The Abdus Salam International Centre for Theoretical Physics

2472-15

Advanced Workshop on Nonlinear Photonics, Disorder and Wave Turbulence

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Logarithmic scaling of critical collapse and strong collapse turbulence

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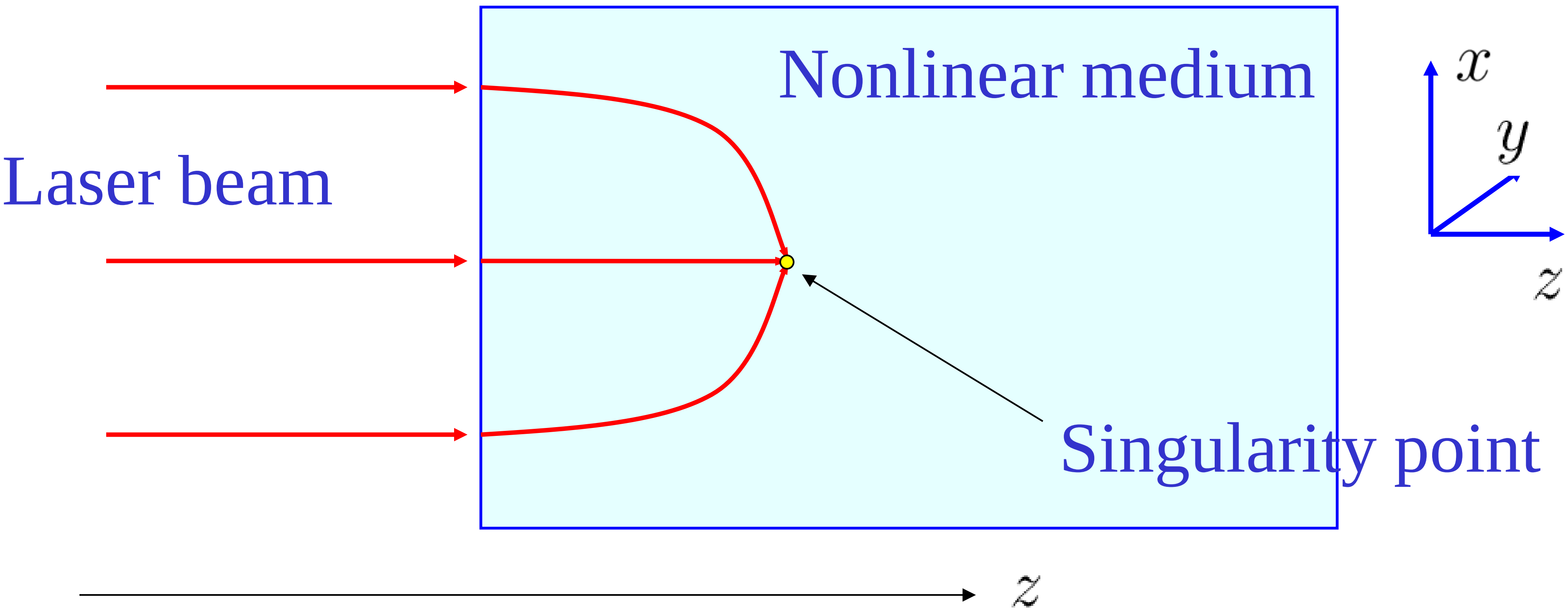
Self-focusing (collapse) of laser beam



$$i\frac{\partial}{\partial z}\psi + \triangle\psi + |\psi|^2\psi = 0$$ - 2D Nonlinear Schrödinger Equation.

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

ψ - amplitude of light

Nonlinear Schrödinger Equation

$$\boxed{i\frac{\partial}{\partial t}\psi + \Delta\psi + |\psi|^2\psi = 0}$$

$$H = \int (|\nabla\psi|^2 - \frac{1}{2}|\psi|^4) d^D\mathbf{r}$$ - Hamiltonian: $$i\psi_t = \frac{\delta H}{\delta\psi^*}$$

$$N = \int |\psi|^2 d^D\mathbf{r}$$ -optical power (in optics) or number of particles (in quntum mechanics)

Conserved Integrals: $\frac{d}{dt}N = \frac{d}{dt}H = 0$

Mean square width: $A \equiv \int |\mathbf{r}|^2 |\psi|^2 d^D \mathbf{r}$

$\boxed{D = 2}$

Virial theorem[1]: $A_{tt} = 8H$

$$\Rightarrow \qquad A = 4Ht^2 + c_1 t + c_2$$

Singularity formation:

$$H < 0 \quad \Rightarrow \quad A\big|_{t \to t_0} \to 0 \quad \Rightarrow \quad \max_{t \to t_0} |\psi| \to \infty$$

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[1] S.N.Vlasov, V.A Petrishchev, and V.I. Talanov (1971)

Collapse turbulence in Nonlinear Schrödinger Equation (NLS)

$$\boxed{i\psi_t + (1 - ia\epsilon)\nabla^2\psi + (1 + i\epsilon|\psi|^s)|\psi|^2\psi = i\epsilon\phi}$$

Dissipation

Forcing

$0 < \epsilon \ll 1$

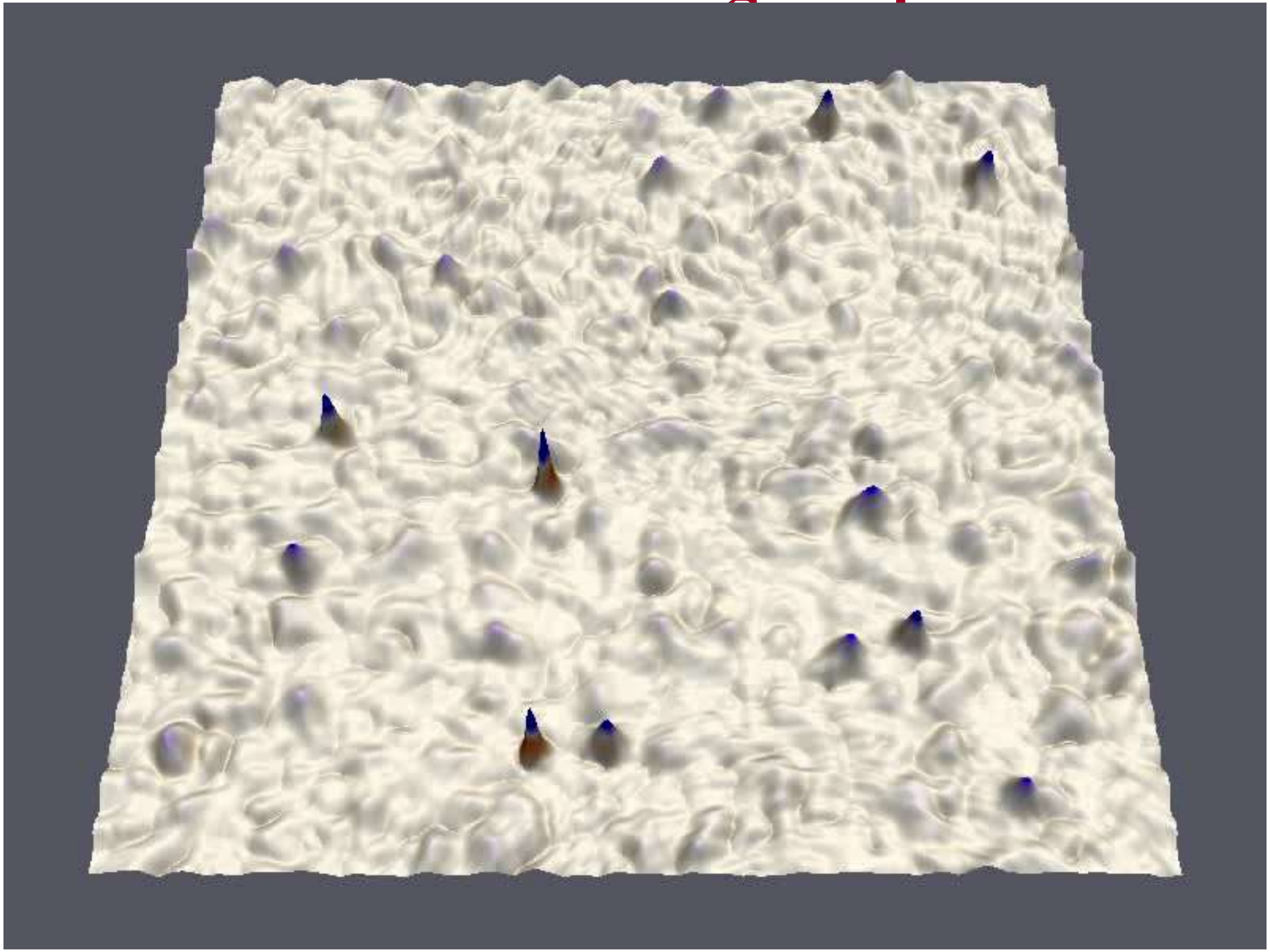
$a \sim 1$ - viscosity coefficient

ϕ - forcing

Particular case: a=0 and s=0 – NLS with two-photon absorbtion

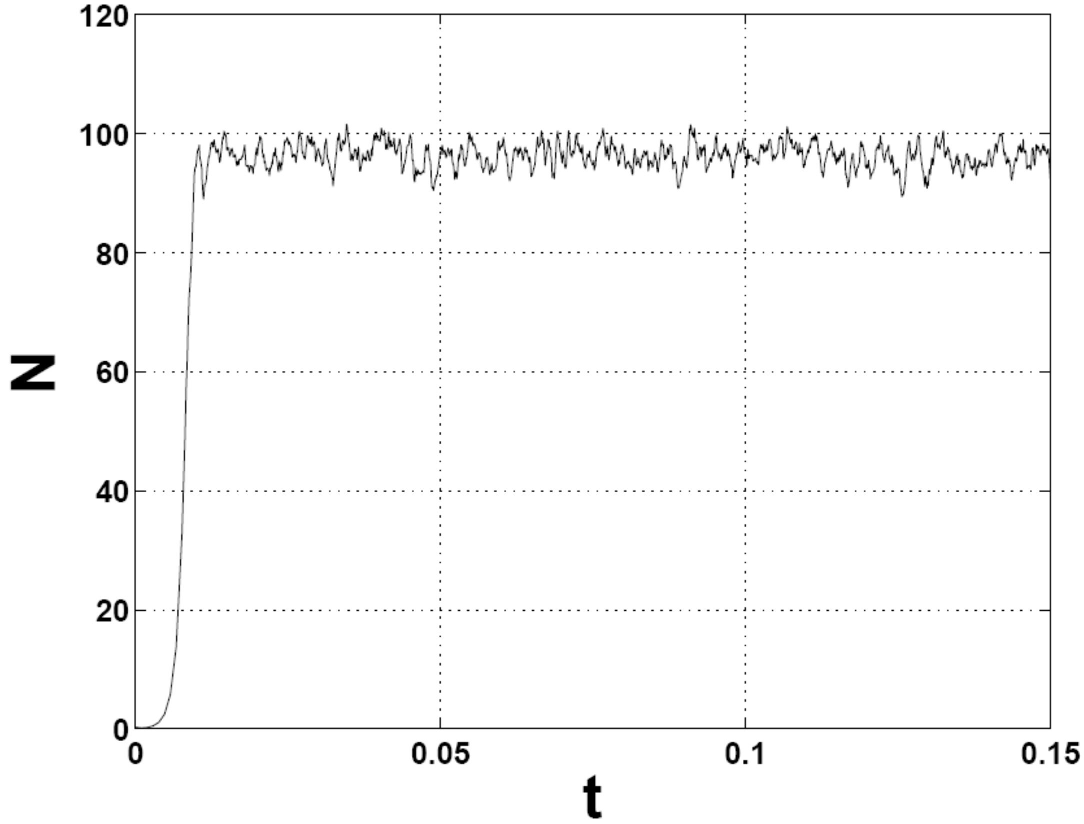
$$i\psi_t + \nabla^2\psi + (1 + i\epsilon)|\psi|^2\psi = 0$$

Collapse turbulence (multiple filamentation or Rogue waves) in 2D Nonlinear Schrödinger Equation

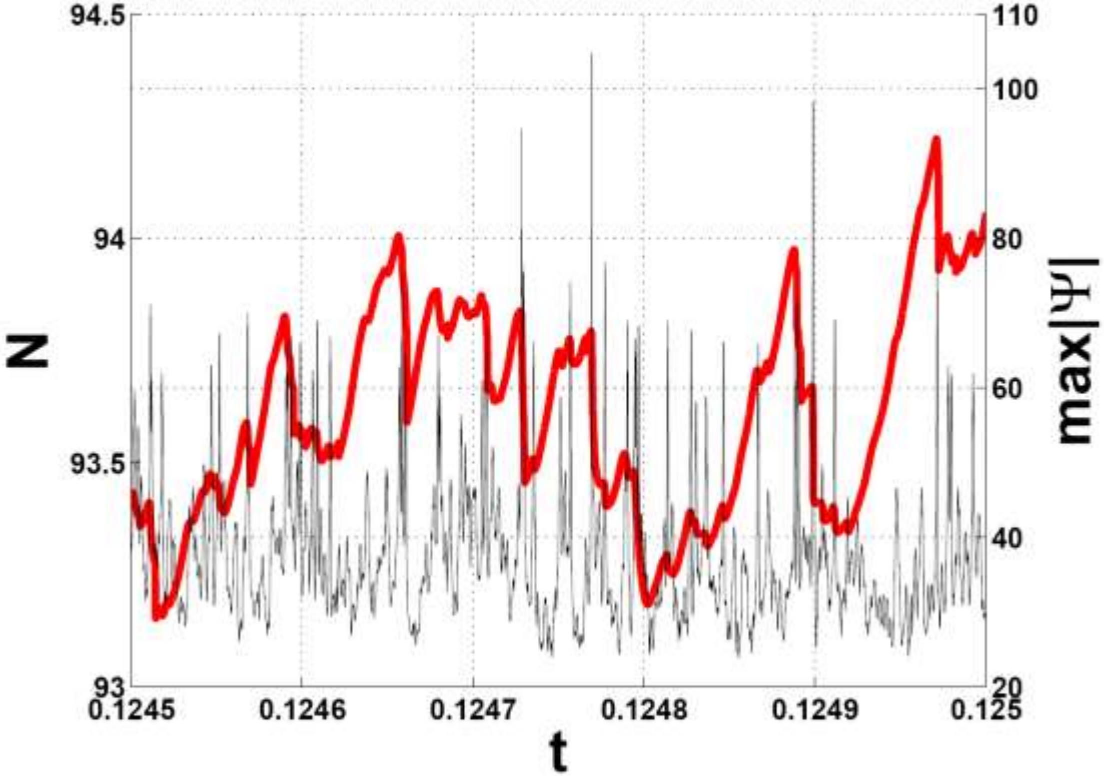


[1]P.M. Lushnikov and N. Vladimirova, Optics Letters **35**, 1695 (2010).
[2]Y. Chung and P.M. Lushnikov, Phys. Rev E, **84**, 036602 (2011).

