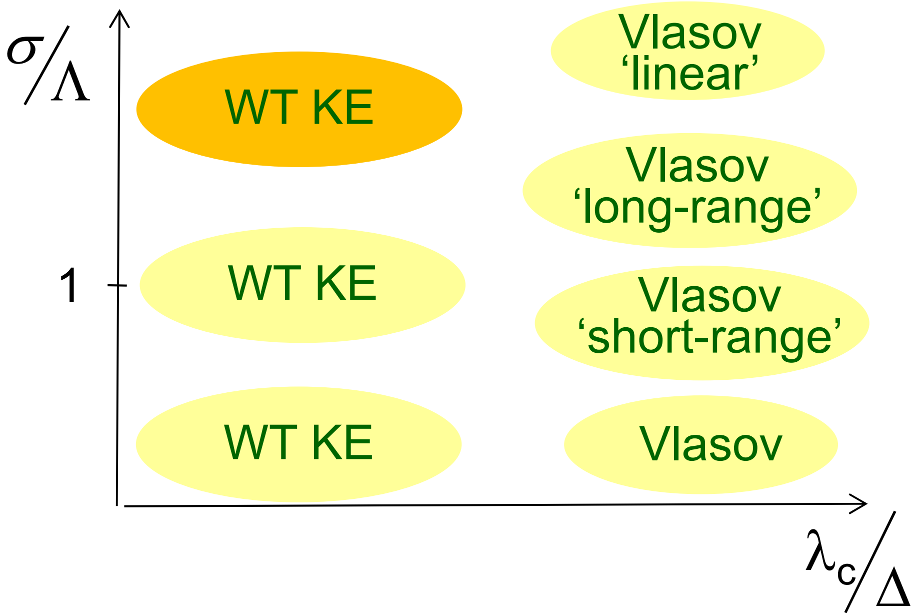
(ii) Quasi-homogeneous statistics



$$\begin{cases} i\partial_z\psi = -\beta\partial_{xx}\psi - \gamma\psi \int_{-\infty}^{+\infty} U(x')\, |\psi|^2(t,x-x')\, dx' \\ \partial_z n_k(x,z) + \partial_k\tilde{\omega}_k(x,z)\,\partial_x n_k(x,z) - \partial_x\tilde{\omega}_k(x,z)\,\partial_k n_k(x,z) = 0 \\ \tilde{\omega}_k(x,z) = \beta k^2 - V(x,z) \\ V(x,z) = \gamma \int U(x-x')N(x',z)\,dx' \\ N(x,z) = \frac{1}{2\pi}\int n_k(x,z)\,dk \end{cases}$$

⟶ **Rigorous proof…?**

Slowing down of thermalization due to a highly nonlocal interaction



→ Campa, Dauxoix, Ruffo – Phys. Rep. **480**, 57 (2009)
Statistical mechanics and dynamics of solvable models with long-range interactions

Slowing down of thermalization due to a highly nonlocal interaction

$$\begin{cases} \partial_z n_{\boldsymbol{k}} = \dfrac{4\pi\gamma^2}{(2\pi)^{2d}} \displaystyle\iiint \mathcal{Q}(n_{\boldsymbol{k}}, n_{\boldsymbol{k}_1}, n_{\boldsymbol{k}_2}, n_{\boldsymbol{k}_3})\ T_{k123}^2\ \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k})\ \delta\left(\beta_s(k_1^2 + k_2^2 - k_3^2 - k^2)\right)\ d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \\ \mathcal{Q}(n_{\boldsymbol{k}}, n_{\boldsymbol{k}_1}, n_{\boldsymbol{k}_2}, n_{\boldsymbol{k}_3}) = n_{\boldsymbol{k}_1} n_{\boldsymbol{k}_2} n_{\boldsymbol{k}_3} n_{\boldsymbol{k}} (n_{\boldsymbol{k}}^{-1} + n_{\boldsymbol{k}_3}^{-1} - n_{\boldsymbol{k}_2}^{-1} - n_{\boldsymbol{k}_1}^{-1}) \\ T_{k123} = \frac{1}{4}\left(\tilde{U}_{12} + \tilde{U}_{13} + \tilde{U}_{k3} + \tilde{U}_{k2}\right) \\ n_k^{RJ} = \dfrac{T}{\beta_s k^2 - \mu} \end{cases}$$

$$\begin{cases} U(\boldsymbol{x}) = \varepsilon U^{(0)}(\varepsilon \boldsymbol{x}) \\ \tilde{U}(\boldsymbol{k}) = \tilde{U}^{(0)}(\boldsymbol{k}/\varepsilon) \\ \boldsymbol{k}_j = \boldsymbol{k} + \varepsilon \boldsymbol{\kappa}_j\ \ (j = 1, 2, 3) \end{cases}$$

$$\partial_z n_{\boldsymbol{k}} = \frac{4\pi\gamma^2\varepsilon^{2d-2}}{(2\pi)^{2d}} \iint \mathcal{Q}(n_{\boldsymbol{k}}, n_{\boldsymbol{k}+\varepsilon\boldsymbol{\kappa}_1}, n_{\boldsymbol{k}+\varepsilon\boldsymbol{\kappa}_2}, n_{\boldsymbol{k}+\varepsilon\boldsymbol{\kappa}_1+\varepsilon\boldsymbol{\kappa}_2})\left[\tilde{U}^{(0)}(\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2) + \tilde{U}^{(0)}(\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2) + 2\tilde{U}^{(0)}(\boldsymbol{\kappa}_2)\right] \times \delta(2\beta_s \boldsymbol{\kappa}_1 \cdot \boldsymbol{\kappa}_2)\ d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2$$

$$\partial_z n_k / n_k \sim 1/\lambda_k$$

$\longrightarrow$ A highly nonlocal interaction slows down the dynamics

$$\lambda_k^{\rm nonloc} \sim \lambda_k^{\rm loc} (\sigma_s/\Lambda_s)^{2d-2}$$

Review: Garnier, Lisak, Picozzi, JOSA B (2012)

Optical wave turbulence

1.- Homogeneous statistics & instantaneous response:

WT kinetic eqn

Thermalization / Anomalous thermalization

Wave condensation in a trap

2.- Inhomogeneous statistics & nonlocal response:

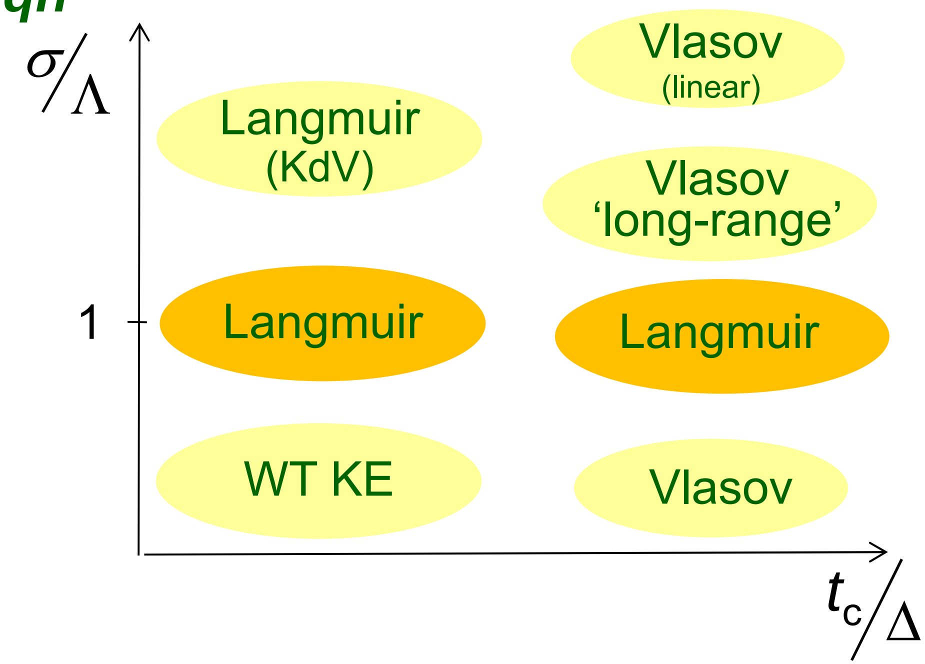
Vlasov kinetic eqn

Effective potential

3.- Noninstantaneous response:

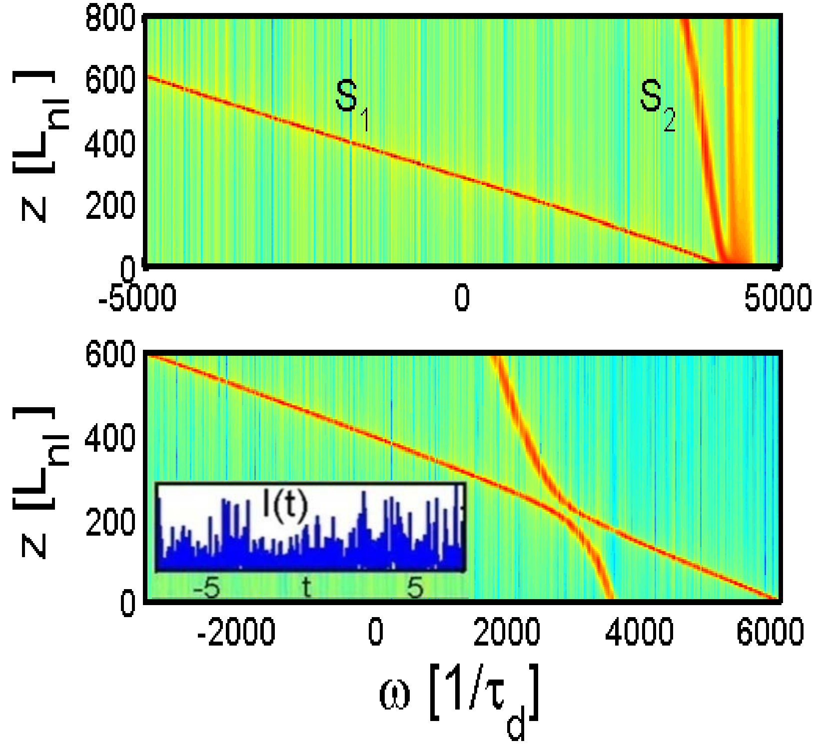
Weak Langmuir turbulence kinetic eqn

Causality condition



Noninstantaneous nonlinearity : Spectral incoherent solitons

$$i\partial_z\psi = -\beta_t\partial_{tt}\psi + \gamma\psi\int_{-\infty}^{+\infty} R(t-t')\ |\psi|^2(z,t')\ dt' \qquad R(t) = 0 \text{ for } t < 0$$



Picozzi, Pitois, Millot, PRL (2008)

Noninstantaneous nonlinearity : Weak Langmuir turbulence

$$i\partial_z\psi = -\beta_t\partial_{tt}\psi + \gamma\psi \int_{-\infty}^{+\infty} R(t-t')\ |\psi|^2(z,t')\ dt' \qquad R(t) = 0 \text{ for } t < 0$$

$$\begin{cases} \tilde{R}(\omega) = \tilde{U}(\omega) + i\,g(\omega) \\ \tilde{U}(\omega) = -\frac{1}{\pi}\mathcal{P}\int \frac{g(\omega')}{\omega'-\omega}\,d\omega' \quad \text{even} \\ g(\omega) = \frac{1}{\pi}\mathcal{P}\int \frac{\tilde{U}(\omega')}{\omega'-\omega}\,d\omega' \quad \text{odd} \end{cases}$$

$$B(t,\tau,z) = \langle \psi(t+\tau/2)\,\psi^*(t-\tau/2)\rangle$$

$$\begin{cases} i\,\partial_z B(t,\tau) = -2\beta\,\partial^2_{t\tau}B(t,\tau) + \gamma\,P(t,\tau) + \gamma\,Q(t,\tau) \\ P(t,\tau) = B(t,\tau)\displaystyle\int \chi(\theta)[N(t-\theta+\tau/2) - N(t-\theta-\tau/2)]d\theta \\ Q(t,\tau) = \displaystyle\int \chi(\theta)[B(t-\theta/2+\tau/2,\theta)B(t-\theta/2,\tau-\theta) \\ \qquad - B(t-\theta/2,\tau+\theta)B(t-\theta/2-\tau/2,-\theta)]d\theta \end{cases}$$

Noninstantaneous nonlinearity : Weak Langmuir turbulence

$$i\partial_z\psi = -\beta_t\partial_{tt}\psi + \gamma\psi\int_{-\infty}^{+\infty} R(t-t')\ |\psi|^2(z,t')\ dt' \qquad R(t) = 0 \text{ for } t<0$$

$$\tilde{R}(\omega) = \tilde{U}(\omega) + i\,g(\omega)$$

$$B(t,\tau,z) = \langle\psi(t+\tau/2)\,\psi^*(t-\tau/2)\rangle$$

$$n_\omega(t,z) = \int_{-\infty}^{+\infty} B(t,\tau,z)\ \exp(-i\omega\tau)\,d\tau$$

$$B(t,\tau,z) = B^{(0)}\left(\varepsilon t,\tau,\varepsilon z\right) + O(\varepsilon) \qquad \varepsilon = t_c/\Delta_t$$

$$\boxed{\partial_z n_\omega(t,z) = \frac{\gamma}{\pi}\,n_\omega(t,z)\int_{-\infty}^{+\infty} g(\omega-\omega')\,n_{\omega'}(t,z)\,d\omega'}$$

$$\mathcal{N} = (2\pi)^{-1}\int n_\omega(t,z)\,d\omega\,dt \qquad \mathcal{S} = (2\pi)^{-1}\int \log[n_\omega(t,z)]\,d\omega\,dt$$

- Reversible: $(z,\omega)\ \to\ (-z,-\omega)$
- $R(t)$ even $\longrightarrow$ $g(\omega)=0$ $\longrightarrow$ 'short-range' Vlasov

Musher, Rubenchik, Zakharov
Weak Langmuir turbulence
Phys. Rep. **252**, 177 (1995)

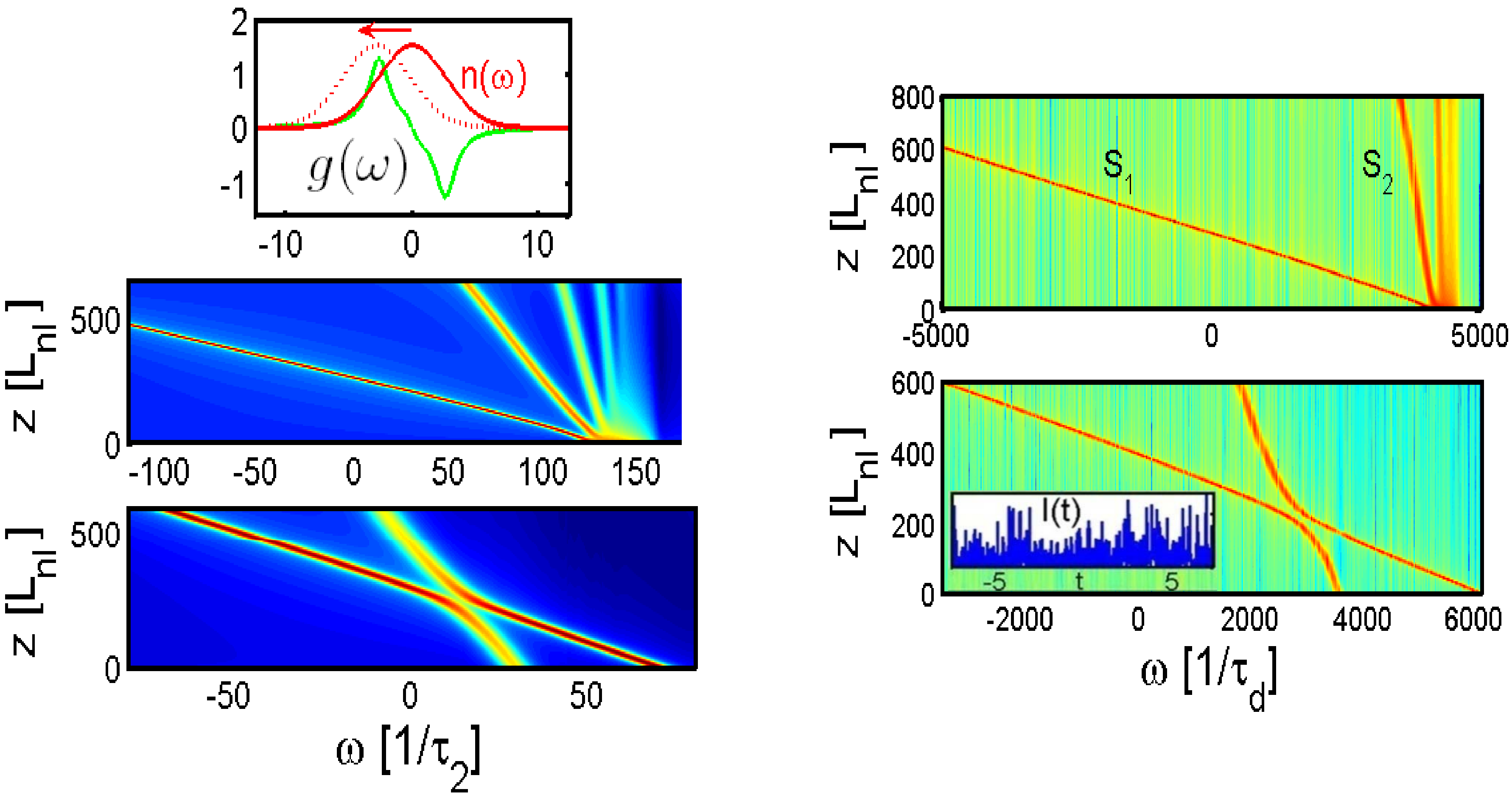
Noninstantaneous nonlinearity : Spectral incoherent solitons

