The Abdus Salam International Centre for Theoretical Physics

2472-16

Advanced Workshop on Nonlinear Photonics, Disorder and Wave Turbulence

15 - 19 July 2013

Wave Turbulence and non gaussian statistics in 1D Nonlinear Schroedinger Equation

Pierre Suret

Laboratoire Phlam, Universite de Lille 1
France

Wave Turbulence and non gaussian statistics in 1D Nonlinear Schrödinger Equation

Pierre Suret, Pierre Walczak, Stéphane Randoux

Laboratoire Phlam, Université de Lille 1, France

Antonio Picozzi

laboratoire Carnot, Université de Bourgogne, Dijon, France

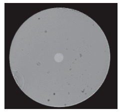
Advanced Workshop on Nonlinear Photonics, Disorder and Wave Turbulence
Trieste, Italy, 15-19 July

Incoherent nonlinear (fiber) optics

Lasers 1960s

Nonlinear optics

Optical fibers 1970s



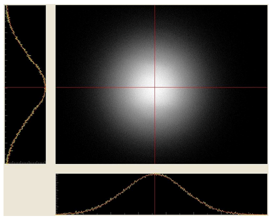
Telecommunications — linear operation

Nonlinear fiber optics — intense cw/pulsed coherent light waves

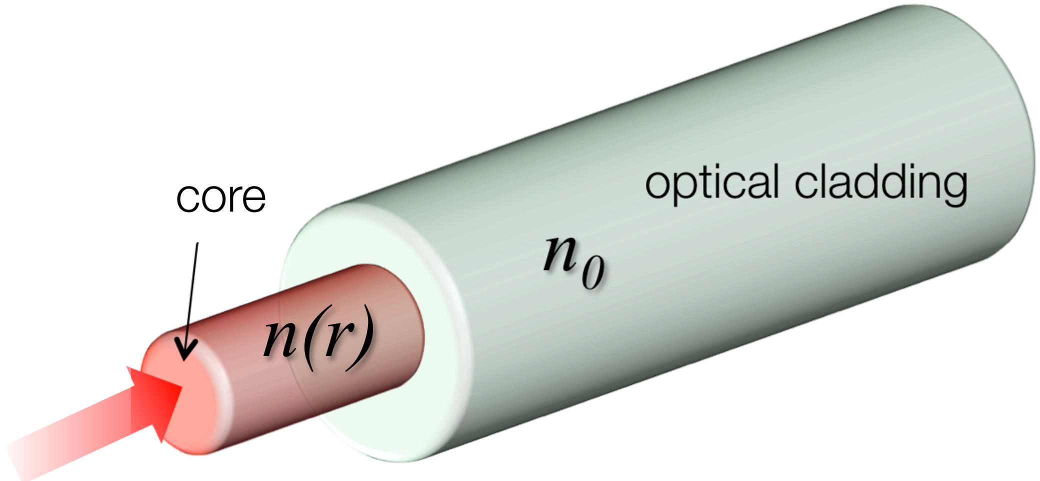
Photonic Crystal fibers 1995

supercontinuum

Single mode (1D) Optical fibers



$\psi(z,t)$



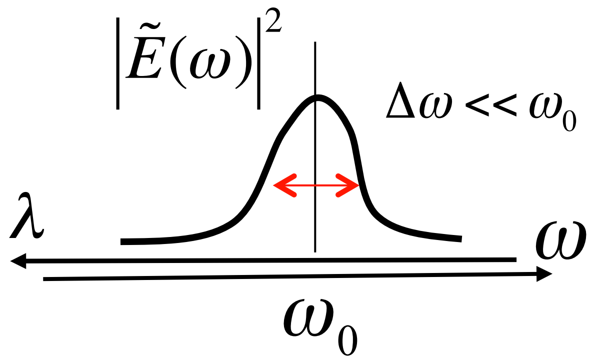
Optical variables …

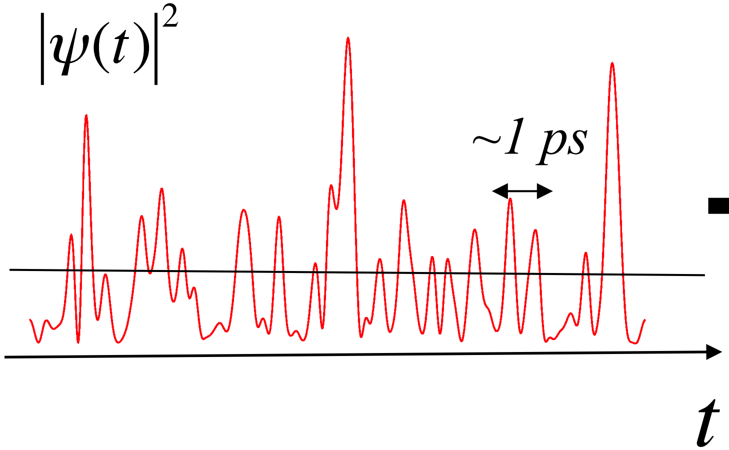
$$E(x,y,z,t) = A(x,y)\psi(z,t)e^{i(\omega_0 t - k_0 z)}$$

slow varying Amplitude

carrier wave

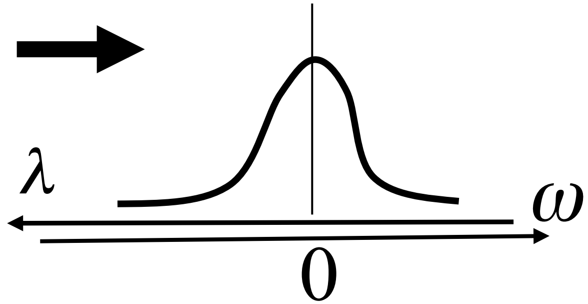
$$\omega_0 \approx 10^{15}\,Hz$$





Slow detector

$$\left\langle \left|\tilde{\psi}(\omega)\right|^2 \right\rangle = n_\omega$$



Optical spectrum = Fourier spectrum = Wave action

power spectral density

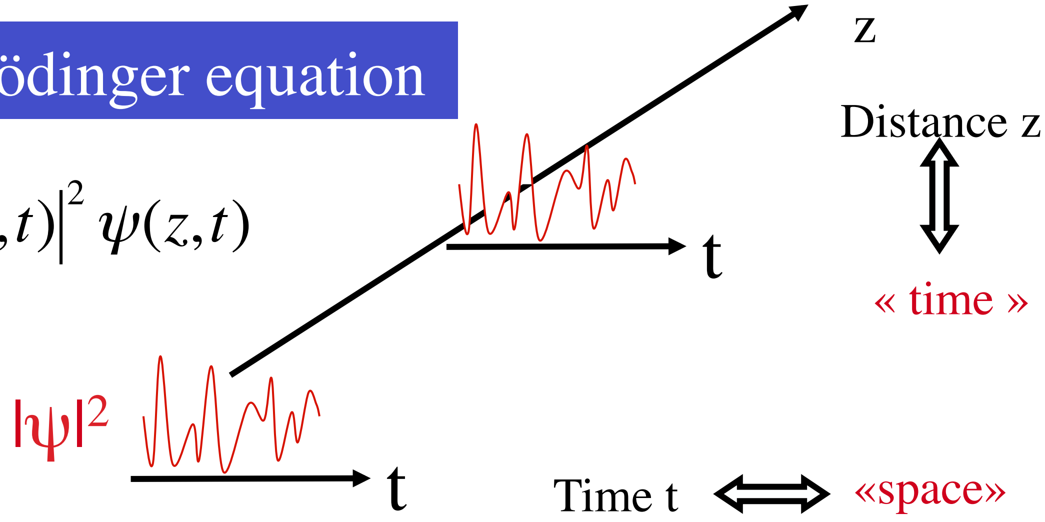
kinetic equation

Propagation of incoherent waves in optical fiber

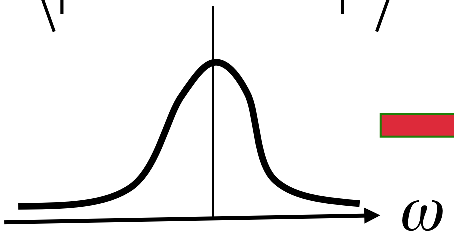
Defocusing 1D Nonlinear Schrödinger equation

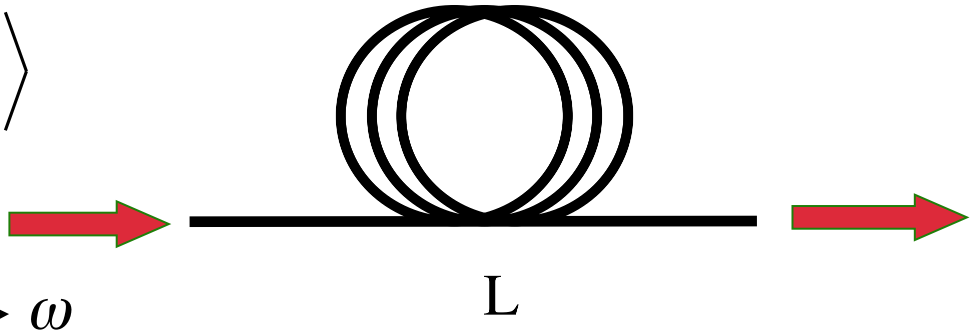
$$i\,\partial_z\,\psi(z,t) = -\,\partial_t^2\,\psi(z,t) + |\psi(z,t)|^2\,\psi(z,t)$$

