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Passive scalar transport in peripheral regions of random flows

V. Lebedev
Landau Institute for Theoretical Physics
Russia

ICTP, Trieste, July 19

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A. Chernykh and V. Lebedev

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How to extract gold particles from sand?

A. Chernykh and V. Lebedev

Advanced workshop on Nonlinear Photonics, Disorder and wave turbulence

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S. Vergeles – Landau Institute

We have in mind dust or chemicals on a surface contacted with turbulent atmosphere or silt on a river bottom. And we are interested in their transport to bulk through the turbulent laminar sublayer. The diffusivity of the particles is assumed to be weak so that turbulence plays the main role in the transport away from the surface.

The transport is described in terms of the pollutant concentration θ. For small concentrations the equation for θ in an external flow is linear:

$$\partial_t\theta + \boldsymbol{v}\nabla\theta = \kappa\nabla^2\theta\,,$$

where $\boldsymbol{v}$ is the flow velocity and κ is the diffusion coefficient. The coefficient is assumed to be small, $\kappa \ll \nu$.

In bulk the mixing time is estimated as λ^{-1} where λ is the Lyapunov exponent, irrespective to the κ value. However, the mixing time near walls is sensitive to the κ value, it can be estimated as $\sqrt{\nu/\kappa}\,\lambda^{-1}$. Besides, the velocity correlation time is estimated as λ^{-1} even near the wall. Thus, the velocity can be treated as short correlated in time for our purposes.

If the velocity is short correlated then closed equations can be derived for the passive scalar correlation functions

$$F_n(t, \boldsymbol{r}_1, \ldots, \boldsymbol{r}_n) = \langle \theta(t, \boldsymbol{r}_1) \ldots \theta(t, \boldsymbol{r}_n) \rangle \,,$$

obtained by averaging over times larger than the velocity correlation time. Let us stress that the situation is strongly anisotropic.

One derives the following equations

$$\partial_t F_n = \kappa \sum_{m=1}^{n} \nabla_m^2 F_n$$
$$+ \sum_{m,k=1}^{n} \sum_{\alpha\beta} \partial_{m\alpha} \left[D_{\alpha\beta}(\boldsymbol{r}_m, \boldsymbol{r}_k) \partial_{k\beta} F_n \right] ,$$

where the object D is expressed via the pair velocity correlation function as

$$D_{\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \int_0^\infty dt \, \langle v_\alpha(t, \boldsymbol{r}_1) v_\beta(0, \boldsymbol{r}_2) \rangle \, .$$

A z-dependence of the eddy diffusion tensor components can be found directly from the proportionality laws $v_x, v_y \propto z$ and $v_z \propto z^2$. Say,

$$D_{zz}(x, y, z_1; x, y, z_2) = \mu z_1^2 z_2^2 \,,$$

where μ is a constant characterizing strength of the velocity fluctuations in the peripheral region.

The equation for the first moment is

$$\partial_t \langle \theta \rangle = \partial_z \left[\mu z^4 \partial_z \langle \theta \rangle \right] + \kappa \partial_z^2 \langle \theta \rangle \, ,$$

Comparing two terms in RHS, one finds a characteristic diffusion length

$$r_{bl} = (\kappa / \mu)^{1/4}.$$

The quantity determines the thickness of the diffusion boundary layer.

We are interested mainly in the passive scalar transport through the region $z \gg r_{bl}$, where the passive scalar is carrying from the diffusive boundary layer to bulk. There we arrive at the proportionality law

$$\langle \theta \rangle \propto z^{-3},$$

that gives the decaying rate of the average θ as z grows.

What about higher moments of θ? If diffusion is negligible outside the diffusive boundary layer then

$$\partial_t \theta = -\boldsymbol{v}\nabla\theta \quad \rightarrow \quad \partial_t(\theta^n) = -\boldsymbol{v}\nabla(\theta^n).$$

Therefore we arrive at the conclusion that

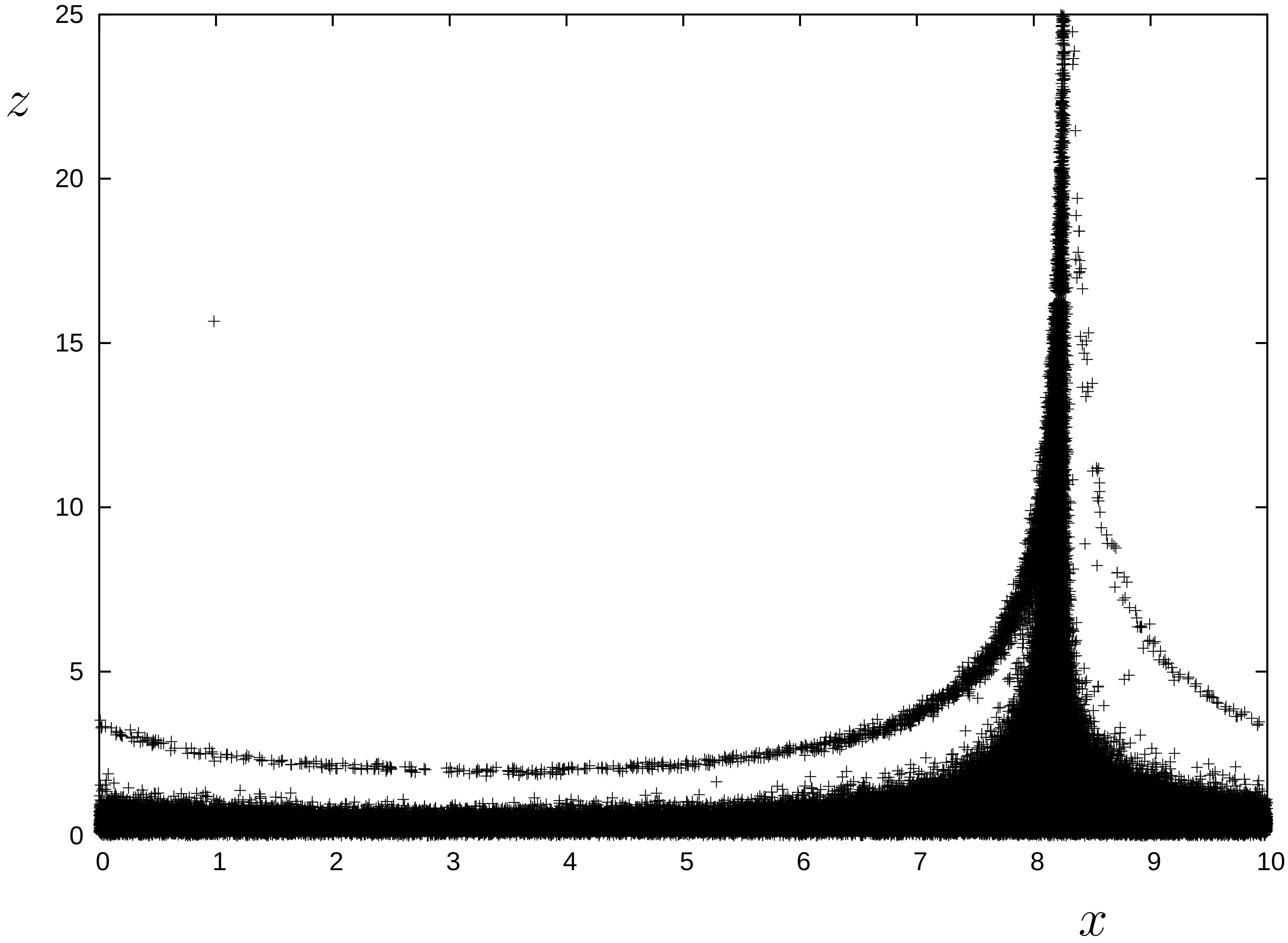
$$\langle\theta^n\rangle \propto z^{-3},$$