$$i\partial_z\psi = -\beta_t\partial_{tt}\psi + \gamma\psi\int_{-\infty}^{+\infty} R(t-t')\ |\psi|^2(z,t')\ dt' \qquad R(t) = 0 \text{ for } t < 0$$

$$\partial_z n_\omega(z) = \frac{\gamma}{\pi} n_\omega(z)\int_{-\infty}^{+\infty} g(\omega-\omega') n_{\omega'}(z) d\omega'$$

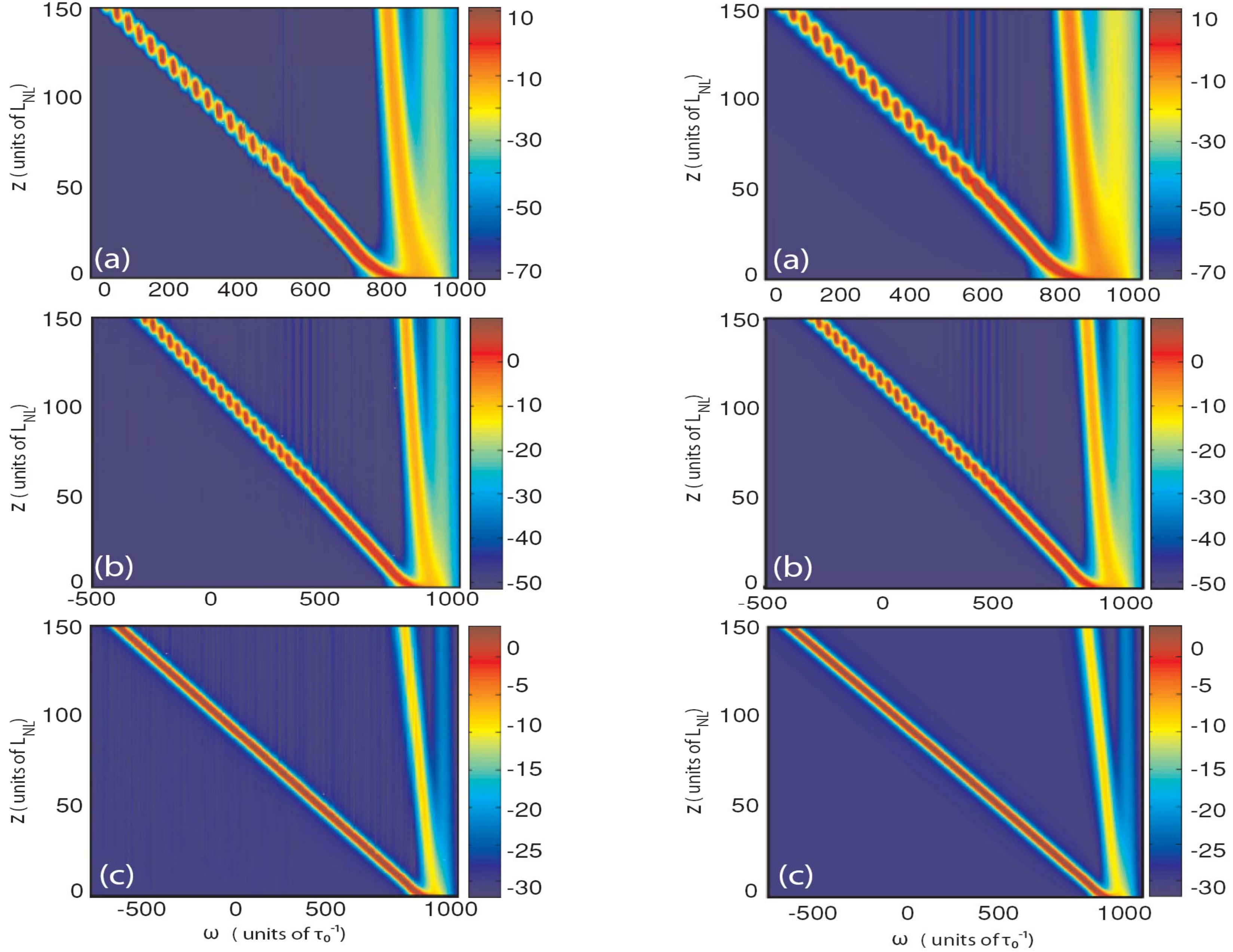
$$N = (2\pi)^{-1}\int n_\omega(z)\, d\omega$$

$$S = \int \ln[n_\omega(z)]\, d\omega$$



Picozzi, Pitois, Millot, PRL (2008)

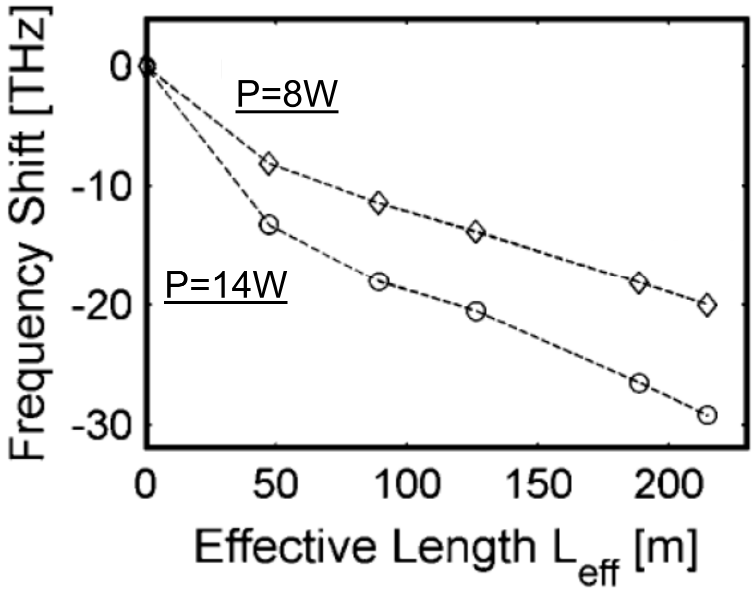
Quantitative agreement: NLS   Langmuir



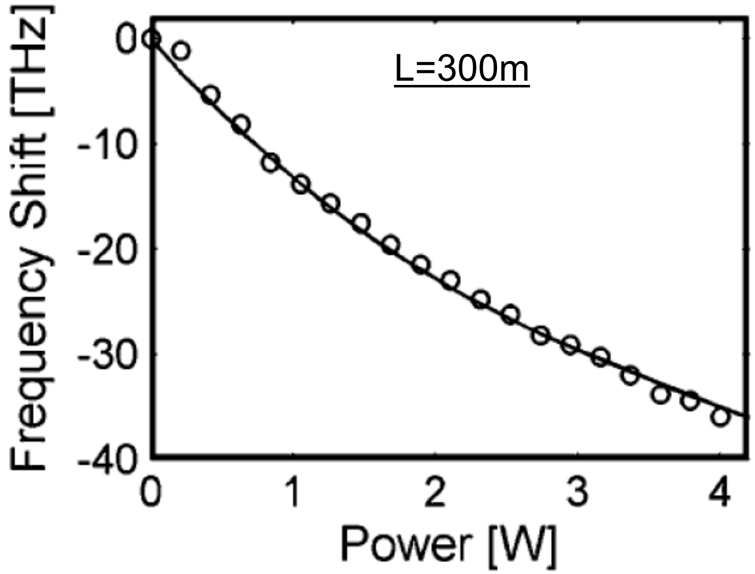
Michel, Kibler, Picozzi, PRA (2011)