Random initial conditions

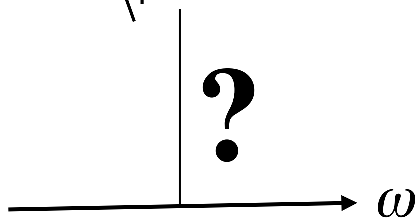


$$\left\langle |\tilde{\psi}(\omega, z=0)|^2 \right\rangle$$





$$\left\langle |\tilde{\psi}(\omega, z=L)|^2 \right\rangle$$



PDF

PDF

No phase-matching conditions in 1D NLS

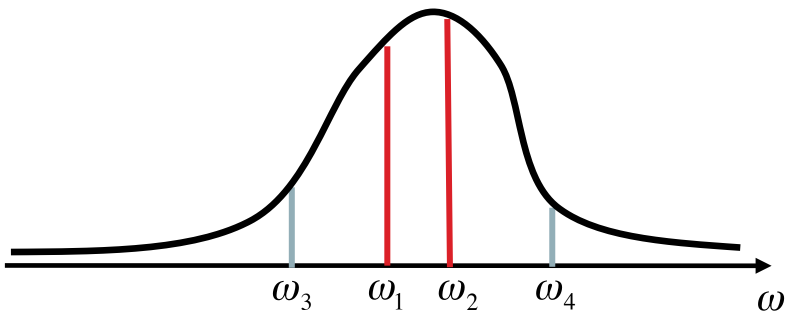
$$i\,\partial_z\,\psi(z,t) = -\,\sigma\,\partial_t^2\,\psi(z,t) + \left|\psi(z,t)\right|^2\psi(z,t)$$

$$\boxed{\omega_1 + \omega_2 = \omega_3 + \omega_4}$$

10^2-10^6 modes

$$\Delta k = k(\omega_1) + k(\omega_2) - k(\omega_3) - k(\omega_4)$$

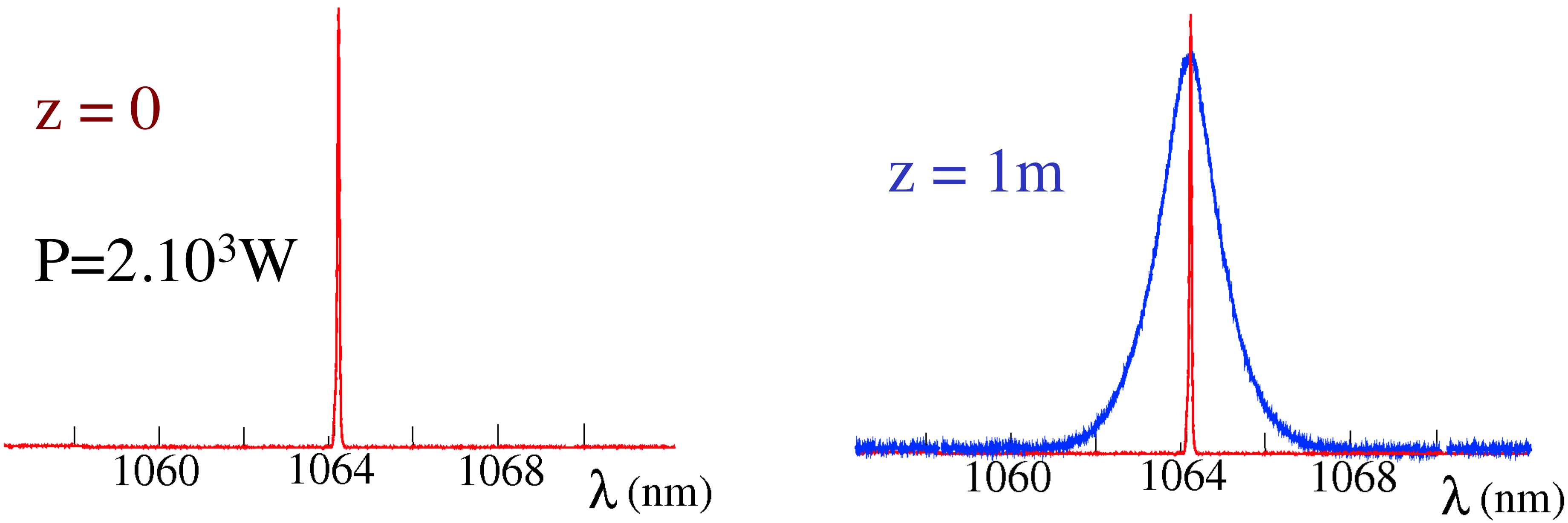
$$k(\omega) = \sigma\omega^2$$



$$\Delta k = 0 \Rightarrow \omega_1 = \omega_3$$

Experiments in defocusing case / strongly nonlinear regime

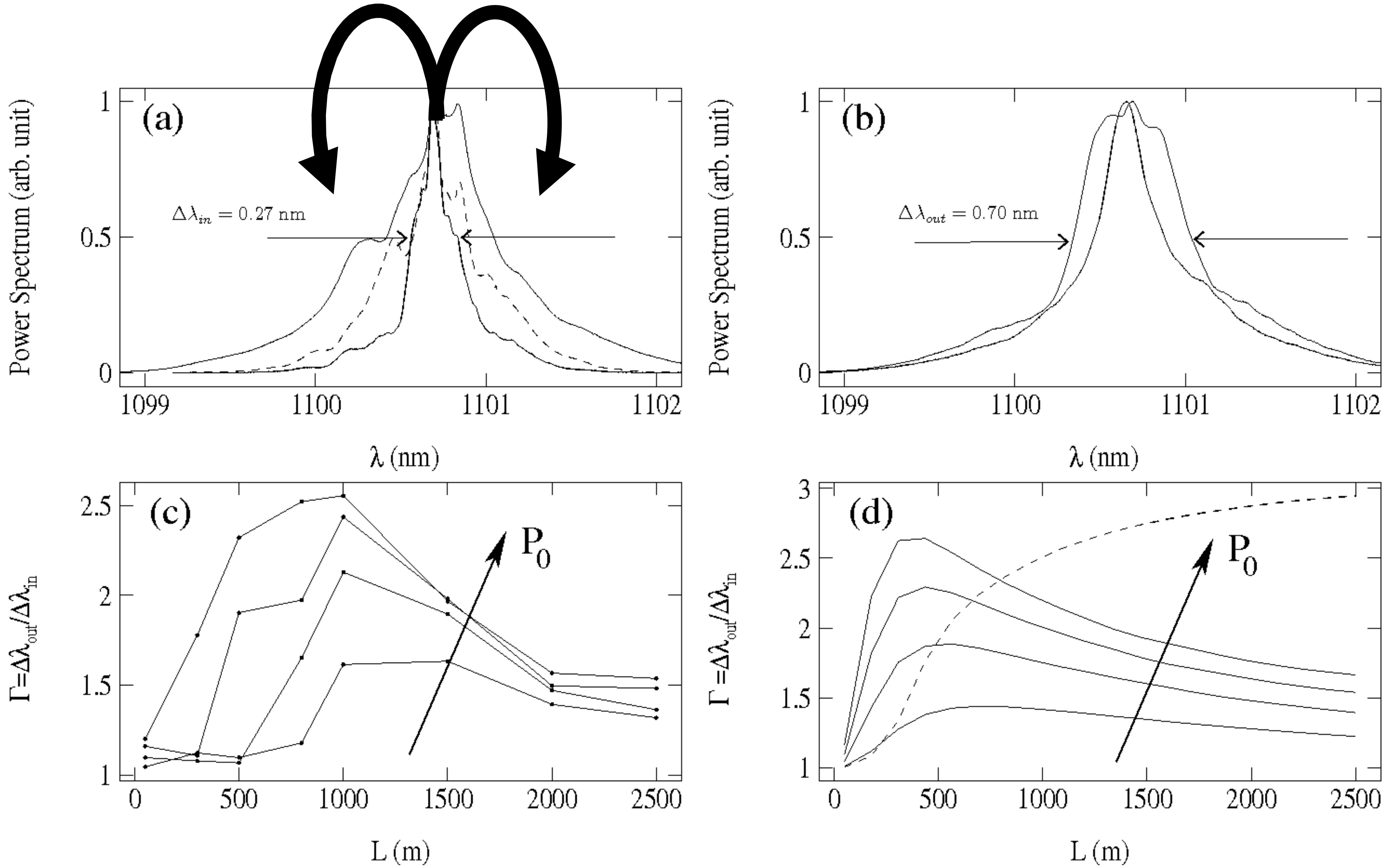
$$\boxed{\frac{H_L}{H_{NL}} \approx 10^{-5}}$$



Experiments and numerical simulations (1D NLS)

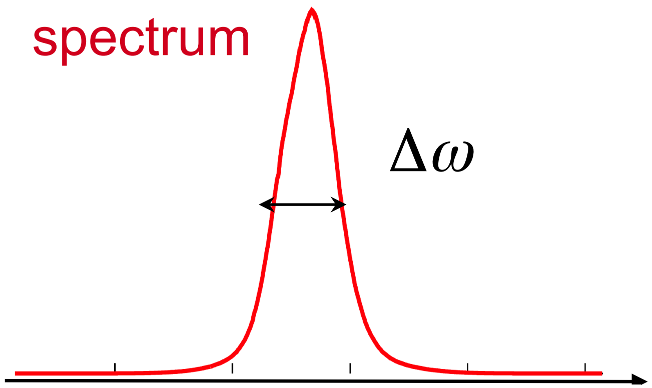
Irreversible evolution toward a steady state