Statistical steady state with forcing in 2D



Zoom in



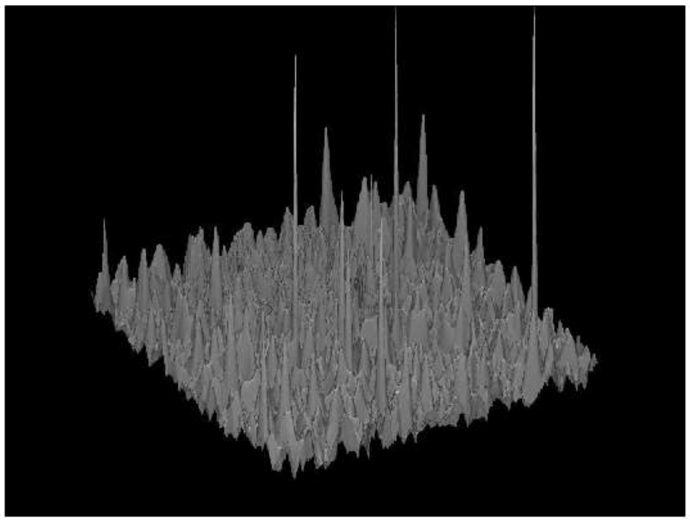
Black curve – $\max|\psi|$

Red curve – number of particles

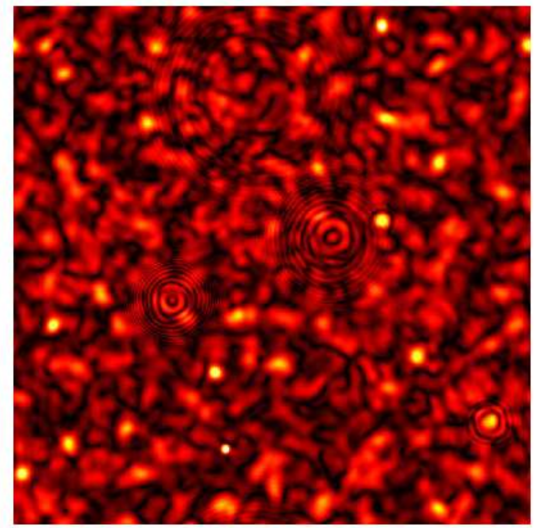
Collapse turbulence of Nonlinear Schrödinger Equation in 2D (Lushnikov and Vladimirova, 2010)

$$i\psi_t + (1 - ia\epsilon)\nabla^2\psi + (1 + i\epsilon)|\psi|^2\psi = i\epsilon\phi$$

(a)

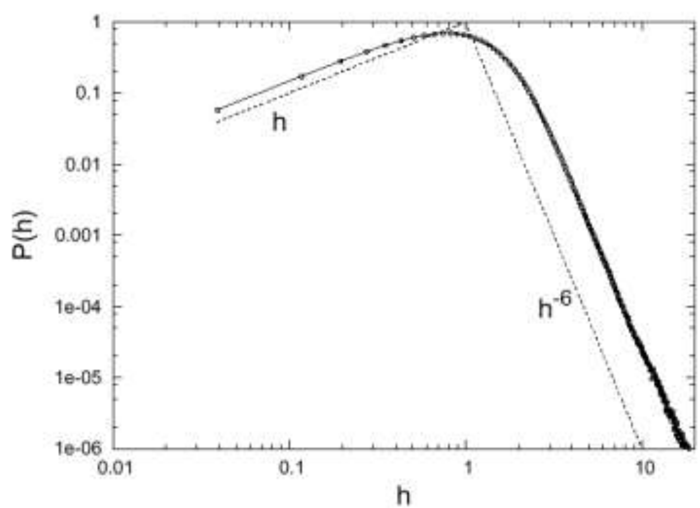


(b)



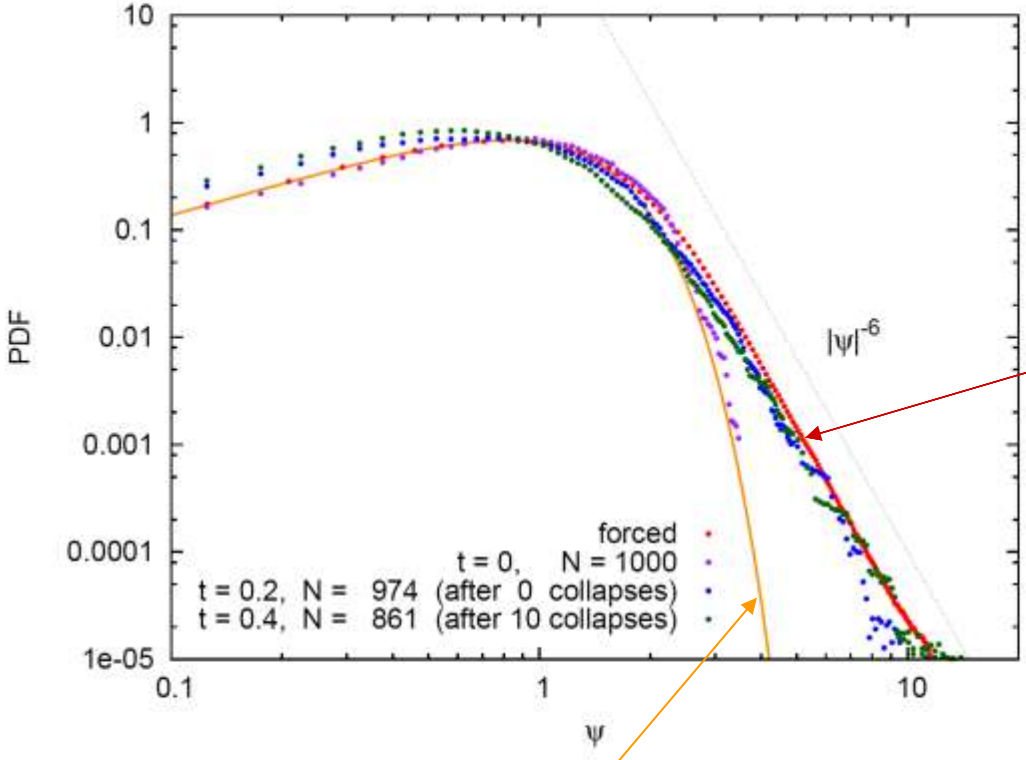
Probability density

$$\mathcal{P}(h) = \frac{\int \delta(|\psi(r,t)| - h)\,dr\,dt}{\int dr\,dt}$$



No forcing case D=2: initial random Gaussian field at t=0 quickly evolves for t>0 into non-Gaussian random field with power law tails of PDF

$$i\psi_t + (1 - ia\epsilon)\nabla^2\psi + (1 + i\epsilon)|\psi|^2\psi = 0$$



Forced limit

Initial random Gaussian field

Spacehomogeneous solution

$$\psi = \psi_0 \exp(i|\psi_0|^{4/D} t)$$

Modulational instability

$$\psi = \left[\psi_0 + \delta\psi \exp(\nu t + ikr)\right] \exp(i|\psi_0|^{4/D} t),$$

$$k_{max}^2 = \frac{2}{D}|\psi_0|^{4/D}$$

Forcing either deterministic $\phi = b\psi, \quad b \sim 1$

or stochastic $\phi = \xi(t, \mathbf{r}),$

$$\langle \xi(t_1, \mathbf{r}_1)\xi^*(t_2, \mathbf{r}_2)\rangle = \delta(t_1 - t_2)\chi(|\mathbf{r}_1 - \mathbf{r}_2|)$$

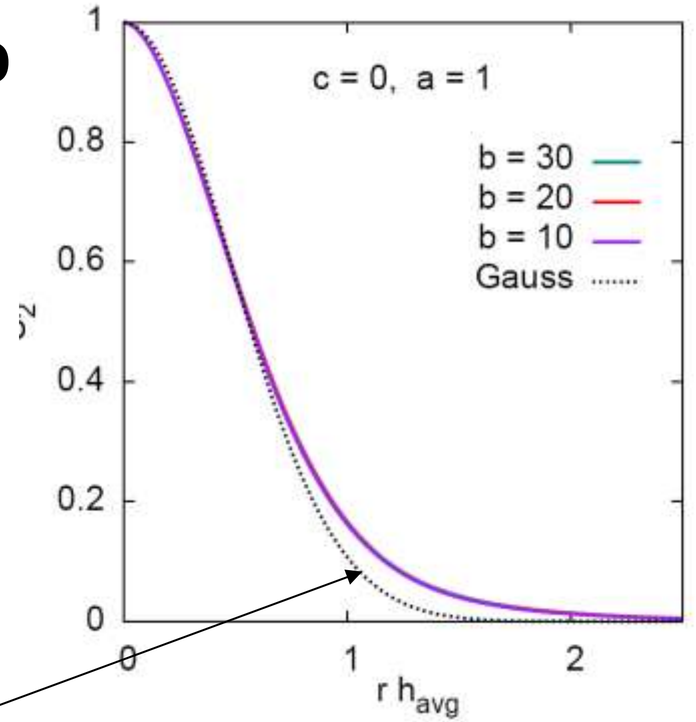
Fluctuations of the background

Correlation length x_{corr} **is well estimated by** $x_{corr} = 1/k_{max}$

$$k_{max}^2 = \frac{2}{D}|\psi_0|^{4/D}$$

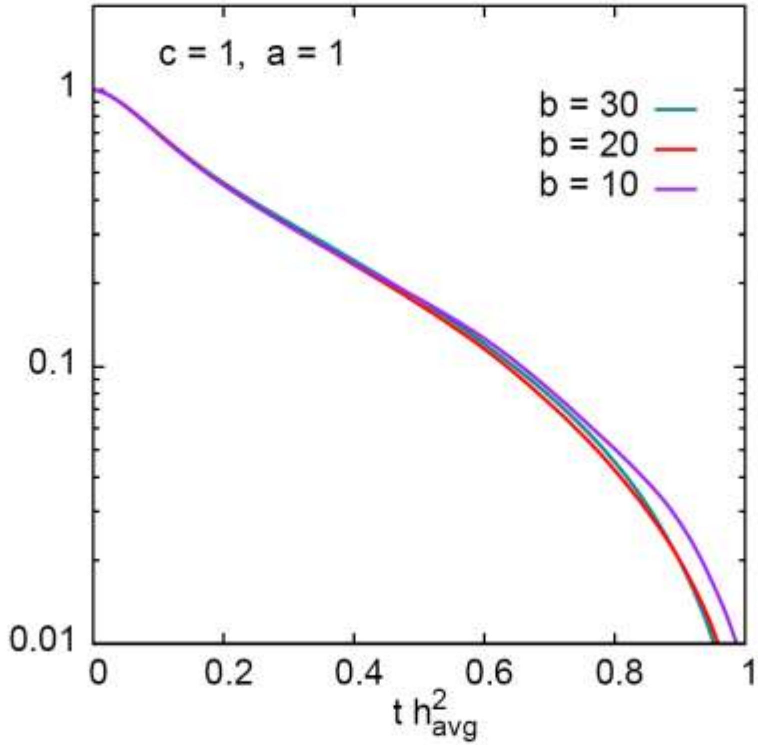
Spatial correlation function is universal for all parameters after rescaling

2D



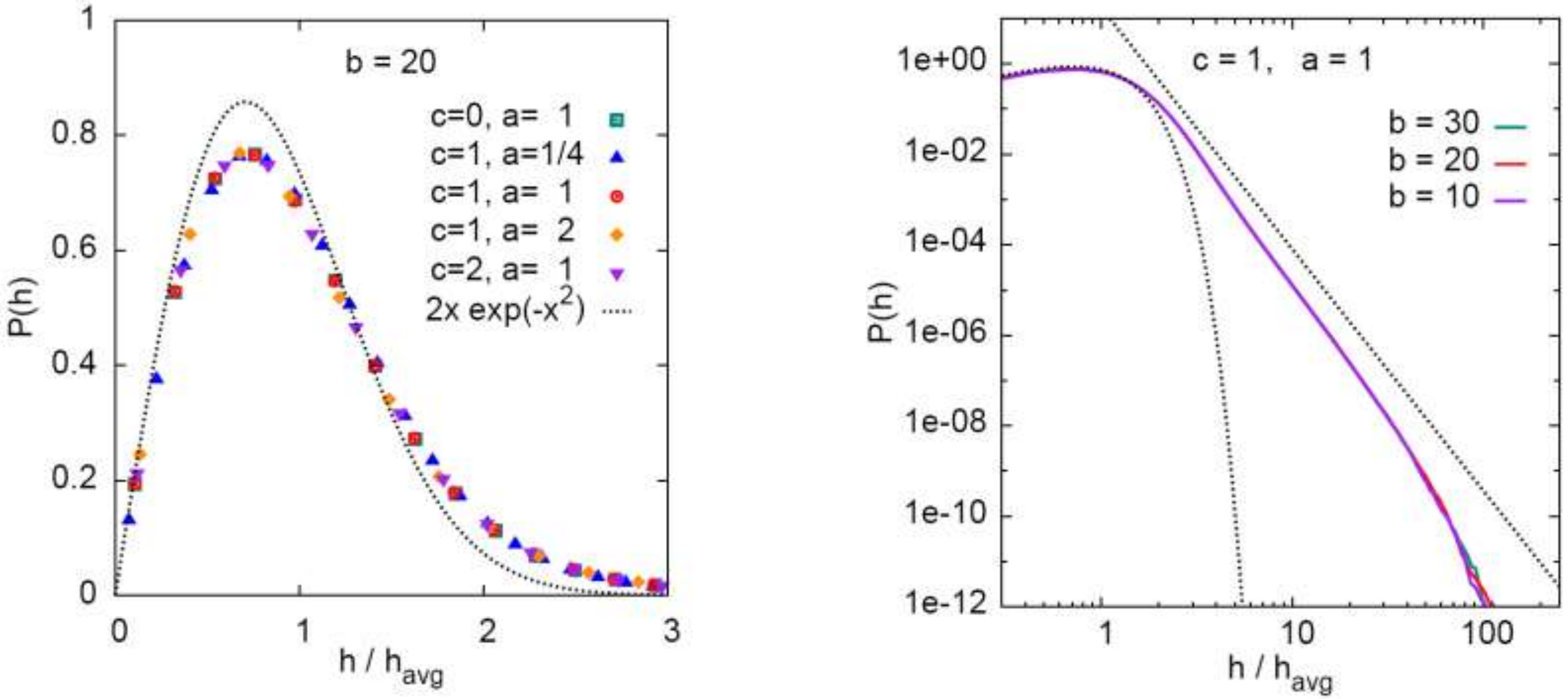
Fit to Gaussian form

Temporal correlation function is also universal for all parameters after rescaling in units of correlation time:



2D

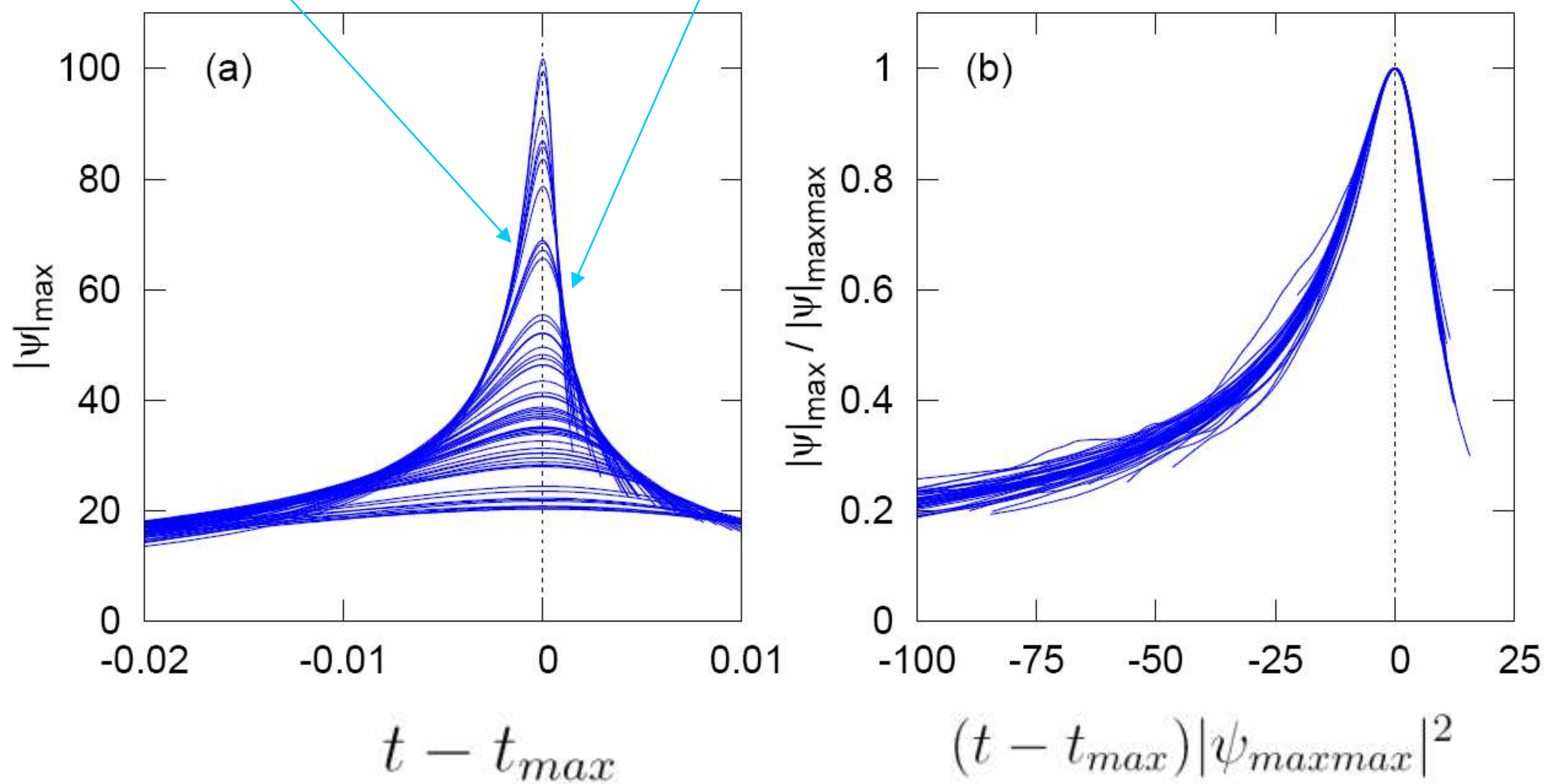
Universality of the probablity density function (PDF)



Universality of multiple collapses in rescaled variables vs. nonrescaled variables

2D

$\propto (t_{max} - t)^{-1/2}$ $\propto (t - t_{max})^{-1}$



Individual collapse

$$|\psi(\mathbf{r},t)| = \frac{1}{L(t)^{D/2}} R_0(\rho), \quad \rho = \frac{r}{L(t)}, \quad r \equiv |\mathbf{r}|$$

Where $R_0(\rho)$ **is the ground state soliton solution**

$$\nabla^2 R_0 - R_0 + R_0^{4/D+1} = 0$$

$\gamma = -L_t L$ **is the slow function of *t***

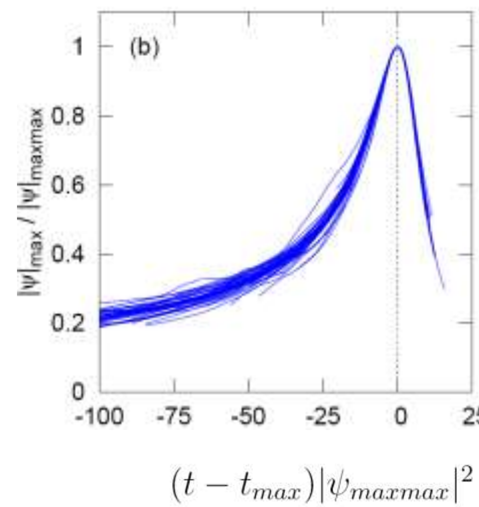
Contribution to PDF from individual collapse

$$\mathcal{P}\left(h|h_{max}\right) \propto \int^{t_{max}} dt \int d\mathbf{r}\delta\left(h - \frac{1}{L(t)^{D/2}} R_0\left(\frac{r}{L(t)}\right)\right)$$

$$\propto \int d\rho\, \rho \int^{L(t_{max})} \frac{dL\, L^{D+1}}{\gamma} \delta\left(h - \frac{1}{L(t)^{D/2}} R_0(\rho)\right)$$

$$\simeq \int \frac{d\rho\, \rho}{\langle\gamma\rangle h^{3+4/D}} [R_0(\rho)]^{2+4/D} \Theta\left(\frac{R_0(0)}{L(t_{max})^{D/2}} - h\right)$$

$$= Const\ h^{-3-4/D}\ \Theta\left(h_{max} - h\right),$$



$\gamma = -L_t L$ **is the slow function of *t***

Contribution to PDF from individual collapse $max|\psi| \equiv h_{max}$

$$\mathcal{P}(h|h_{max}) \simeq Const\frac{1}{h^{3+4/D}}\,\Theta(h_{max}-h)$$

$$\begin{aligned}\mathcal{P}(h) &= \int dh_{max}\mathcal{P}(h|h_{max})\mathcal{P}_{max}(h_{max}) \\ &\simeq Const\ h^{-3-4/D}\int dh_{max}\Theta(h_{max}-h)\mathcal{P}_{max}(h_{max}) \\ &= Const\ h^{-3-4/D}H_{max}(h),\end{aligned}$$

Where $H_{max}(h) = \int_h^\infty \mathcal{P}_{max}(h_{max})dh_{max}$ **is the probability to have collapse with maximum amplitude above** h

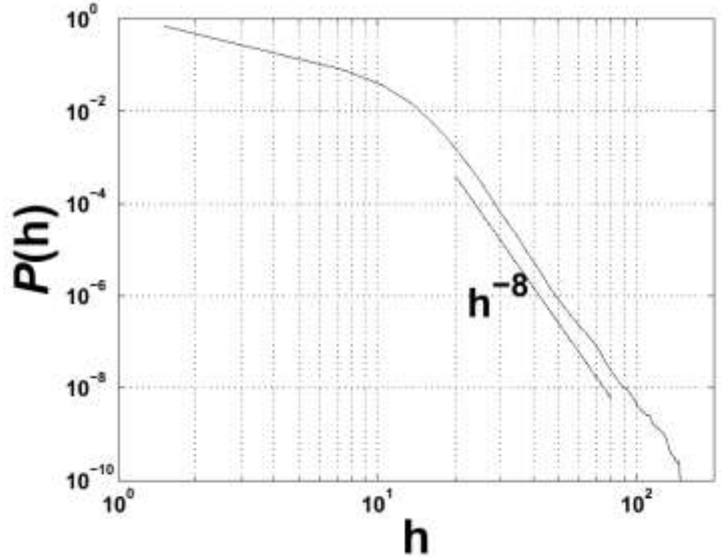
Numerical PDF for $|\psi|$

$\mathcal{P}(h) \propto \frac{1}{h^{4+4/D}}$ $\Rightarrow$ **expect**

$H_{max}(h) \propto h^{-1}$

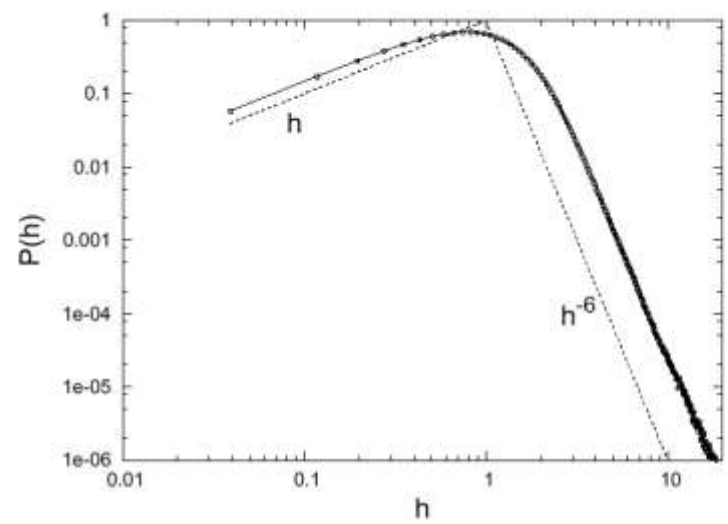
1D:

$\mathcal{P}(h) \propto \frac{1}{h^8}$



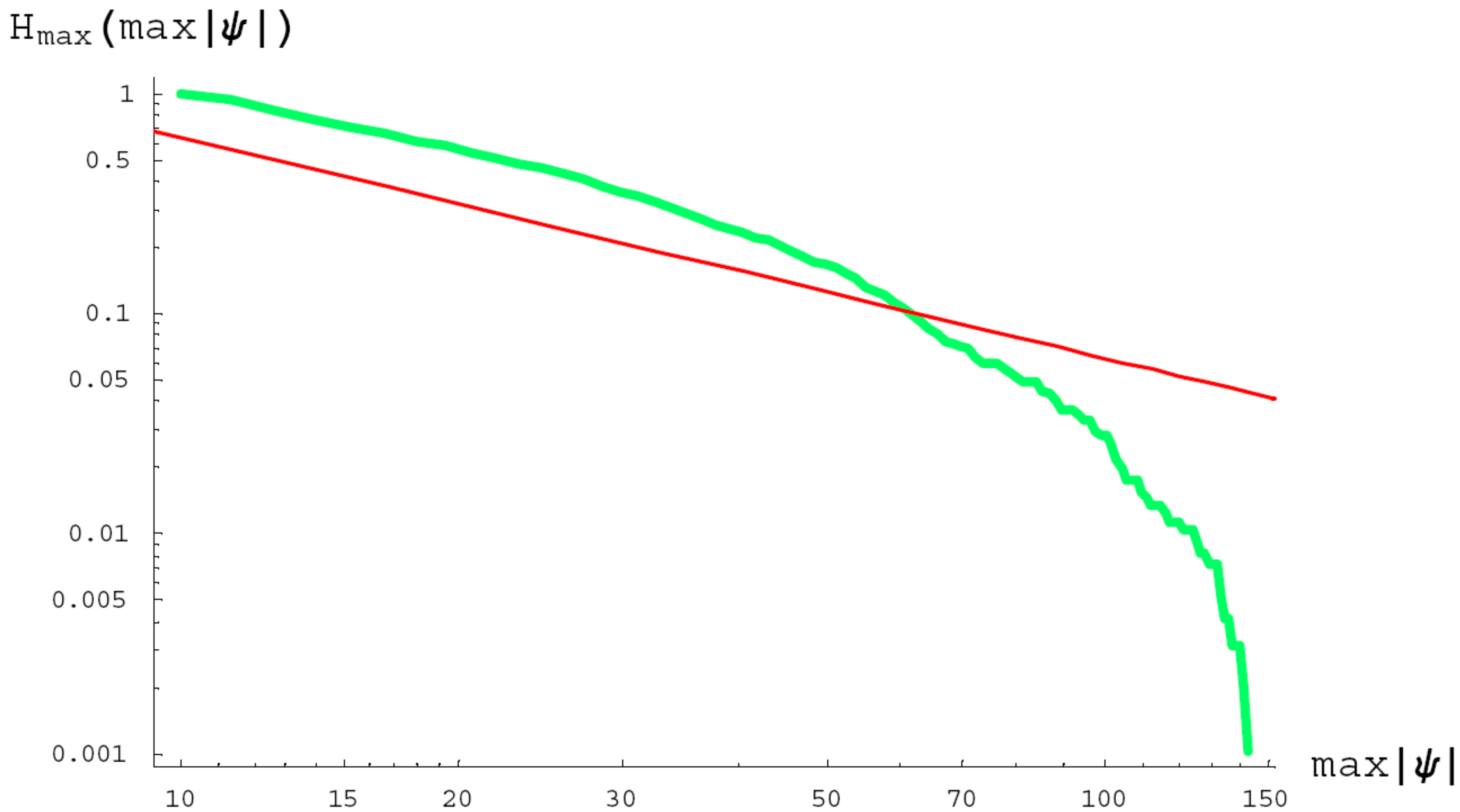
2D:

$\mathcal{P}(h) \propto \frac{1}{h^6}$



2D

Probability to have collapse with maximum amplitude above $\max|\psi|$ from optimal background fluctuations

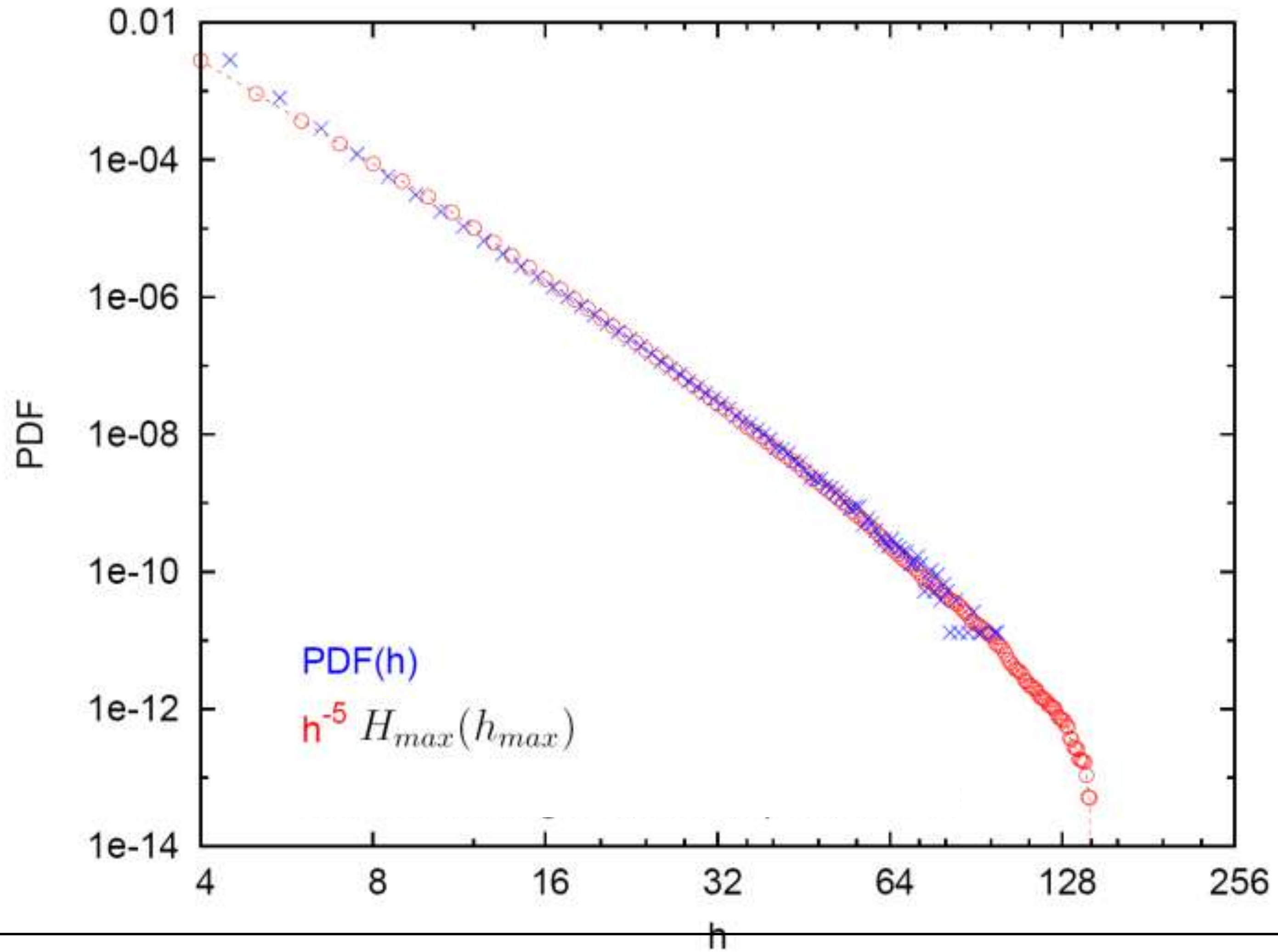


- ***1/h***

- **numerics**

2D

Probability to have collapse with maximum amplitude above $\max|\psi|$ **combined with contribution from single collapse vs. full numerical PDF**[1]



[1]P.M. Lushnikov and N. Vladimirova, Optics Letters, 35, 1967 (2010).

Critical collapse of 2D Nonlinear Schrödinger Equation: Self-similar solution near singularity

$$\boxed{i\frac{\partial}{\partial t}\psi + \Delta\psi + |\psi|^2\psi = 0}$$

$$\Psi(\mathbf{x}, t) = \Psi(r, t), \quad r = (x^2 + y^2)^{1/2}$$

$$\Psi(r, t) \simeq \frac{1}{L} V(\rho) e^{i\tau + iLL_t\rho^2/4}, \quad L \to 0,$$

$$\rho = \frac{r}{L}, \quad \tau = \int_0^t \frac{dt'}{L^2(t')},$$

Soliton solution of NLS:

$$\Delta V - V + |V|^2 V = 0$$

LogLog law[1]:

$$\boxed{L = \left(2\pi \frac{t_c - t}{\ln|\ln(t_c - t)|}\right)^{1/2}}$$

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[1] G. Fraiman (1985); M. Landman, G. Papanicolaou, C. Sulem, and P. Sulem (1987); A. Dyachenko, A. Newell, A. Pushkarev and V.E. Zakharov (1992).

A little of history of 2D NLS collapse

- **1962** G.A. Askaryan: Self-focusing of laser beam

- **1964** R.Y. Chiao, E. Garmire and C.H. Townes: self-trapping of laser beam, Townes soliton, collapse above threshold

- **1970** V.I. Talanov: lens transform

- **1971** S.N.Vlasov, V.A Petrishchev, and V.I. Talanov: Virial theorem and exact proof of collapse formation

- **1985** E.A. Kuznetsov and S.K. Turitsyn: conformal symmetry and Noether theorem

- **1985** G. Fraiman : “almost” log-log scaling of collapse

- **1987** M. Landman, G. Papanicolaou, C. Sulem, and P. Sulem: log-log scaling of collapse

- **1992** A. Dyachenko, A. Newell, A. Pushkarev and V.E. Zakharov: collapse turbulence

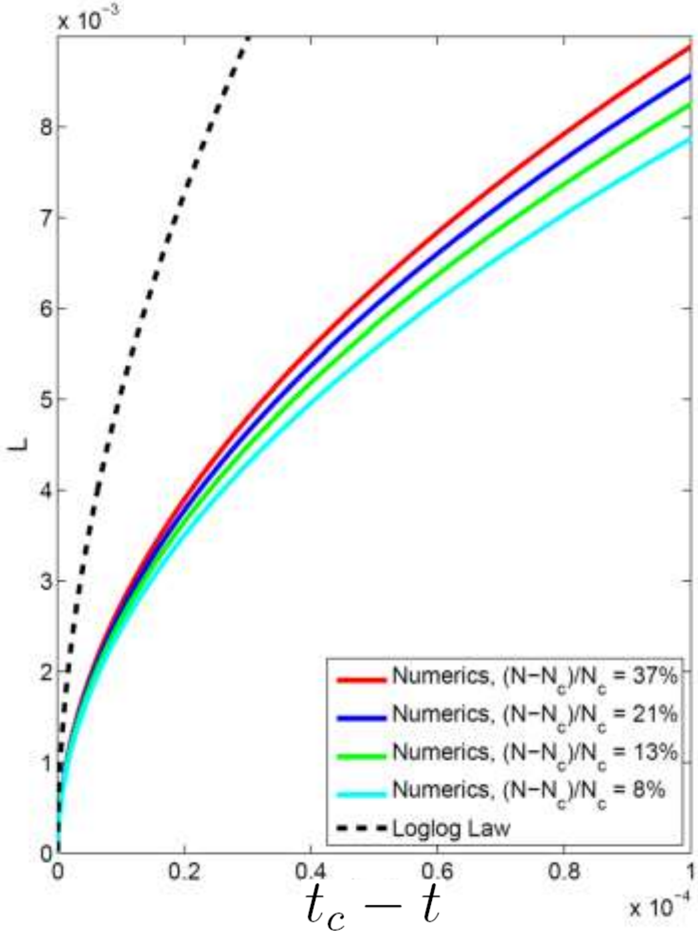
- **1993** V.M. Malkin: collapse in terms of the excess of number of particles above critical

- **2006** F. Merle and P. Raphael: exact proof of existence of log-log scaling

But simulations failed to confirm log-log law in a convincing way[1] **although the exact proof of the existence of log-log scaling was given**[2]

Example of NLS simulations:

***L(t)* depends on initial conditions**



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[1]See e.g.: N.E. Kosmatov, V.F. Shvets and V.E. Zakharov (1991);
G. D. Akrivis, V.A. Dougalis, O.A. Karakashian, and W.R. McKinney (2003).
[2]F. Merle and P. Raphael (2006).

Singularity Formation in Keller-Segel Model of Bacterial Aggregation

E. Coli clusters in Petri dish

Reduced Keller-Segel equation

$$\boxed{\begin{aligned}\frac{\partial \rho}{\partial t} &= \Delta \rho - \nabla \left(\rho \nabla c\right)\\ 0 &= \Delta c + \rho.\end{aligned}}$$

ρ - bacterial density

c - concentration of chemoattractant

Also describes the collapse of the Brownian gas of self-gravitating particles (ρ is density and c is the gravitational potential).

Collapse classification for Keller-Segel eqn

M.P. Brenner, P. Constantin, L.P. Kadanoff et. al. (1999).

$D<2$ – global existence of solution
$D=2$ – critical collapse
$D>2$ – supercritical collapse

For $D=2$ critical number of cells $N_c = 8\pi$

For $N>N_c$ positive-definite quantity $$A = \int r^2 \rho dx$$
Turns to zero in a finite time which means formation of collapse

$D=2$. Self-similar collapsing solution of Keller-Segel equation[1]:

$$\rho = \frac{1}{L(t)^2}\frac{8}{(1+y^2)^2},$$

$$c = -2\ln(1+y^2),$$

$$y = \frac{r}{L(t)},$$

$$L(t) = 2e^{-\frac{2+\gamma}{2}}\sqrt{t_c - t}\,e^{-\sqrt{-\frac{\ln(t_0-t)}{2}}} \times \left[1 + O\left(\frac{\ln\left[-\ln\left(t_c - t\right)\right]}{\sqrt{-2\ln\left(t_c - t\right)}}\right)\right], \qquad \gamma = 0.577216\ldots$$

Much bigger correction to $\sqrt{t_c - t}$ compare with *LogLog* in NLS!

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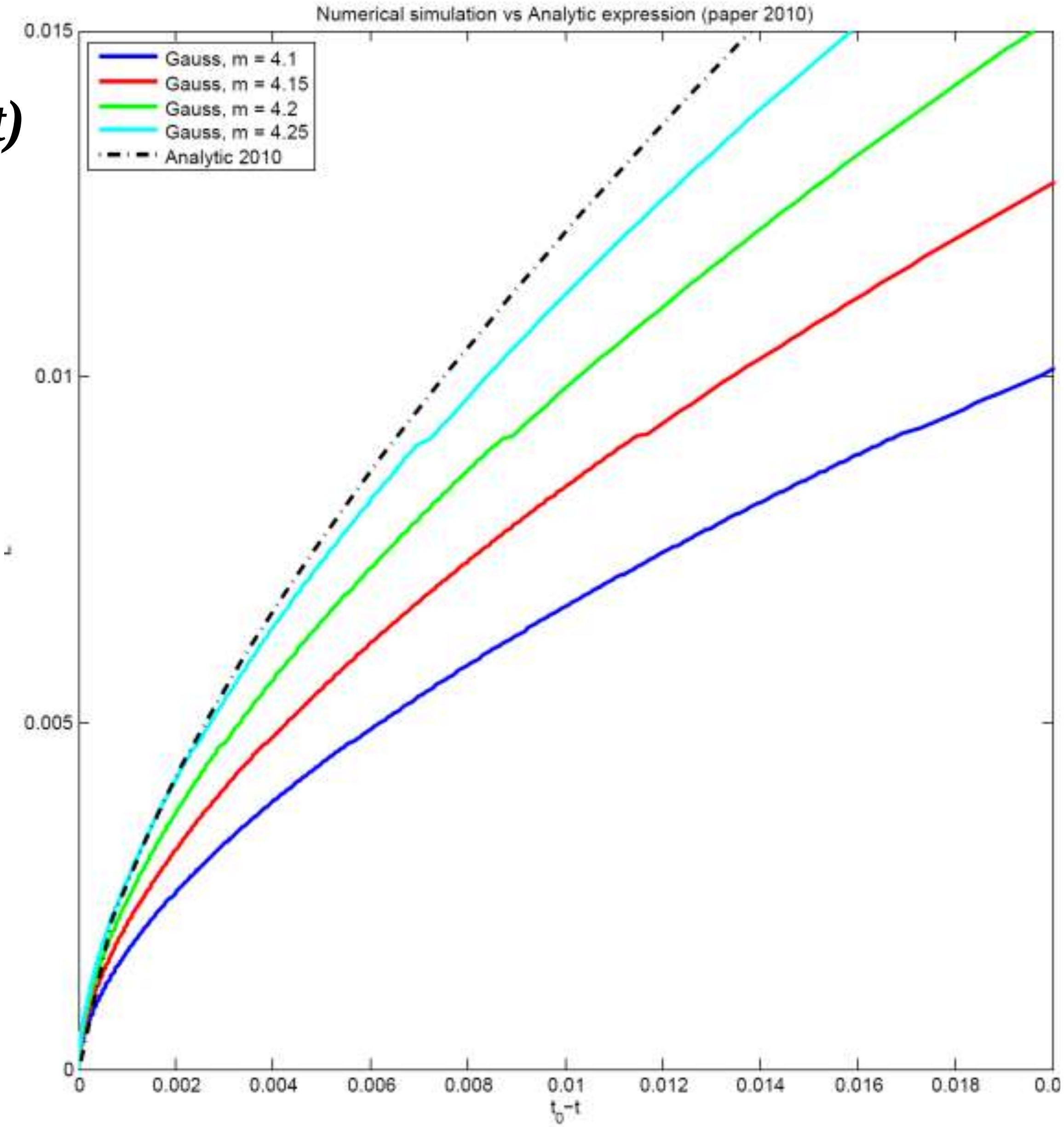
[1] P.M. Lushnikov, Phys. Lett. A (2010). S.I. Dejak, P.M. Lushnikov, Y.N. Ovchinnikov, and I.M. Sigal. Physica D 241 1245-1254 (2012).