Spectral incoherent solitons: Experiment

- Incoherent wave source with a spectral width of 20 THz (at λ=630nm):
  - → ASE of a dye amplifier pumped by frequency-doubled Nd:YAG laser (5ns – 25Hz)
- Use of a conventional PM optical fiber (loss~2.3km^{-1})



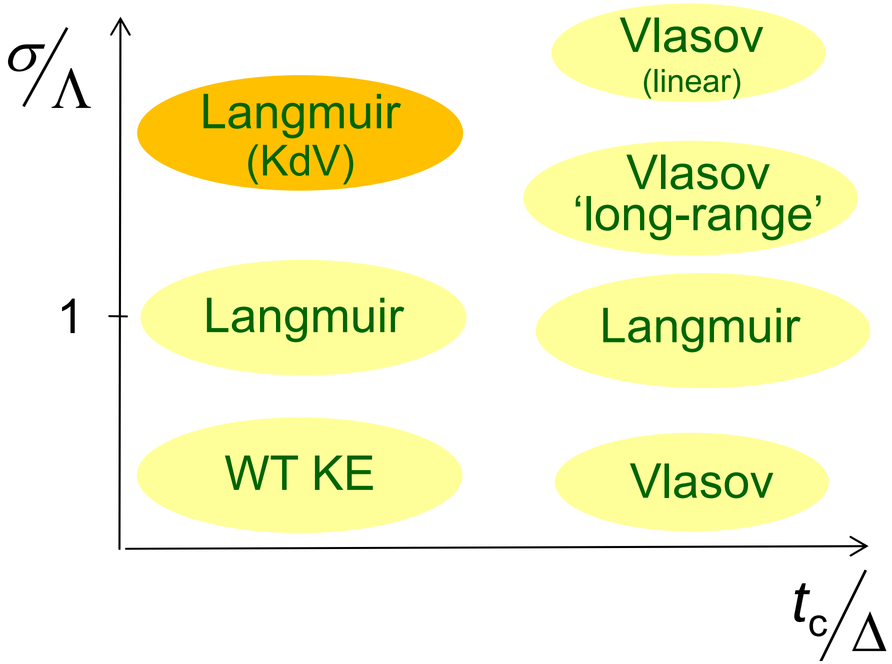
Comparison with numerics

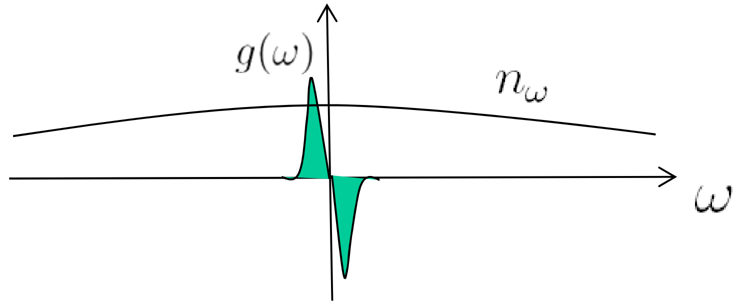


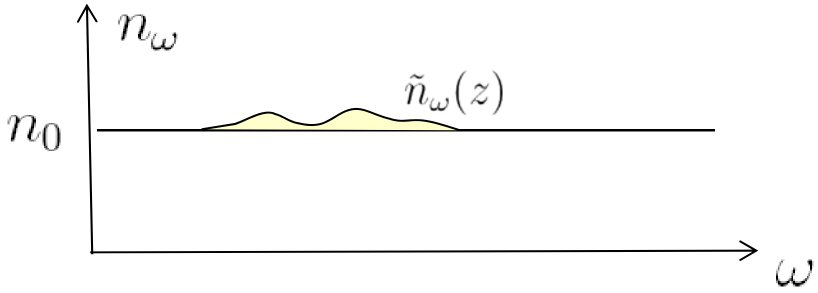
$$\begin{cases} i\partial_z \psi = -\beta\, \partial_{tt}\psi + \gamma\, \psi \displaystyle\int_{-\infty}^{+\infty} \chi(t')\, |\psi|^2(z, t-t')\, dt', \\ \partial_z n(z,\omega) = \dfrac{\gamma}{\pi}\, n(z,\omega) \displaystyle\int_{-\infty}^{+\infty} \widetilde{\chi}_{\rm I}(\omega-\omega')\, n(z,\omega')\, d\omega' \end{cases}$$

Picozzi, Pitois, Millot, PRL (2008)

Korteweg-de Vries eqn







$$\partial_z n_\omega(z) = \frac{\gamma}{\pi} n_\omega(z) \int_{-\infty}^{+\infty} g(\omega - \omega') n_{\omega'}(z) d\omega'$$

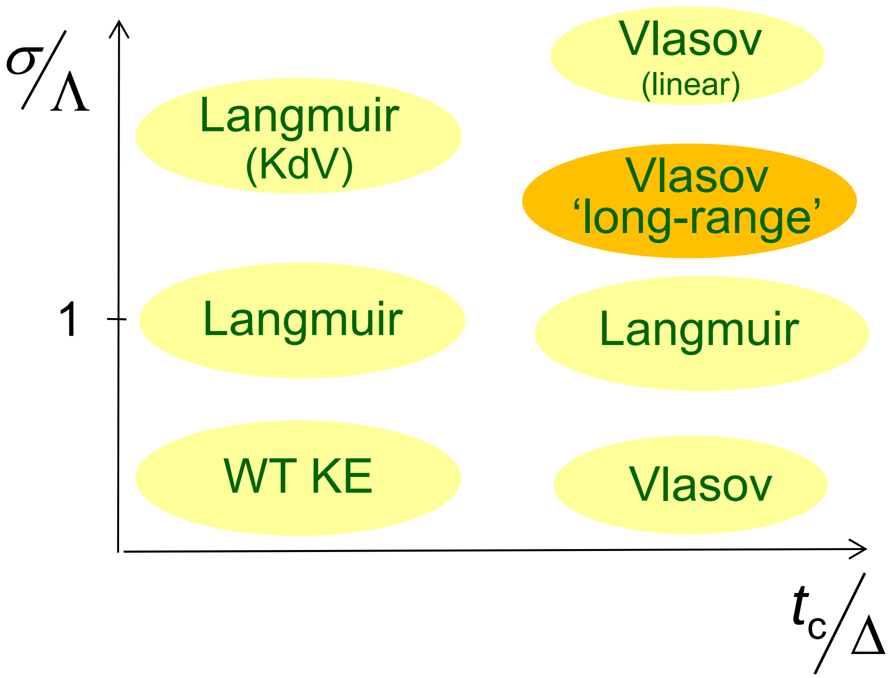
$$\varepsilon = \Delta\omega_g / \Delta\omega \ll 1 \qquad n_\omega(z) = n_0 + \tilde{n}_\omega(z)$$

$$\tilde{n}_\omega(z) = \varepsilon^2 \tilde{n}_\omega^{(0)}(\varepsilon^2 z) + O(\varepsilon^4) \qquad g(\omega) = g^{(0)}\left(\frac{\omega}{\varepsilon}\right)$$

$$\boxed{\partial_z \tilde{n}_\omega(z) - \frac{\gamma n_0 g_1}{\pi} \partial_\omega \tilde{n}_\omega(z) = \frac{\gamma g_1}{\pi} \tilde{n}_\omega(z) \partial_\omega \tilde{n}_\omega(z) + \frac{\gamma n_0 g_3}{6\pi} \partial_\omega^3 \tilde{n}_\omega(z)}$$

$$g_1 = -\int_{-\infty}^{+\infty} \omega g(\omega) d\omega, \qquad g_3 = -\int_{-\infty}^{+\infty} \omega^3 g(\omega) d\omega$$

