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2472-6

Advanced Workshop on Nonlinear Photonics, Disorder and Wave Turbulence

15 - 19 July 2013

Emission and capture of light in random media: Euclidean random matrix theory

S.E. Skipetrov

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Collaborations: A. Goetschy and I.M. Sokolov

Université Joseph Fourier GRENOBLE

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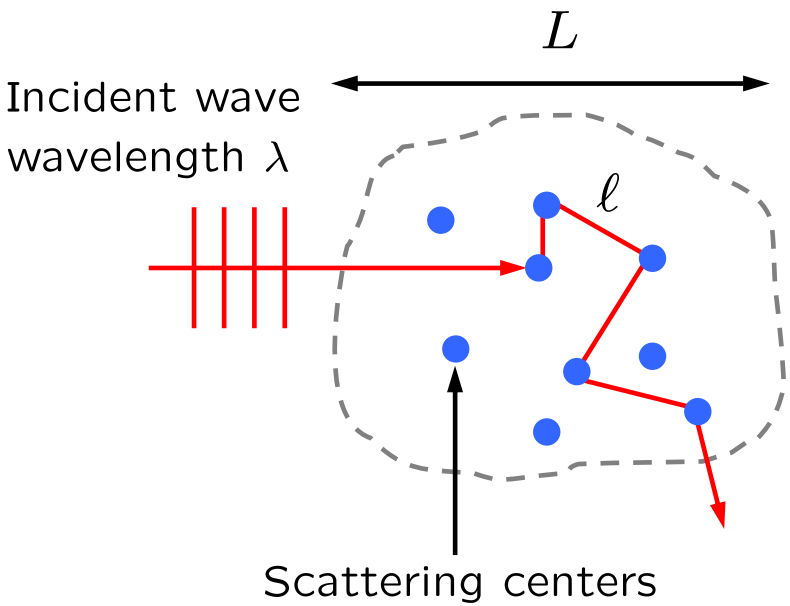
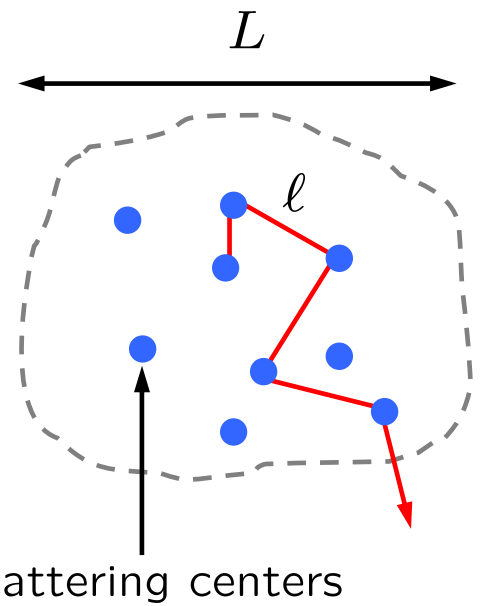
cnrs

The problem: waves in 3D random media

How long does a wave stay inside the medium ?

How does the medium emit waves ?

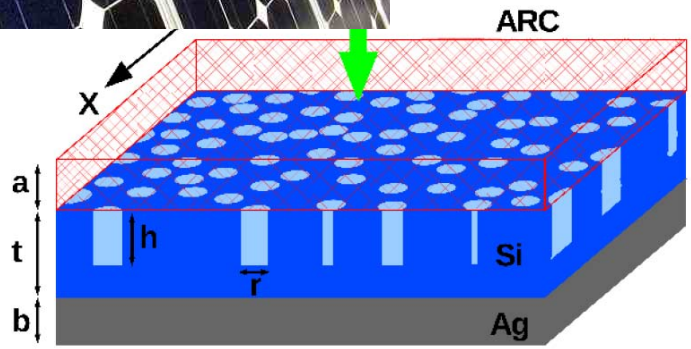




We propose a new random matrix theory approach to deal with these problems

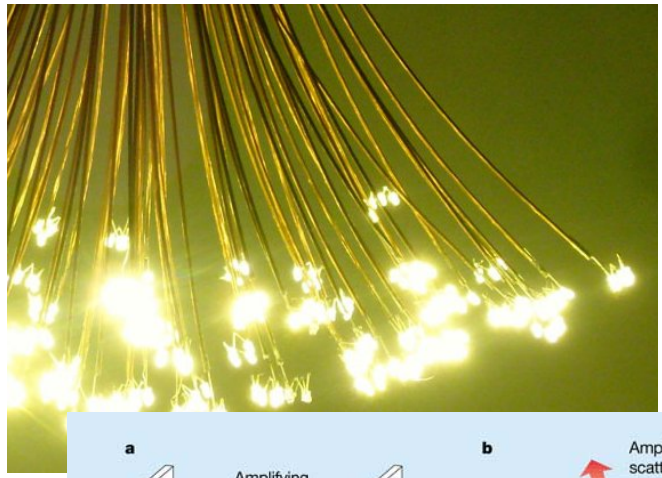
Why is it important ?

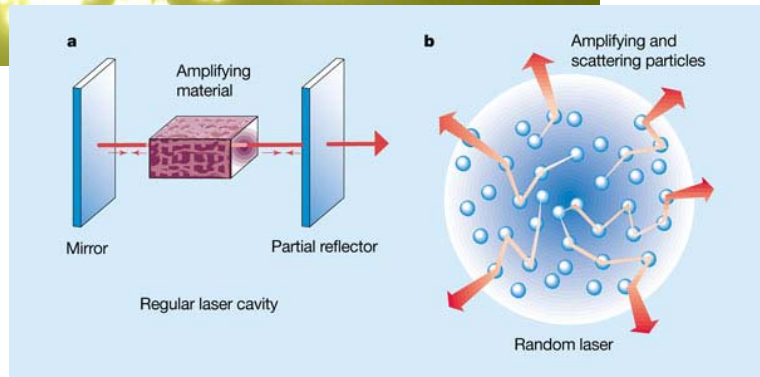
Disorder for solar cells



F. Pratesi *et al.*,
Optics Express **21**, A460 (2013)

New sources of light

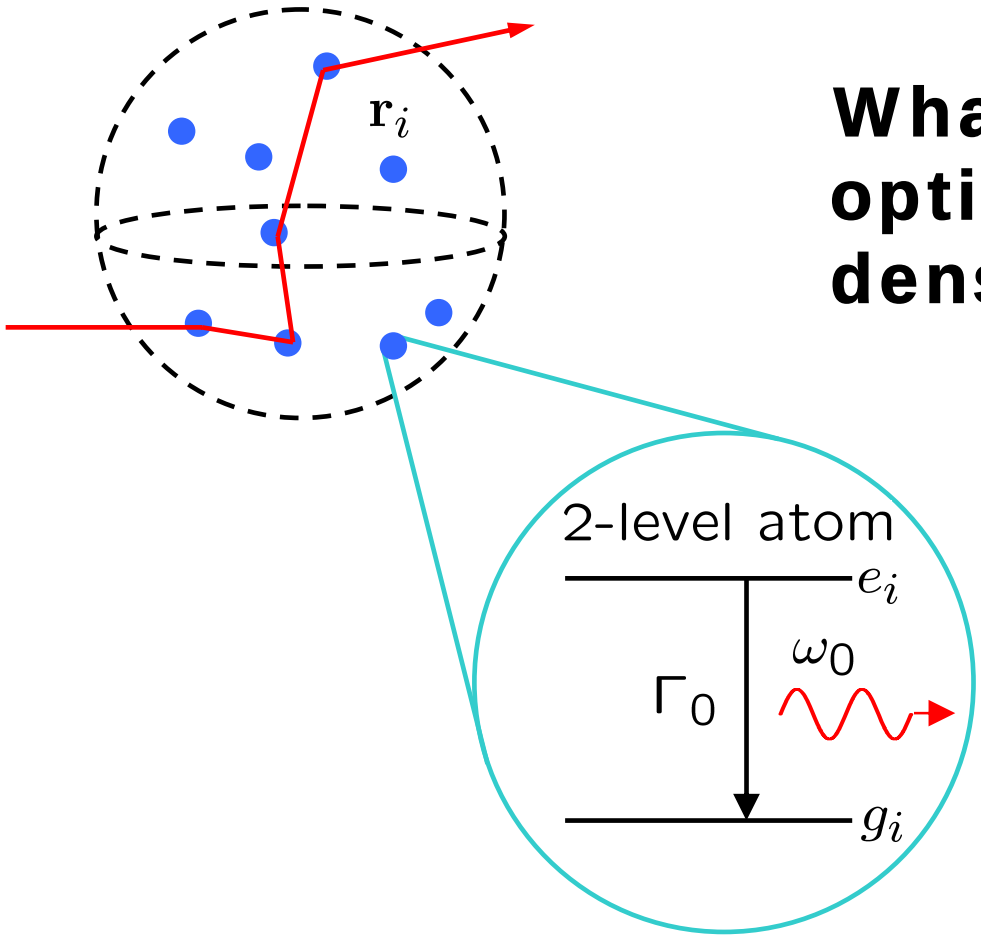




D. Wiersma,
Nature **406**, 132 (2000)

The model

N randomly distributed atoms

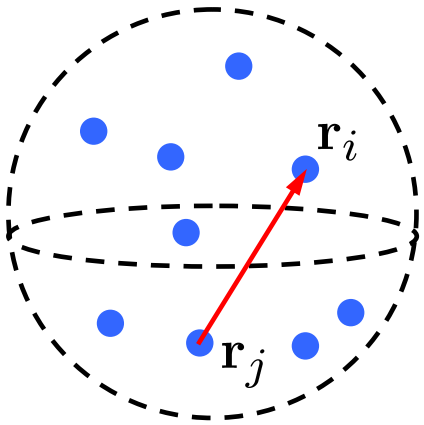


(atom = point scatterer)

What are the collective optical properties of dense atomic clouds?

Green's matrix

N randomly distributed atoms



Propagation of light from atom *j* to atom *i* is described by the **Green's matrix** $\hat{G}$

$$r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$$
$$k_0 = \omega_0/c$$

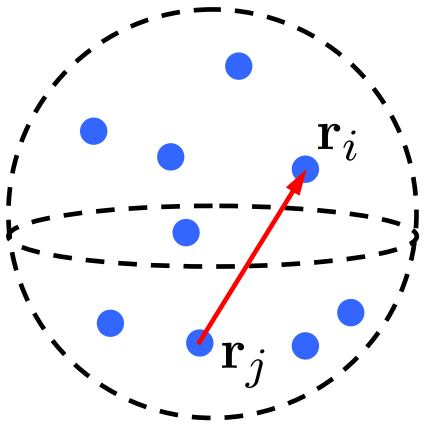
$$G_{e_i e_j} = \mathrm{i}\delta_{e_i e_j} + (1 - \delta_{e_i e_j})\frac{e^{\mathrm{i}k_0 r_{ij}}}{k_0 r_{ij}}$$ **(scalar approximation)**

$$G_{e_{im} e_{jm'}} = \mathrm{i}\delta_{e_{im} e_{jm'}} - \frac{2}{\Gamma_0}(1 - \delta_{e_{im} e_{jm'}}) \sum_{\mu,\nu} d^{\mu}_{e_{im} g_i} d^{\nu}_{g_j e_{jm'}} \frac{e^{\mathrm{i}k_0 r_{ij}}}{\hbar r_{ij}^3}$$
$$\times \left\{ \delta_{\mu\nu} \left[1 - \mathrm{i}k_0 r_{ij} - (k_0 r_{ij})^2\right] - \frac{r^{\mu}_{ij} r^{\nu}_{ij}}{r_{ij}^2} \left[3 - 3\mathrm{i}k_0 r_{ij} - (k_0 r_{ij})^2\right] \right\}$$

(full vector case)

Green's matrix

N randomly distributed atoms



$$r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$$
$$k_0 = \omega_0/c$$

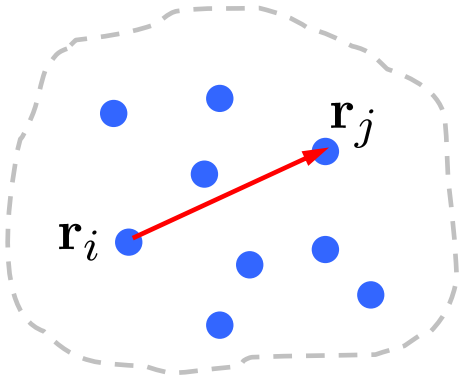
Green's matrix $\hat{G}$ describes the propagation of light between pairs of atoms

Eigenvalue problem:

$$\hat{G}\boldsymbol{\psi}_n = \Lambda_n \boldsymbol{\psi}_n$$

Analysis of the eigenvalue problem will allow us to understand collective properties of large ensembles of atoms (= point scatterers)

Green's matrix as a Euclidean random matrix



$i, j = 1, \ldots N$

Mézard, Parisi and Zee
Nucl. Phys. B **559**, 689
(1999)

$N \times N$ matrix $F_{ij} = f(\mathbf{r}_i, \mathbf{r}_j)$ ← Arbitrary function

$= f(\mathbf{r}_i - \mathbf{r}_j)$

$= f(|\mathbf{r}_i - \mathbf{r}_j|)$

Eigenvalue problem:

$$\hat{F}\boldsymbol{\psi}_n = \Lambda_n \boldsymbol{\psi}_n$$

Λ_n — eigenvalues

$\boldsymbol{\psi}_n = \{\psi_n^i\}$ — eigenvectors

$n = 1, \ldots, N$

We will be interested in the random Green's matrix:

$$f(\mathbf{r}_i, \mathbf{r}_j) = G(\mathbf{r}_i, \mathbf{r}_j), \quad G_{ij} = G(\mathbf{r}_i, \mathbf{r}_j) = \frac{e^{\mathrm{i}k_0|\mathbf{r}_i - \mathbf{r}_j|}}{k_0|\mathbf{r}_i - \mathbf{r}_j|}$$

(scalar approximation)