Comparison with numerics for Keller-Segel equation:

L(t)



$t_c - t$

L(t) **from numerics with different masses**

L(t) **from 2010 analytic**

Corrected *L(t)* scaling[1]:

$$L(t) = 2e^{-\frac{2+\gamma}{2}}\sqrt{t_c - t}\exp\left\{-\sqrt{-\frac{\ln\beta(t_c - t)}{2}} + \frac{-1 + b\ln x}{2x} + \frac{-1 + 2b + 2\tilde{M}(1 - b\ln x)}{4x^2} + O\left(\frac{1}{x^2}\right) + O\left(\frac{(\ln x)^2}{x^3}\right)\right\}$$

Here

$$x = \sqrt{-2\ln\beta(t_c - t)} - \tilde{M},$$

$$\tilde{M} = -2 - \gamma + \ln 2,$$

$$b = 1 + \frac{\pi^2}{6},$$

$$\beta = 2\exp\left\{2l^* - \frac{\tilde{M}^2}{2}\right\},$$

$$l^* = -\ln L_0 - \frac{1}{4}\ln^2 a_0 + \frac{\tilde{M} + 1}{2}\ln a_0 - \frac{b}{2}\left(\ln\ln\frac{1}{a_0} + \frac{1}{\ln\frac{1}{a_0}}\right),$$

$$L_0 := L(t_0), \quad a_0 := a(t_0) = -L(t_0)\partial_t L(t_0).$$

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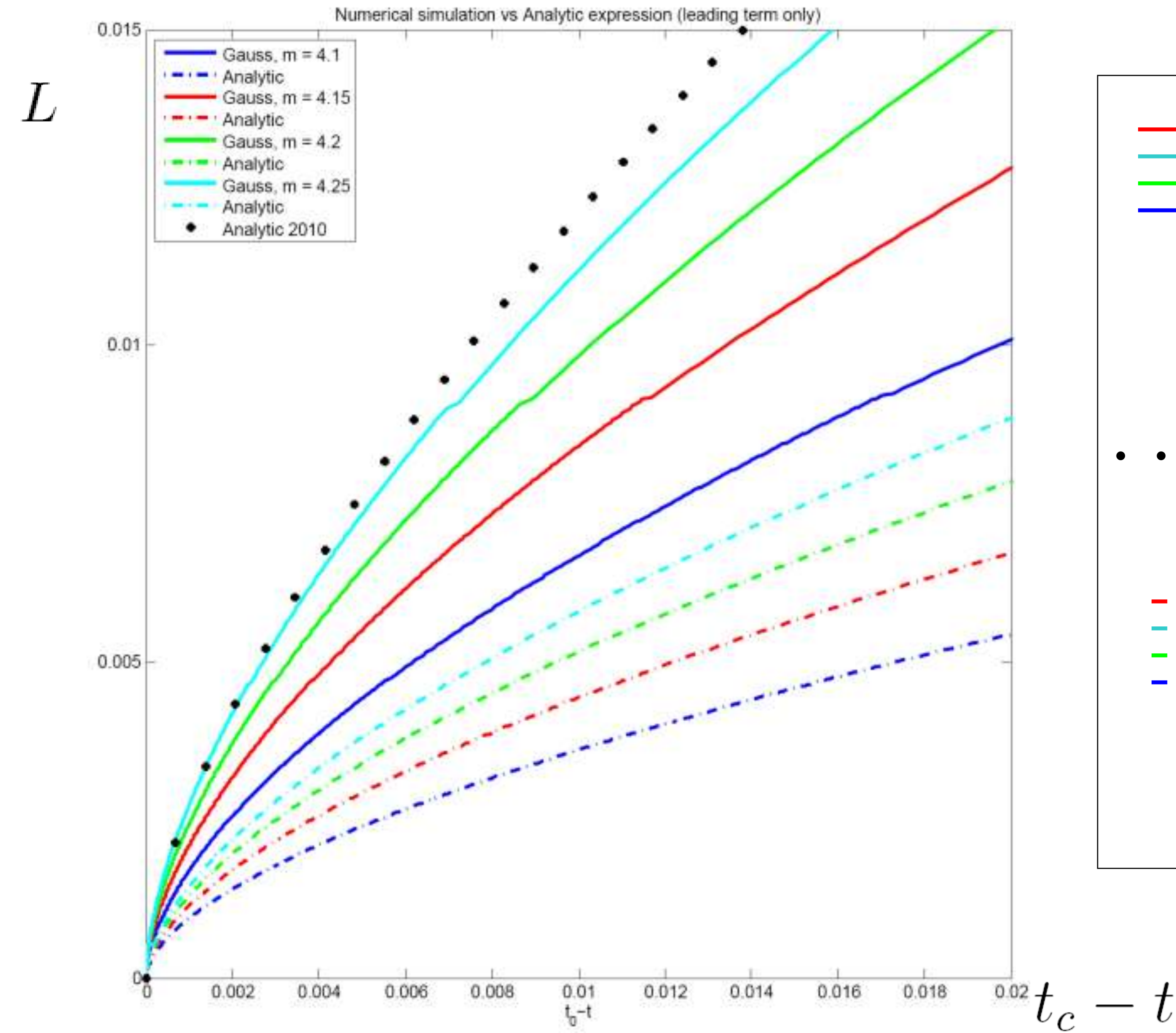
[1] S.A. Dyachenko, P.M. Lushnikov and N. Vladimirova, AIP Conf. Proc. 1389, 709-712 (2011);
S.A. Dyachenko, P.M. Lushnikov and N. Vladimirova, Nonlinearity (2013).

Numerics vs. analytic up to O(1/x) terms (S. Dyachenko, P. Lushnikov and N. Vladimirova, 2011):

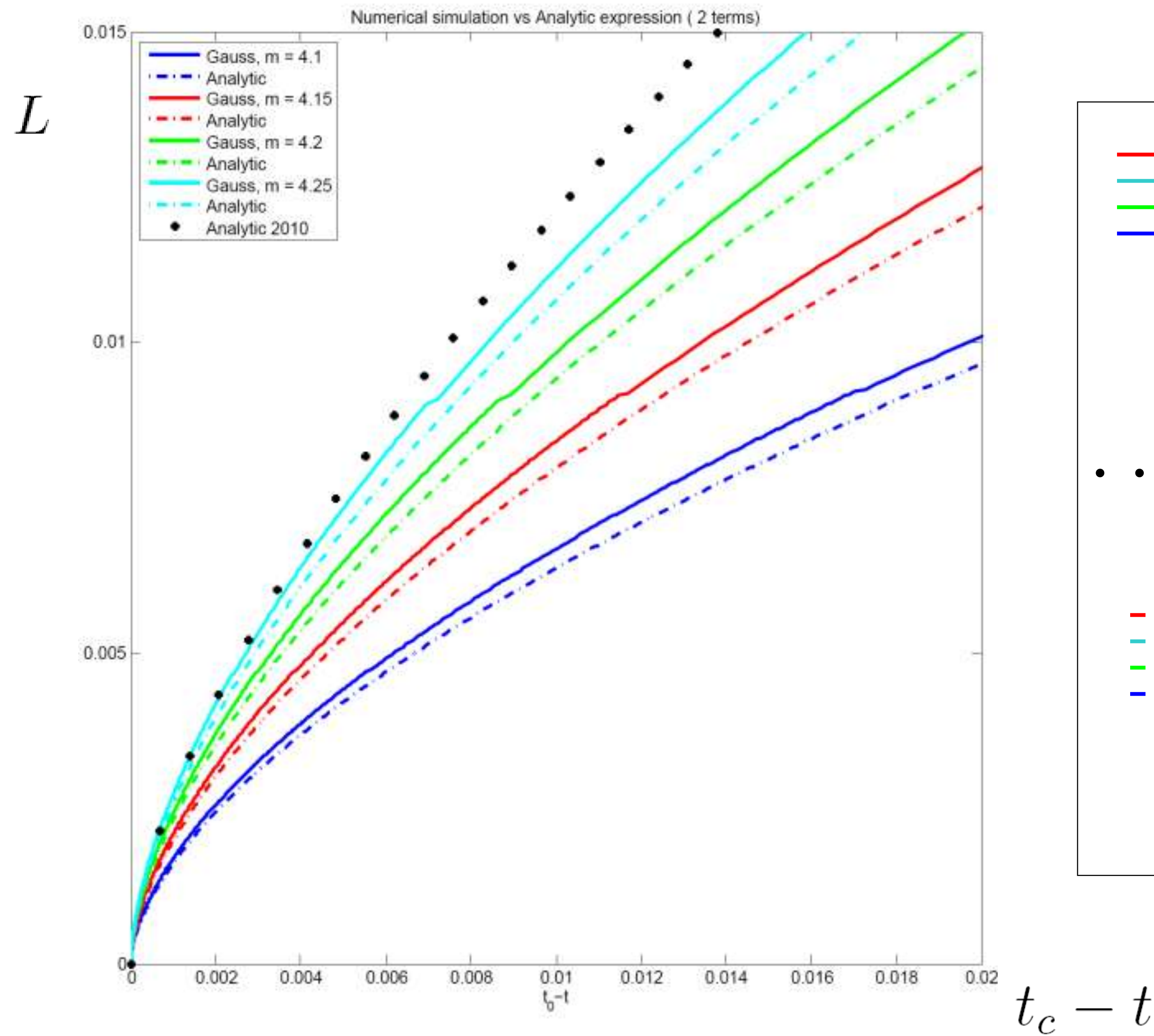


L(t) **from numerics with different masses**

L(t) **from 2010 analytic**

L(t) **from up to O(1/x) terms**

Numerics vs. analytic up to O(1/x²) terms

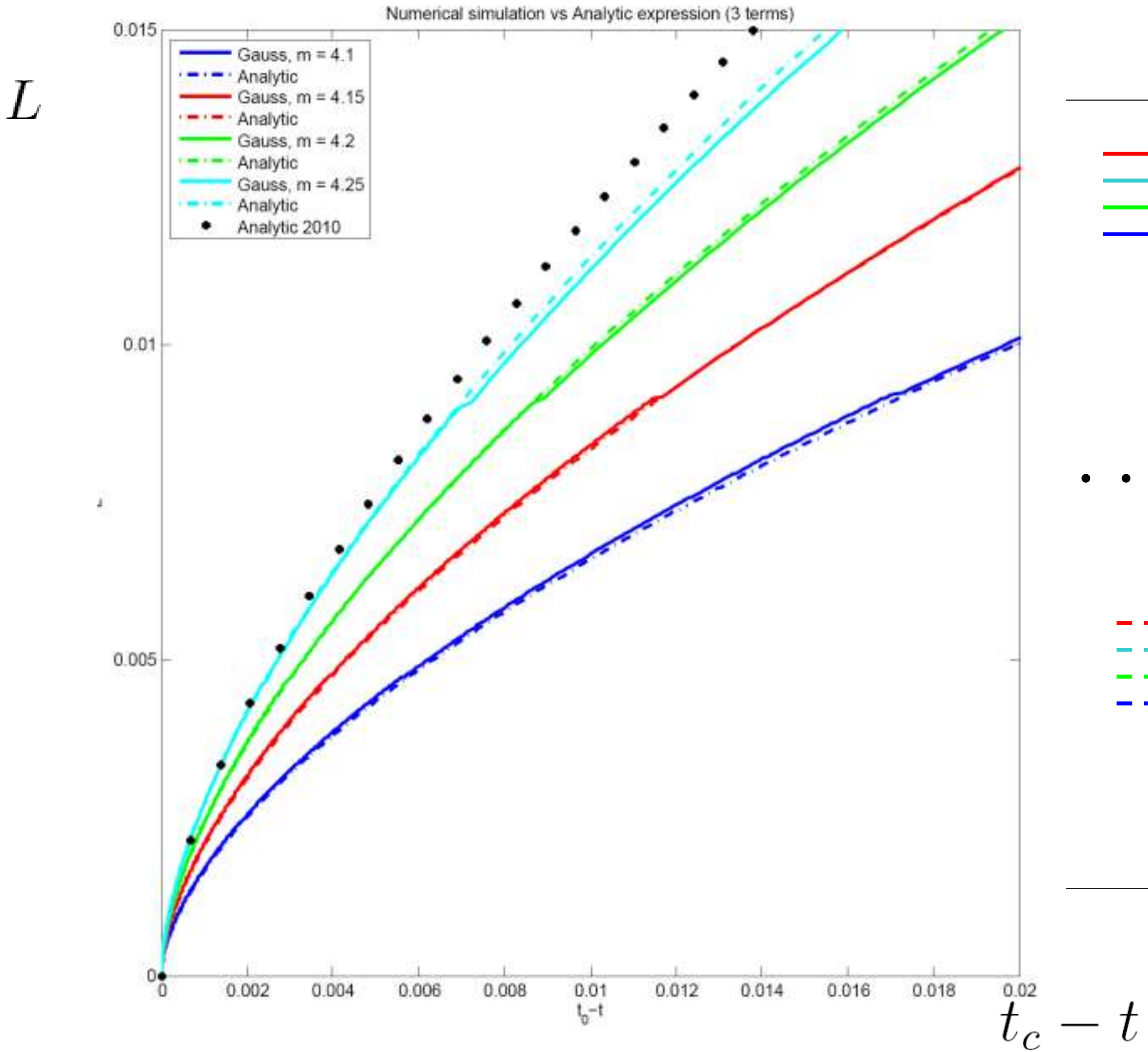


L(t) **from numerics with different masses**

L(t) **from 2010 analytic**

L(t) **from up to O(1/x²) terms**

Numerics vs. analytic up to O(1/x³) terms



L(t) **from numerics with different masses**

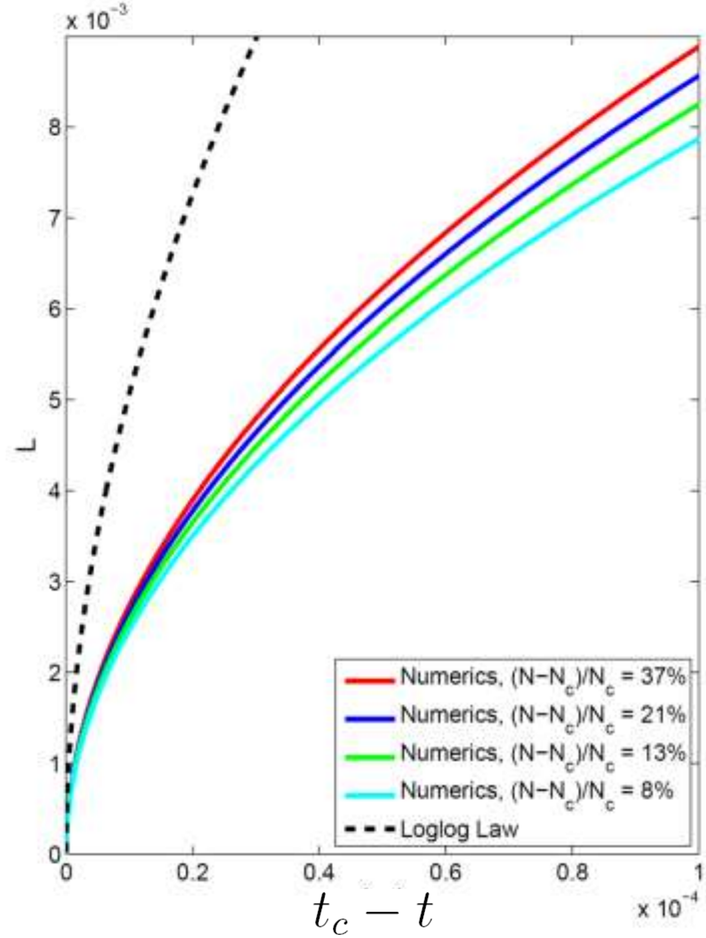
L(t) **from 2010 analytic**

L(t) **from up to O(1/x³) terms**

But simulations failed to confirm log-log law in a convincing way[1] although the exact proof of the existence of log-log scaling was given[2]