Temporal long-range Vlasov eqn



Temporal long-range Vlasov eqn

$$\begin{cases} i\partial_z\psi = -\beta_t\partial_{tt}\psi + \gamma\psi \displaystyle\int_{-\infty}^{+\infty} R(t-t')\ |\psi|^2(z,t')\ dt' \\ R(t) = \varepsilon R^{(0)}(\varepsilon t) \qquad B(t,\tau,z) = B^{(0)}(\varepsilon t,\tau,\varepsilon z) + O(\varepsilon) \\ B^{(0)}(\varepsilon t,\tau,\varepsilon z) = (2\pi)^{-1}\int n_\omega^{(0)}(\varepsilon t,\varepsilon z)\exp(i\omega\tau)\,d\omega \end{cases}$$

$$\begin{cases} \partial_z n_\omega(t,z) + \partial_\omega \tilde{k}_\omega(t,z)\,\partial_t n_\omega(t,z) - \partial_t\tilde{k}_\omega(t,z)\,\partial_\omega n_\omega(t,z) = 0 \\ \tilde{k}_\omega(t,z) = k(\omega) + V(t,z) \qquad k(\omega) = \beta_t\omega^2 \\ V(t,z) = \gamma \displaystyle\int R(t-t')N(t',z)\,dt' \qquad N(t,z) = B(t,\tau=0,z) = (2\pi)^{-1}\int n_\omega(t,z)\,d\omega \end{cases}$$

$$\mathcal{N} = (2\pi)^{-1}\iint n_\omega(t,z)d\omega dt \qquad \mathcal{M} = \iint f[n]d\omega dt$$

$\underline{R(t)\ \text{is causal}}$

$$\tilde{R}(\omega) = \tilde{U}(\omega) + ig(\omega) \qquad \begin{cases} U(t) = \dfrac{1}{2\pi}\displaystyle\int \tilde{U}(\omega)\exp(i\omega t)\,d\omega, \\ G(t) = \dfrac{i}{2\pi}\displaystyle\int g(\omega)\exp(i\omega t)\,d\omega. \end{cases}$$

$$\partial_z n_\omega(t,z) + \partial_\omega k_\omega(t,z)\,\partial_t n_\omega(t,z) - \partial_t V_U(t,z)\,\partial_\omega n_\omega(t,z) = \partial_t V_G(t,z)\,\partial_\omega n_\omega(t,z)$$

$$V_U(t,z) = \gamma\int U(t-t')N(t',z)\,dt' \qquad V_G(t,z) = \gamma\int G(t-t')N(t',z)\,dt'$$

$$\mathcal{H}_{Vl} = \iint k(\omega)\,n_\omega(t,z)\,dt d\omega + \tfrac{1}{2}\int V_U(t,z)N(t,z)\,dt$$

‘Long-range’ incoherent modulational instability

$$i\partial_z A = \beta\partial_{tt} A + \gamma A \int_{-\infty}^{+\infty} R(t-t')\ |A|^2(z,t')\ dt' = 0$$

$$\partial_z n_\omega(t,z) + \partial_\omega \tilde{k}_\omega(t,z)\,\partial_t n_\omega(t,z) - \partial_t \tilde{k}_\omega(t,z)\,\partial_\omega n_\omega(t,z) = 0$$

$$\tilde{k}_\omega(t,z) = k(\omega) + V(t,z) \qquad k(\omega) = \beta\omega^2 \qquad \beta > 0$$

$$V(t,z) = \gamma \int R(t-t') N(t',z)\,dt' \qquad N(t,z) = B(t,\tau=0,z) = (2\pi)^{-1} \int n_\omega(t,z)\,d\omega$$

Stability analysis:

$$n_\omega(t,z) = n_\omega^0 + \epsilon_\omega(t,z) \quad \epsilon_\omega(t,z) \ll n_\omega^0$$

$$\partial_z \epsilon_\omega(t,z) - 2\beta\omega\partial_t\epsilon_\omega(t,z) - \gamma\partial_\omega n_\omega^0 \int dt'\partial_t R(t-t') \int d\omega\epsilon_\omega(t,z) = 0$$

Fourier-Laplace transform

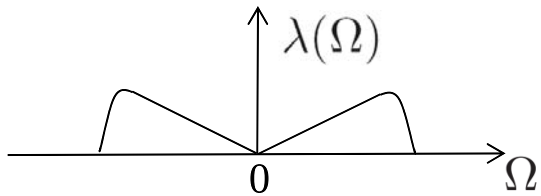
$$\tilde{\epsilon}_\omega(\Omega,\lambda) = \int_0^\infty dz \int_{-\infty}^{+\infty} dt\ \exp(-\lambda z - i\Omega t)\,\epsilon_\omega(t,z)$$

$$-1 = 2\gamma\beta\Omega^2\tilde{R}(\Omega)\int_{-\infty}^{+\infty}\frac{n_\omega^0}{(2\beta\Omega\omega + i\lambda)^2}\,d\omega \qquad n_\omega^0 = N_0\Delta\omega/[\pi(\omega^2 + \Delta\omega^2)]$$

$$\boxed{\lambda(\Omega) = |\Omega|\left(\sqrt{2\beta\gamma N_0\tilde{R}(\Omega)} - 2\beta\Delta\omega\right)}$$

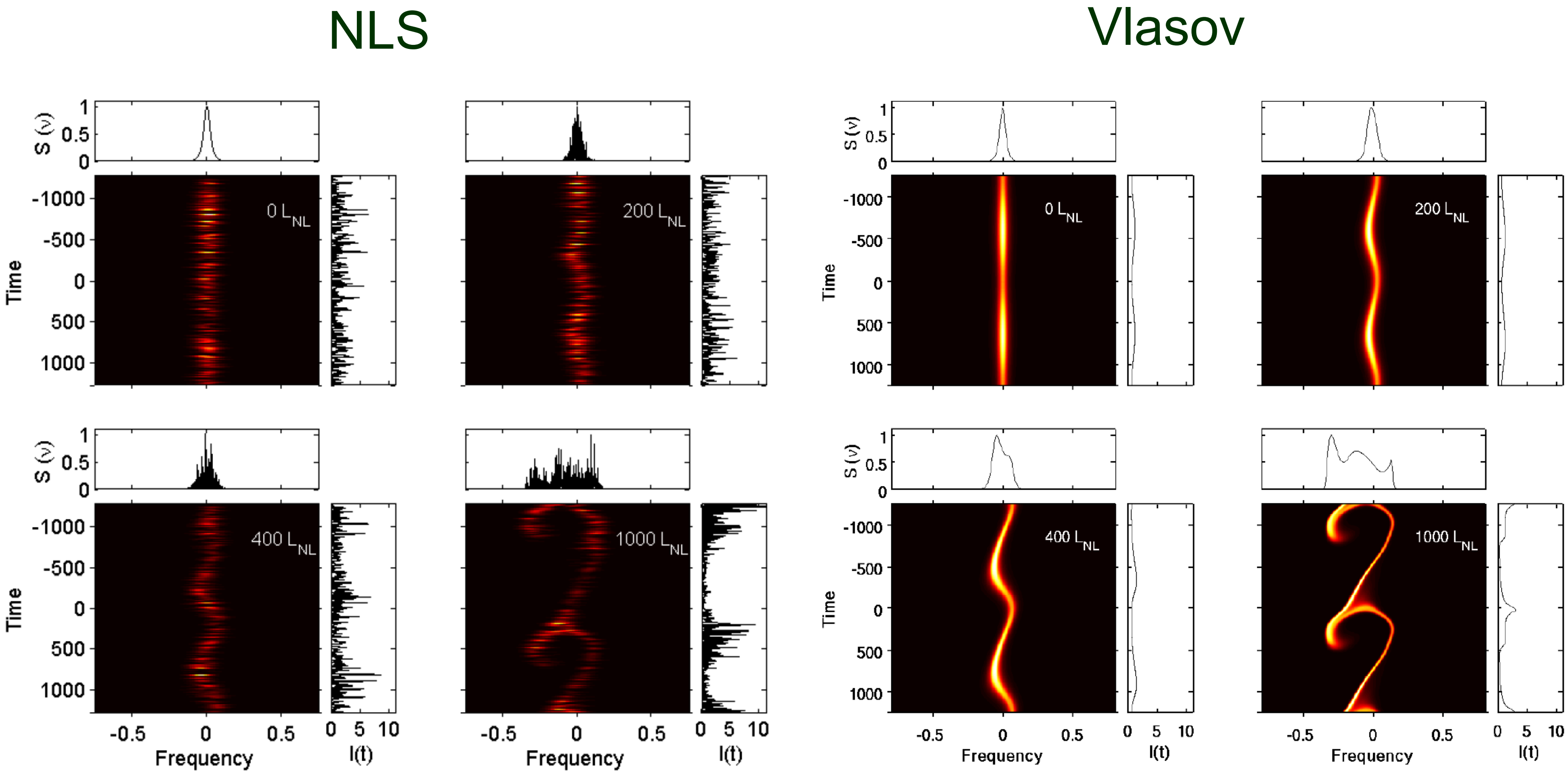
‘Landau damping’

$$\begin{cases}\tilde{R}(\omega) = \tilde{U}(\omega) + ig(\omega)\\ \Re[\lambda(\Omega)] \text{ is even}\end{cases}$$