as well !

However, numerics shows that at $z \gg r_{bl}$ the moments are characterized by the exponents η_n

$$\langle \theta^n \rangle \propto z^{-\eta_n},$$

different from 3! That implies that diffusion is relevant at the scales. How it can be? To answer the question we turn to numerics.


z
25
20
15
10
5
0
0
1
2
3
4
5
6
7
8
9
10
x

One can define the passive scalar correlation length l (along the wall), that can be found by balance of the molecular and the eddy diffusion along the wall:

$$l \sim \sqrt{\kappa/\mu}\ z^{-1}.$$

The quantity is of order of r_{bl} at $z \sim r_{bl}$ and diminishes as z grows.

To exclude the effect of the molecular diffusion, we introduce an integral of the passive scalar field

$$\Theta(t,z) = A^{-1}\int dx\ dy\ \theta(t,x,y,z)\,,$$

where A is the area of the surface and z is its separation from the wall. Obviously $\langle\Theta\rangle \propto z^{-3}$. What about high-order moments?

Assuming that the passive scalar correlation length is smaller than the velocity one, we can derive

$$\partial_t \langle \Theta^n \rangle = \mu \left[z^4 \partial_z^2 + 4nz^3 \partial_z + 4n(n-1) z^2 \right] \langle \Theta^n \rangle \, .$$

The equation leads to the scaling

$$\langle \Theta^n \rangle \propto z^{-\zeta_n}, \qquad \zeta_n = 2n - 1/2 + \sqrt{2n + 1/4} \, .$$

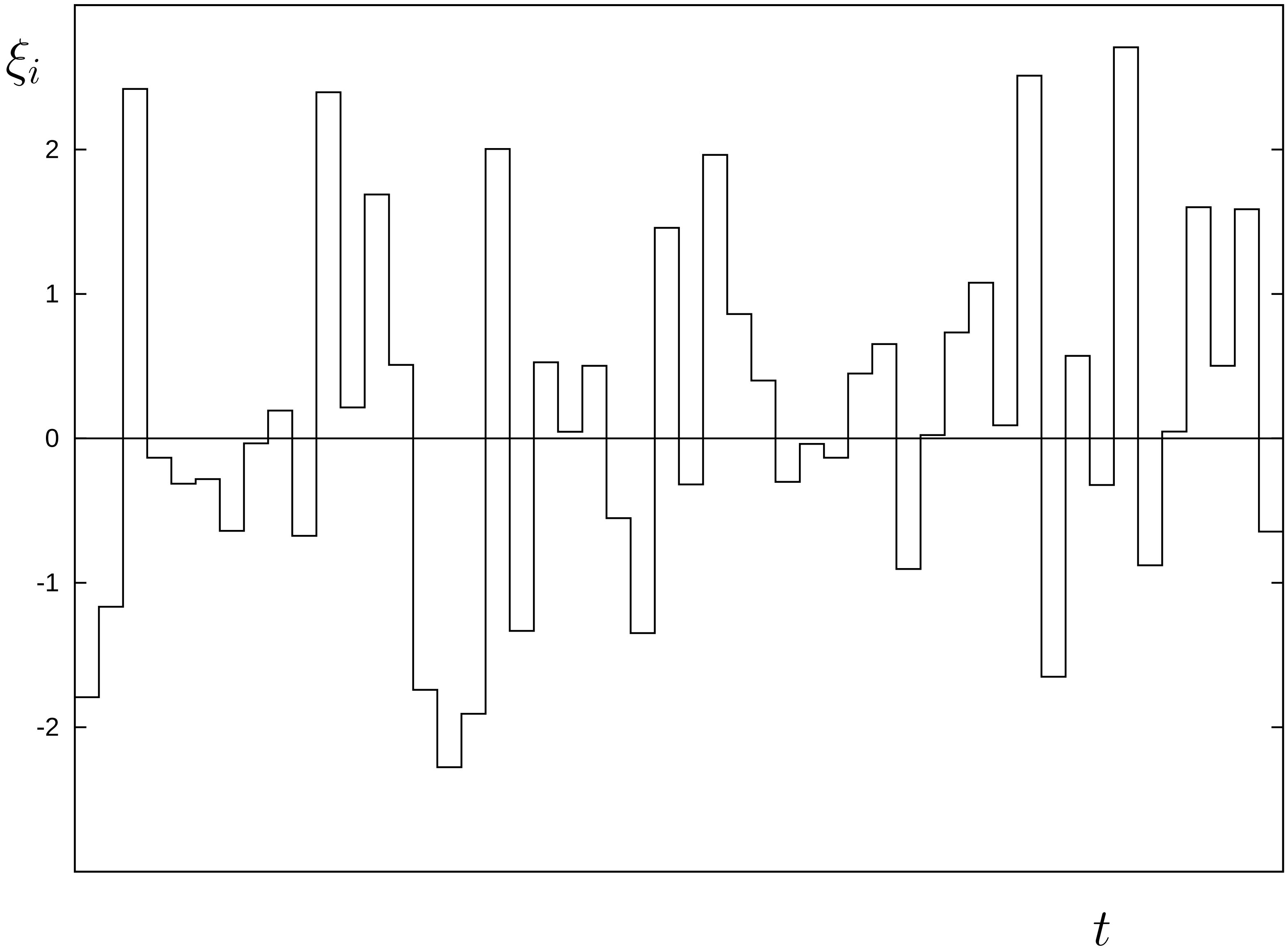
Obviously, $\zeta_n > 3$. What does it mean? That means that only a small part of the tongues reaches (for a finite time) infinity delivering an amount of particles to bulk. And the particle flux to bulk is related to such events. Typically, the tongue is pulled upto some finite z and then goes back, tilts and nestles upto the diffusive boundary layer.