Green's matrix as a Euclidean random matrix

$$\hat{G}\boldsymbol{\psi}_n = \Lambda_n \boldsymbol{\psi}_n$$

Analytic theory for the eigenvalue distribution of an arbitrary non-Hermitian Euclidean matrix in the limit of large size $N \to \infty$

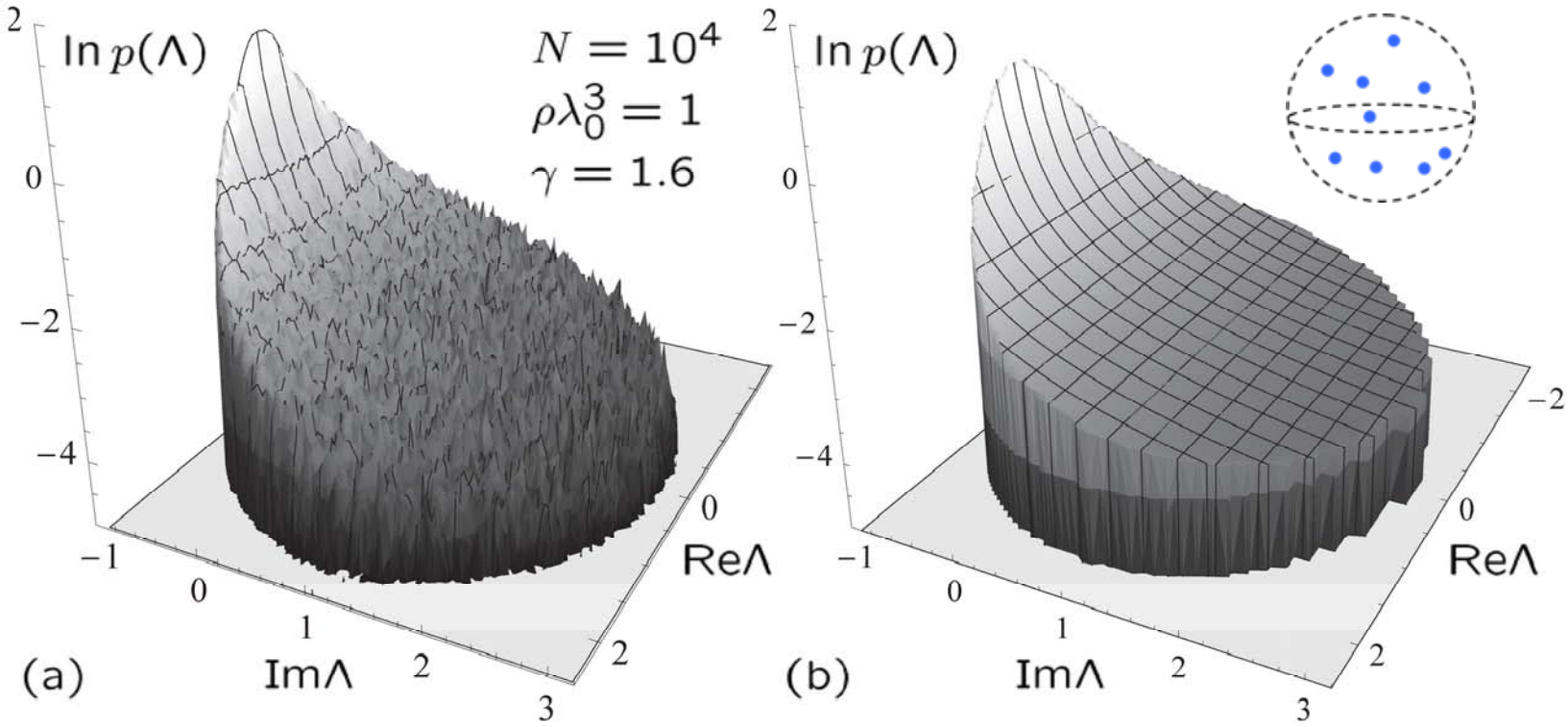
J. Phys. A: Math. Theor. **44**, 065102 (2011)
Phys. Rev. E **84**, 011150 (2011)
EPL **96**, 34005 (2011)
arXiv:1303.2880

Main advances of our approach with respect to previous theories of Euclidean random matrices:

- Non-perturbative in density ρ or $1/\rho$
- Full account for the finite sample size and the break-down of translational invariance
- General equations derived for non-Hermitian matrices

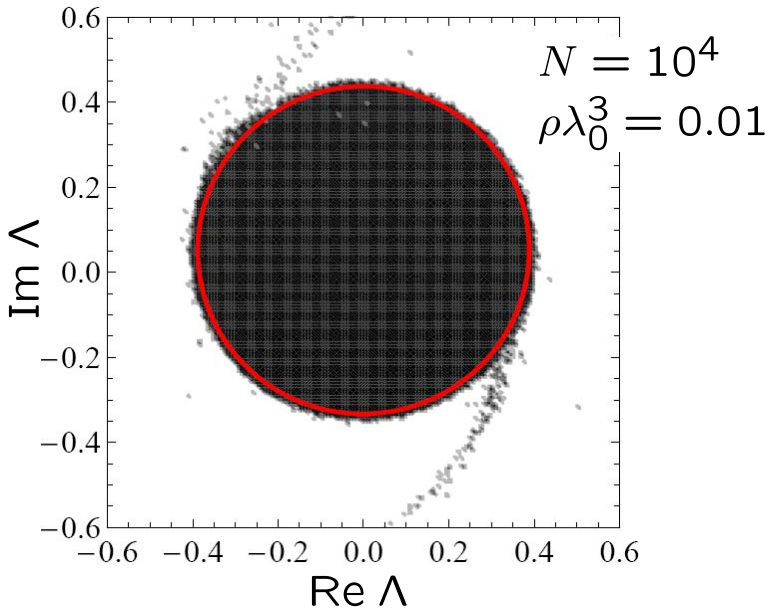
Eigenvalue density (scalar)

Numerical simulations **Analytic theory**



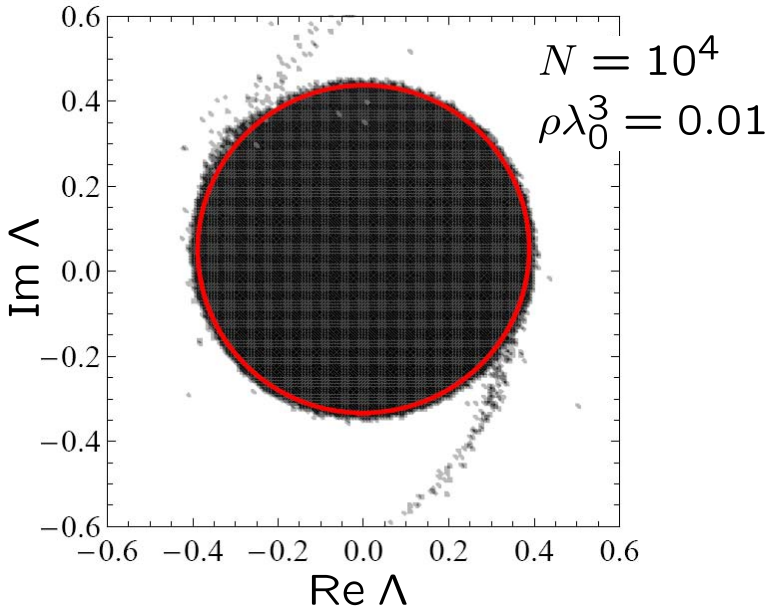
A. Goetschy & S.E. Skipetrov, PRE **84**, 011150 (2011)
arXiv:1303.2880

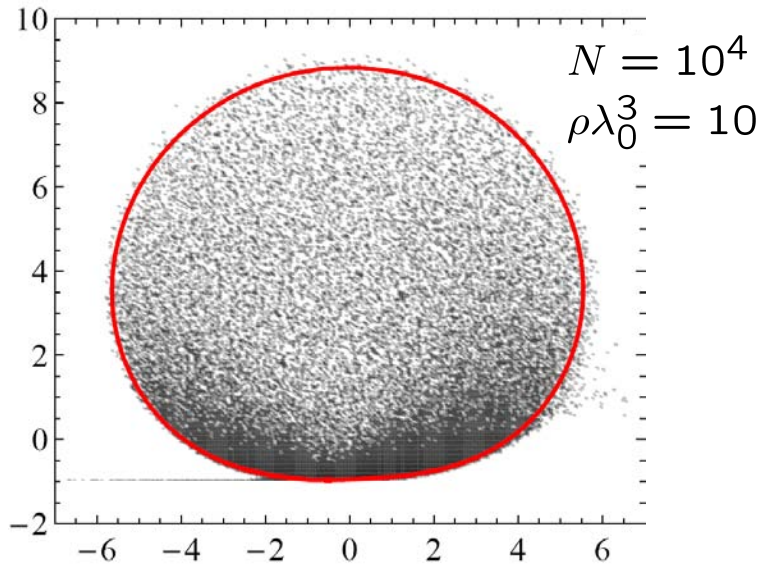
Borderline of the eigenvalue domain (scalar)



A. Goetschy & S.E. Skipetrov, PRE **84**, 011150 (2011)

Borderline of the eigenvalue domain (scalar)





A. Goetschy & S.E. Skipetrov, PRE **84**, 011150 (2011)

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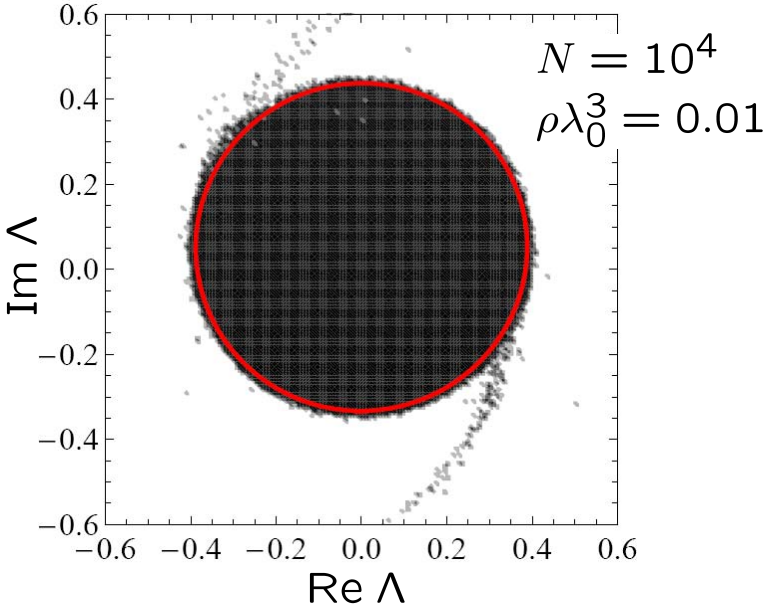