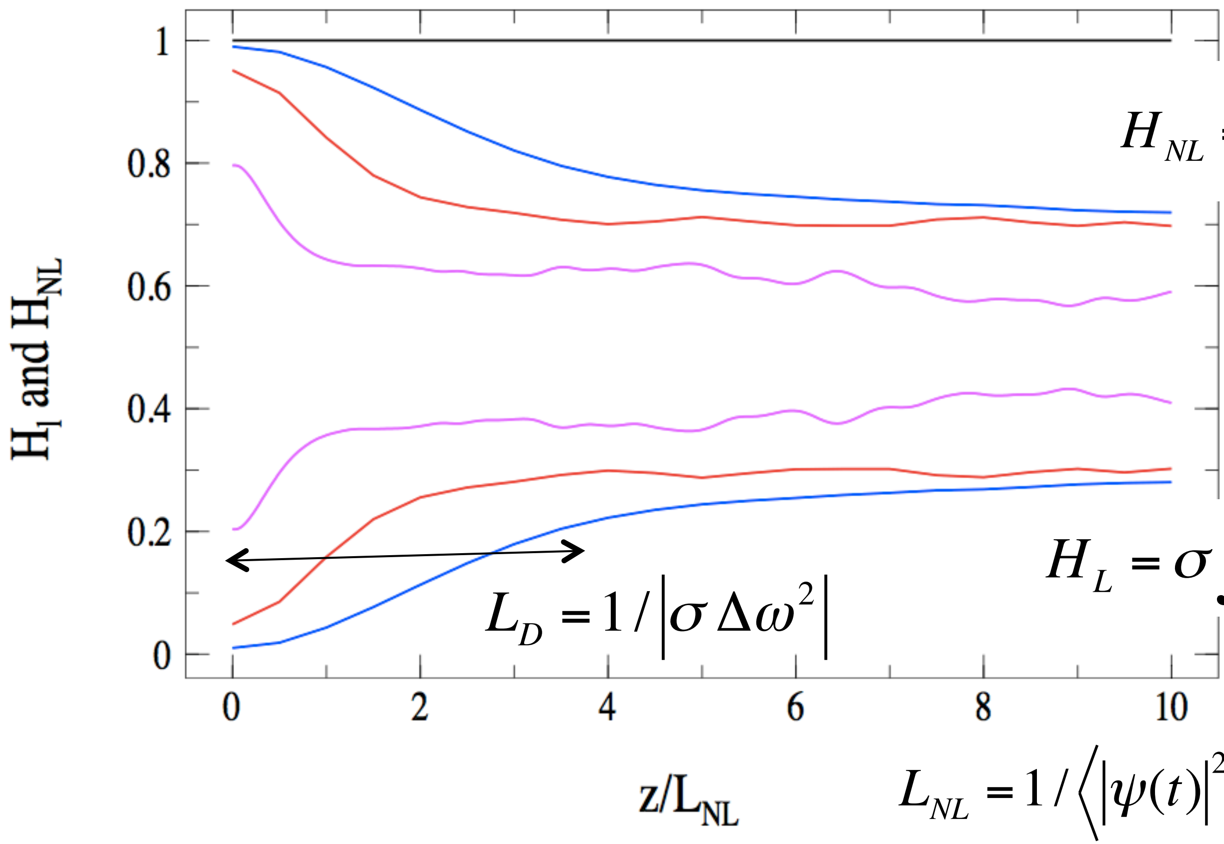
defocusing case : no BF



B. Barviau, S. Randoux, and P. Suret , Optics Letters, **31**, pp. 1696-1698 (2006)

$$i\,\partial_z\psi(z,t) = -\sigma\,\partial_t^2\psi(z,t) + |\psi(z,t)|^2\,\psi(z,t)$$





$$H_{NL} = \frac{1}{2}\int |\psi(t)|^4\,dt$$

$$H_L = \sigma\int \omega^2\,|\tilde{\psi}(\omega)|^2\,d\omega$$

$$L_{NL} = 1/\left\langle |\psi(t)|^2 \right\rangle$$

« Wave turbulence » in 1D NLS : non resonant interactions

1D NLS $$i\,\partial_z \psi(z,t) = -\sigma\,\partial_t^2 \psi(z,t) + |\psi(z,t)|^2\,\psi(z,t)$$

$\sigma = \pm 1$

$$\left\langle \tilde{\psi}(z,\omega)\,\tilde{\psi}^*(z,\omega')\right\rangle = n_\omega(z)\,\delta(\omega-\omega')$$

$$\left\langle \tilde{\psi}(z,\omega_1)\,\tilde{\psi}(z,\omega_2)\,\tilde{\psi}^*(z,\omega_3)\,\tilde{\psi}^*(z,\omega_4)\right\rangle = J_{1,2}^{3,4}(z)\,\delta(\omega_1+\omega_2-\omega_3-\omega_4)$$

$$\left\langle \tilde{\psi}(\omega_1)\,\tilde{\psi}(\omega_2)\,\ldots\,\tilde{\psi}^*(\omega_m)\,\tilde{\psi}^*(\omega_n)\right\rangle = \mathcal{F}\left[n_1, n_2, \ldots n_m n_n\right]$$

$$\frac{\partial n_{\omega_1}(z)}{\partial z} = \frac{1}{\pi}\int\!\!\int\!\!\int d\omega_{2-4}\, Im\left[J_{1,2}^{3,4}(z)\right]\delta(\omega_1+\omega_2-\omega_3-\omega_4),$$

$$\frac{\partial J_{1,2}^{3,4}(z)}{\partial z} - i\,\Delta k\; J_{1,2}^{3,4}(z) = \frac{i}{\pi}\mathcal{N}(z)$$

$$\mathcal{N}(z) = n_{\omega_1}(z)n_{\omega_2}(z)n_{\omega_3}(z)n_{\omega_4}(z)\left[\frac{1}{n_{\omega_2}(z)} + \frac{1}{n_{\omega_1}(z)} - \frac{1}{n_{\omega_4}(z)} - \frac{1}{n_{\omega_3}(z)}\right]$$

« Wave turbulence » in 1D NLS

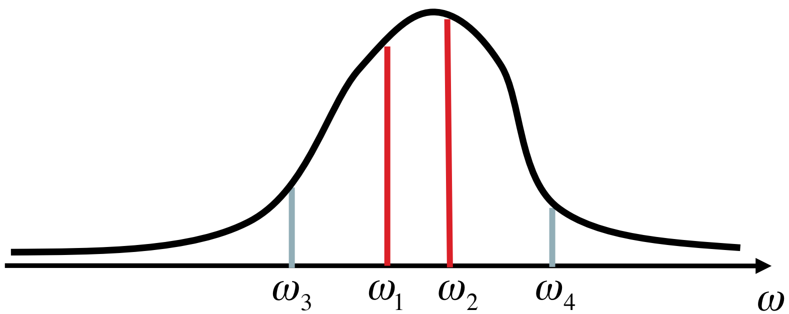
$$\boxed{z >> \frac{1}{\Delta k}}$$

$$\frac{\partial n_{\omega_1}(z)}{\partial z} = \frac{1}{\pi}\int\int\int d\omega_{2-4}\mathcal{N}(z)\delta(\omega_1+\omega_2-\omega_3-\omega_4)\delta(\Delta k)$$

$$\boxed{\omega_1+\omega_2=\omega_3+\omega_4}$$

$$\Delta k = k(\omega_1)+k(\omega_2)-k(\omega_3)-k(\omega_4)$$

$$k(\omega)=\sigma\omega^2$$



$$\Delta k = 0 \Rightarrow \omega_1 = \omega_3$$

$$\boxed{\partial_z\, n(\omega,z)=0}$$

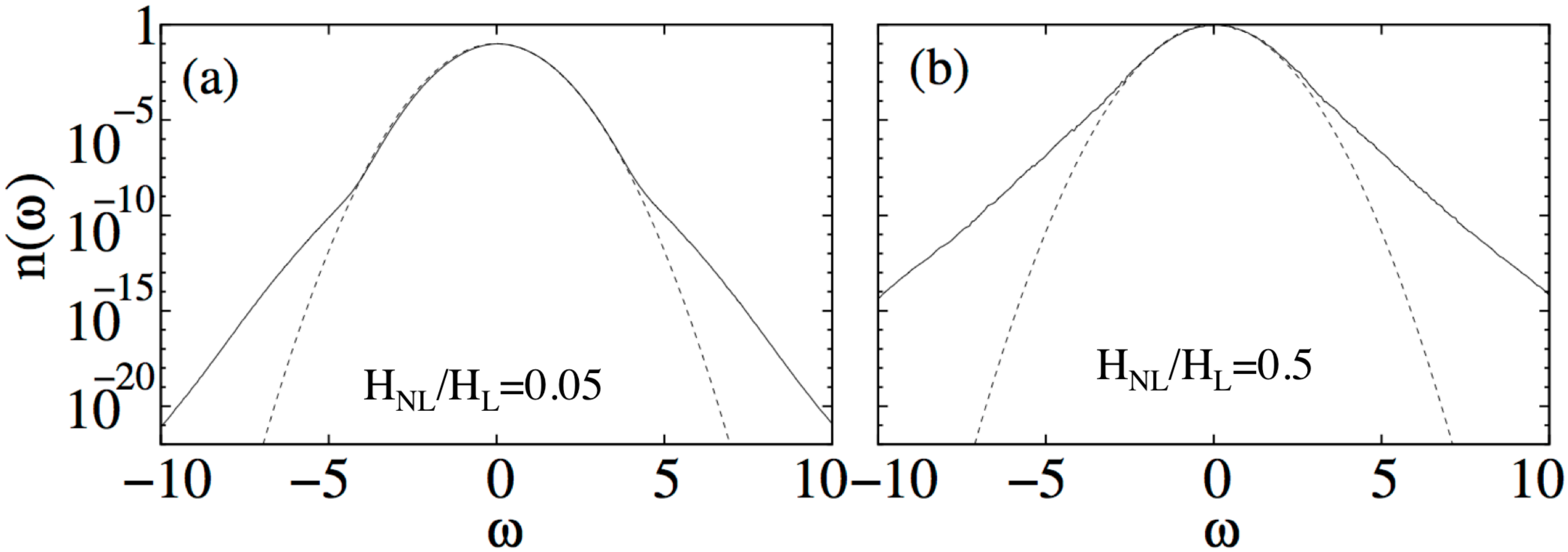
But in optical fiber experiments : short « time » are crucial !