We conducted Lagrangian simulations where dynamics of a large number of particles subjected to flow advection and Langevin forces is examined. The set of the particles is used instead of the passive scalar field θ, that can be treated as density of the particles. A big advantage of the approach is its applicability to a number of space dimensions d.

In our scheme a particle trajectory $\varrho(t)$ obeys the equation

$$\partial_t \varrho = v(t, \varrho) + \chi(t),$$

where the first term represents the particle advection and the second term represents the Langevin force. The variables χ are independent for different particles whereas the velocity is the same.

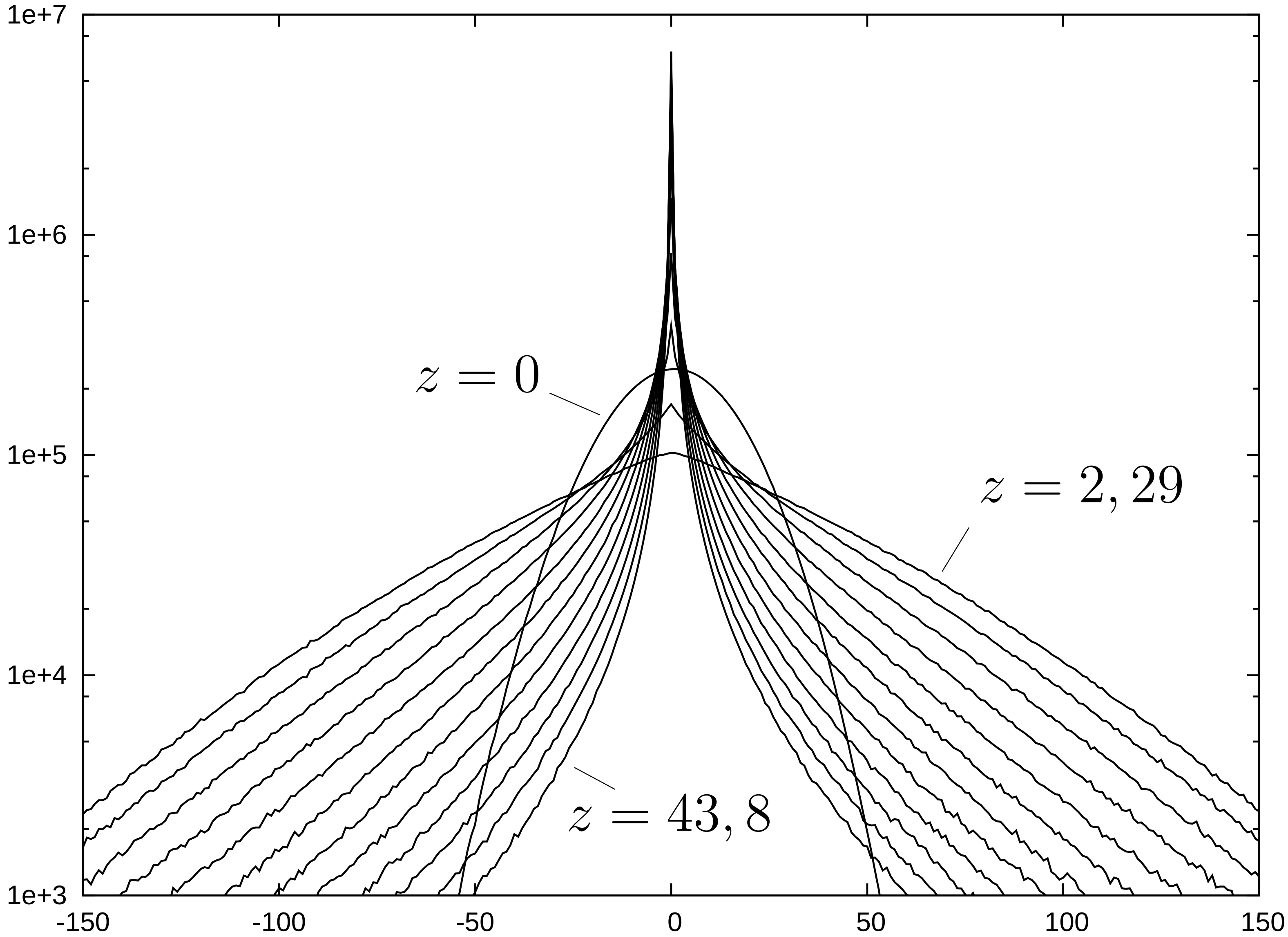


ξ_i
2
1
0
-1
-2
t

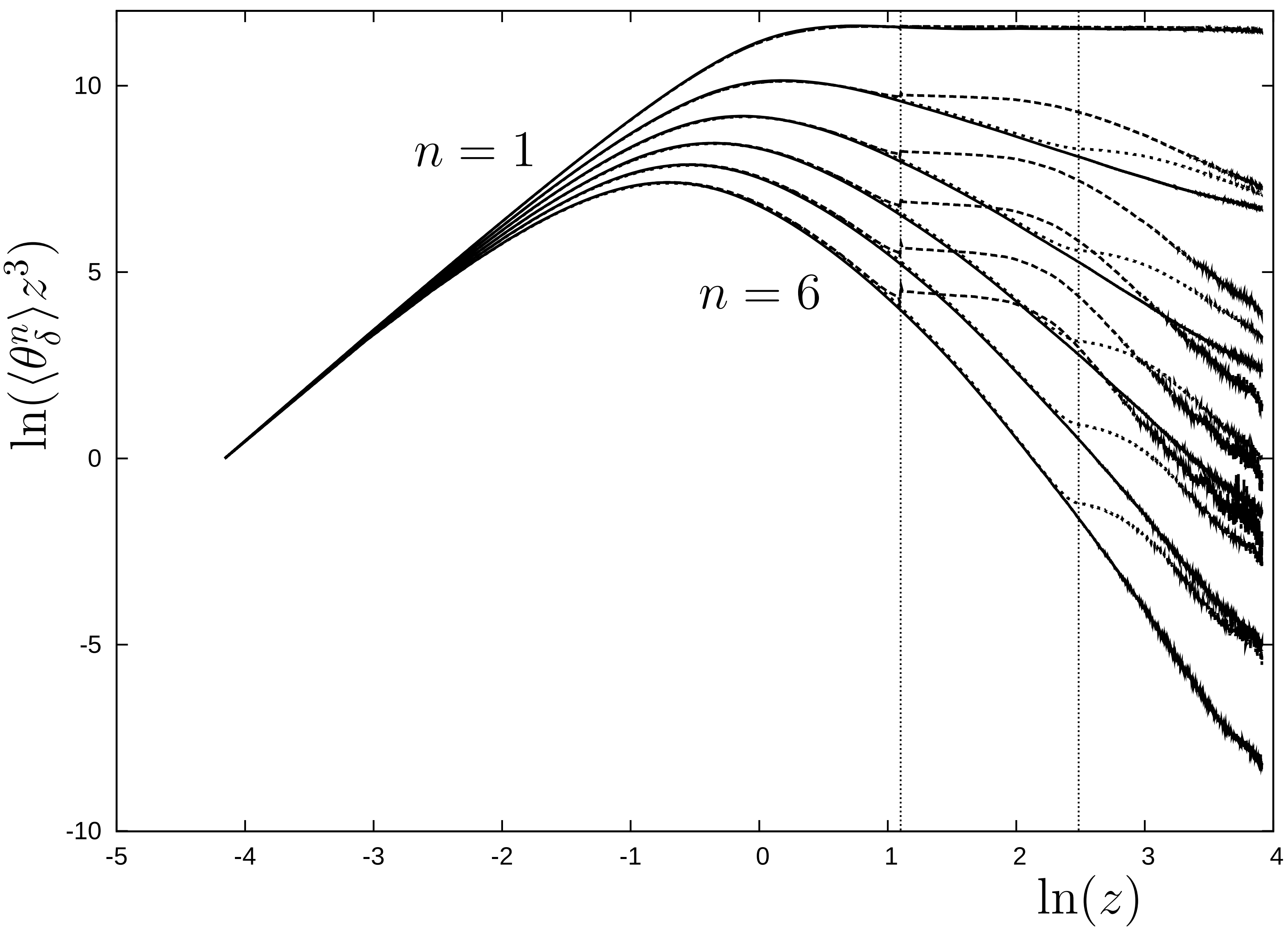
To establish principal qualitative features of the process, we perform mainly $2d$ simulations. The setup is periodic in x and the velocity in majority of runs was

$$v_x = z\left(\xi_1 \cos\frac{2\pi x}{L} + \xi_2 \sin\frac{2\pi x}{L}\right)\frac{L}{\pi},$$
$$v_z = z^2\left(\xi_1 \sin\frac{2\pi x}{L} - \xi_2 \cos\frac{2\pi x}{L}\right),$$

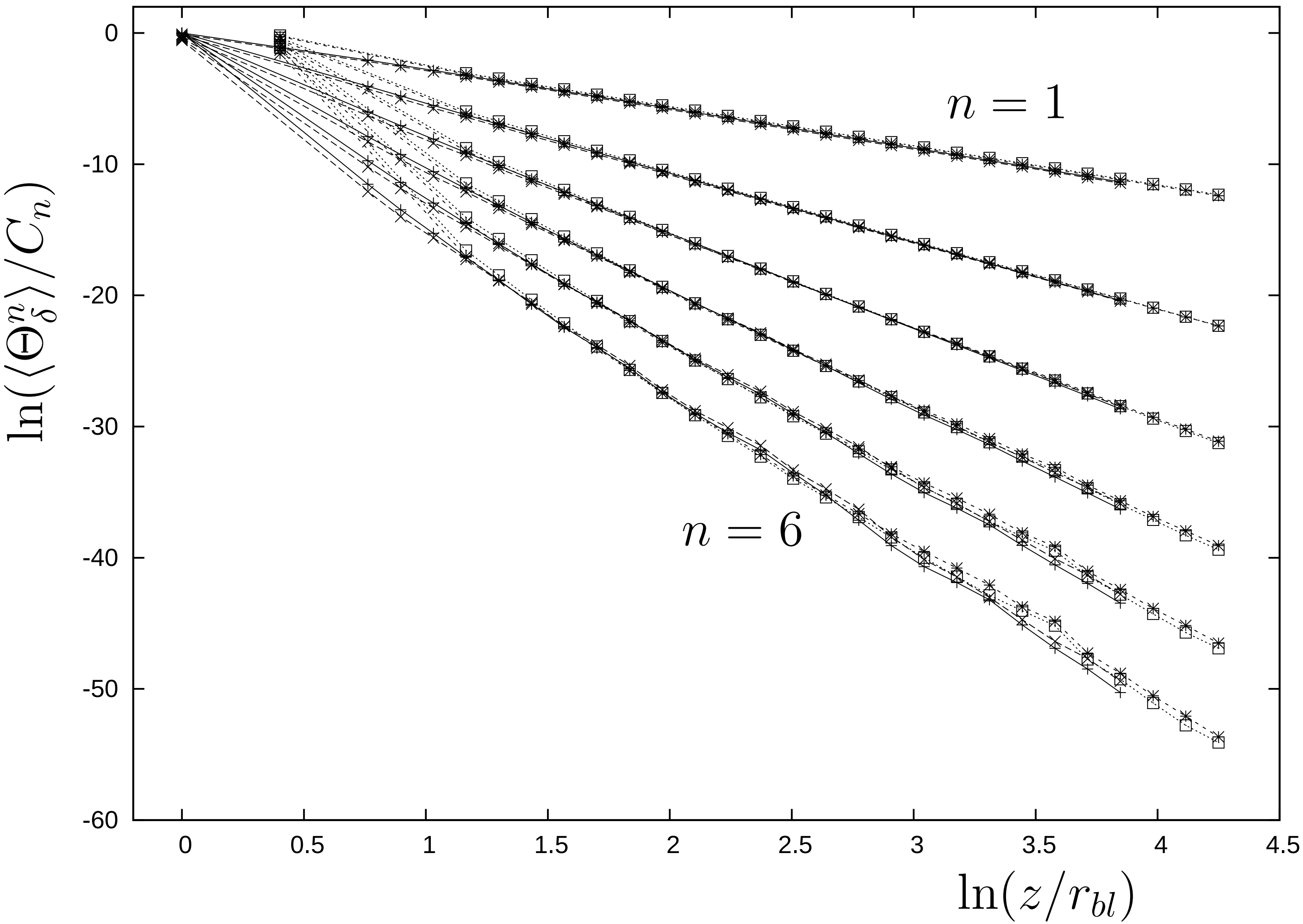
where ξ_1 and ξ_2 are independent random functions of time.