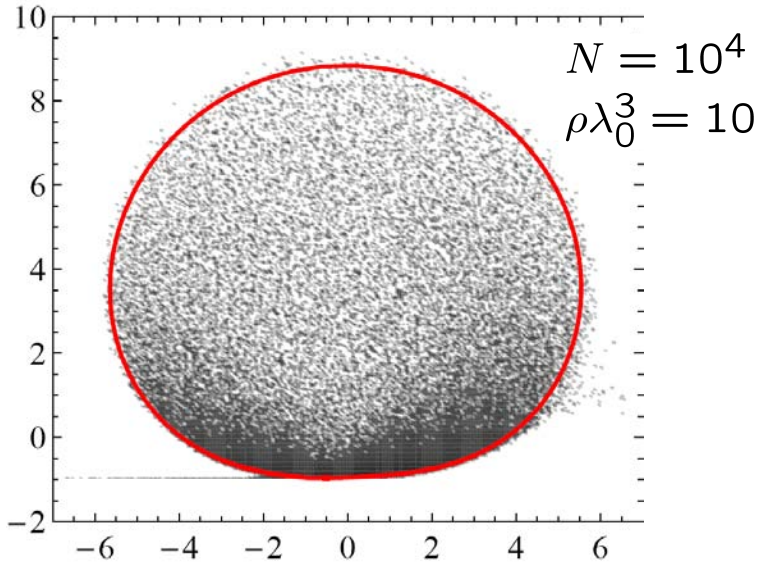

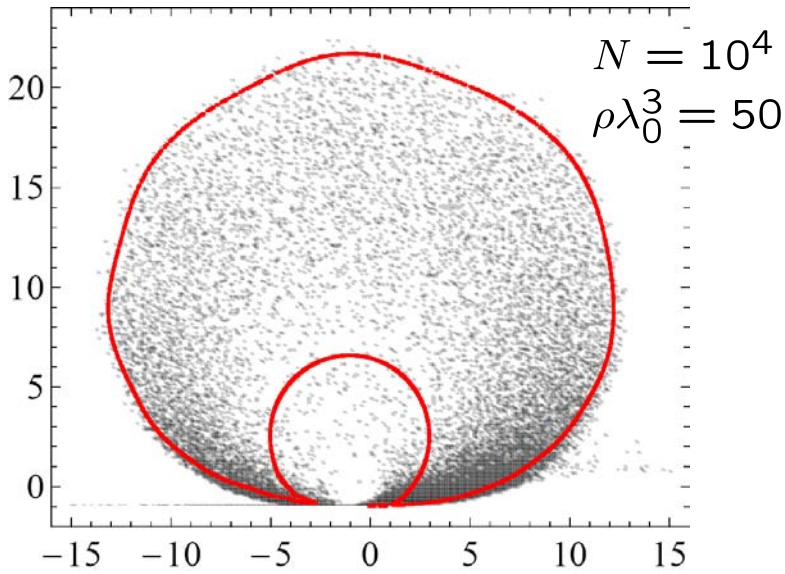

A. Goetschy & S.E. Skipetrov, PRE **84**, 011150 (2011)

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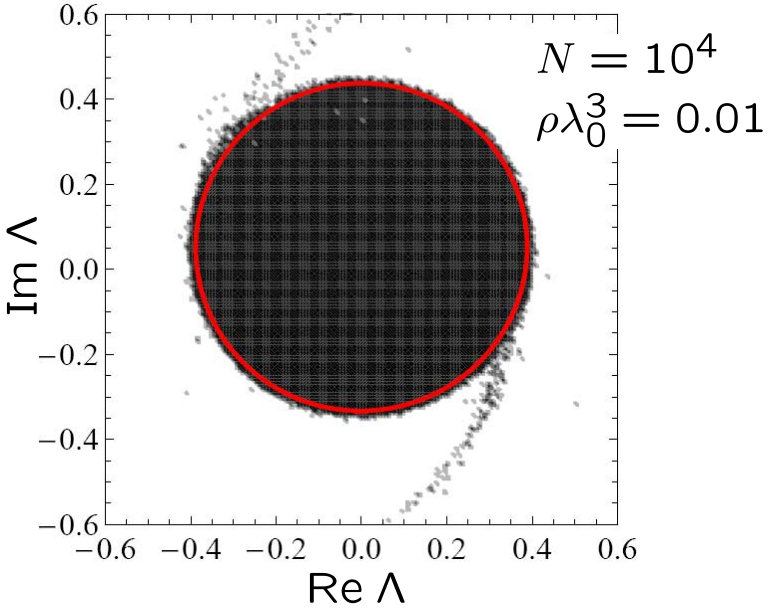
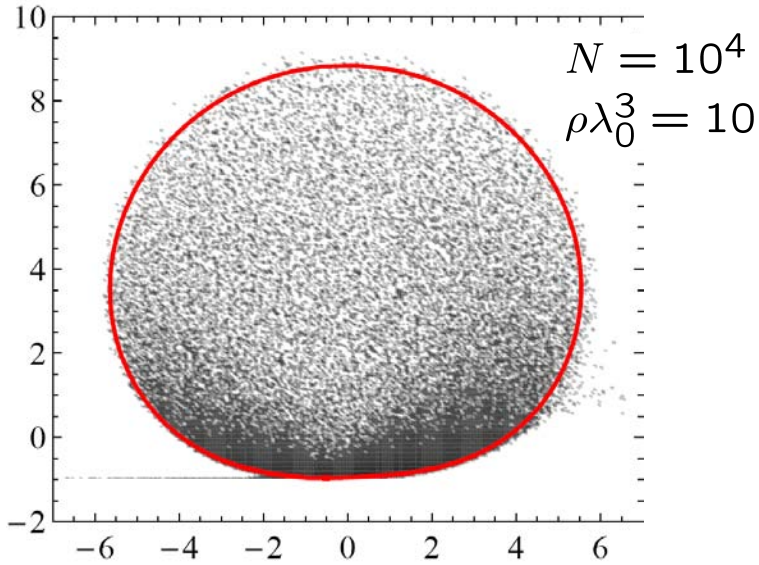
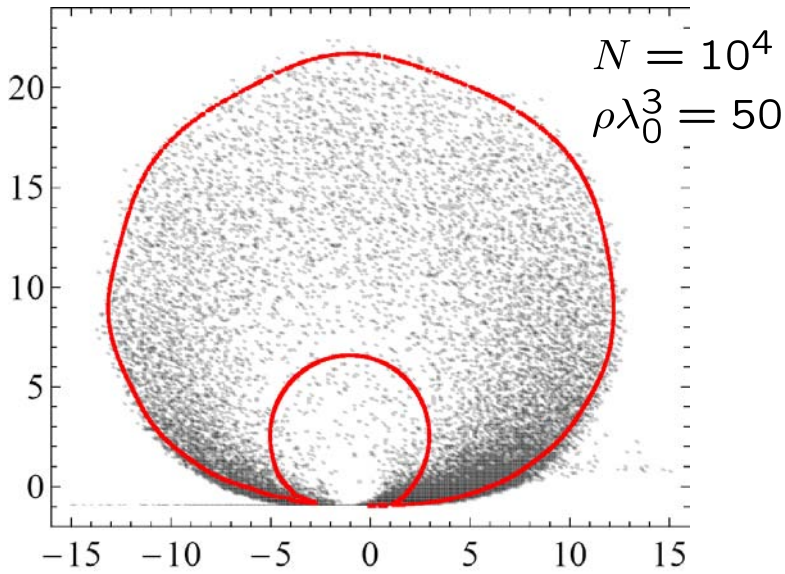
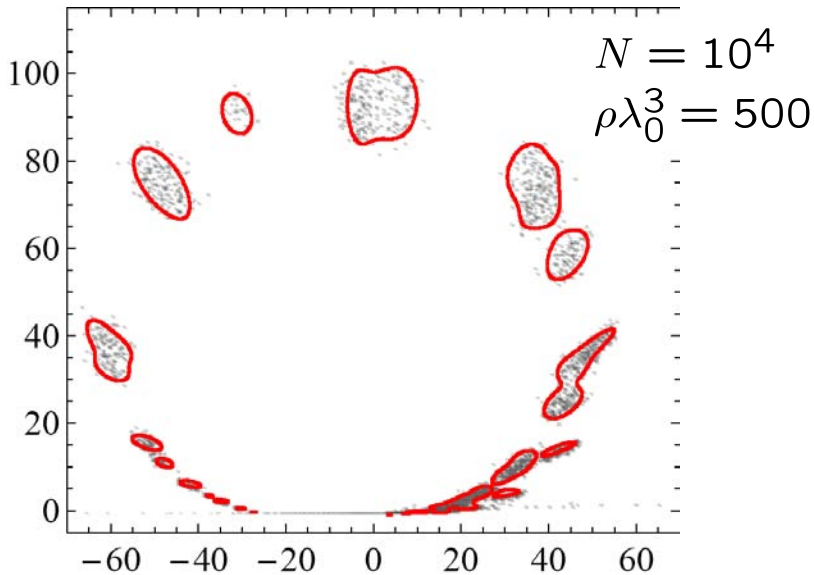


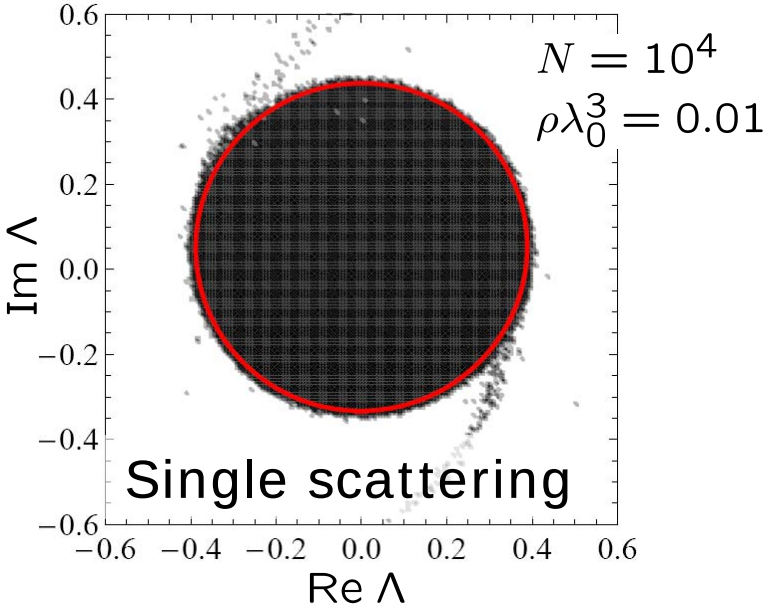


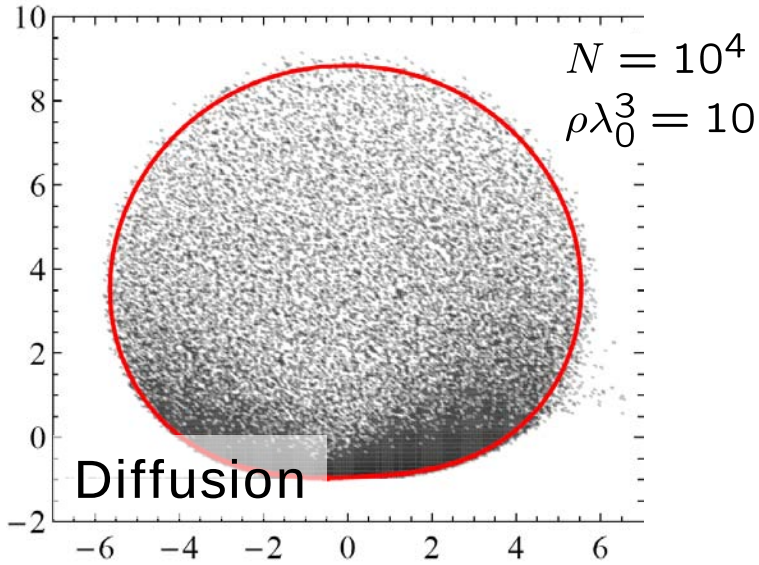


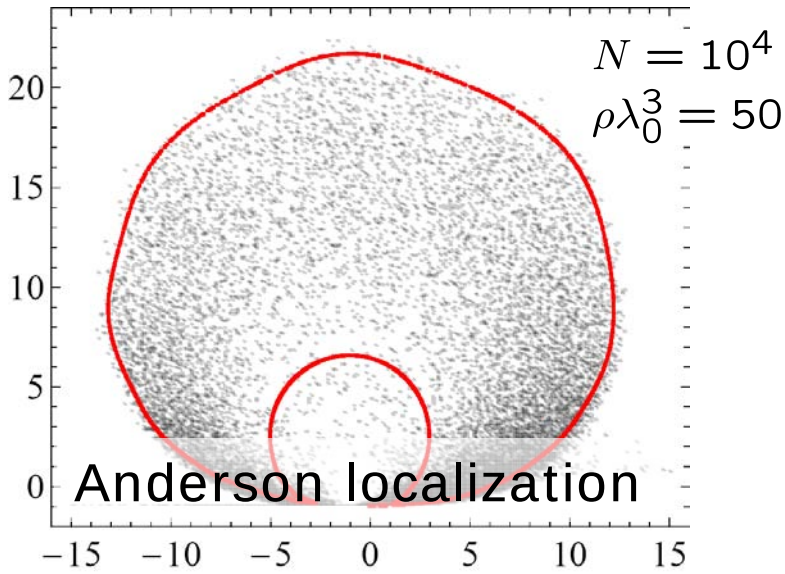


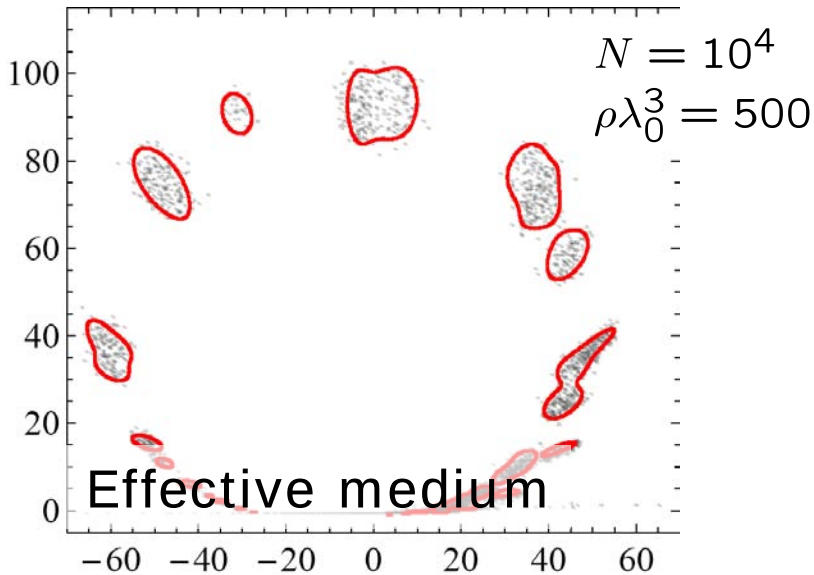
A. Goetschy & S.E. Skipetrov, PRE **84**, 011150 (2011)

Borderline of the eigenvalue domain (scalar)





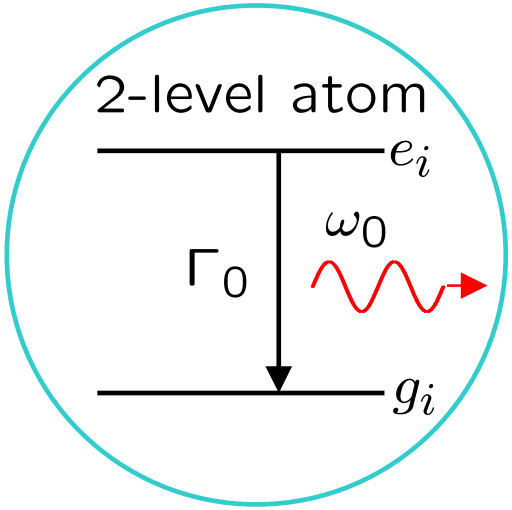




A. Goetschy & S.E. Skipetrov, PRE **84**, 011150 (2011)

Collective spontaneous emission

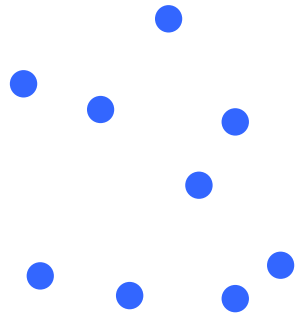
One atom



Frequency of emission $= \omega_0$

Decay rate $= \Gamma_0$

N identical atoms



N excited states with random emission frequencies $\omega_n \neq \omega_0$ and decay rates $\Gamma_n \neq \Gamma_0$

Eigenvalues:

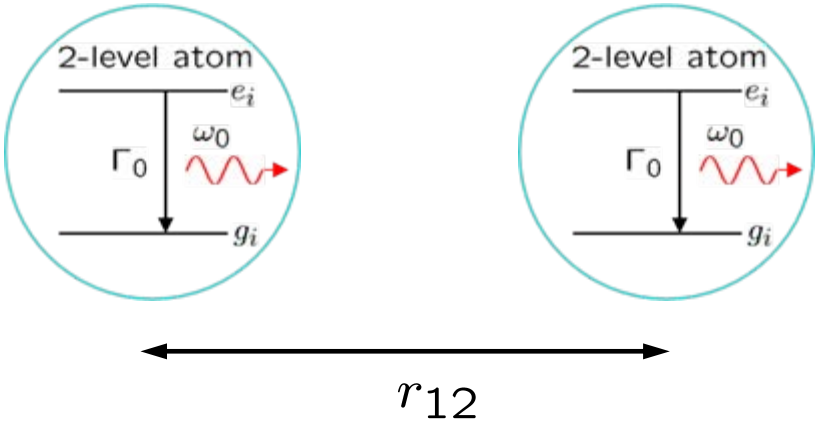
$\Lambda_n = \mathrm{Re}\Lambda_n + i\mathrm{Im}\Lambda_n$

Frequency

Decay rate

Collective spontaneous emission by 2 atoms

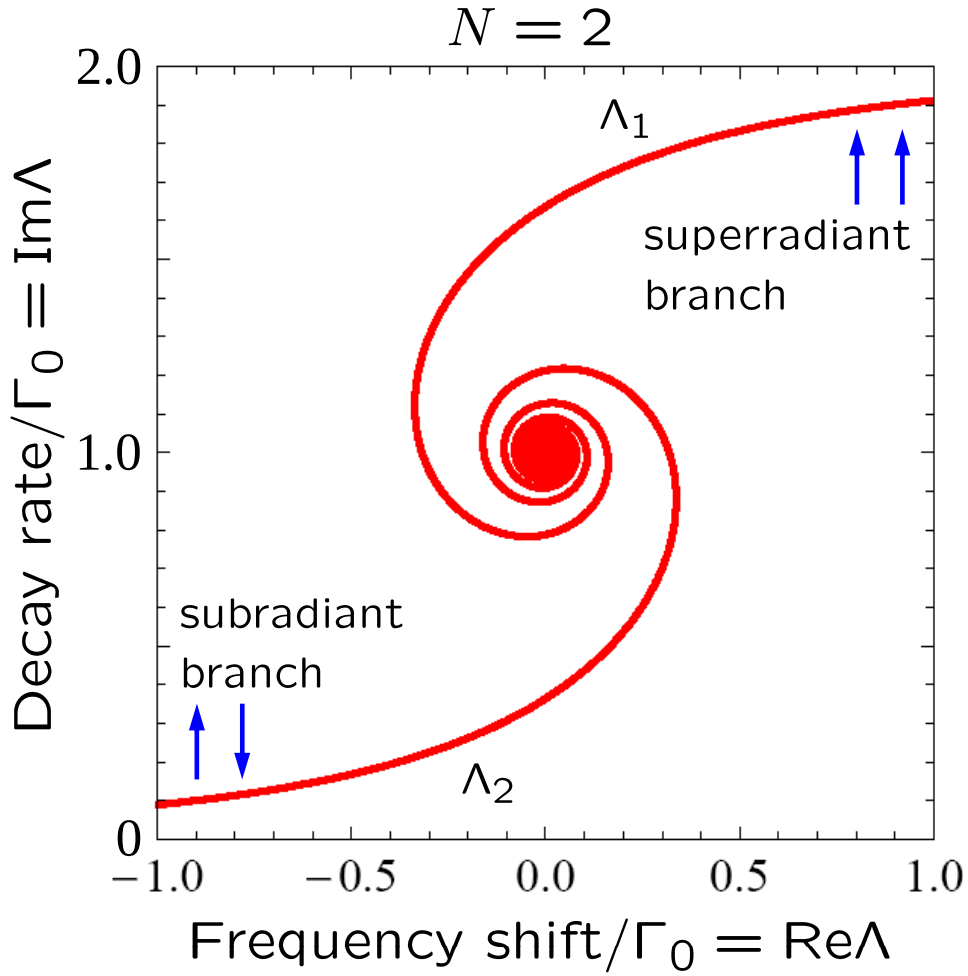
2 identical atoms



Scalar Green's function:

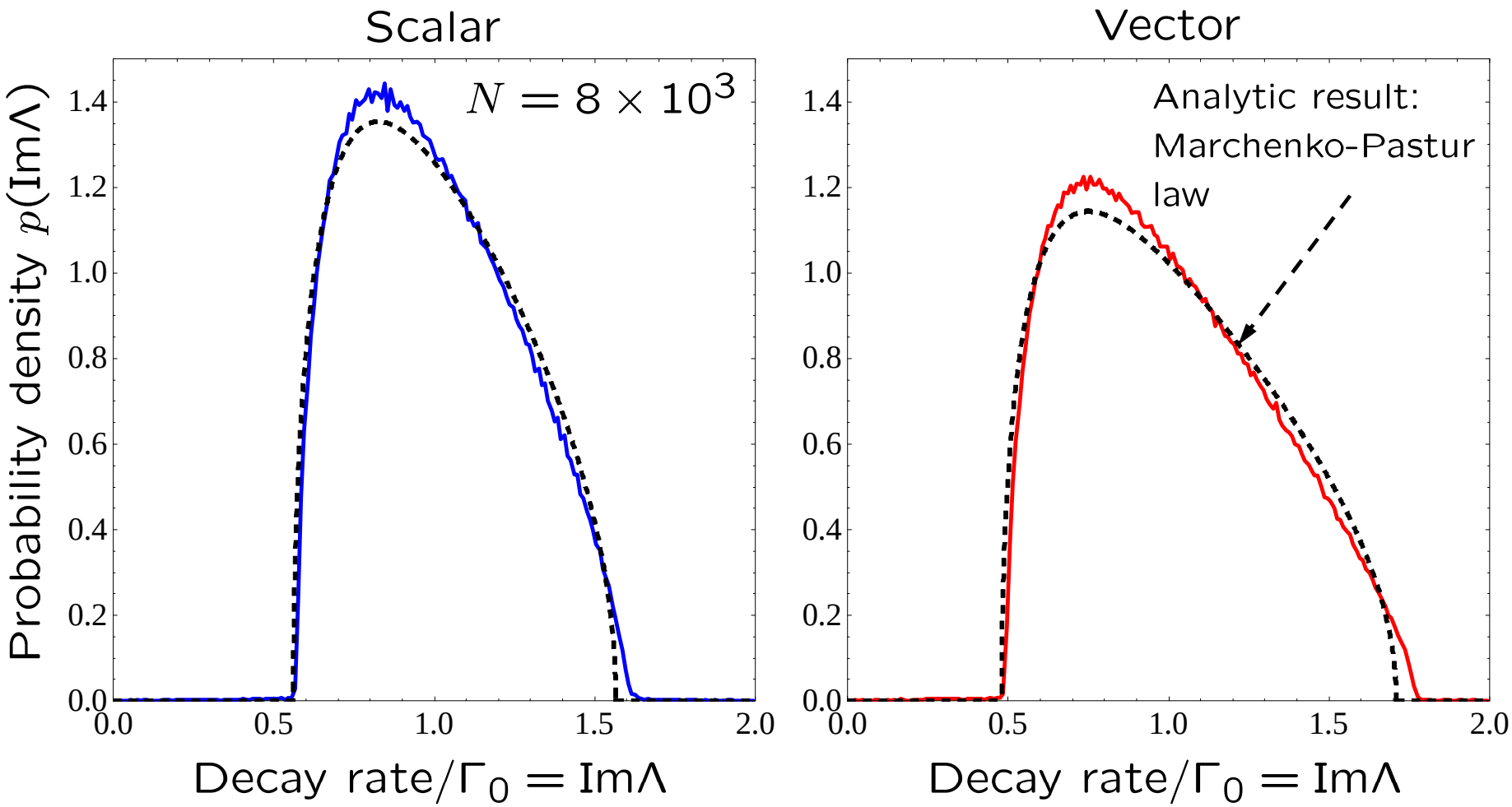
$$G_{e_i e_j} = \mathrm{i}\delta_{e_i e_j} + (1 - \delta_{e_i e_j}) \frac{e^{\mathrm{i}k_0 r_{ij}}}{k_0 r_{ij}}$$

$$\Lambda_{1,2} = \pm G_{12}$$
$$\Psi_{1,2} = (\pm 1, 1)$$



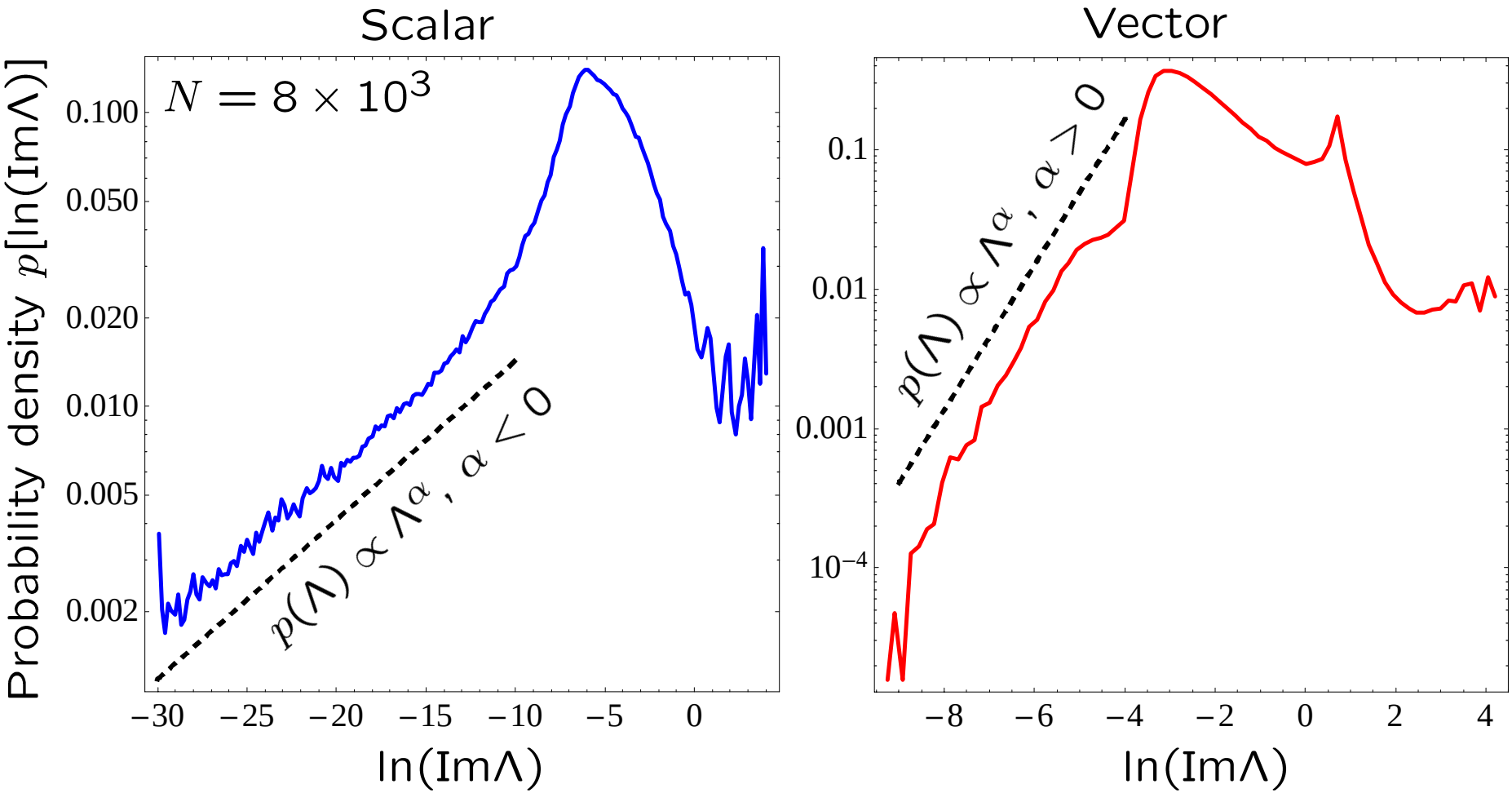
Collective spontaneous emission by *N* atoms