Sergei Yu. Annenkov and Victor Shrira, Journal of Fluid Mechanics, **449**, p. 341, (2001)
S. Annenkov and V. Shrira, Doklady Physics, **49**, 6, p. 389 (2004)
P. Janssen, J. Phys. Oceanography, **33**, 864, 2002
D.B.S. Soh *et al.,* Opt. Express **18**, 22393-22405 (2010)
P. Suret *et al.*, Opt. Express **19**, 17852-17863 (2011)

Numerical simulations : NLS

$$i\,\partial_z \psi(z,t) = -\sigma\,\partial_t^2 \psi(z,t) + |\psi(z,t)|^2\,\psi(z,t)$$

$$\boxed{n_\omega^0 = n_\omega(z=0) = n_0\,\exp\left[-\left(\frac{\omega}{\Delta\omega}\right)^2\right]}$$



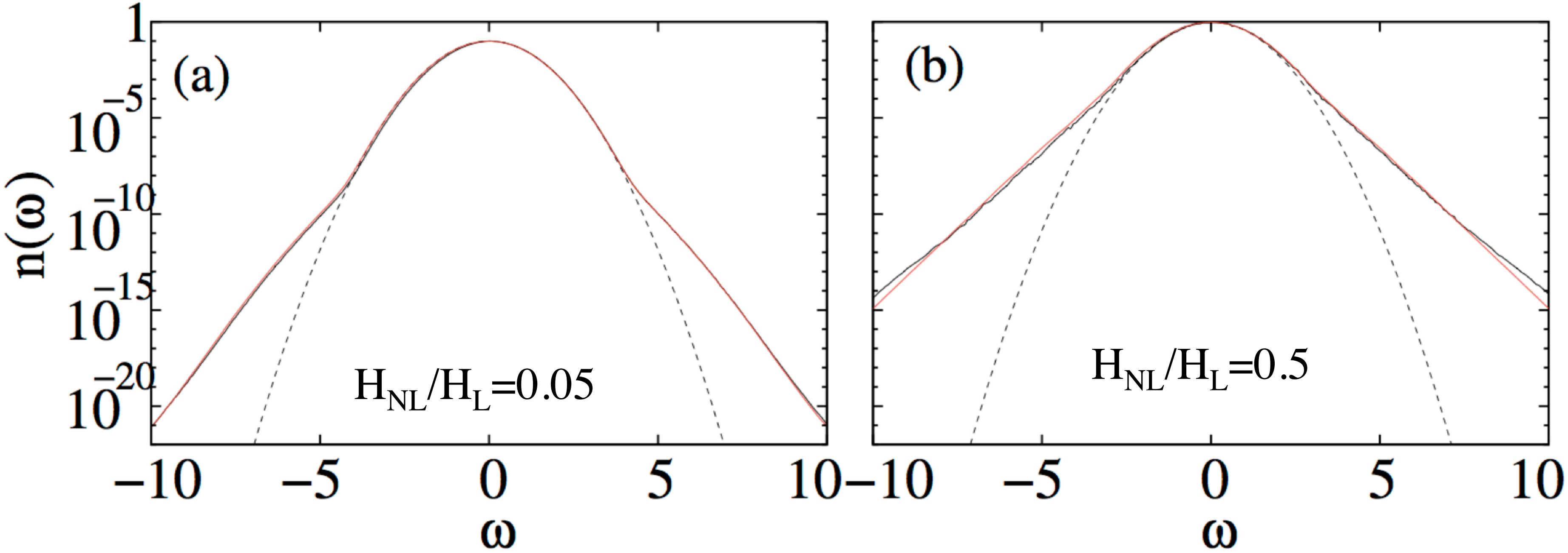
Numerical simulations : NLS / kinetic equations

$$\frac{\partial n_{\omega_1}(z)}{\partial z} = \frac{1}{\pi}\int\int\int d\omega_{2-4}\, Im\left[J^{3,4}_{1,2}(z)\right]\delta(\omega_1+\omega_2-\omega_3-\omega_4),$$

$$\frac{\partial J^{3,4}_{1,2}(z)}{\partial z} - i\,\Delta k\; J^{3,4}_{1,2}(z) = \frac{i}{\pi}\mathcal{N}(z)$$

$$\boxed{n^0_\omega = n_\omega(z=0) = n_0\,\exp\left[-\left(\frac{\omega}{\Delta\omega}\right)^2\right]}$$

$$\mathcal{N}(z) = n_{\omega_1}(z)n_{\omega_2}(z)n_{\omega_3}(z)n_{\omega_4}(z)\left[\frac{1}{n_{\omega_2}(z)}+\frac{1}{n_{\omega_1}(z)}-\frac{1}{n_{\omega_4}(z)}-\frac{1}{n_{\omega_3}(z)}\right]$$



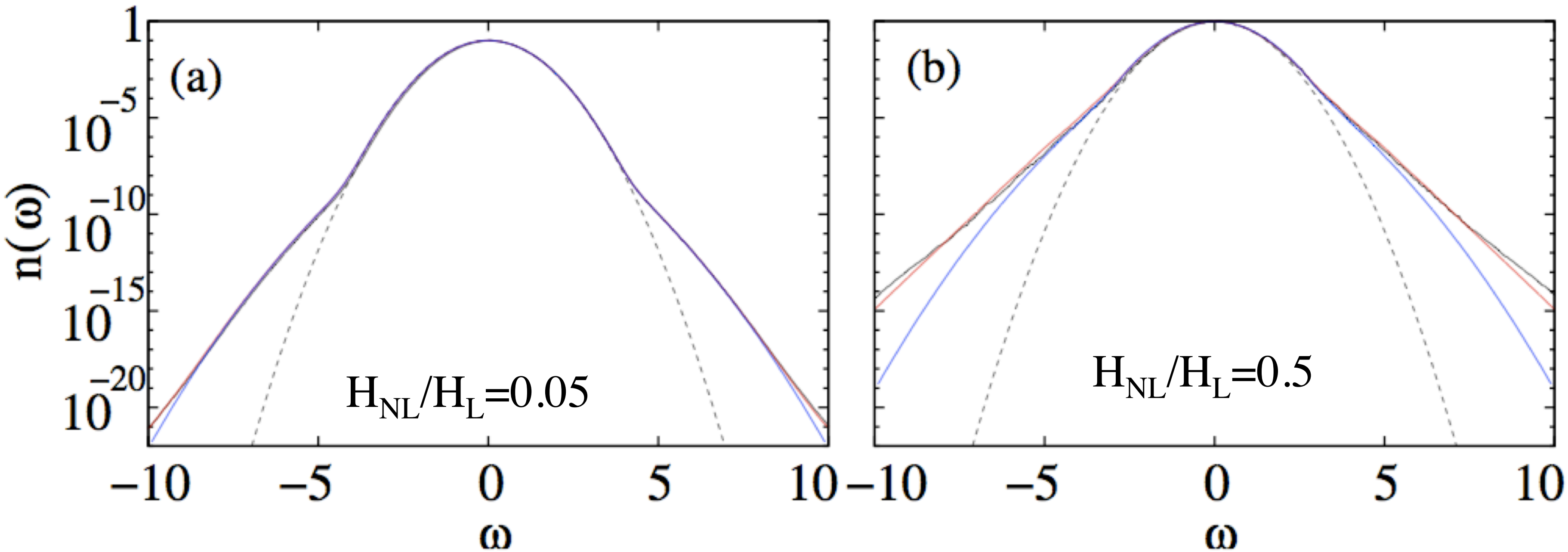
Numerical simulations : a strong approximation

$$\frac{\partial n_{\omega_1}(z)}{\partial z} = \frac{1}{\pi}\int\int\int d\omega_{2-4}\, Im\left[J_{1,2}^{3,4}(z)\right]\delta(\omega_1+\omega_2-\omega_3-\omega_4),$$

$$\frac{\partial J_{1,2}^{3,4}(z)}{\partial z} - i\,\Delta k\; J_{1,2}^{3,4}(z) = \frac{i}{\pi}\boxed{N(z=0)}$$

$$n_\omega^0 = n_\omega(z=0) = n_0 \exp\left[-\left(\frac{\omega}{\Delta\omega}\right)^2\right]$$

$$\mathcal{N}(z) = n_{\omega_1}(z)n_{\omega_2}(z)n_{\omega_3}(z)n_{\omega_4}(z)\left[\frac{1}{n_{\omega_2}(z)}+\frac{1}{n_{\omega_1}(z)}-\frac{1}{n_{\omega_4}(z)}-\frac{1}{n_{\omega_3}(z)}\right]$$



Good approximation $\mathcal{N}(z) = \mathcal{N}(z=0)$!!