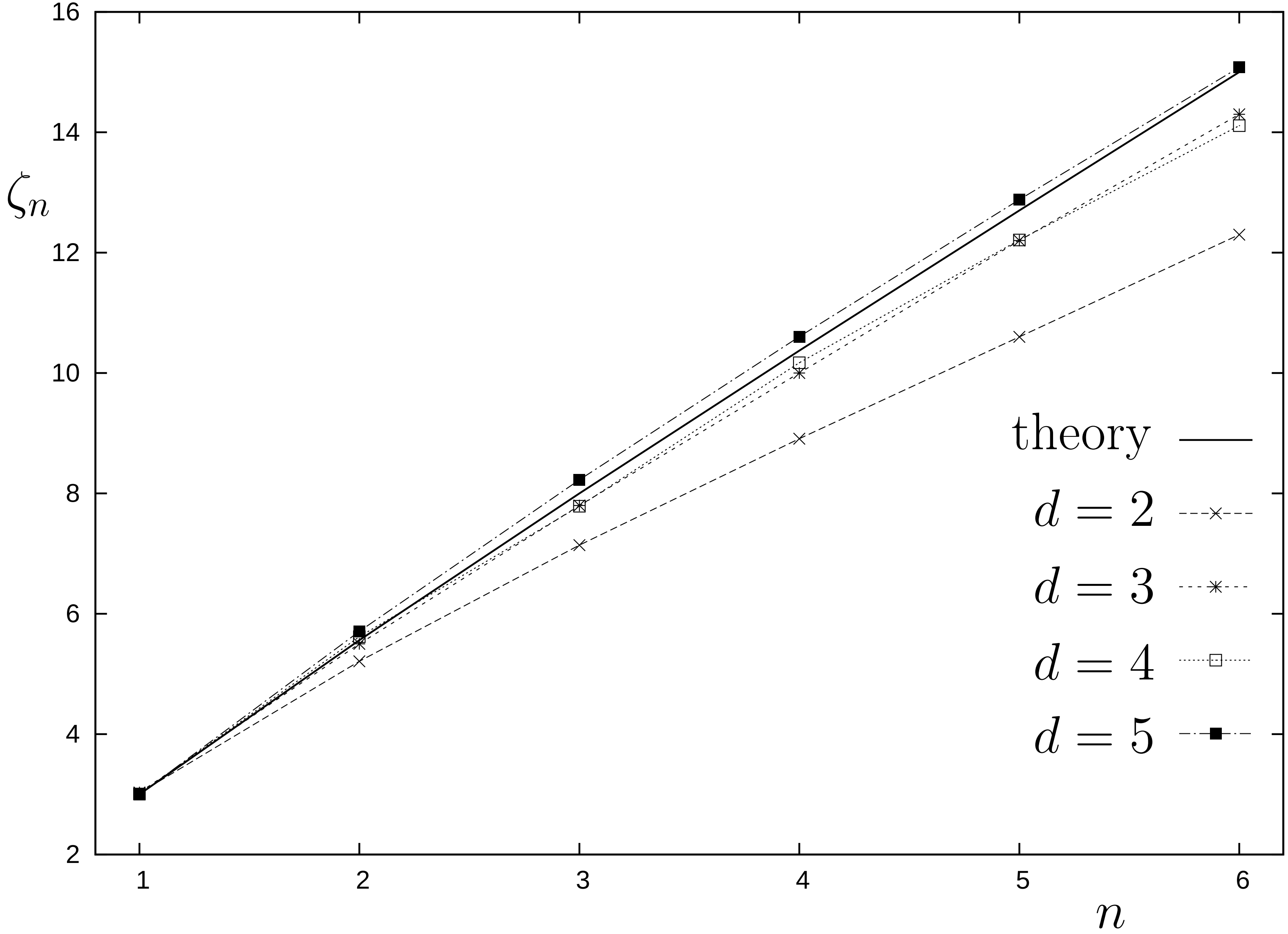
1e+7
1e+6
1e+5
1e+4
1e+3
-150
-100
-50
0
50
100
150
$z = 0$
$z = 2,29$
$z = 43,8$



$n = 1$
$n = 6$
$\ln(\langle\theta_\delta^n\rangle z^3)$
$\ln(z)$
10
5
0
-5
-10
-5
-4
-3
-2
-1
0
1
2
3
4