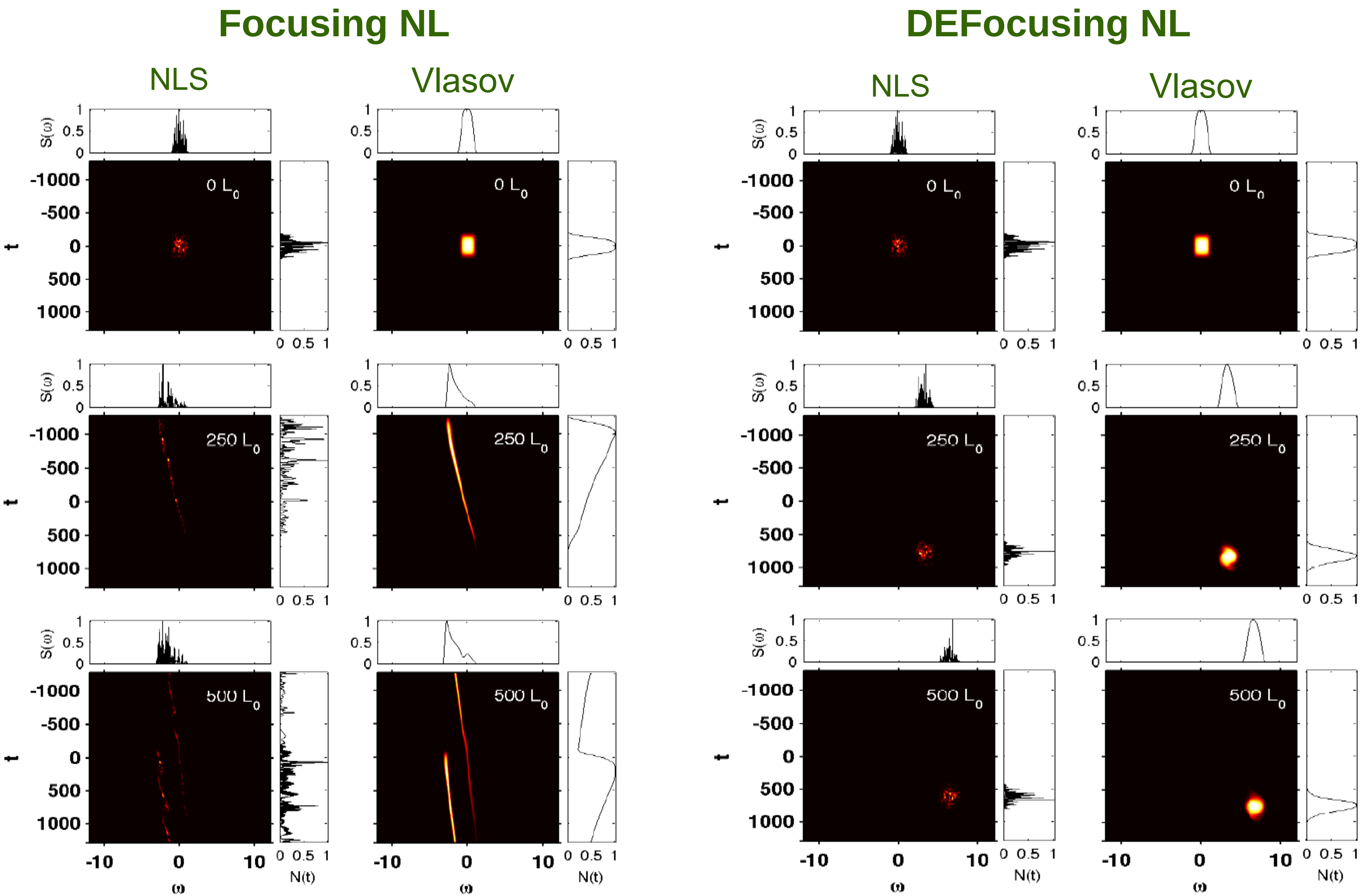
Kibler, Michel, Garnier, Picozzi, OL (2012)

‘Long-range’ incoherent modulational instability



Ok for the linear regime… → Nonlinear regime ?

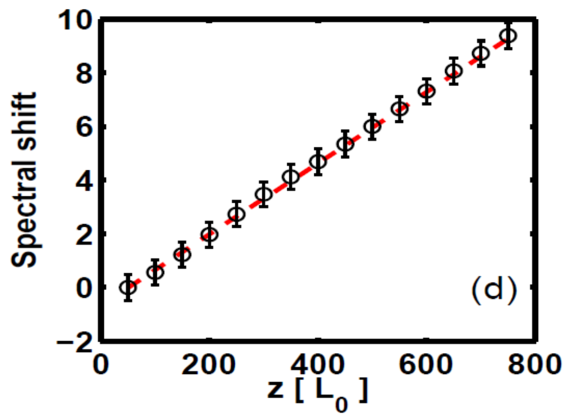
Incoherent solitons in the defocusing regime



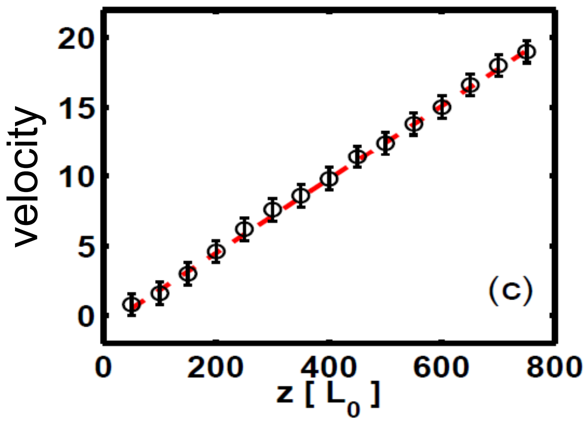
Michel, Kibler, Garnier, Picozzi PRA (2012)