Example of NLS simulations:

L(t) **depends on initial conditions**



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[1]See e.g.: N.E. Kosmatov, V.F. Shvets and V.E. Zakharov (1991);
G. D. Akrivis, V.A. Dougalis, O.A. Karakashian, and W.R. McKinney (2003).
[2]F. Merle and P. Raphael (2006).

Can we develop perturbation theory for 2D NLS collapse beyond log-log scaling?

Answer is Yes

Start from the review of the standard theory

Blow-up variables

$$\rho = \frac{r}{L}, \quad \tau = \int^t \frac{dt'}{L^2(t')}$$

and lens transform

$$\psi(r,t) = \frac{1}{L} V(\rho,\tau) e^{i\tau + iLL_t\rho^2/4}$$

$$\Rightarrow \quad iV_\tau + \nabla^2 V - V + |V|^2 V + \frac{\beta}{4}\rho^2 V = 0,$$

where $\boxed{\beta = -L^3 L_{tt}}$ **- adiabatically slow small parameter**

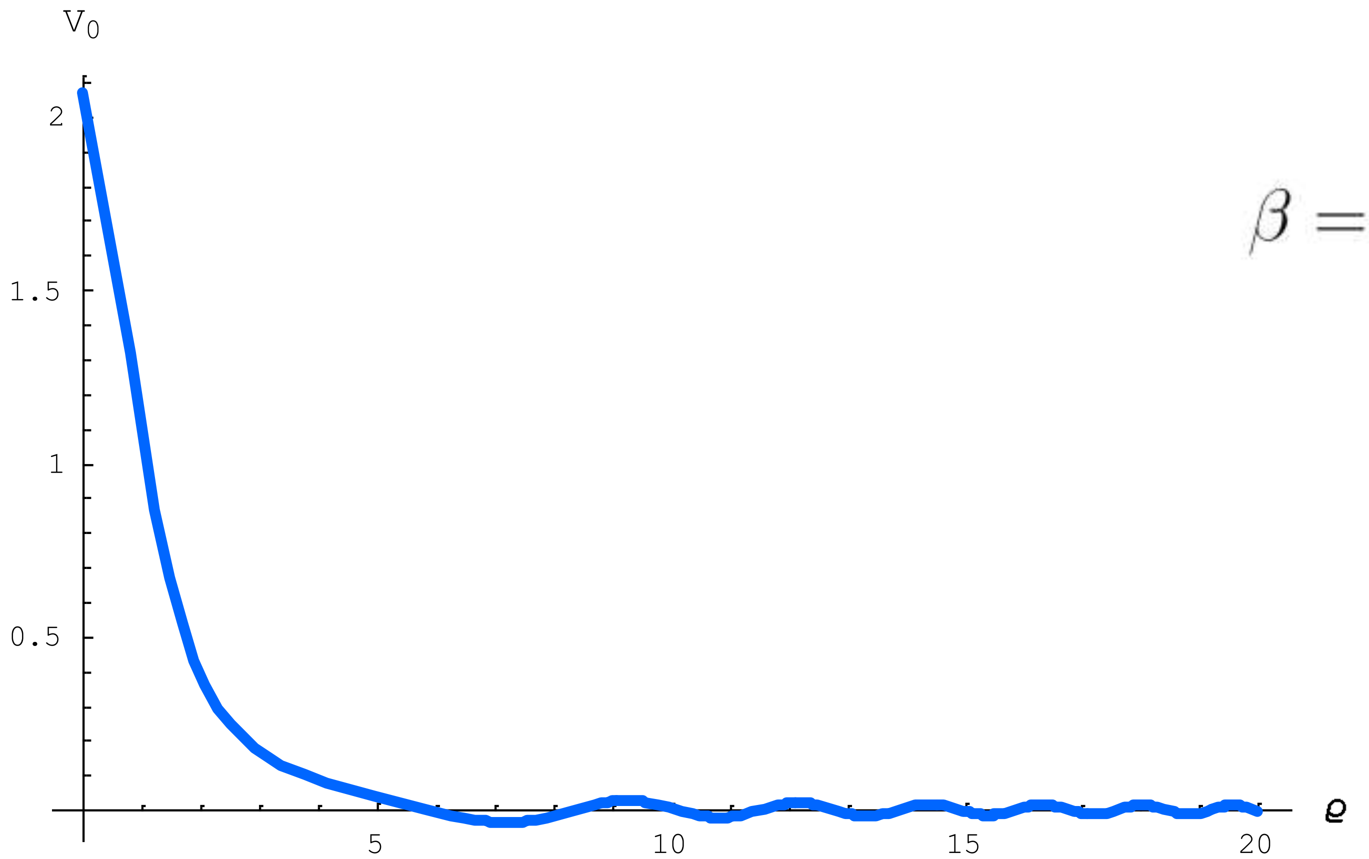
$$\beta \ll 1$$

Looking for solution in the form

$$V = V_0 + V_1 + \dots$$

In adiabatic approximaiton of slow β

$$\nabla^2 V_0 - V_0 + |V_0|^2 V_0 + \frac{\beta}{4}\rho^2 V_0 = 0$$

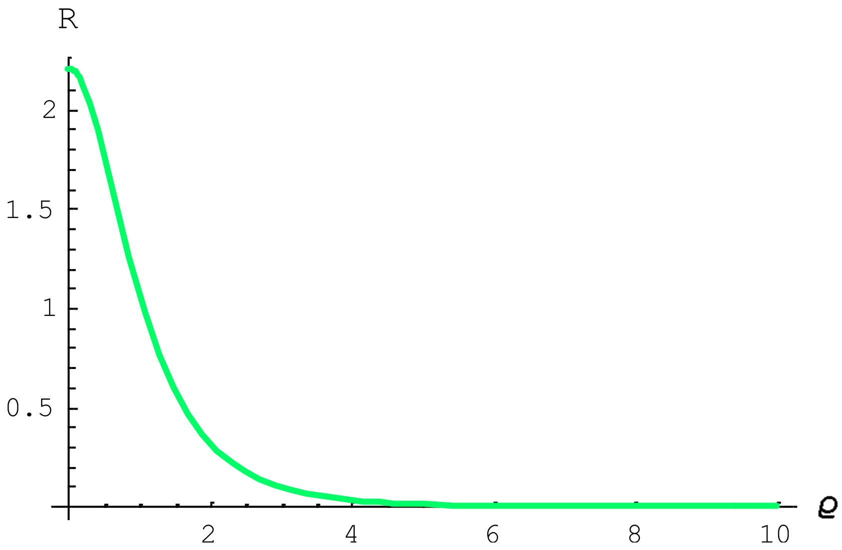


$\beta = 0.2$

Approximation through ground state soliton $R(\rho)$

$$V_0 = R(\rho) + \beta \frac{\partial V_0}{\partial \beta}\bigg|_{\beta=0} + O(\beta^2)$$

$$-R + \nabla^2 R + R^3 = 0$$



V_1 - has the imaginary part because of slow dependence of β on τ :

$$i\frac{\partial V_0}{\partial \tau} = i\beta_\tau \frac{\partial V_0}{\partial \beta}$$

Also we need to make sure that V has only outgoing waves for $\rho \to \infty$

In analogy with Gamov α-decay theory introduce nonself-adjoint problem

$$\nabla^2 \tilde{V}_0 - \tilde{V}_0 + |\tilde{V}_0|^2 \tilde{V}_0 + \frac{\beta}{4}\rho^2 \tilde{V}_0 - i\nu(\beta)\tilde{V}_0 = 0,$$

where ν can be determined from the balance of the norm of V as

$$\nu \sim e^{-\pi/\sqrt{\beta}}$$

$$\Rightarrow \beta_\tau = -\tilde{M} \exp\left[-\tfrac{\pi}{\beta^{1/2}}\right], \tilde{M} = 45.056\ldots$$

Basic ordinary differential equation (ODE) system of the standard theory:

$$
\begin{cases}
\beta_\tau = -\tilde{M} \exp\left[-\frac{\pi}{\beta^{1/2}}\right], \tilde{M} = 45.056\ldots, \\
L^3 L_{tt} = -\beta, \\
\tau = \int_0^t \frac{dt'}{L^2(t')}
\end{cases}
$$

Asymptotic solution near collapse time t_c :

$$
\boxed{L = \left(2\pi \frac{t_c - t}{\ln|\ln(t_c - t)|}\right)^{1/2}}
$$

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[1]G. Fraiman (1985); M. Landman, G. Papanicolaou, C. Sulem, and P. Sulem (1987); A. Dyachenko, A. Newell, A. Pushkarev and V.E. Zakharov (1992); V. F. Malkin (1993).

But simulations failed to confirm log-log law in a convincing way[1] **although the exact proof of the existence of log-log scaling was given**[2]

Example of NLS simulations:

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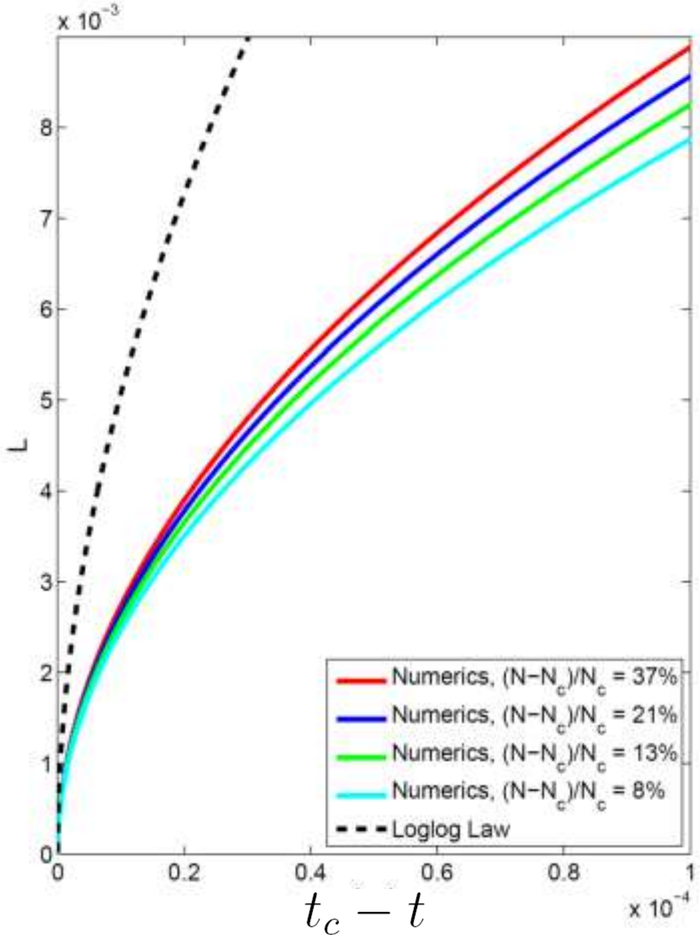