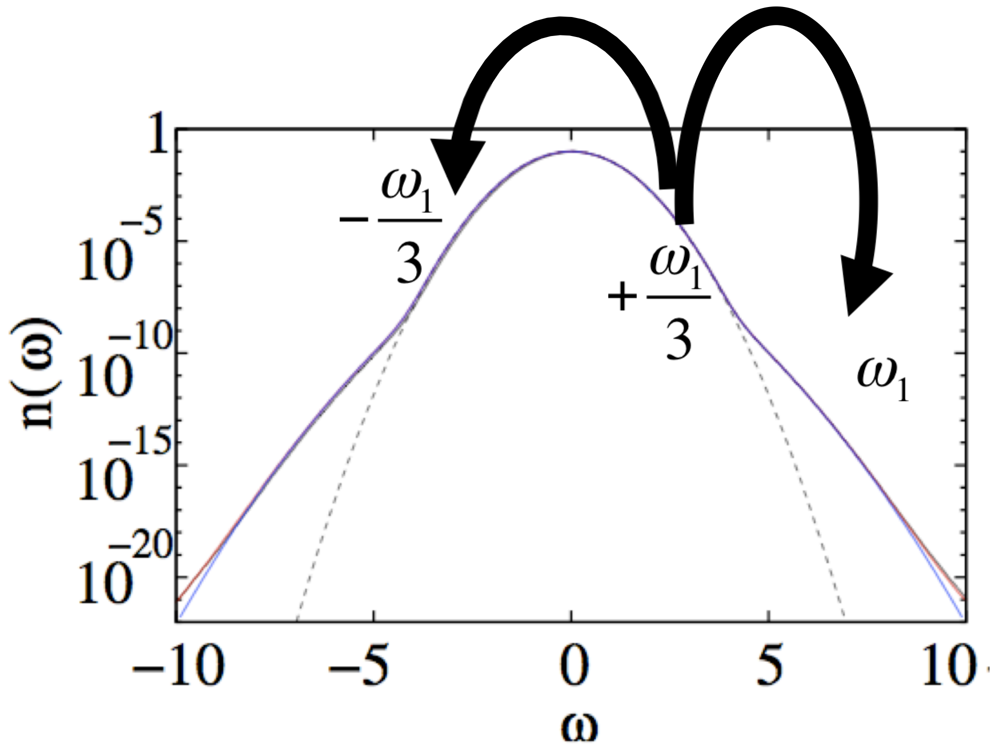
$$\frac{\partial n_{\omega_1}(z)}{\partial z} \simeq \frac{1}{\pi^2} \int\int d\omega_{3-4}\, n^0_{\omega_3}\, n^0_{\omega_4}\, n^0_{\omega_3+\omega_4-\omega_1} \frac{\sin(\Delta k\, z)}{\Delta k}$$

$$\boxed{n^0_{\omega_3} n^0_{\omega_4} n^0_{\omega_3+\omega_4-\omega_1} = e^{-(\omega_3)^{2n}} e^{-(\omega_4)^{2n}} e^{-(\omega_3+\omega_4-\omega_1)^{2n}}}$$

Dominant contribution

$$\boxed{\omega_3 \simeq \omega_4 \simeq \frac{\omega_1}{3},\ \omega_2 \simeq -\frac{\omega_1}{3}}$$

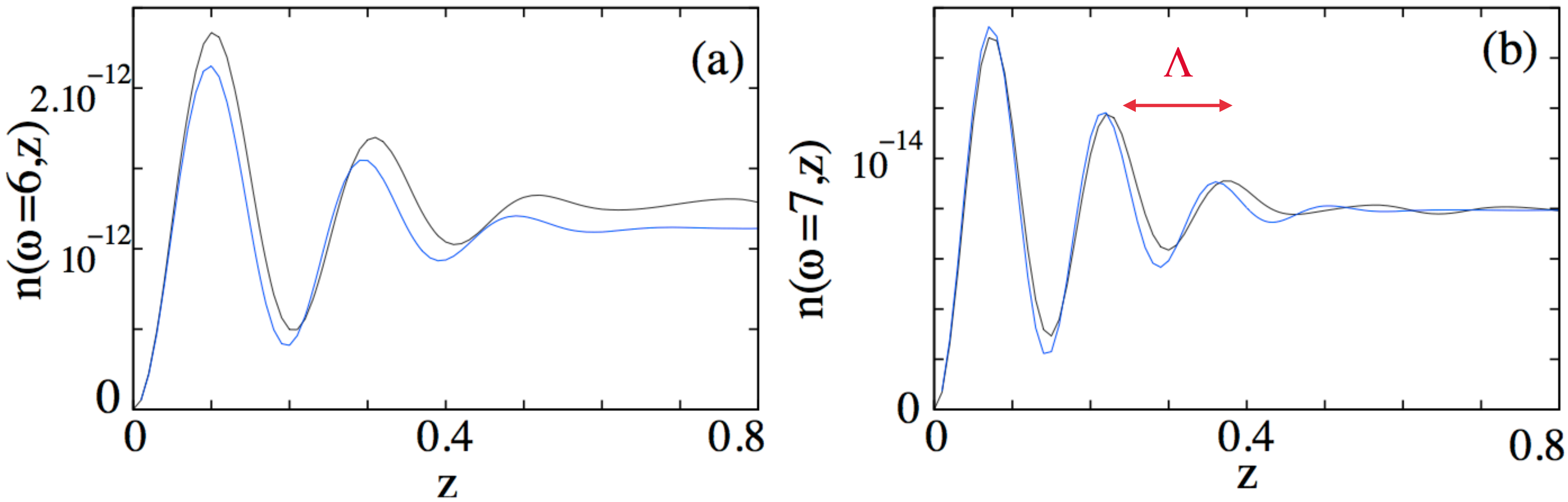
$$\boxed{\overline{\Delta k} = \frac{8}{9}\sigma\omega_1^2}$$



Damped oscillations toward steady state

$$\frac{\partial n_{\omega_1}(z)}{\partial z} \simeq \frac{n_0^3}{\sqrt{3}\pi}\frac{9}{8\omega_1^2}\exp\left(\frac{-\omega_1^2}{3}\left(1+\frac{8\,z^2}{9}\right)\right)\sin\left(\frac{8\,\omega_1^2\,z}{9}\right)$$

comparison with numerical integration 1D NLS

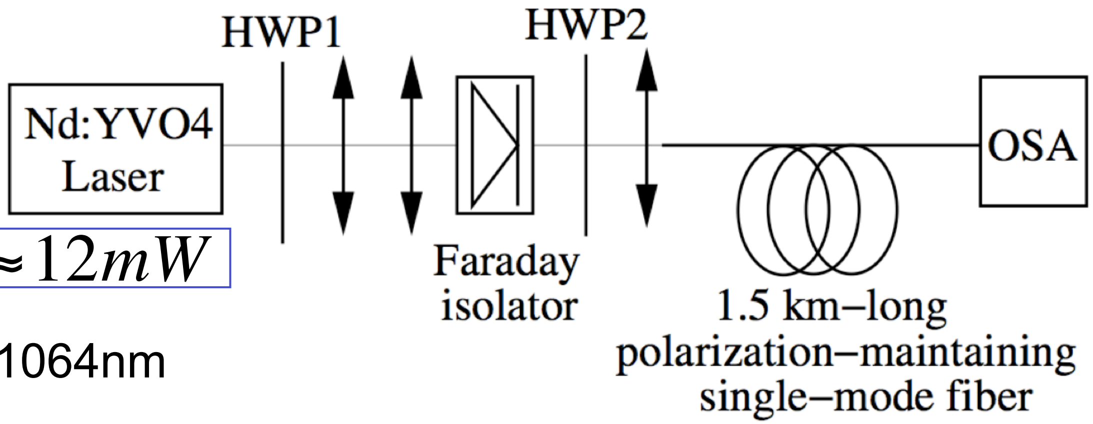


The period of oscillations

is given by the dominant contribution

$$\Lambda = \frac{2\pi}{\overline{\Delta k}} \approx \frac{2\pi}{\frac{8}{9}\omega_1^2}$$