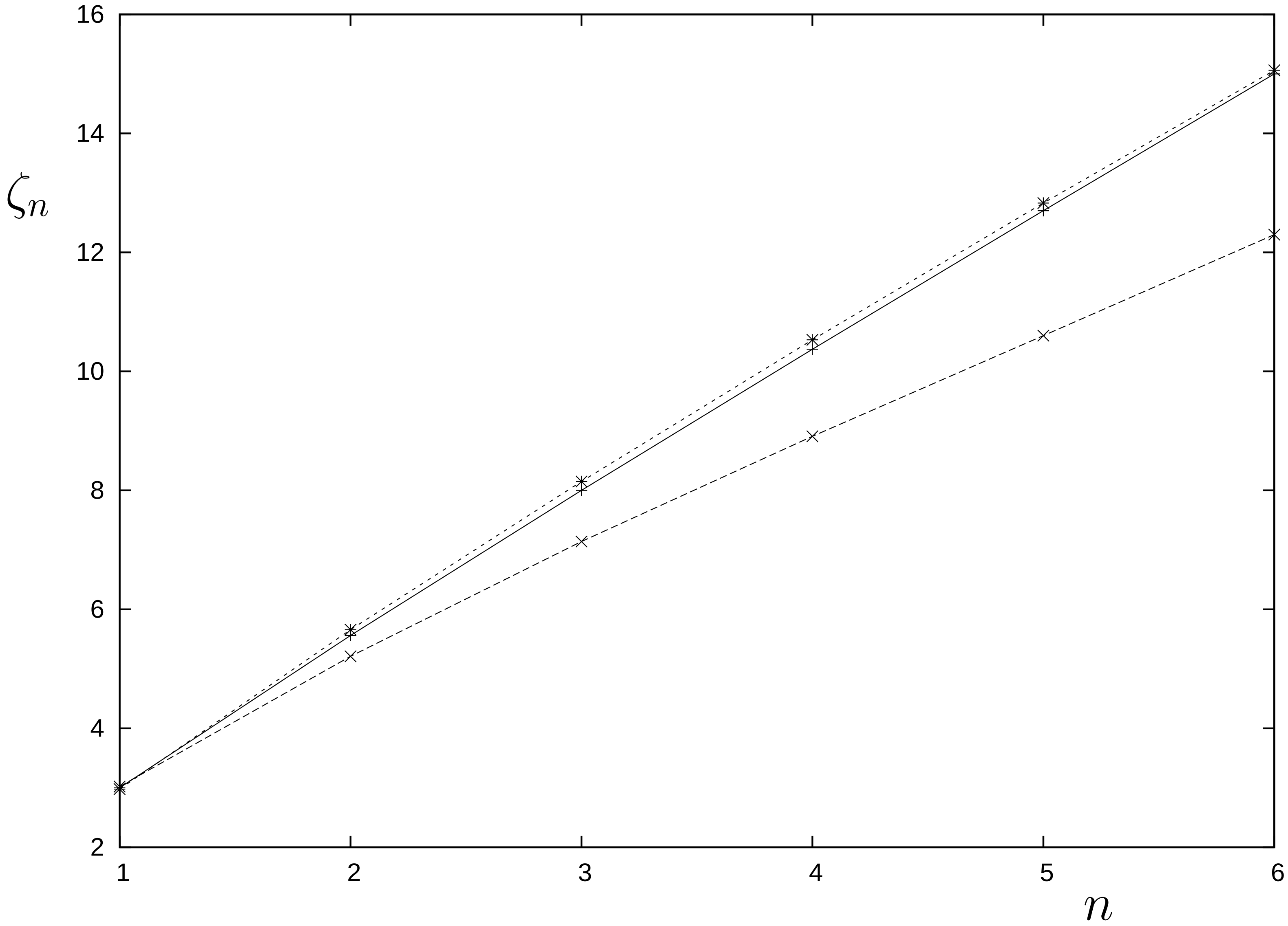
$n = 1$
$n = 6$
$\ln(\langle \Theta_\delta^n \rangle / C_n)$
$\ln(z/r_{bl})$
0
-10
-20
-30
-40
-50
-60
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5



16
14
12
10
8
6
4
2
1
2
3
4
5
6
ζ_n
n
theory
$d = 2$
$d = 3$
$d = 4$
$d = 5$

Numerics reveal deviations of the scaling exponents from the analytical predictions that are related to an existence of long correlations along the wall that can be produced by the multi-fold structures. That leads to increasing moments in comparison with the short correlated case. That is why the scaling exponents are smaller than the theoretical values.

The deviations naturally diminish as d grows. However, the effect is a consequence of the artificial fact that the velocity correlation length coincides with the size system. Let us consider $d = 2$ and make the velocity correlation length smaller than L by using a mixture of the ninth and the eleventh harmonics. Then we arrive at a good agreement with numerics.



16
14
12
10
8
6
4
2
1
2
3
4
5
6
ζ_n
n

Inertial particles are governed by the equation

$$\tau \frac{d\boldsymbol{u}}{dt} + \boldsymbol{u} = \boldsymbol{v} + \boldsymbol{\xi}.$$

Here, $\boldsymbol{u}$ is the particle velocity, $\boldsymbol{v}$ is the flow velocity, $\boldsymbol{\xi}$ is Langevin force and τ is the particle relaxation time associated with its inertial properties.

If the diffusion is neglected then the equation for the one-particle probability density is

$$\partial_t \rho = -u\partial_z \rho + \partial_u (u\rho) + z^4 \partial_u^2 \rho,$$

where we put $\mu = \tau = 1$. We will be looking for stationary distribution of the probability density that corresponds to zero probability flux to z-infinity:

$$\int_{-\infty}^{+\infty} du\ u\rho = 0.$$

For $u \ll z$ the equation is reduced to

$$\partial_u(u\rho) + z^4 \partial_u^2 \rho = 0,$$

that have the solution

$$\rho \propto z^{-6} \exp\left(-\frac{u^2}{2z^4}\right).$$

It is realized at $z \ll 1$ since $u \sim z^2$.

For $u \gg z$ the equation is reduced to

$$u\partial_z\rho = z^4\partial_u^2\rho.$$

After the self-similar substitution $\rho = z^{-5a}h(\xi)$ where $\xi = u/z^{5/3}$ we obtain

$$h'' + (5/3)\xi^2 h' + 5a\xi h = 0.$$

It can be solved in terms of the confluent hypergeometric function.