Incoherent solitons in the defocusing regime

Causality → spectral shift



Dispersion → const. acceleration

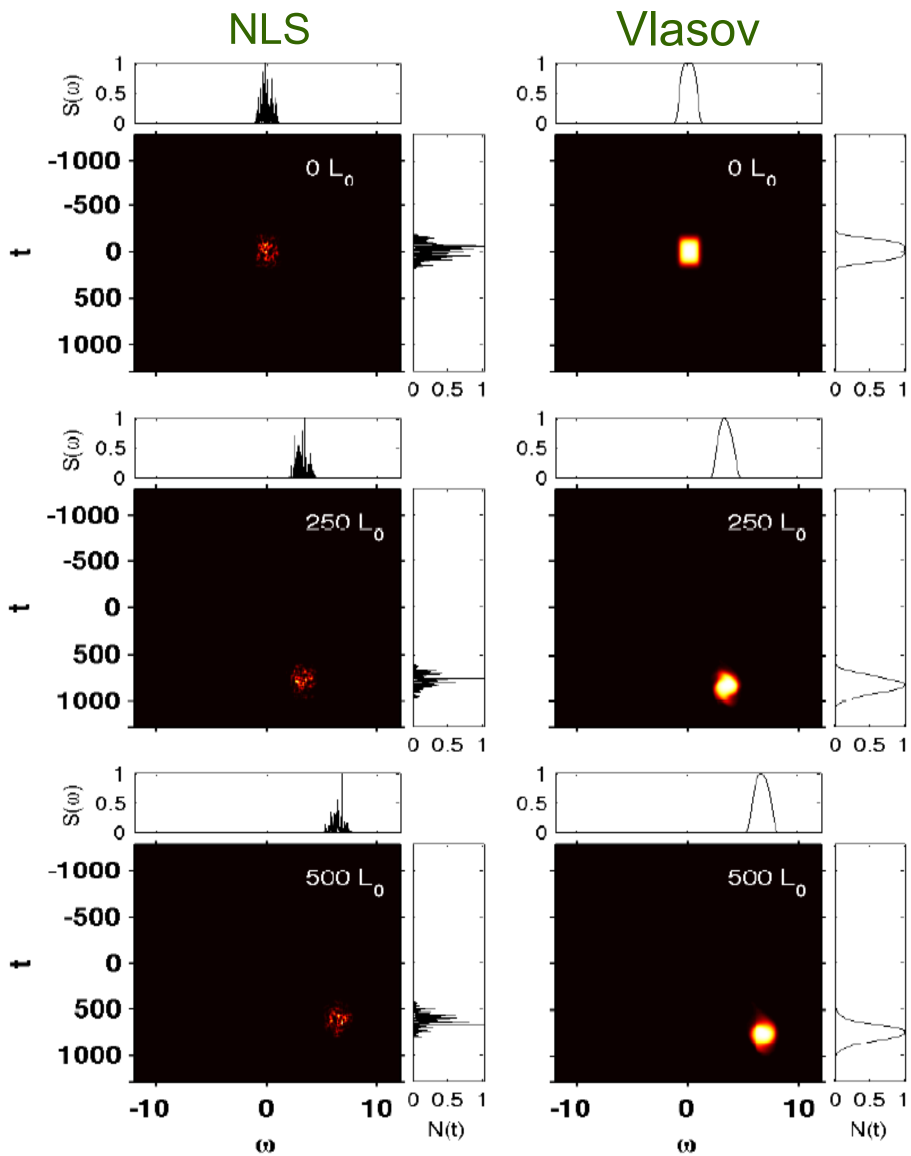


$$k(\omega) = \omega^2$$

$$\downarrow$$

$$\partial_\omega k(\omega) = 2\omega$$

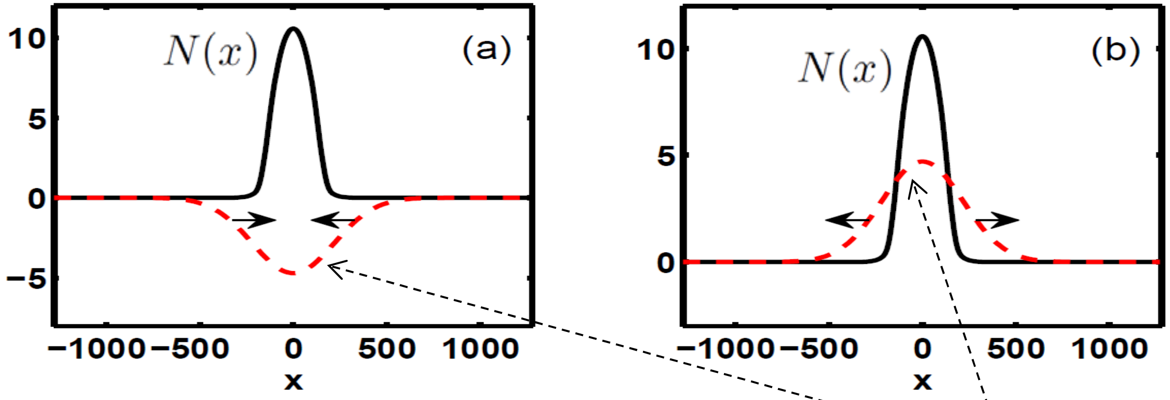
DEFocusing NL



Incoherent solitons in the defocusing regime

Spatial case:

Focusing **DEFocusing**



$$V(x) = -\gamma U * N$$

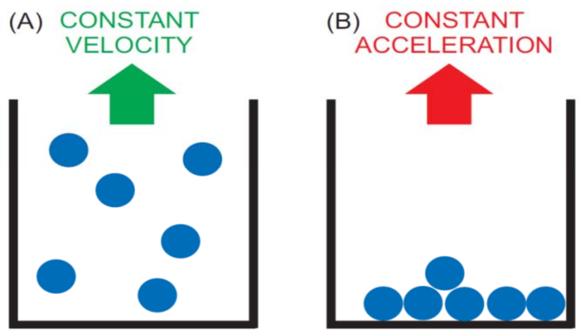
Temporal case ?

Non-inertial frame of reference: $\xi = z, \quad \tau = t - \alpha\beta z^2, \quad \Omega = \omega - \alpha z$

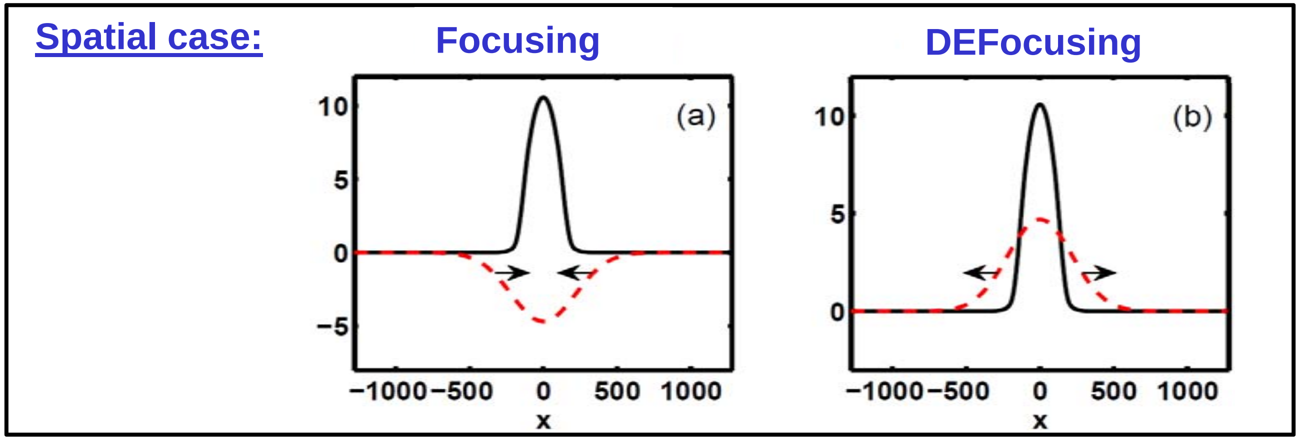
$$\partial_\xi n_\omega(t,z) + 2\beta\Omega\partial_\tau n_\Omega(\tau,\xi) - \partial_\tau V_{\text{eff}}(\tau,\xi)\,\partial_\Omega n_\Omega(\tau,\xi) = 0$$

$$\longrightarrow V_{\text{eff}}(\tau,\xi) = V(\tau,\xi) + \alpha\tau$$

Analogy with an elevator:



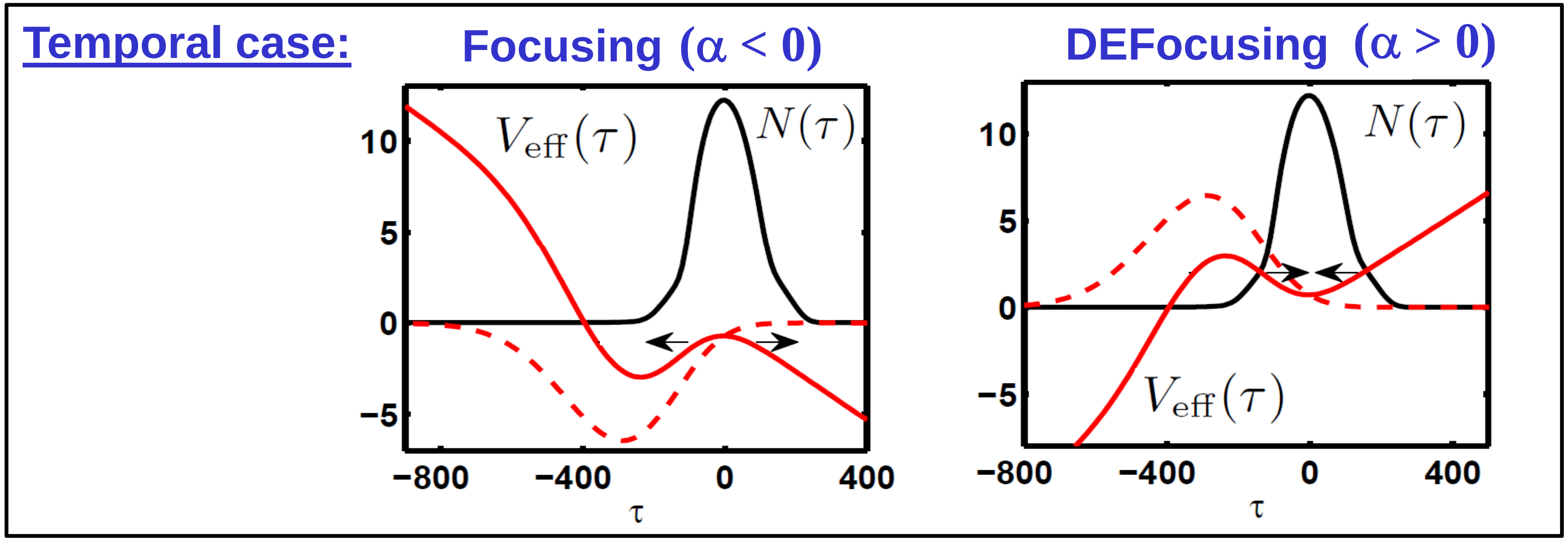
Incoherent solitons in the defocusing regime