Low density $\rho/k_0^3 = 10^{-4}$

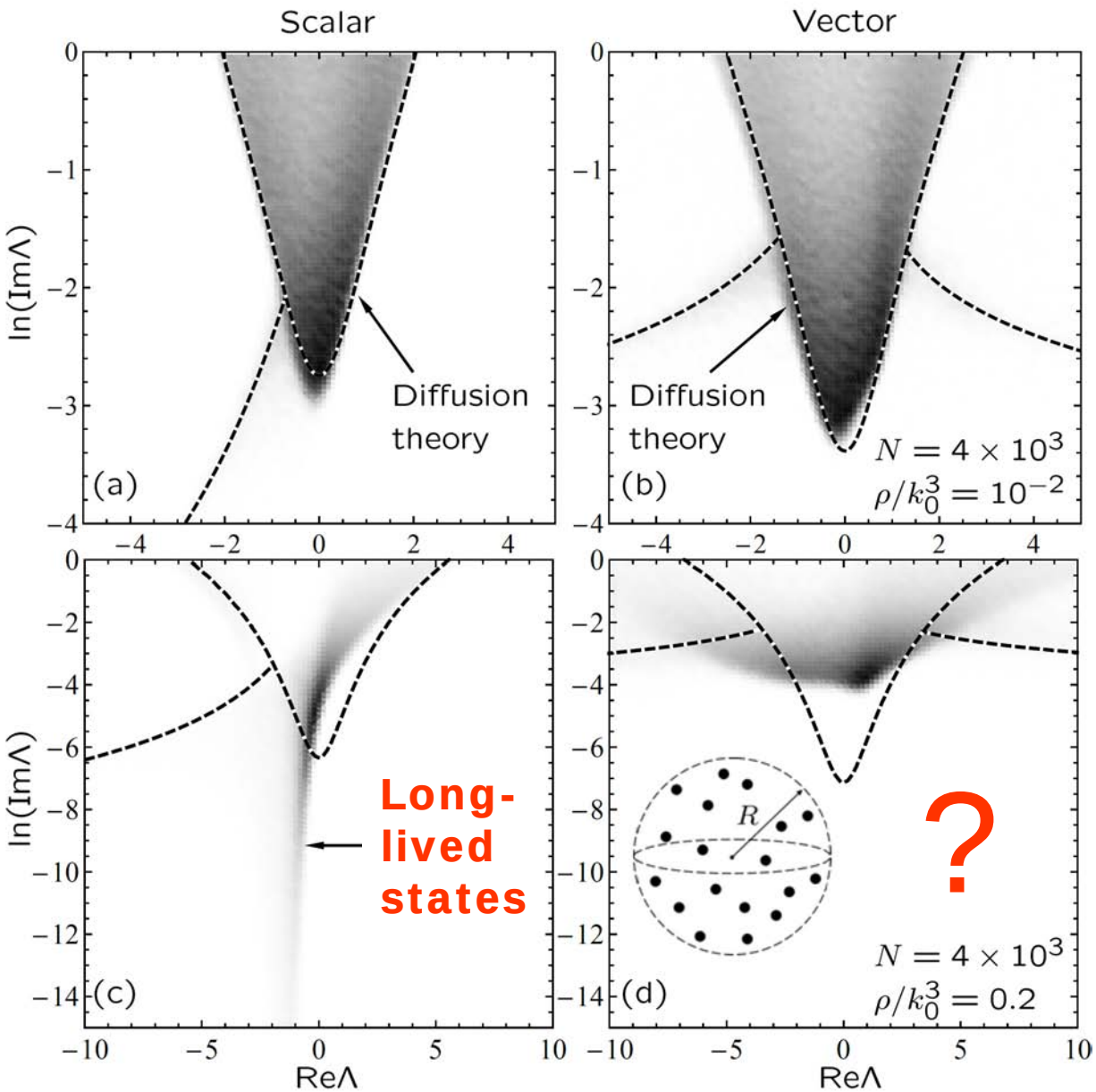


Collective spontaneous emission by *N* atoms

High density $\rho/k_0^3 = 1$



Collective spontaneous emission by *N* atoms



Low density of atoms

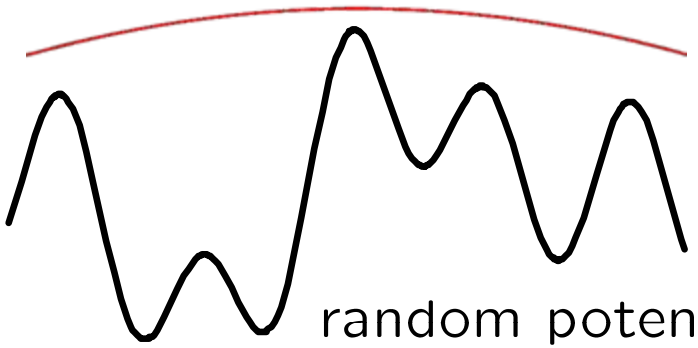
High density of atoms

S.E. Skipetrov & I.M. Sokolov, arXiv:1303.4655

Anderson localization

= appearance of spatially localized states due to disorder

high-energy
extended eigenstate $\psi_n(x)$



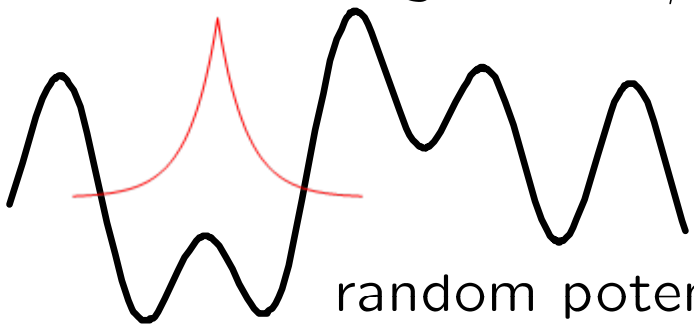
random potential $V(x)$

Inverse participation ratio

$$\mathrm{IPR}_n = \frac{\sum_{i=1}^{N} |\psi_n(\mathbf{r}_i)|^4}{\left(\sum_{i=1}^{N} |\psi_n(\mathbf{r}_i)|^2\right)^2} \sim \frac{1}{M}$$

for a state (mode)
localized on M atoms

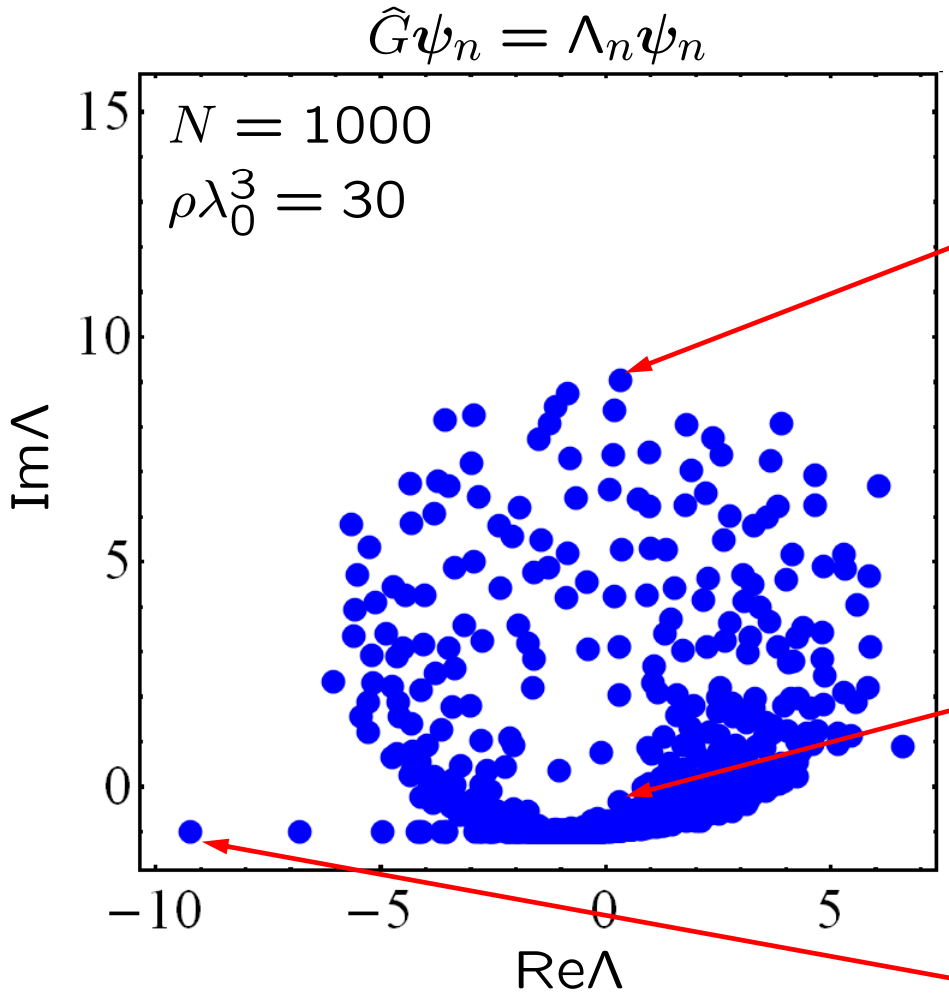
low-energy
localized eigenstate $\psi_n(x)$

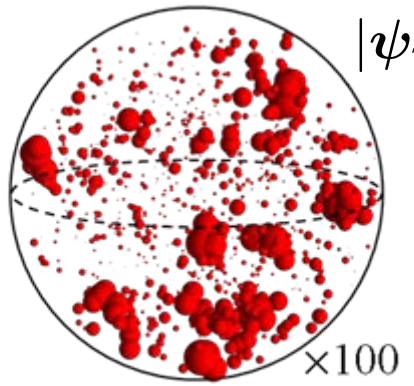


random potential $V(x)$

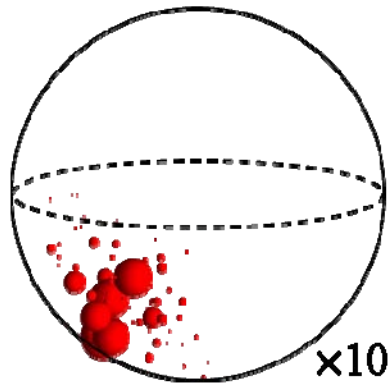
P.W. Anderson,
Phys. Rev. **109**, 1492 (1958)

Typical eigenvectors of the scalar Green's matrix

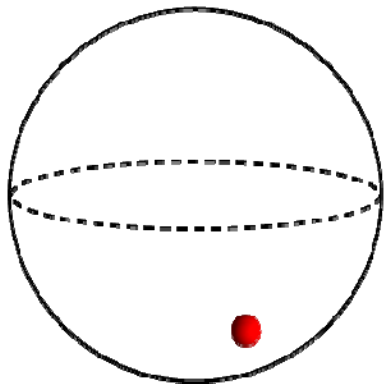




Extended



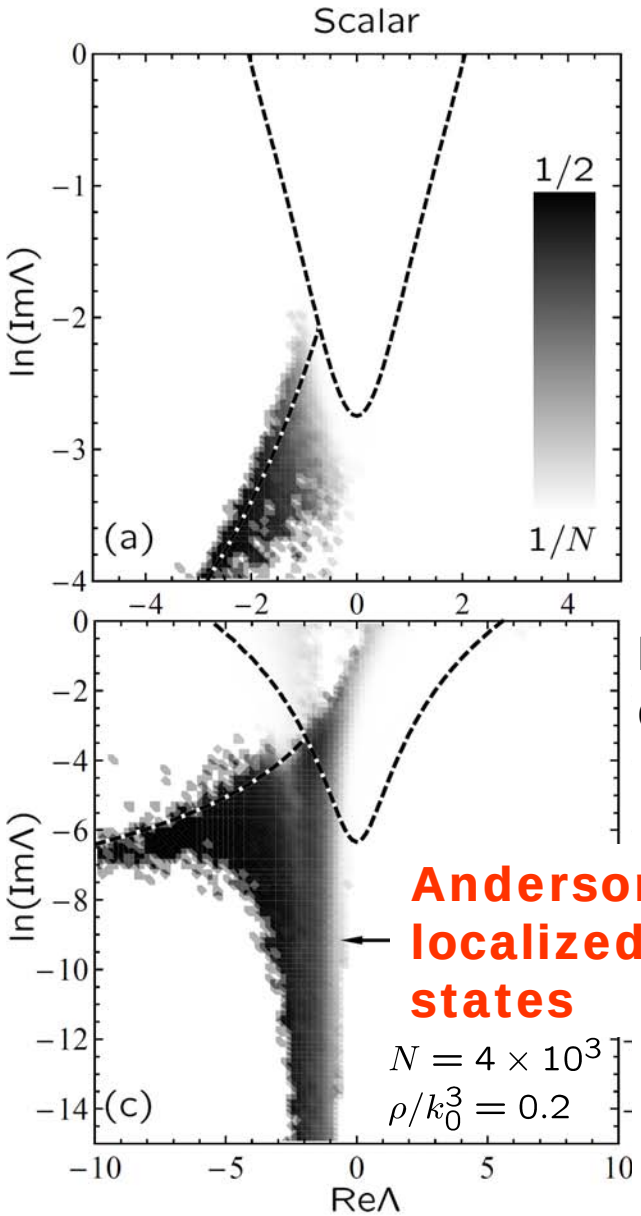
Localized



Localized on 2 points

A. Goetschy & S.E. Skipetrov,
EPL **96**, 34005 (2011)

Inverse participation ratio



Low density of atoms

High density of atoms

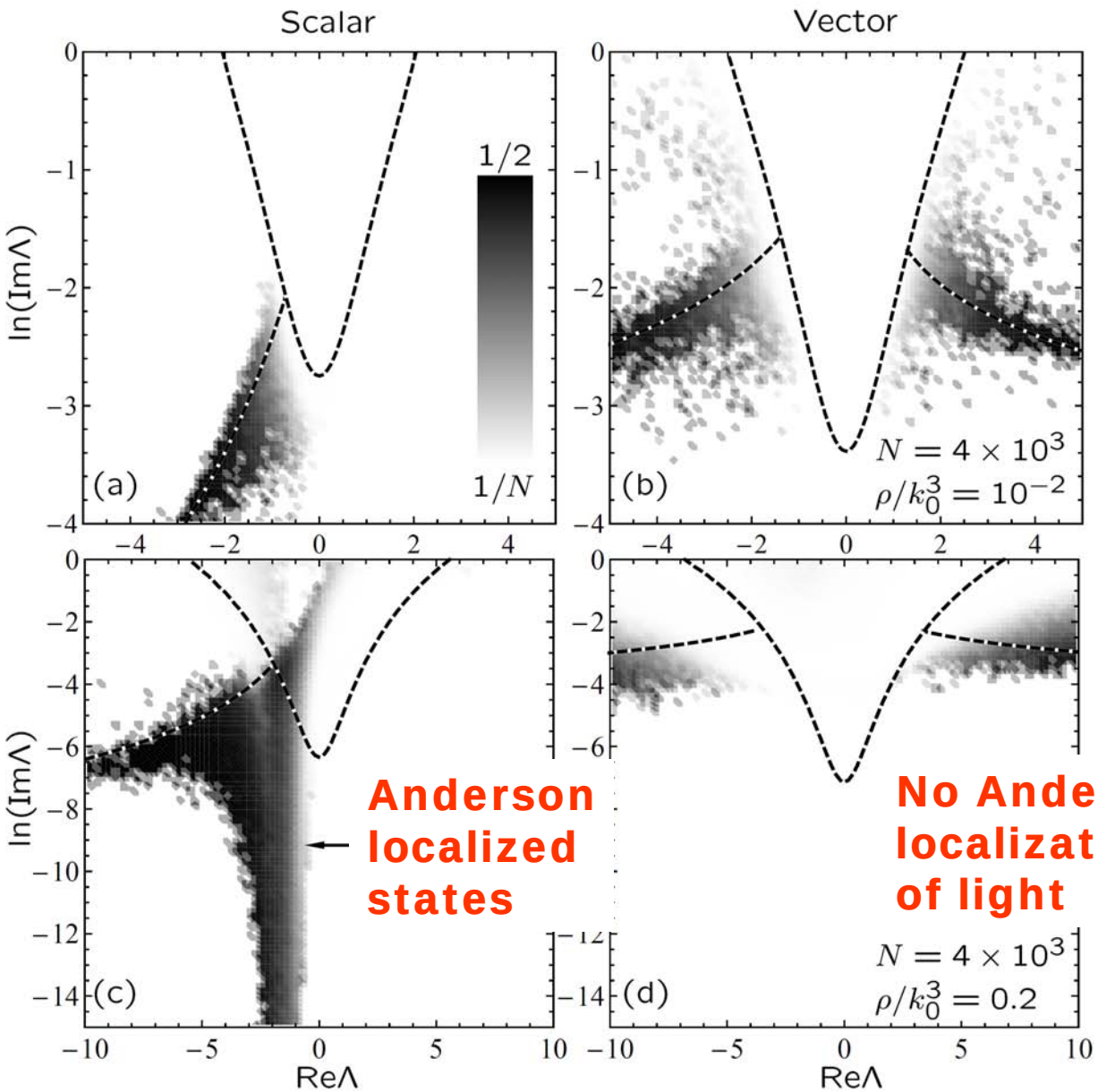
Anderson localized states

$$\mathrm{IPR}_n = \frac{\sum_{i=1}^{N} |\psi_n(\mathbf{r}_i)|^4}{\left(\sum_{i=1}^{N} |\psi_n(\mathbf{r}_i)|^2\right)^2} \sim \frac{1}{M}$$