Both asymptotics are zero if

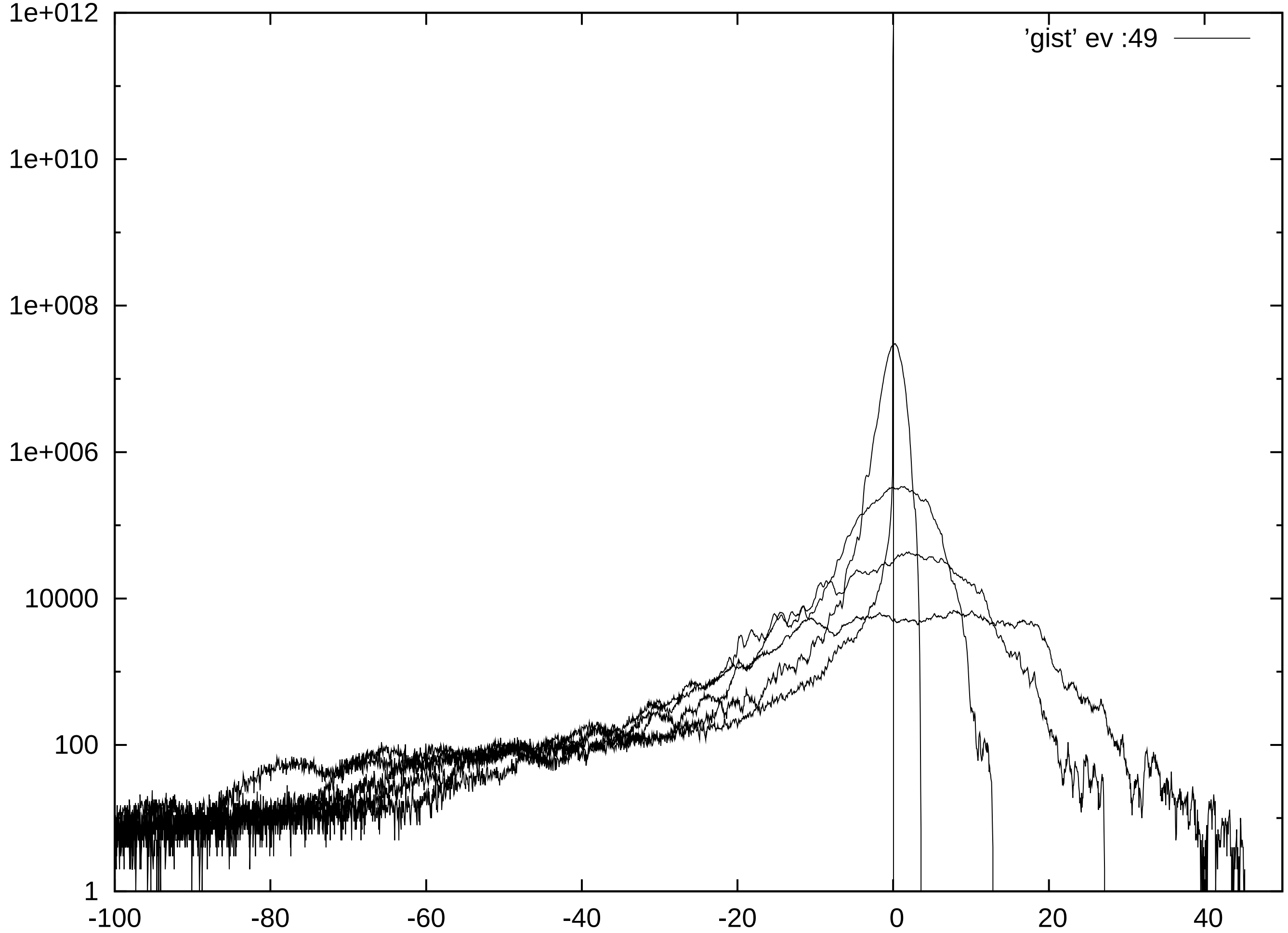
$$a = -\frac{1}{6} + n \to 5/6.$$

Then at large negative u: $\rho \propto |u|^{-5/2}$.