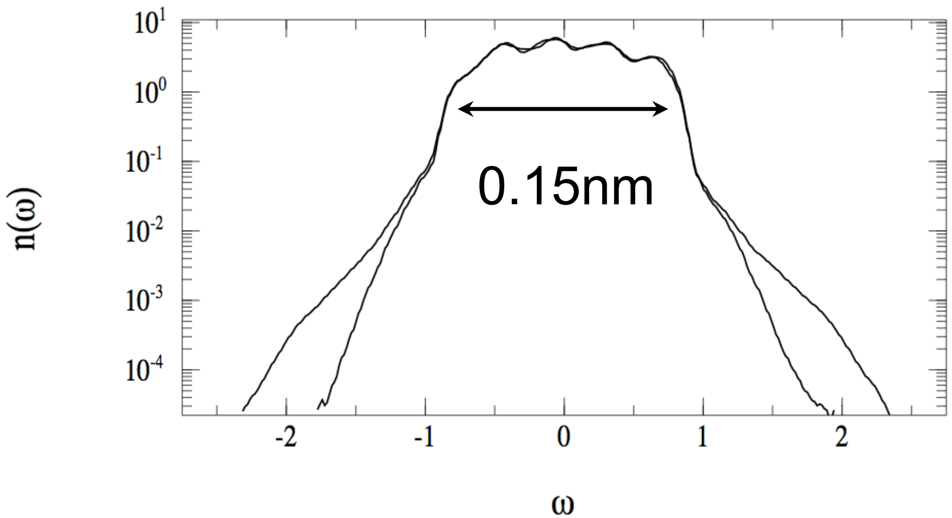
Experiments



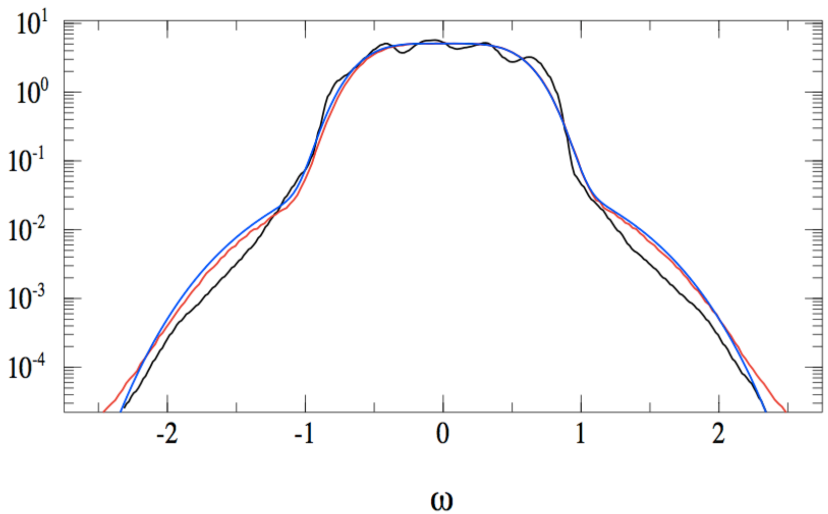
$\boxed{P \approx 12mW}$

λ=1064nm

Experiments



Experiments / 1DNLS / kinetic eq.



P. Suret *et al.*, Opt. Express **19**, 17852-17863 (2011)

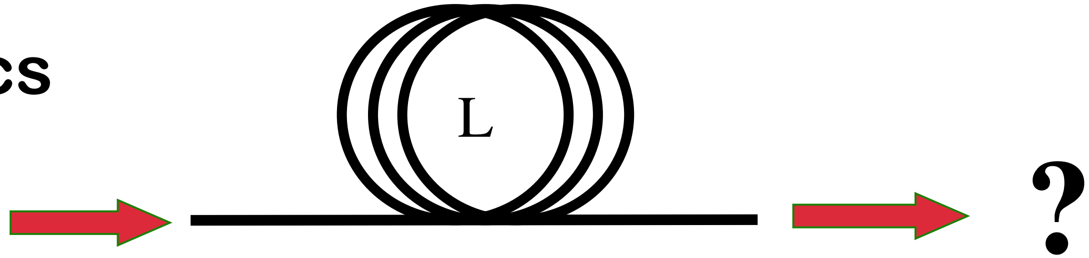
Evolution of the PDF (1D defocusing NLS)

$$i\,\partial_z\,\psi(z,t) = -\,\partial_t^2\,\psi(z,t) + |\psi(z,t)|^2\,\psi(z,t)$$

Gaussian Statistics

$$\mathrm{Re}\left[\psi(z=0,t)\right]$$

$$\mathrm{Im}\left[\psi(z=0,t)\right]$$

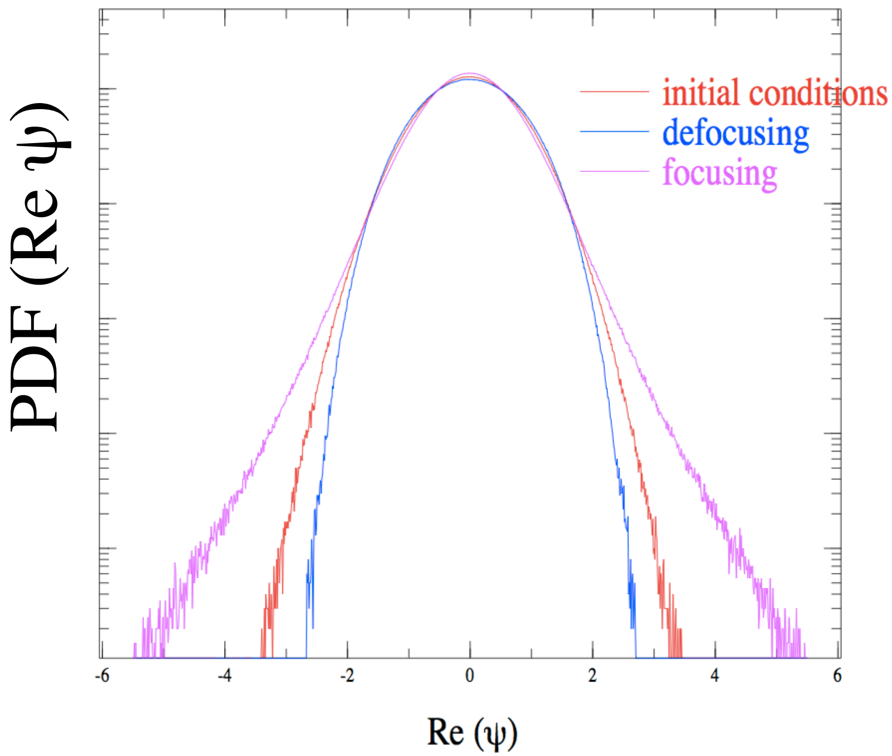


- Hydrodynamics

P. Janssen, J. Phys. Oceanogrphy 2003
N. Mori, J. Phys. Oceanogrphy 2011

- Optics (spatial)

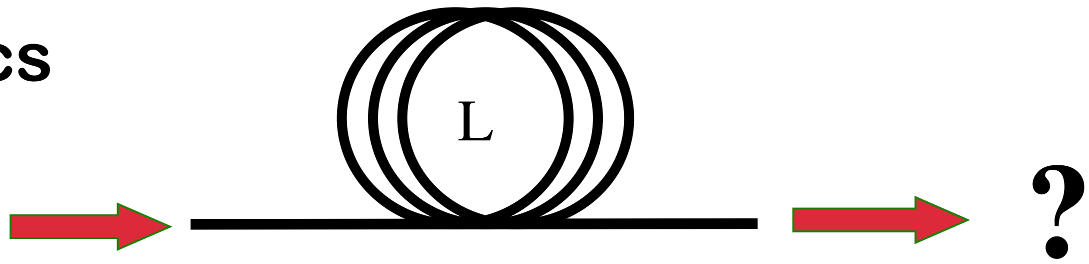
Y. Bromberg et al., Nature Photonics, 2010