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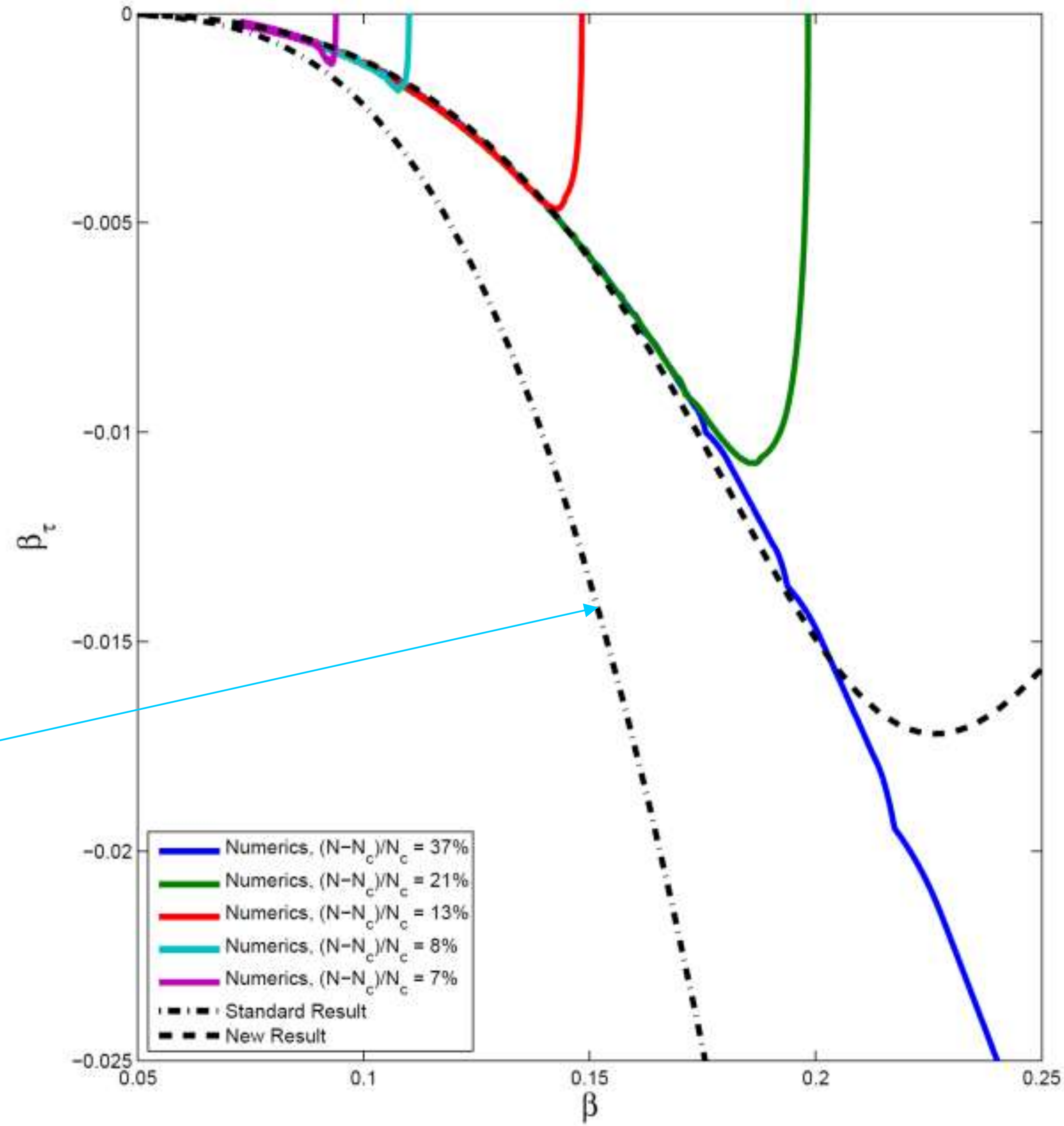
[1]See e.g.: N.E. Kosmatov, V.F. Shvets and V.E. Zakharov (1991); G. D. Akrivis, V.A. Dougalis, O.A. Karakashian, and W.R. McKinney (2003).

[2]F. Merle and P. Raphael (2006).

Modifying the standard theory

L(t) **is not universal but**
$\beta_\tau(\beta)$ is universal:

$$\beta_\tau = -\tilde{M} \exp\left[-\frac{\pi}{\beta^{1/2}}\right]$$



Look at

$$\nabla^2 \tilde{V}_0 - \tilde{V}_0 + |\tilde{V}_0|^2 \tilde{V}_0 + \frac{\beta}{4}\rho^2 \tilde{V}_0 - i\nu(\beta)\tilde{V}_0 = 0,$$

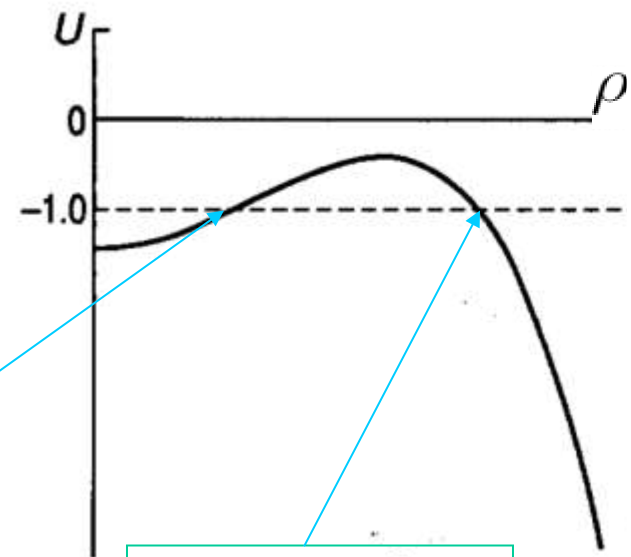
as the Schrödinger equation with the effective potential *U*:

$$U(\rho) = -|\tilde{V}_0|^2 + \frac{\beta}{4}\rho^2$$

and complex eigenvalue *E*:

$$E = -1 - i\nu(\beta)$$

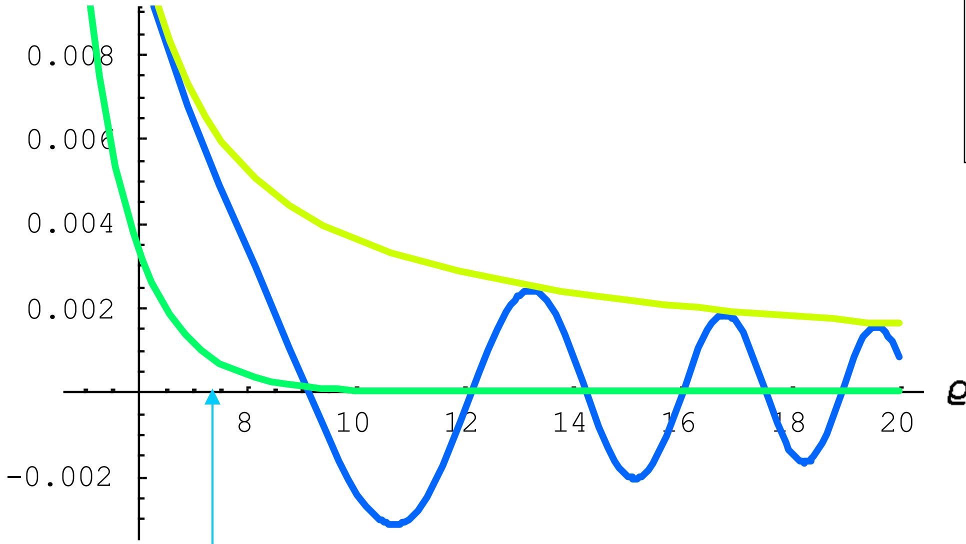
$\Rightarrow$ **2 turning points ρ_a and ρ_b of WKB:**



$$\boxed{\rho_a \sim 1}$$

$$\boxed{\rho_b \simeq \frac{2}{\beta^{1/2}}}$$

Solution near ρ_b



$|V|$ - from numerics

V_0 – soliton with β

R – ground state soliton with $\beta = 0$

$\rho_b \simeq 7.4$

$\beta = 0.073$

$V_0 \simeq |V|$ to the left from ρ_b

Oscillating tail is given by the linear combination of confluent hypergometric functions of the first and second kinds:

$$c_1 e^{-\frac{i}{4}\sqrt{\beta}\rho^2}\ {}_1F_1\left(\frac{1}{2}+i\frac{1}{2\sqrt{\beta}};1;i\sqrt{\beta}\rho^2\right)+c_2 e^{-\frac{i}{4}\sqrt{\beta}\rho^2}U\left(\frac{1}{2}+i\frac{1}{2\sqrt{\beta}};1;i\sqrt{\beta}\rho^2\right).$$

Matching asymptotics and using WKB give

$$V_0(\beta,\rho)=\frac{2^{1/2}A_R}{\beta^{1/4}}e^{-\frac{\pi}{2\beta^{1/2}}}\frac{1}{\rho}\cos\left[\frac{\beta^{1/2}}{4}\rho^2-\beta^{-1/2}\ln\rho+\phi_0\right],\quad \rho\gg\rho_b$$

Here $A_R\equiv 3.52$ **is determined by the asymptotic of ground state soliton** $R_0(\rho)=\frac{A_R}{\rho^{1/2}}e^{-\rho},\quad \rho\gg 1$

$\Rightarrow$ **Asymptotics of complex solution**

$$V(\beta,\rho)=\frac{2^{1/2}A_R}{-\beta^{1/4}}e^{-\frac{\pi}{2\beta^{1/2}}}\frac{1}{\rho}\exp\left[i\frac{\beta^{1/2}}{4}\rho^2-i\beta^{-1/2}\ln\rho-i\phi_0\right],\quad \rho\gg\rho_b.$$

Introducing the number of particles to the left of the second turning point

$$N_b = \int_{r<\rho_b L} |\psi|^2 d\mathbf{r} = 2\pi \int_{\rho<\rho_b} |V|^2 \rho d\rho.$$

and balancing the flux of particles through that point

$$\frac{dN_b}{d\tau} = \rho\left[iV^* V_\rho + c.c.\right]|_{\rho=\rho_b}, \qquad \frac{dN_b}{d\tau} = \beta_\tau \frac{dN_b}{d\beta}$$

$\Rightarrow$ **New basic ODE system**

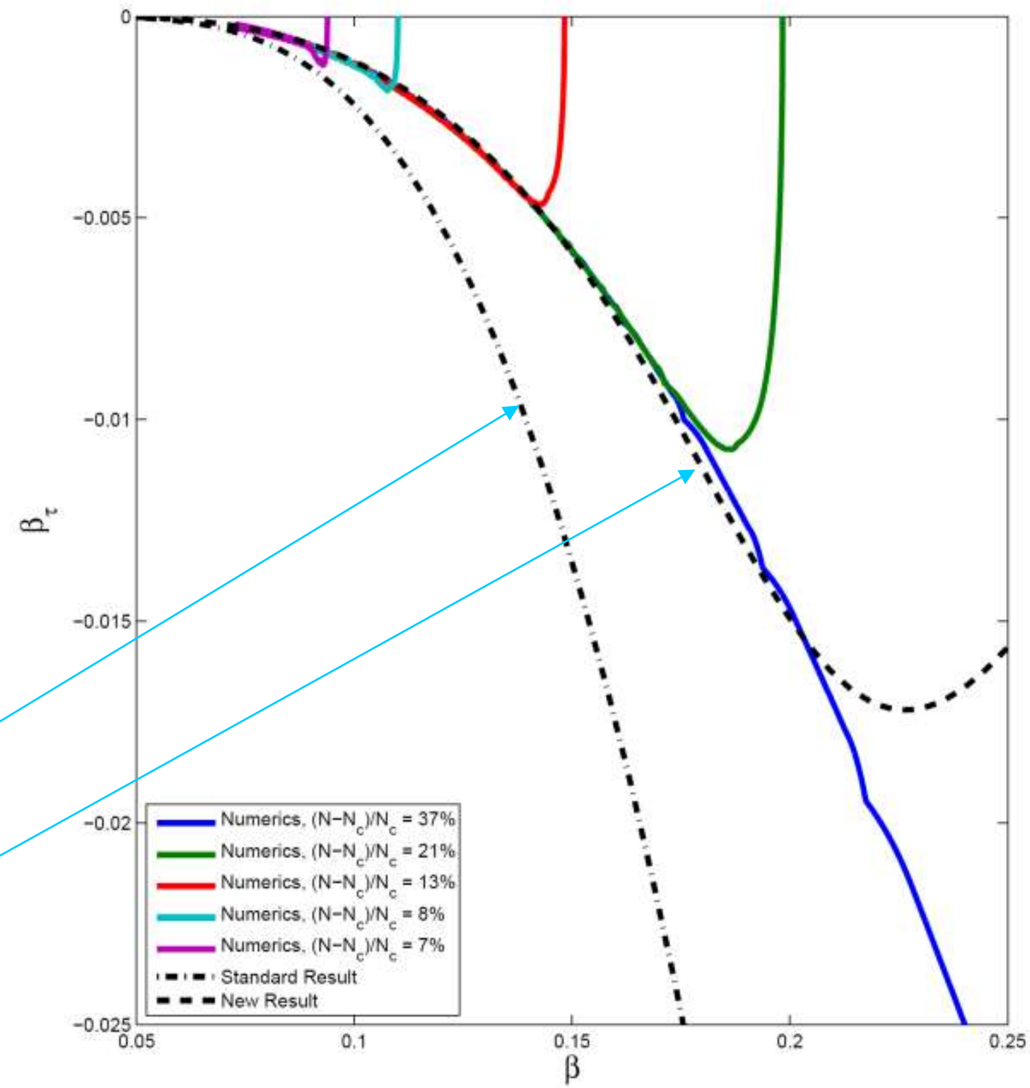
$$\begin{cases} \beta_\tau = -\tilde{M}\left[1 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + c_5\beta^5 + O(\beta^6)\right]^{-1} \exp\left[-\frac{\pi}{\beta^{1/2}}\right], \\ L^3 L_{tt} = -\beta, \\ \tau = \int_0^t \frac{dt'}{L^2(t')} \end{cases}$$

Here $$\frac{dN_b}{d\beta} = M\left[1 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + c_5\beta^5 + O(\beta^6)\right]$$

$$c_1 = 4.793,\ c_2 = 52.37,\ c_3 = 296.99,\ c_4 = -4660.87,\ c_5 = 10540.4$$

Returning to previous Figure

L(t) **is not universal but**
$\boldsymbol{\beta_\tau(\beta)}$ **is universal:**



$$\beta_\tau = -\tilde{M}\exp\left[-\frac{\pi}{\beta^{1/2}}\right]$$

$$\beta_\tau = -\tilde{M}\left[1 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + c_5\beta^5 + O(\beta^6)\right]^{-1}\exp\left[-\frac{\pi}{\beta^{1/2}}\right]$$