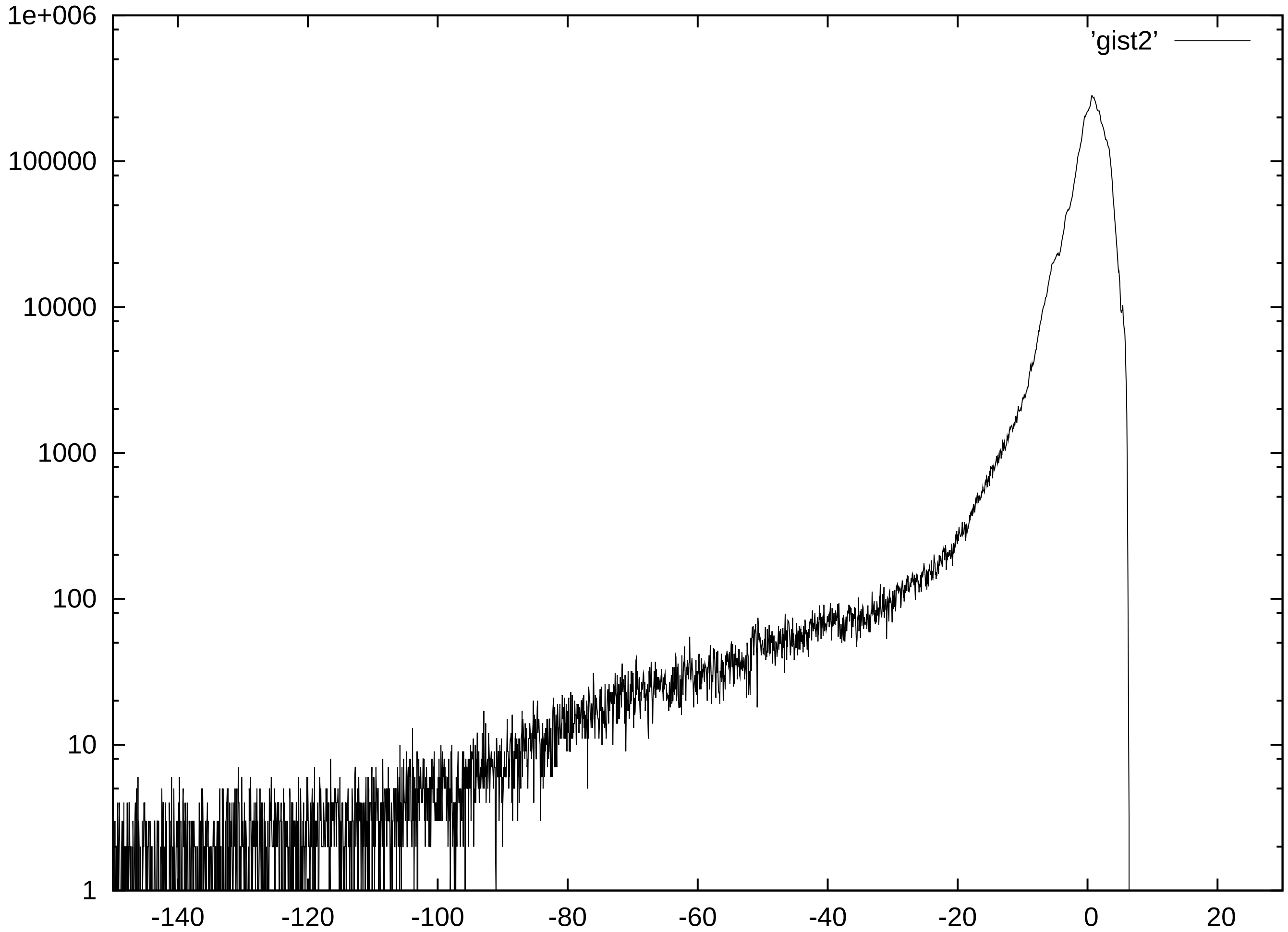
At large positive u

$$\rho \propto \exp\left(-\frac{5}{9}\frac{u^3}{z^5}\right).$$

Highly non-symmetric distribution.



1e+012
1e+010
1e+008
1e+006
10000
100
1
-100
-80
-60
-40
-20
0
20
40
'gist' ev :49



1e+006
100000
10000
1000
100
10
1
-140
-120
-100
-80
-60
-40
-20
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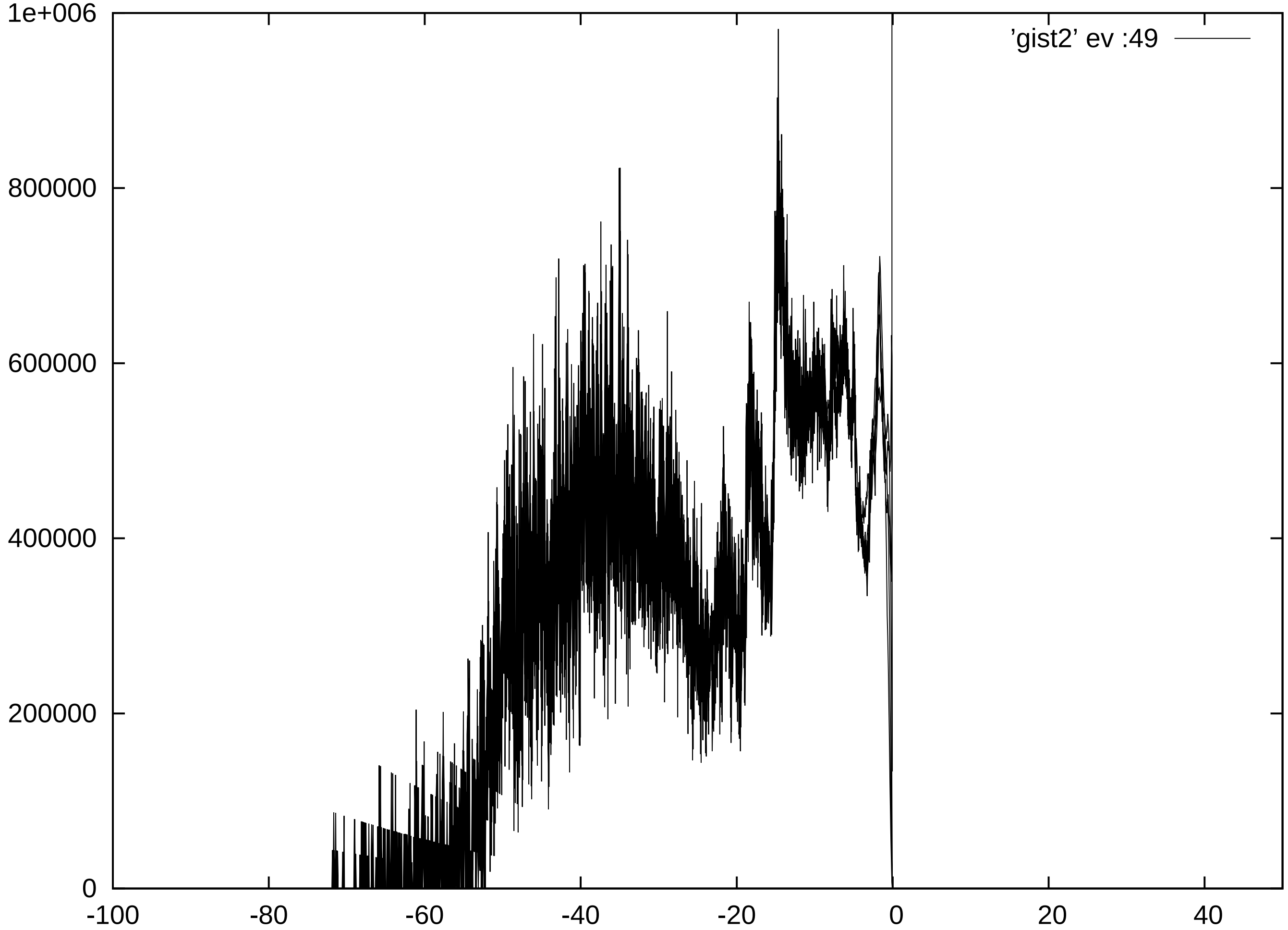
20
'gist2'


1e+006
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600000
400000
200000
0
-100
-80
-60
-40
-20
0
20
40
'gist2' ev :49