for a state (mode) localized on M atoms

S.E. Skipetrov & I.M. Sokolov, arXiv:1303.4655

Inverse participation ratio



Low density of atoms

$$\mathrm{IPR}_n = \frac{\sum_{i=1}^{N} |\psi_n(\mathbf{r}_i)|^4}{\left(\sum_{i=1}^{N} |\psi_n(\mathbf{r}_i)|^2\right)^2} \sim \frac{1}{M}$$

for a state (mode) localized on M atoms

High density of atoms

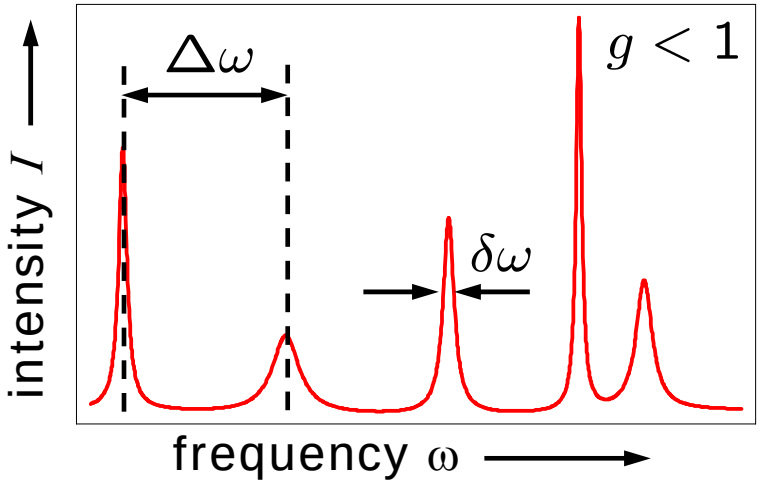
Anderson localized states

No Anderson localization of light

S.E. Skipetrov & I.M. Sokolov, arXiv:1303.4655

Dimensionless conductance

Spectrum of a disordered system



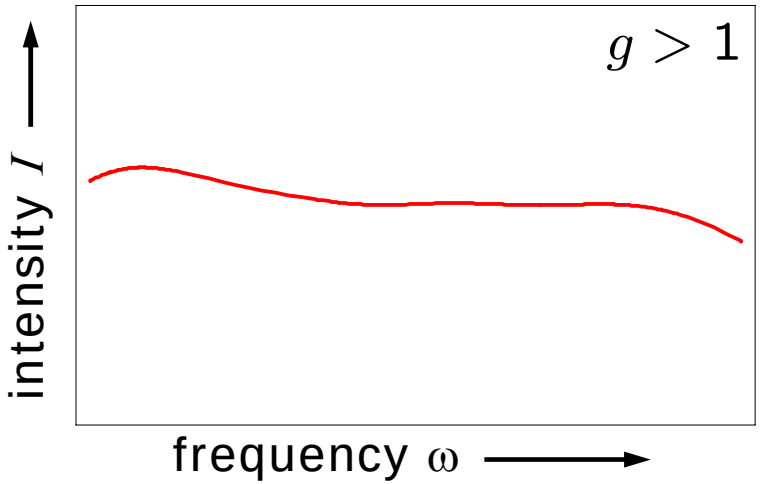
$\delta\omega$ - "mode width"

$\Delta\omega$ - "mode spacing"

$g = \frac{\delta\omega}{\Delta\omega}$ - Thouless parameter

(dimensionless conductance)

The same with mode widths x 10



Thouless criterium
of Anderson localization:

$$g < 1$$

Scaling theory of Anderson localization

Main idea: Study how g evolves with sample size L

If the modes are extended, g grows with L

If the modes are localized, g decreases with L

At the critical point $g = g_c$ is independent of L

This reasoning can be quantified by the function

$$\beta(g) = \frac{\partial \ln g}{\partial \ln L}$$

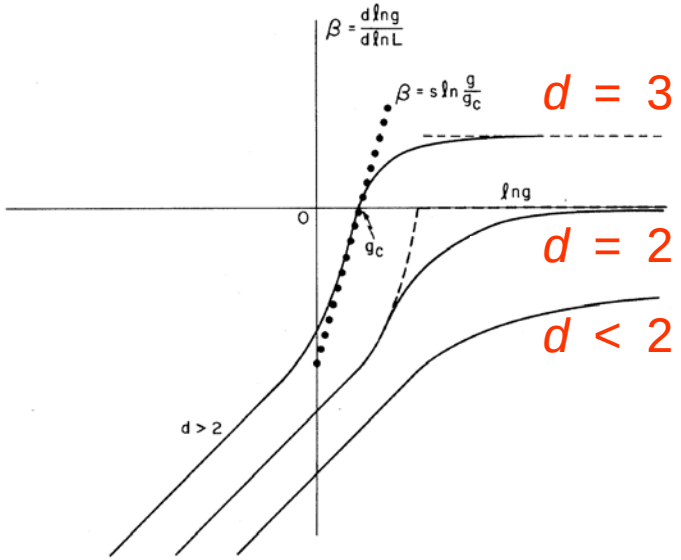
> 0 if extended eigenstates

< 0 if localized eigenstates

E. Abrahams *et al.*, PRL **42**, 673 (1979)

Scaling theory of Anderson localization

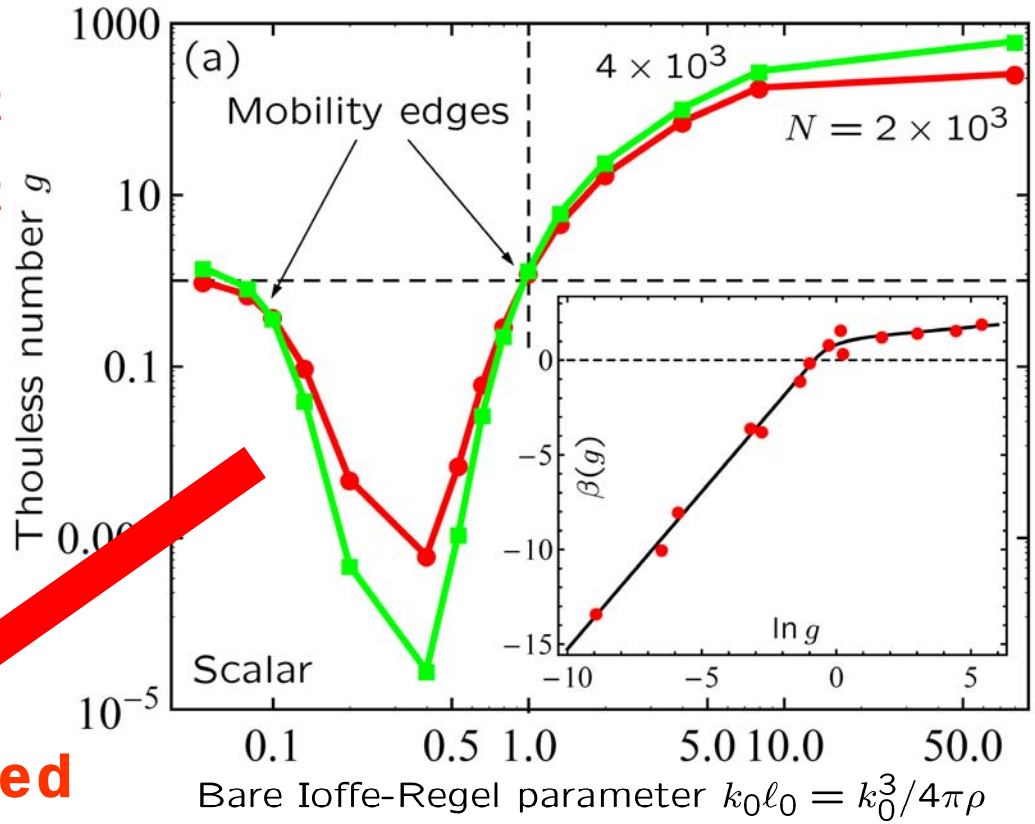
E. Abrahams *et al.* (1979):



$$\beta(g) = \frac{\partial \ln g}{\partial \ln L}$$

Localization as expected

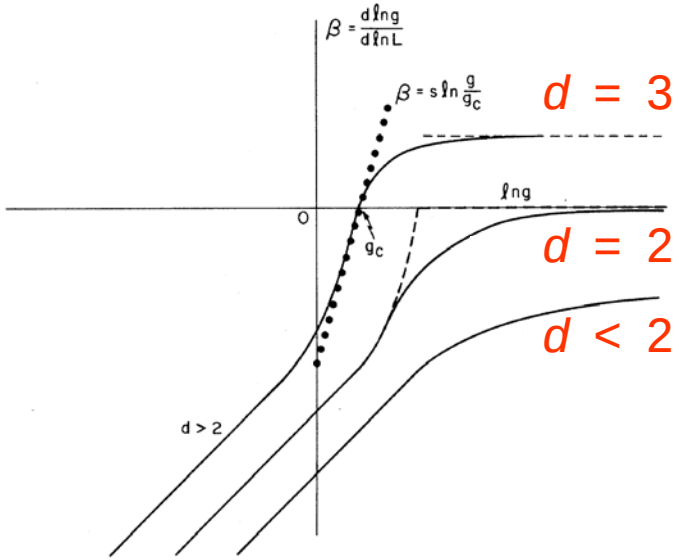
Scalar model $d = 3$



S.E. Skipetrov & I.M. Sokolov, arXiv:1303.4655

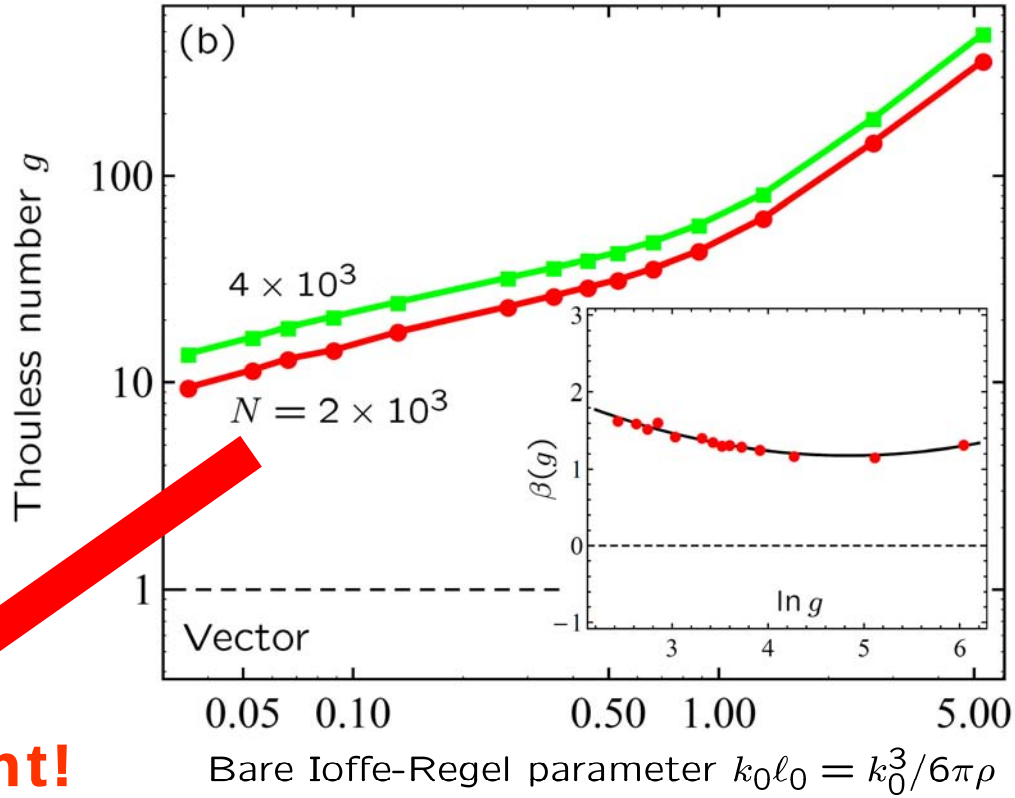
Scaling theory of Anderson localization

E. Abrahams *et al.* (1979):



$$\beta(g) = \frac{\partial \ln g}{\partial \ln L}$$

Vector model $d = 3$

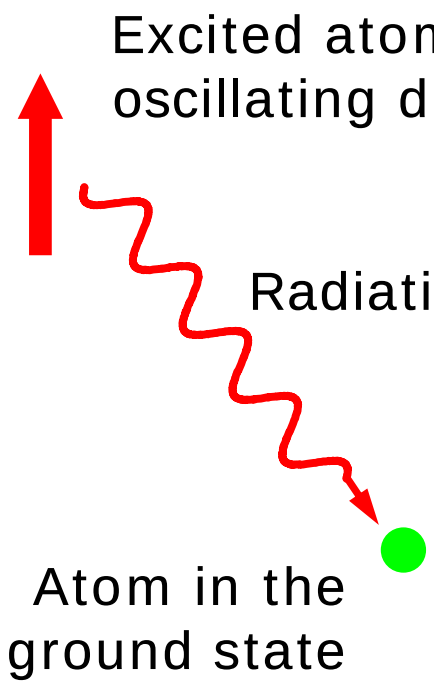


No localization of light!