Finding asymptotic of a new basic ODE system

$$\begin{cases} \beta_\tau = -\tilde{M}\left[1 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + c_5\beta^5 + O(\beta^6)\right]^{-1}\exp\left[-\frac{\pi}{\beta^{1/2}}\right], \\ L^3 L_{tt} = -\beta, \\ \tau = \int_0^t \frac{dt'}{L^2(t')} \end{cases}$$

$$-\ln\frac{L}{L_0} = \frac{2\pi^3 e^x}{\tilde{M}}\left[\frac{1}{x^4} + \frac{4}{x^5} + \frac{20+\pi^2 c_1}{x^6} + \frac{120+6\pi^2 c_1}{x^7} + \frac{840+42\pi^2 c_1+\pi^4 c_2}{x^8} + \frac{6720+336\pi^2 c_1+8\pi^4 c_2}{x^9}\right.$$
$$+\frac{60480+3024\pi^2 c_1+72\pi^4 c_2+\pi^6 c_3}{x^{10}} + \frac{604800+30240\pi^2 c_1+720\pi^4 c_2+10\pi^6 c_3}{x^{11}}$$
$$+\frac{6652800+332640\pi^2 c_1+7920\pi^4 c_2+110\pi^6 c_3+\pi^8 c_4}{x^{12}} + \frac{79833600+3991680\pi^2 c_1+95040\pi^4 c_2+1320\pi^6 c_3+12\pi^8 c_4}{x^{13}}$$
$$\left.+\frac{1037836800+51891840\pi^2 c_1+1235520\pi^4 c_2+17160\pi^6 c_3+156\pi^8 c_4+\pi^{10} c_5}{x^{14}} + O\left(\frac{1}{x^{15}}\right)\right]$$

$$x = \frac{\pi}{\beta^{1/2}}$$

$$\tau = \int_0^t \frac{dt'}{L^2(t')} \quad \Rightarrow$$

$$t_c - t = \int_t^{t_c} dt = \int_\tau^\infty L^2 d\tau = \int_\beta^0 L^2 \frac{d\tau}{d\beta} d\beta$$

$$= -\int_\beta^0 L^2 \frac{1}{\tilde{M}} \left[1 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + c_5\beta^5 + O(\beta^6)\right] \exp\left[\frac{\pi}{\beta^{1/2}}\right] d\beta$$

Using $\beta(L)$ from the inversion of previous expression and inverting that equation

$$\Rightarrow$$

Asymptotic of new basic ODE system[1]

$$\begin{cases} \beta_\tau = -\tilde{M}\left[1 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + c_5\beta^5 + O(\beta^6)\right]^{-1} \exp\left[-\frac{\pi}{\beta^{1/2}}\right], \\ L^3 L_{tt} = -\beta, \\ \tau = \int_0^t \frac{dt'}{L^2(t')} \end{cases}$$

$$L = \left(\frac{2\pi(t_c - t)}{\ln A - 4\ln 3 + 4\ln\ln A}\right)^{1/2}\left[1 + \frac{2(1 + 4\ln 3 - 4\ln\ln A)}{(\ln A)^2}\right.$$
$$\left. + \frac{14 - 48\ln\ln A + 48(\ln\ln A)^2 + 48\ln 3 - 96(\ln A)(\ln 3) + 48(\ln 3)^2 + \frac{1}{2}\pi^2 c_1}{(\ln A)^3} + O\left(\frac{(\ln\ln A)^3}{(\ln A)^4}\right)\right]$$

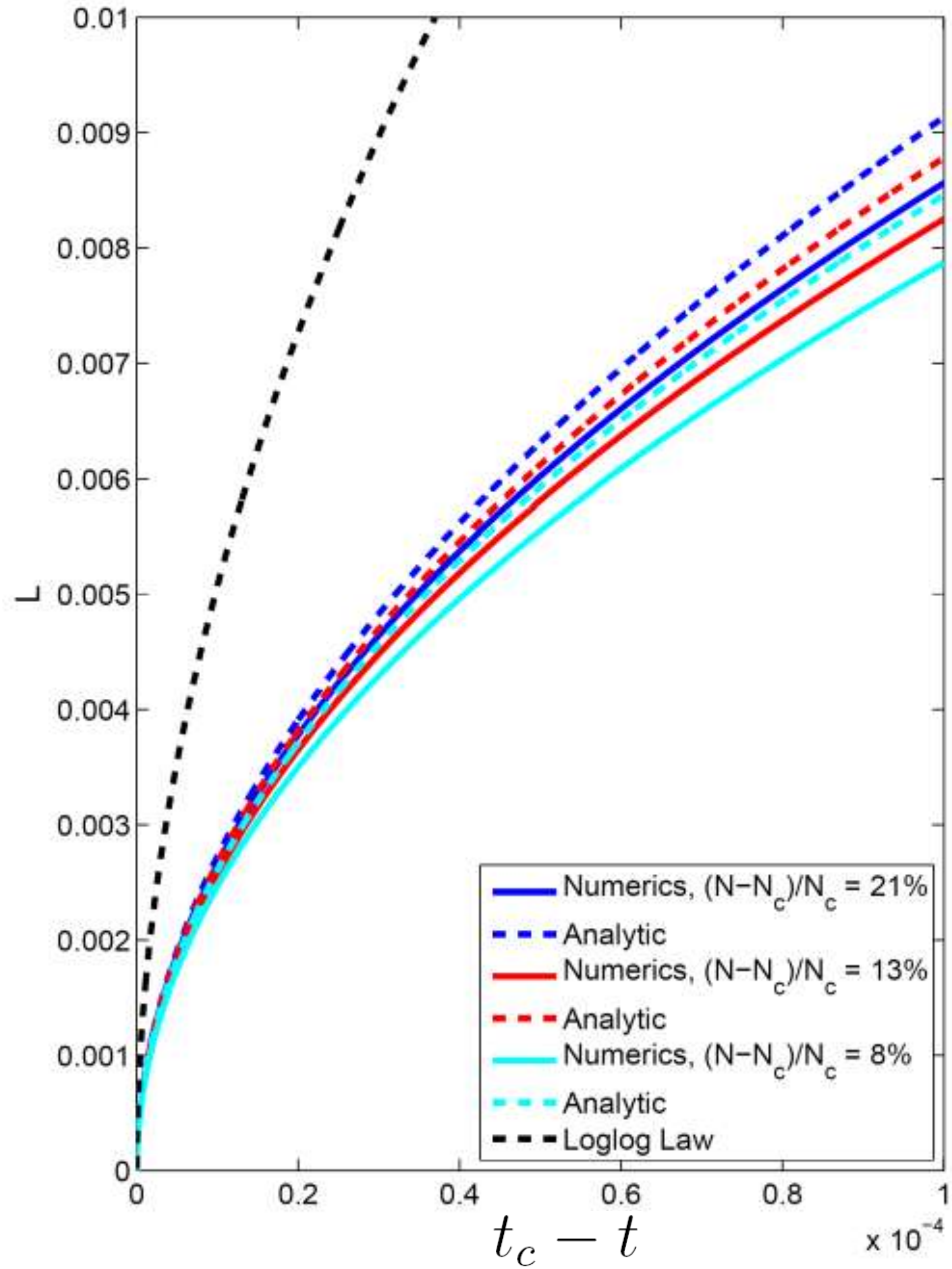
$$A = -3^4\frac{\tilde{M}}{2\pi^3}\ln\left[[2\pi(t_c - t)]^{1/2}\frac{e^{-a_0}}{L(t_0)}\right], \quad \tilde{M} = 44.773\ldots, \quad \beta_0 = \beta(t_0),\ c_1 = 4.793\ldots, c_2 = 52.37\ldots$$

$$a_0 = \frac{e^{\frac{\pi}{\sqrt{\beta_0}}}}{\tilde{M}}\left(\frac{2\beta_0^2}{\pi} + \frac{8\beta_0^{5/2}}{\pi^2} + \frac{2\beta_0^3\left(20 + \pi^2 c_1\right)}{\pi^3} + \frac{12\beta_0^{7/2}\left(20\pi^3 + \pi^5 c_1\right)}{\pi^7} + \frac{2\beta_0^4\left(840\pi^3 + 42\pi^5 c_1 + \pi^7 c_2\right)}{\pi^8}\right)$$

[1] P.M. Lushnikov, S.A. Dyachenko and N. Vladimirova, Phys. Rev. A (2013).

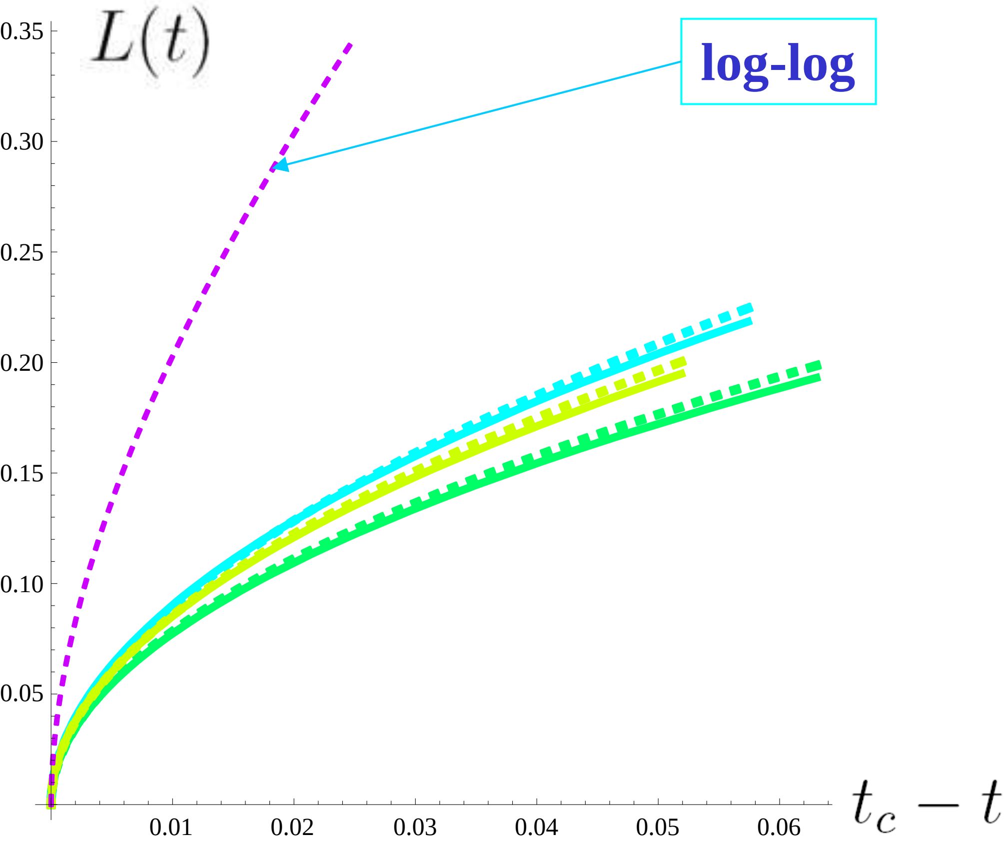
Simulations vs. analytic

$$L = \left(\frac{2\pi(t_c - t)}{\ln A - 4\ln 3 + 4\ln\ln A} \right)^{1/2}$$



Simulations vs next order analytic

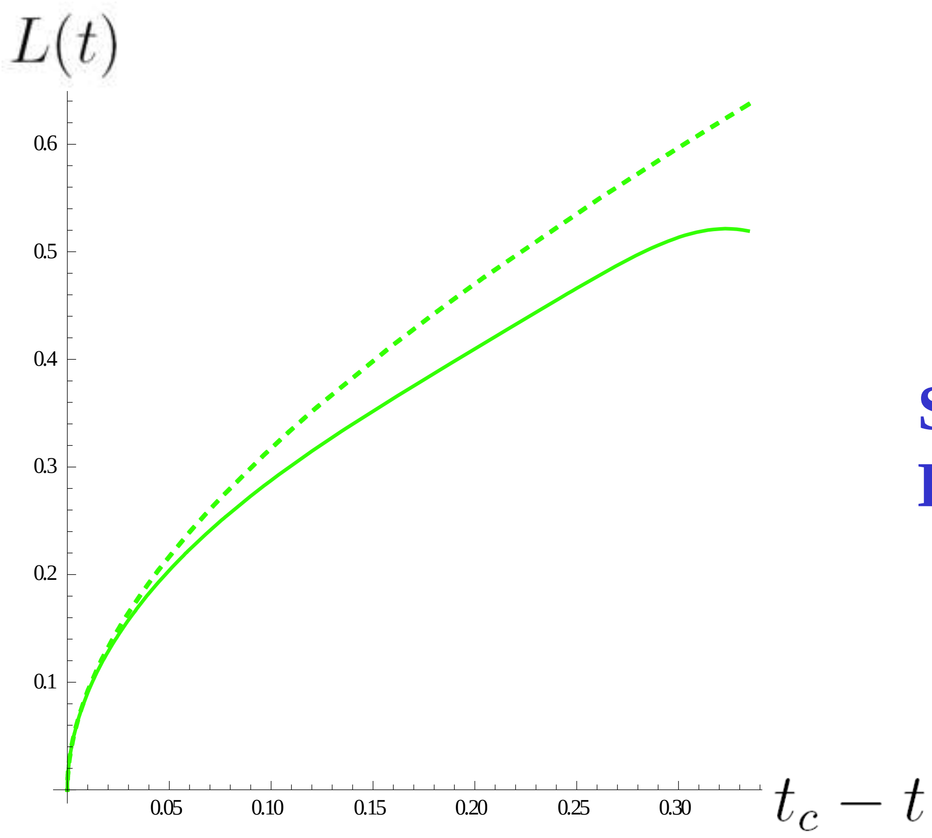
$$L = \left(\frac{2\pi(t_c - t)}{\ln A - 4\ln 3 + 4\ln\ln A}\right)^{1/2}\Bigg[1 + \frac{2(1 + 4\ln 3 - 4\ln\ln A)}{(\ln A)^2}$$
$$+\frac{14 - 48\ln\ln A + 48(\ln\ln A)^2 + 48\ln 3 - 96(\ln A)(\ln 3) + 48(\ln 3)^2 + \frac{1}{2}\pi^2 c_1}{(\ln A)^3} + O\left(\frac{(\ln\ln A)^3}{(\ln A)^4}\right)\Bigg]$$



$$A = -3^4 \frac{\tilde{M}}{2\pi^3} \ln\left[[2\pi(t_c - t)]^{1/2} \frac{e^{-a_0}}{L(t_0)}\right]$$

Solid – numerics
Dashed - analytics

Simulations vs. analytic – larger interval starting from the initial Gaussian pulse



Solid – numerics
Dashed - analytics

Conclusion

- Strong turbulence in NLS is determined by near singular collapses. Background fluctuations seed collapse. Relation between amplitude and correlation length of these fluctuations is universal. Tails of PDF for are dominated by the strong turbulence in NLS is determined by near singular collapses.

-Self-similar solution of Nonlinear Schrödinger equation

- Nonperturbative modification of standard leading order loglog scaling allows detailed numerical verification of self-similar solution. New analytical scaling is in excellent agreement with simulations for amplitudes only **3 times** above the initial laser pulse amplitude

- Standard loglog scaling dominates only for amplitudes above

$$10^{10^{100}} = 10^{\text{Googol}} = \text{Googolplex}$$