Experiments in single mode fibers with narrow spectrum

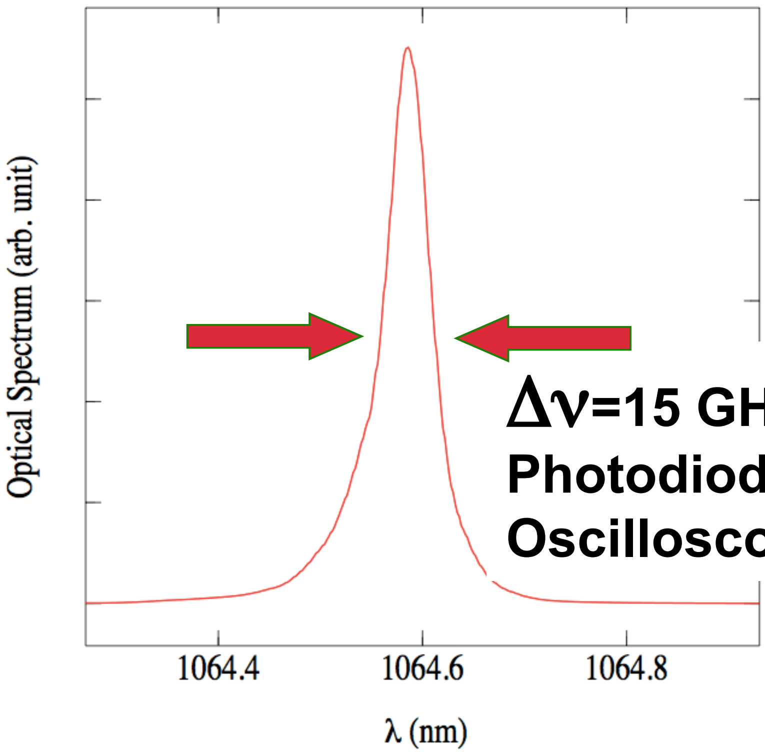
Gaussian Statistics

$$\mathrm{Re}\left[\psi(z=0,t)\right]$$
$$\mathrm{Im}\left[\psi(z=0,t)\right]$$

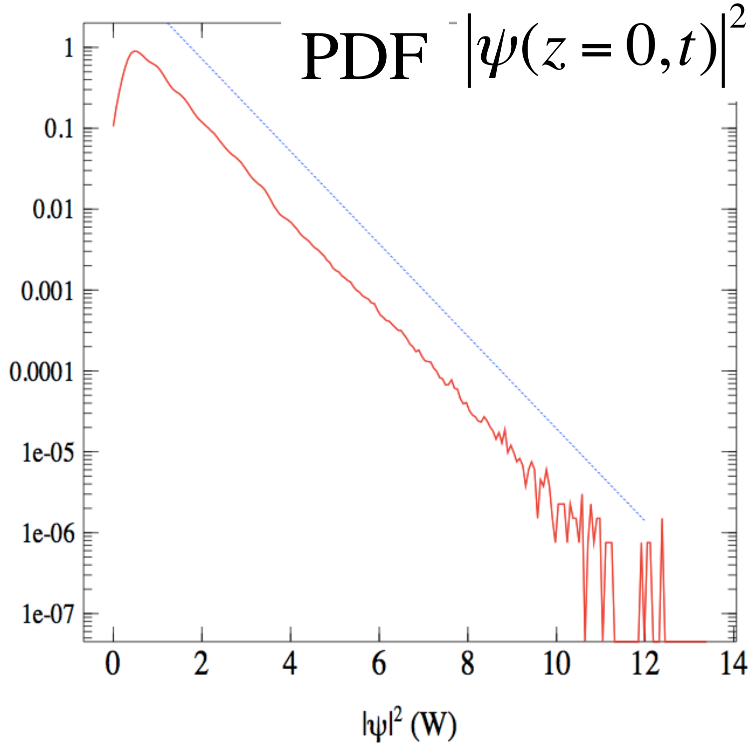


Exponential Statistics

PDF $\left|\psi(z=0,t)\right|^2$



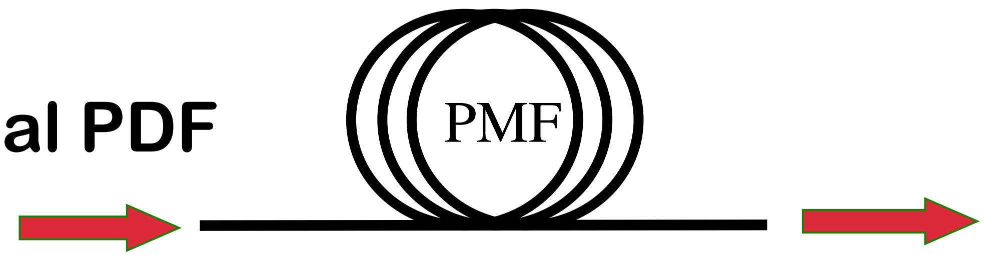
Δν=15 GHz
Photodiode 12 GHz
Oscilloscope 36 GHz



P. Suret, P. Walczak and S. Randoux, to be submitted to Phys. Rev.

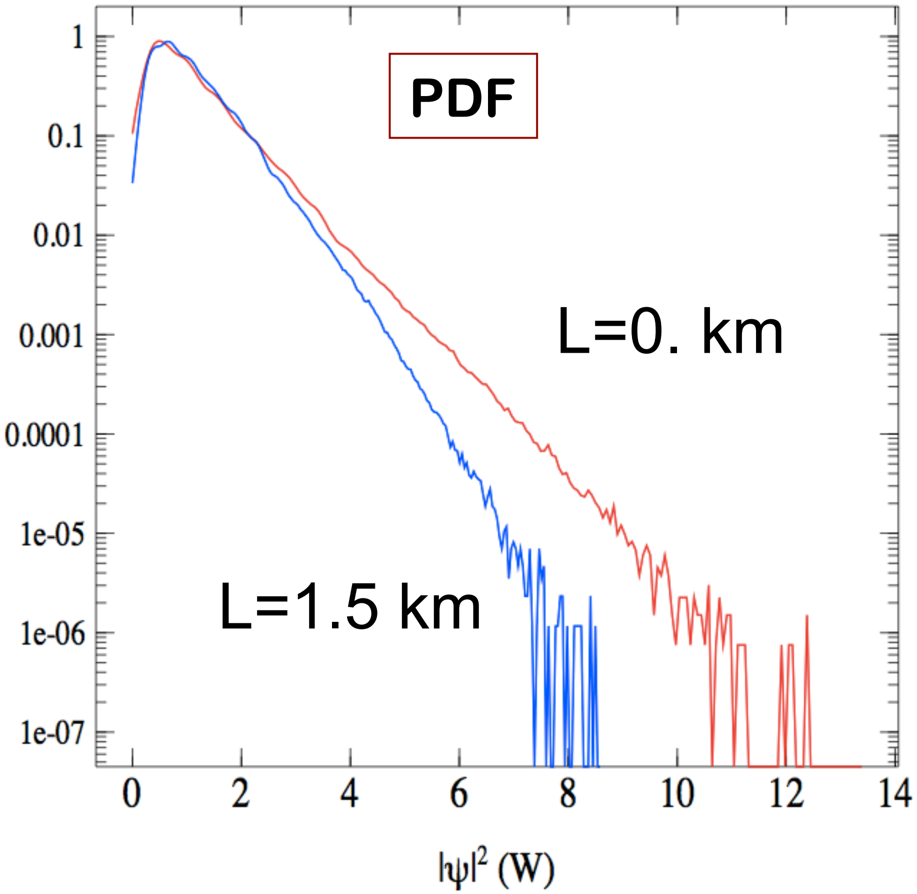
Experiments in single mode fibers

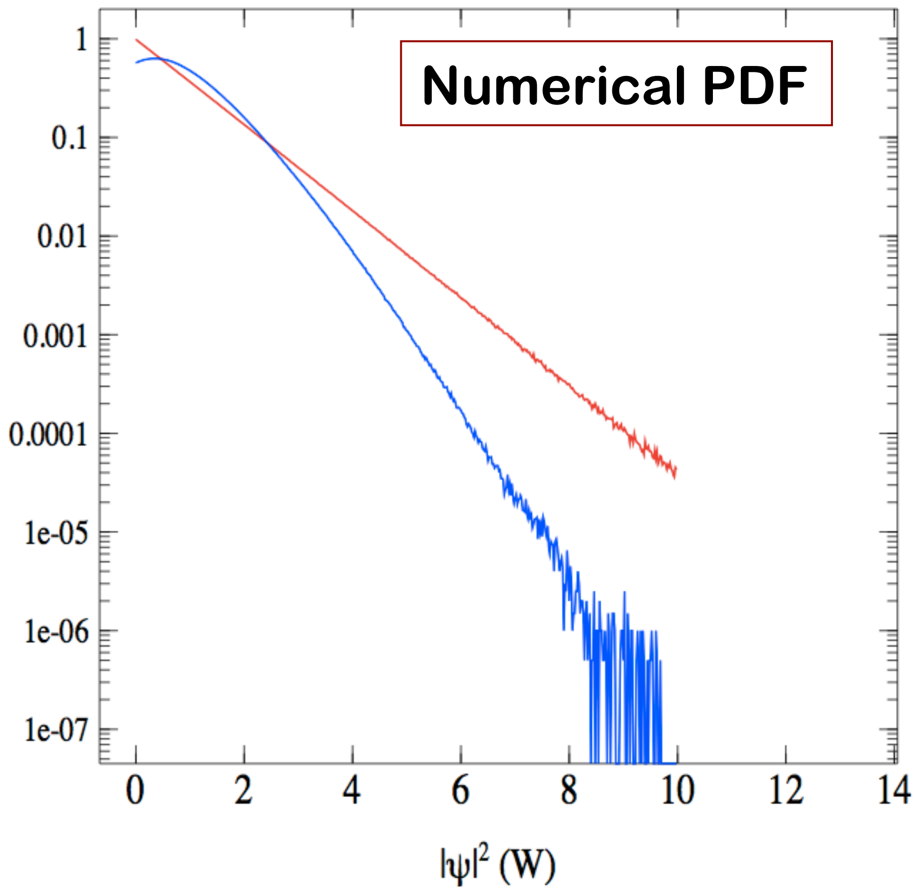
Z=0
Exponential PDF
P=0.3W



Low tail

PDF





P. Suret, P. Walczak and S. Randoux, to be submitted to Phys. Rev.

Usual wave thermalization

$$i\,\partial_z\,\psi_1(z,t) = -\,\beta_1\,\partial_t^2\,\psi_1(z,t) + \left(|\psi_1|^2 + \kappa\,|\psi_2|^2\right)\psi_1(z,t)$$

$$i\,\partial_z\,\psi_2(z,t) = -\,\beta_2\,\partial_t^2\,\psi_2(z,t) + \left(|\psi_2|^2 + \kappa\,|\psi_1|^2\right)\psi_2(z,t)$$