S.E. Skipetrov & I.M. Sokolov, arXiv:1303.4655

Coupling between atoms: classical picture

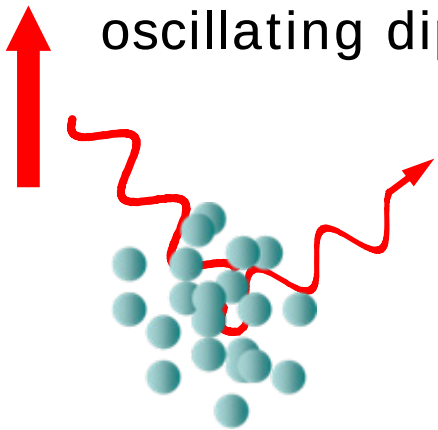
Radiative coupling



Coupling between atoms: classical picture

Radiative coupling

Excited atom =
oscillating dipole



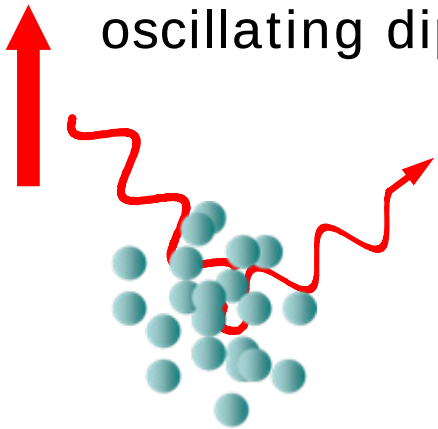
Atom in the
ground state

Long-range, affected by disorder

Coupling between atoms: classical picture

Radiative coupling

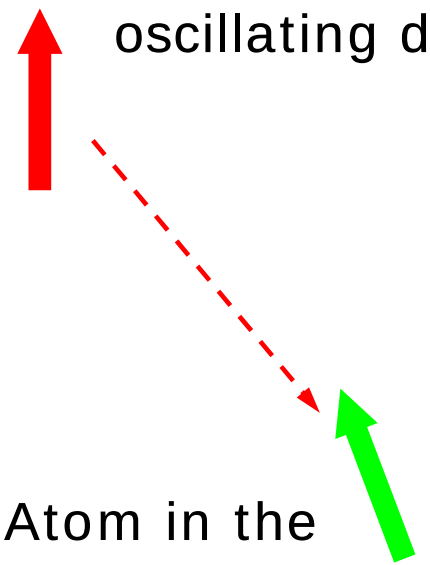
Excited atom = oscillating dipole



Atom in the ground state

Nonradiative dipole coupling

Excited atom = oscillating dipole



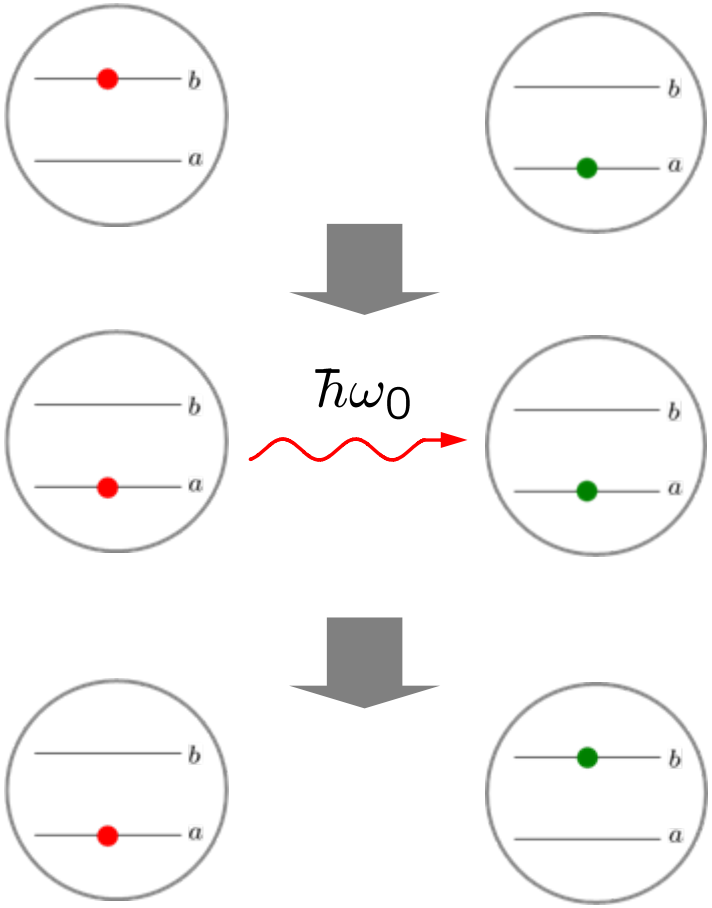
Atom in the ground state acquires an induced dipole moment

Long-range, affected by disorder

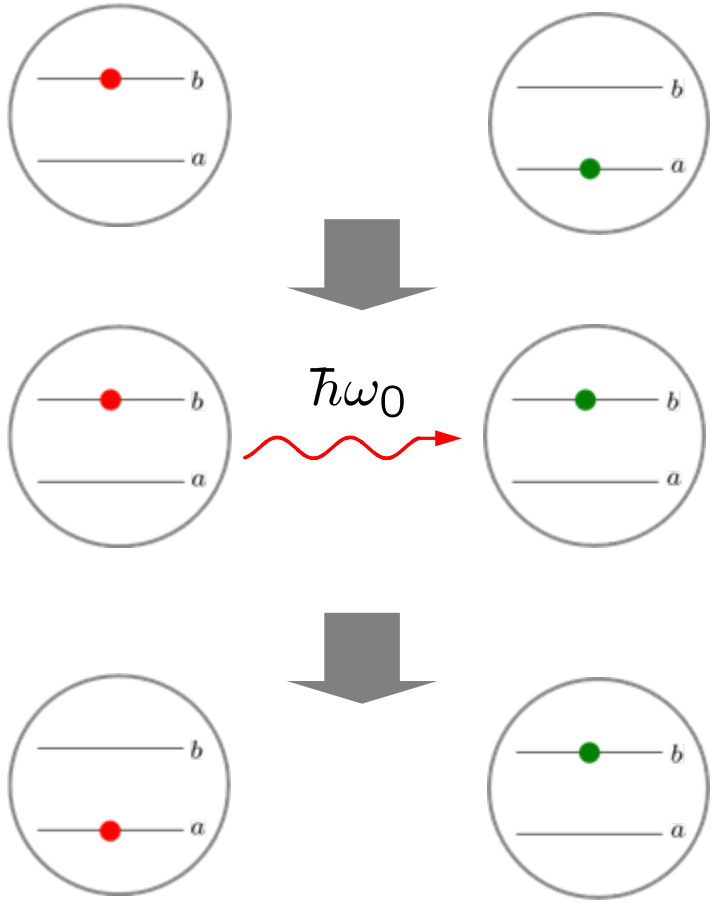
Short-range, unaffected by disorder

Coupling between atoms: quantum picture

Radiative coupling
= exchange of photons



Nonradiative dipole coupling
= exchange of virtual photons

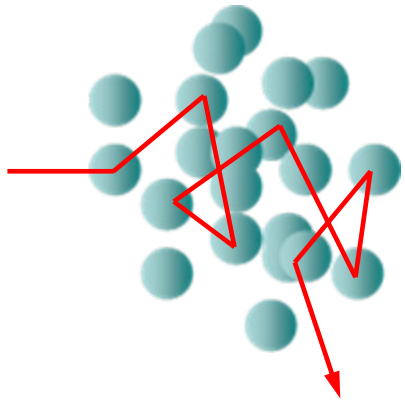


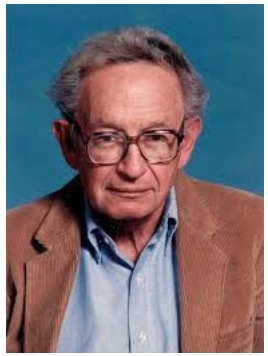
Förster against Anderson

Nonradiative dipole coupling = exchange of **virtual** photons
is also known as **Förster resonance energy transfer (FRET)**

Theodor Förster

Photon random walk





Philip Anderson

Nonradiative transport:

$$D = D_0 \left[1 + \text{const} \left(\frac{\rho}{k_0^3} \right)^3 \right]$$

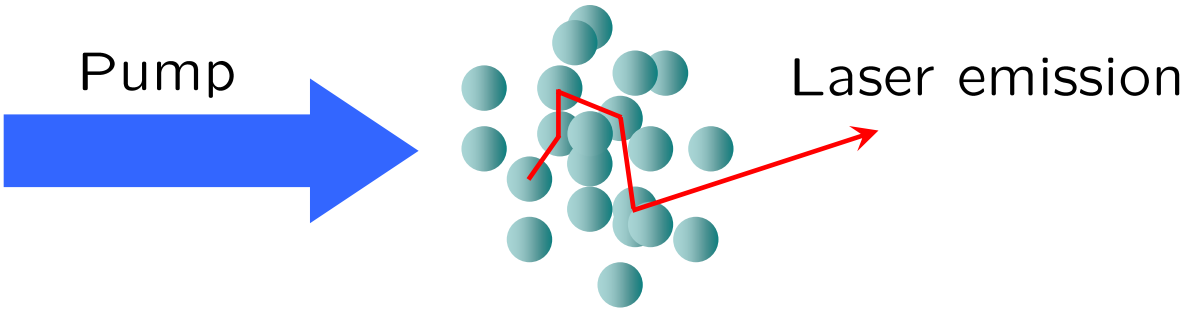
$$\frac{\rho}{k_0^3} \ll 1$$

Weak localization:

$$D = D_0 \left[1 - \text{const} \left(\frac{\rho}{k_0^3} \right)^2 \right]$$

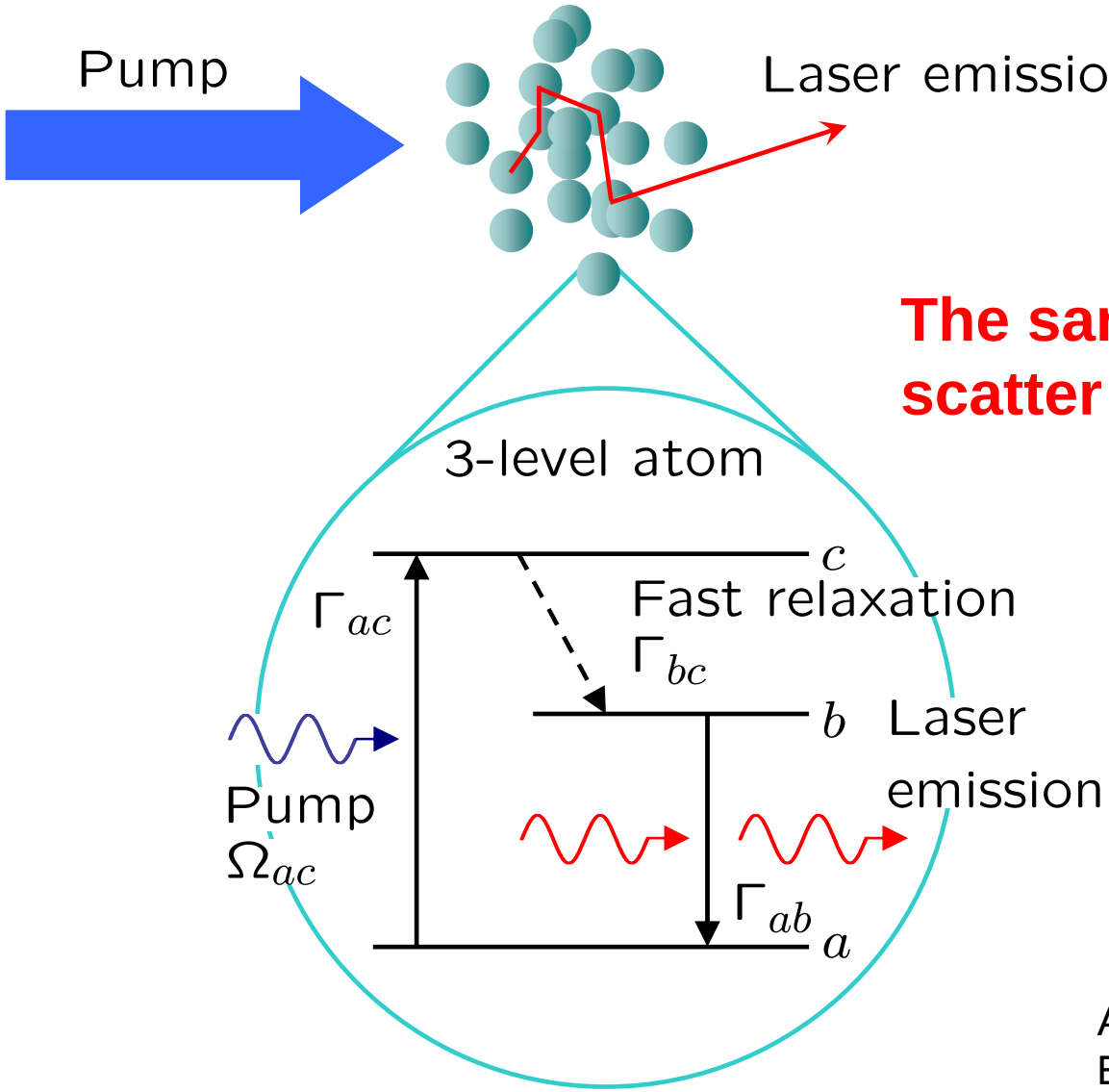
Th.M. Nieuwenhuizen *et al.*, Phys. Lett. A **184**, 360 (1994)