Non-inertial frame of reference: $\xi = z, \quad \tau = t - \alpha\beta z^2, \quad \Omega = \omega - \alpha z$

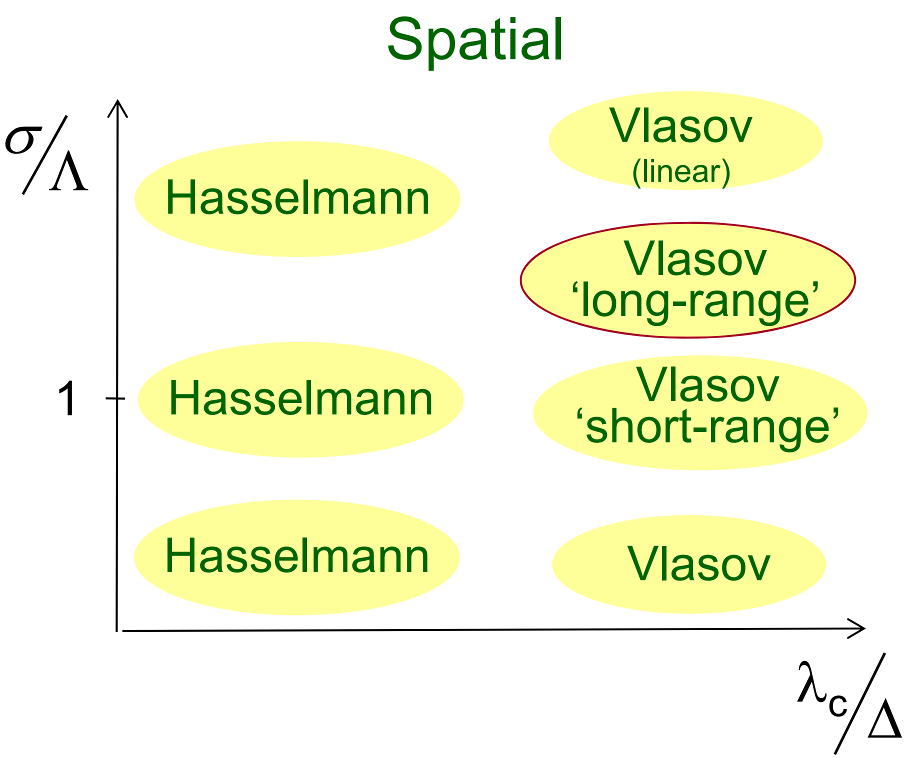
$$\partial_\xi n_\omega(t,z) + 2\beta\Omega\partial_\tau n_\Omega(\tau,\xi) - \partial_\tau V_{\text{eff}}(\tau,\xi)\,\partial_\Omega n_\Omega(\tau,\xi) = 0$$

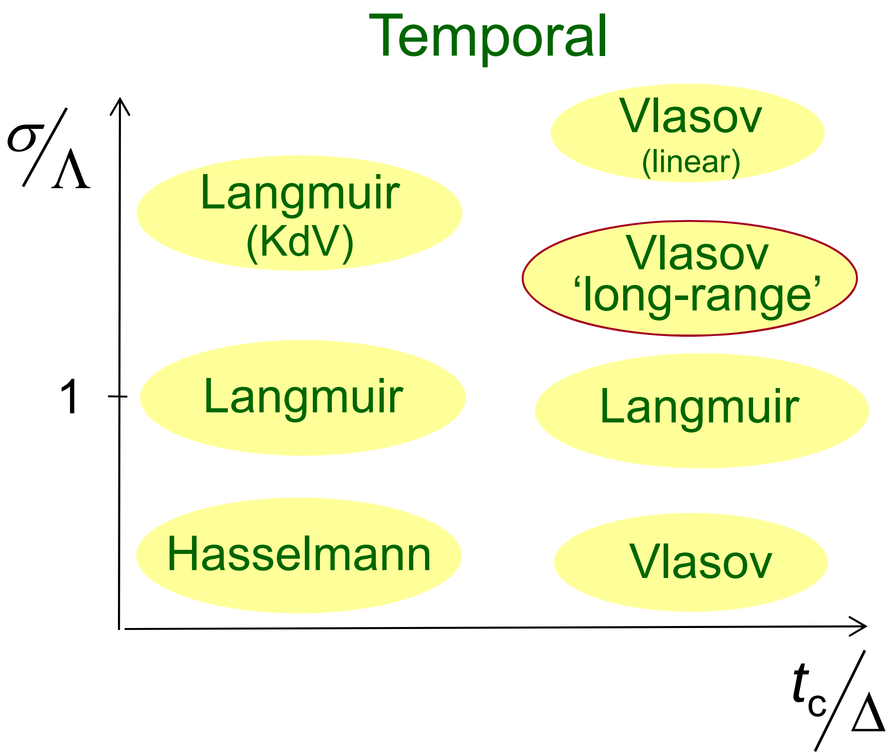
$$\longrightarrow V_{\text{eff}}(\tau,\xi) = V(\tau,\xi) + \alpha\tau \qquad V(\tau) = -\gamma R * N$$



Michel, Kibler, Garnier, Picozzi PRA (2012)

Summary





Review:

Toward a wave turbulence formulation of statistical nonlinear optics

Garnier, Lisak, Picozzi JOSA B (2012)

Extension to disordered systems…?

$$i\partial_t\psi = -\nabla^2\psi + V(\boldsymbol{r})\psi + |\psi|^2\psi \qquad \psi(\boldsymbol{r},t) = \sum_m c_m(t)\, u_m(\boldsymbol{r})\, \exp(-i\omega_m t)$$

$$\begin{cases} \partial_t n_{\boldsymbol{\kappa}}(t) = \dfrac{4\pi\gamma^2}{\omega_0^6}\displaystyle\iiint d\boldsymbol{\kappa}_1 d\boldsymbol{\kappa}_2 d\boldsymbol{\kappa}_3 \delta(\omega_{\boldsymbol{\kappa}_1} + \omega_{\boldsymbol{\kappa}_3} - \omega_{\boldsymbol{\kappa}_2} - \omega_{\boldsymbol{\kappa}}) \\ \qquad \times |\tilde{W}_{\boldsymbol{\kappa}\boldsymbol{\kappa}_1\boldsymbol{\kappa}_2\boldsymbol{\kappa}_3}|^2 n_{\boldsymbol{\kappa}} n_{\boldsymbol{\kappa}_1} n_{\boldsymbol{\kappa}_2} n_{\boldsymbol{\kappa}_3} \left(n_{\boldsymbol{\kappa}}^{-1} + n_{\boldsymbol{\kappa}_2}^{-1} - n_{\boldsymbol{\kappa}_1}^{-1} - n_{\boldsymbol{\kappa}_3}^{-1}\right) \\ \qquad + \dfrac{8\pi}{\omega_0^2}\displaystyle\int d\boldsymbol{\kappa}_1 \delta(\omega_{\boldsymbol{\kappa}_1} - \omega_{\boldsymbol{\kappa}}) |\tilde{U}_{\boldsymbol{\kappa}\boldsymbol{\kappa}_1}(\mathbf{n})|^2 (n_{\boldsymbol{\kappa}_1} - n_{\boldsymbol{\kappa}}) \\ \tilde{W}_{\boldsymbol{\kappa}\boldsymbol{\kappa}_1\boldsymbol{\kappa}_2\boldsymbol{\kappa}_3} = \displaystyle\int u_{\boldsymbol{\kappa}}^*(\boldsymbol{r}) u_{\boldsymbol{\kappa}_1}(\boldsymbol{r}) u_{\boldsymbol{\kappa}_2}^*(\boldsymbol{r}) u_{\boldsymbol{\kappa}_3}(\boldsymbol{r})\, d\boldsymbol{r} \\ \tilde{U}_{\boldsymbol{\kappa}\boldsymbol{\kappa}_1}(n) = \dfrac{1}{\omega_0^2}\displaystyle\int d\boldsymbol{\kappa}'\, \tilde{W}_{\boldsymbol{\kappa}\boldsymbol{\kappa}_1\boldsymbol{\kappa}'\boldsymbol{\kappa}'}\, n_{\boldsymbol{\kappa}'} \end{cases}$$

$\longrightarrow$ Disorder prevents thermalization?
Thermalization and condensation on Anderson modes?

$\longrightarrow$ **In collaboration with C. Conti (Rome) and C. Michel (Nice)**