$\boxed{\kappa = 2}$

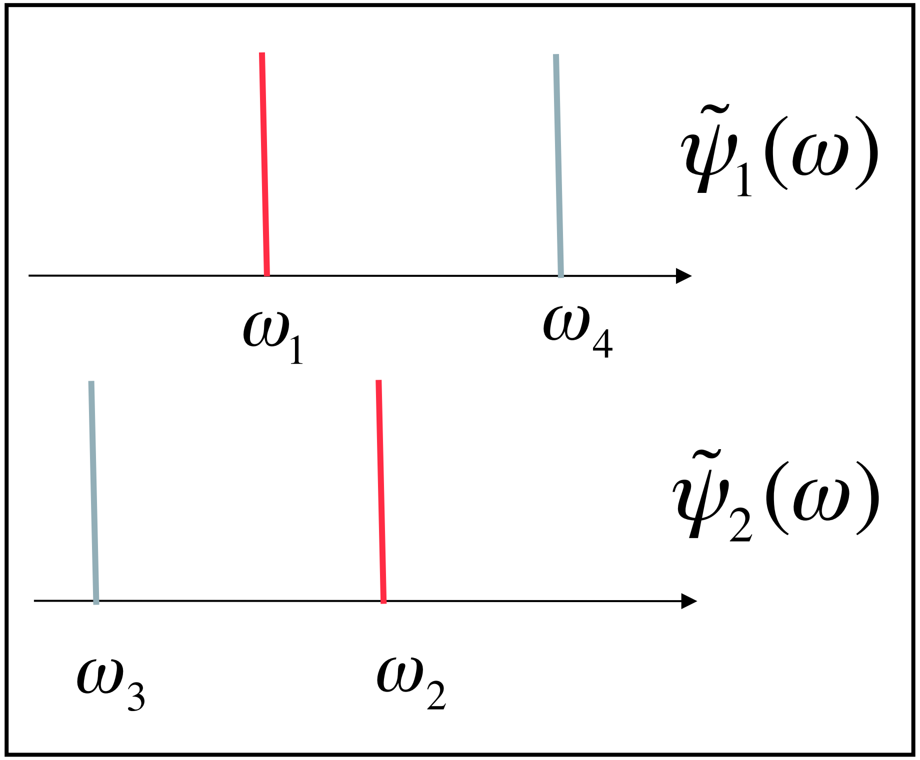
$\boxed{\beta_1 \neq \beta_2}$

Cross 4 Waves mixing (κ)

Energy $\boxed{\omega_1 + \omega_2 = \omega_3 + \omega_4}$

Phase-Matching

$$\boxed{\beta_1\,\omega_1^{\,2} + \beta_2\,\omega_2^{\,2} = \beta_2\,\omega_3^{\,2} + \beta_1\,\omega_4^{\,2}}$$



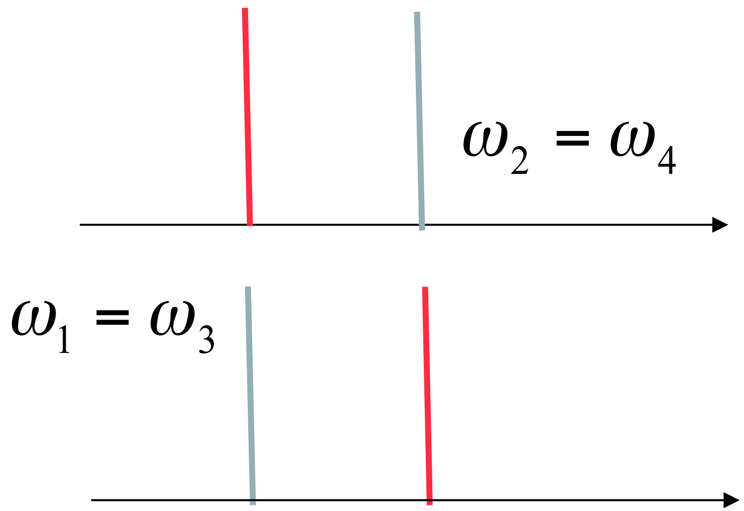
Thermodynamical equilibrium (Rayleigh-Jeans)

Anomalous Thermalization

$$i\,\partial_z\,\psi_1(z,t) = -\,\beta\,\partial_t^2\,\psi_1(z,t) + \left(|\psi_1|^2 + \kappa\,|\psi_2|^2\right)\psi_1(z,t)$$
$$i\,\partial_z\,\psi_2(z,t) = -\,\beta\,\partial_t^2\,\psi_2(z,t) + \left(|\psi_2|^2 + \kappa\,|\psi_1|^2\right)\psi_2(z,t)$$

$$\boxed{\beta_1 = \beta_2}$$

Degenerate resonances conditions



$$\tilde{\psi}_1(\omega)$$

$$\tilde{\psi}_2(\omega)$$

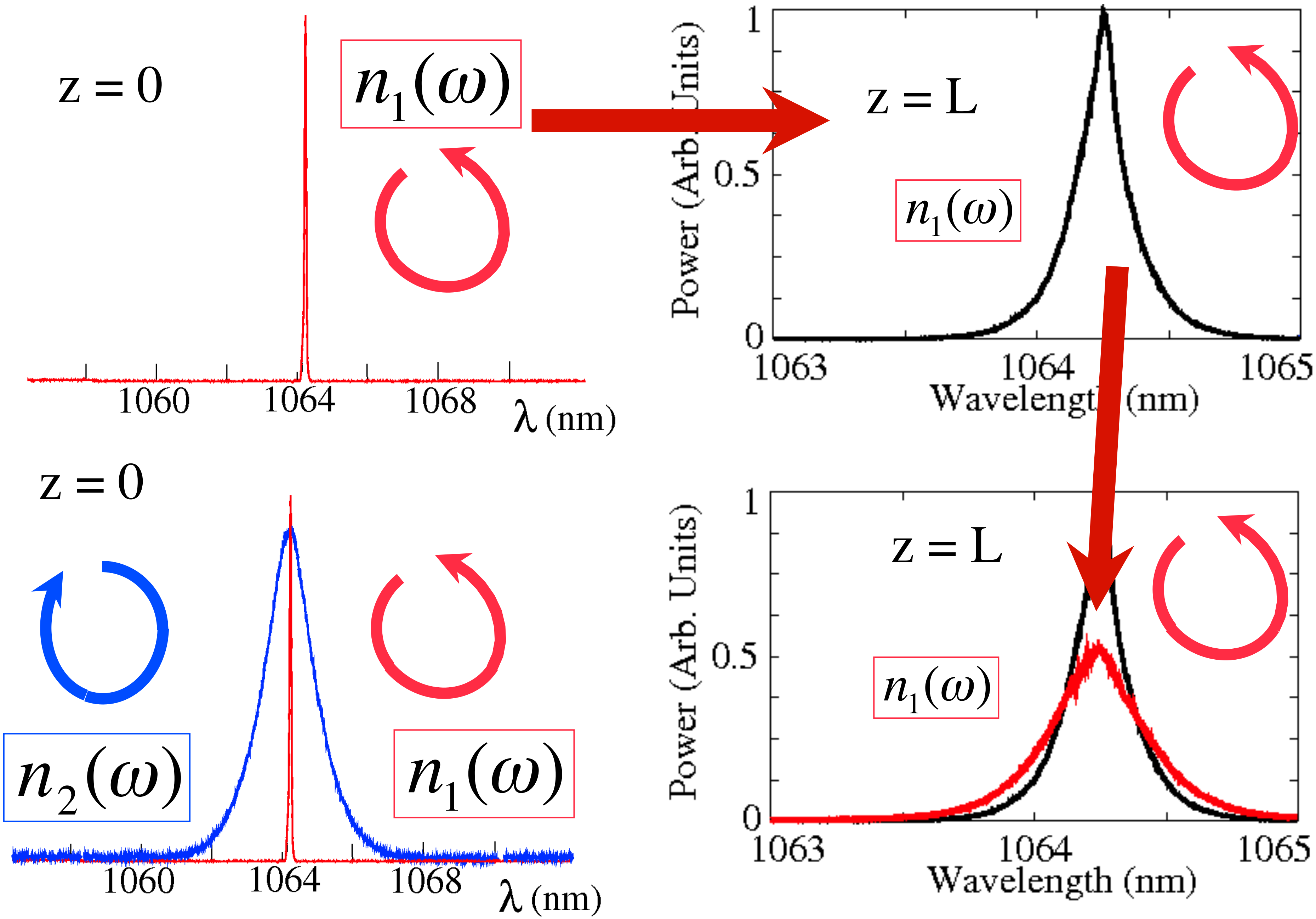
J_ω LOCAL invariant **(for each ω)** $J_\omega = n_1(\omega,z) + n_2(\omega,z)$ $\quad \dfrac{\partial J_\omega}{\partial z} = 0$

NO energy equipartition

equilibrium state preserves a memory of the initial condition

Suret *et al.*, PRL **104**, 054101 (2010)
C. Michel *et al.*, Opt. Lett. **35**, 2367-2369 (2010)
C. Michel *et al.* Lett. In Math. Phys., **96**, p 415 (2011)

Experiments



Incoherent waves and phase-matching conditions

Resonances conditions $\omega_1 + \omega_2 = \omega_3 + \omega_4$

$$\Delta k = \left(k(\omega_1) + k(\omega_2)\right) - \left(k(\omega_3) + k(\omega_4)\right) = 0$$

Integrability

Wave thermalization
(condensation,
Rayleigh-Jeans distribution)
V.E. Zakharov, V. L'vov and G. Falkovich, Springer, 1992

Anomalous Wave thermalization
Suret et al. Phys. Rev. Lett, **104**, 054101, 2010

Turbulence in Integrable Systems
V.E. Zakharov, Studies in Applied Mathematics,
122, 3, (2009)

Resonances

Conclusion

WTT applied to 1D-NLS equation (propagation in single mode fiber)

Theory / Experiments

Anomalous Thermalization

Suret *et al.*, PRL **104**, 054101 (2010)
C. Michel *et al.*, Opt. Lett. **35**, 2367-2369 (2010)
C. Michel *et al.* Lett. In Math. Phys., **96**, p 415 (2011)

Integrable case : transient regime (FWM without phase-matching)

P. Suret *et al.*, Opt. Express **19**, 17852-17863 (2011)

Open Questions

It does not matter how integrability is broken ?

Theoretical framework (Fourier space / random initial condition) ?

- (quasi-) kinetic equations (weakly nonlinear regime)
- Highly nonlinear regime : IST+random initial conditions ?