Random laser: the principle



V.S. Letokhov, Sov. Phys. JETP **26**, 835 (1968)

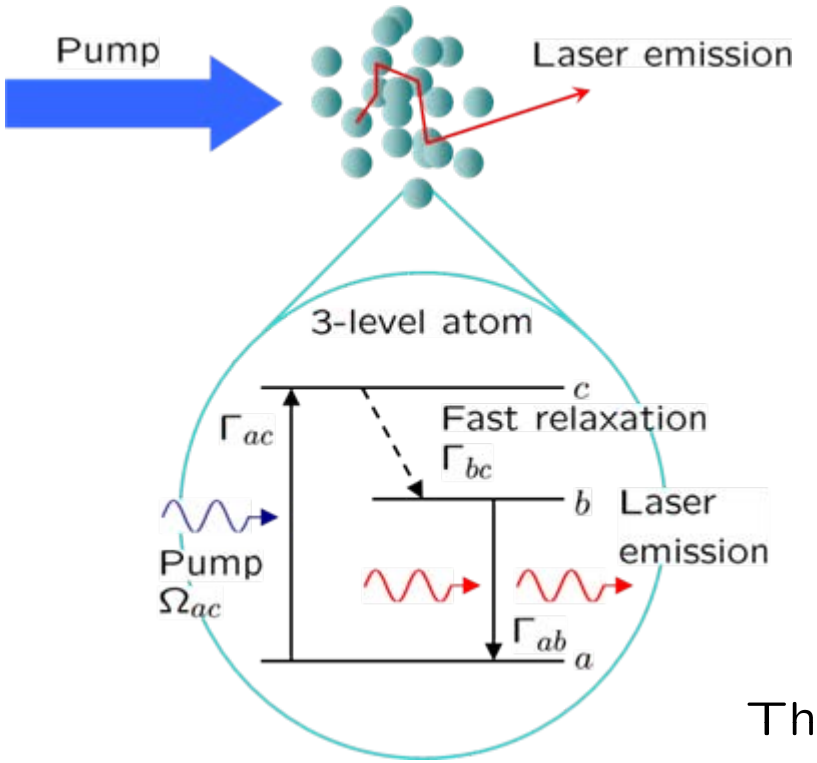
Random laser: the principle



The same atoms scatter and amplify light !

A. Goetschy & S.E. Skipetrov, EPL **96**, 34005 (2011)

Random laser: threshold condition



Laser threshold:

$$\Lambda_n = \frac{1}{t}$$

An eigenvalue of the matrix $\hat{G}$:

$$\hat{G}\vec{\Psi}_n = \Lambda_n \vec{\Psi}_n$$

The dimensionless scattering matrix of a single atom:

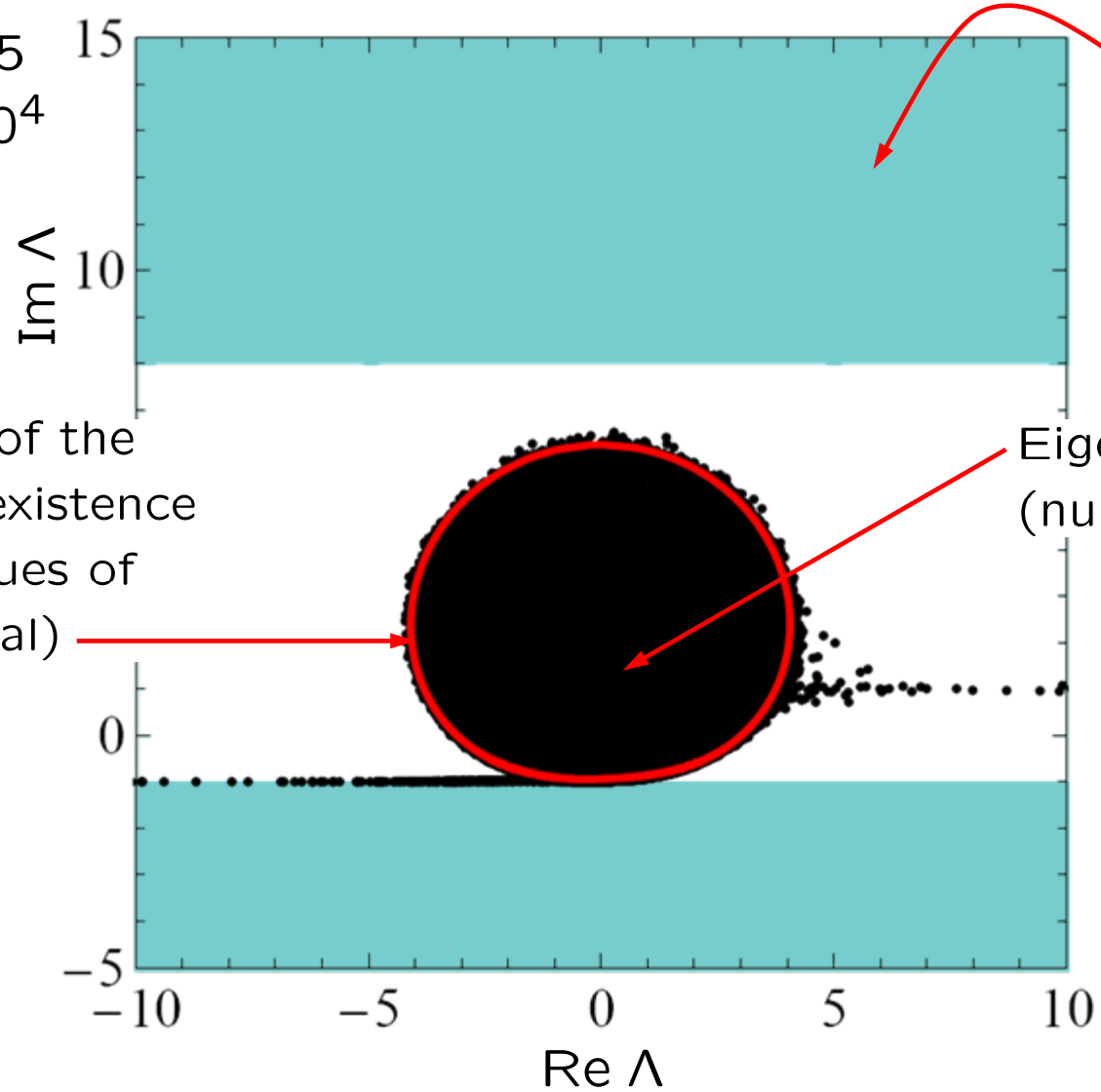
$$t(\omega) = \frac{1}{2} \times \frac{W-1}{W+1} \times \frac{1}{\omega - \omega_0 + \frac{\mathrm{i}}{2}(W+1)}$$

The true scattering matrix is $\tilde{t} = (4\pi/k_0)t$

A. Goetschy & S.E. Skipetrov,
EPL **96**, 34005 (2011)

Random laser: below threshold

$\rho\lambda_0^3 = 5$
$N = 10^4$



Domain spanned by $1/t(\omega)$ when ω and the pump W are varied

Borderline of the domain of existence of eigenvalues of $\hat{G}$ (analytical)

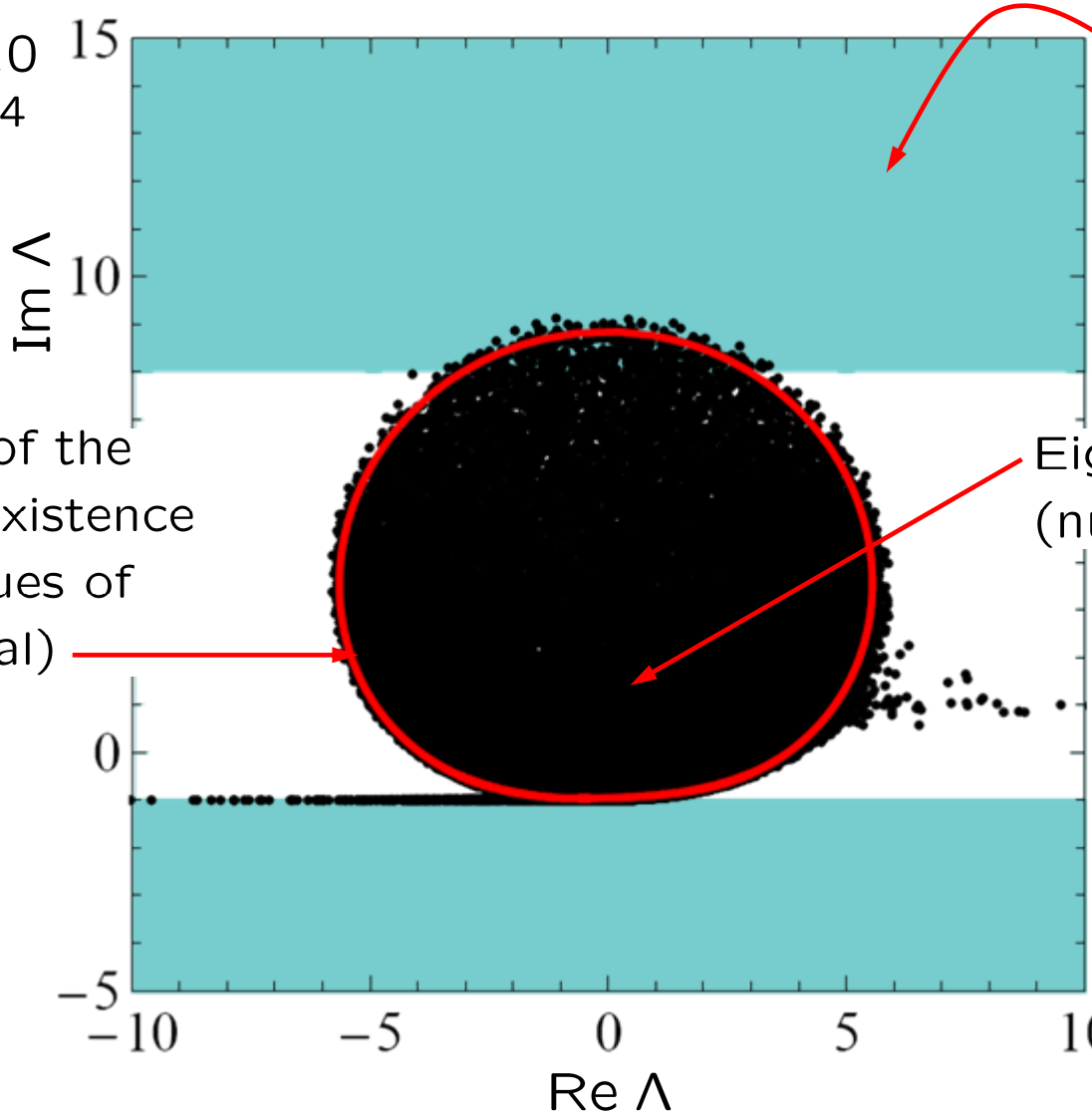
Eigenvalues of $\hat{G}$ (numerical)

Laser threshold:
$\Lambda = \frac{1}{t}$

A. Goetschy & S.E. Skipetrov, EPL **96**, 34005 (2011)

Random laser: above threshold

$\rho\lambda_0^3 = 10$
$N = 10^4$



Domain spanned by $1/t(\omega)$ when ω and the pump W are varied

Borderline of the domain of existence of eigenvalues of $\hat{G}$ (analytical)

Eigenvalues of $\hat{G}$ (numerical)

Laser threshold:
$\Lambda = \frac{1}{t}$

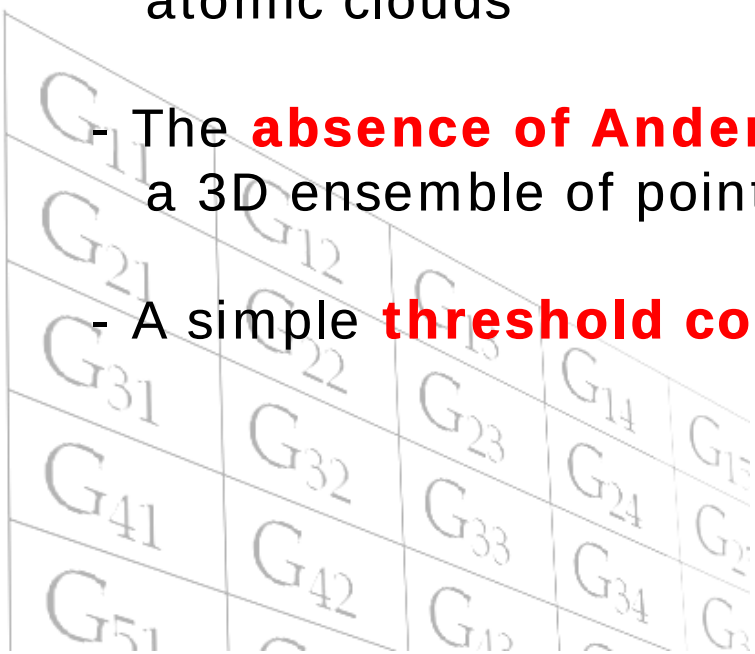
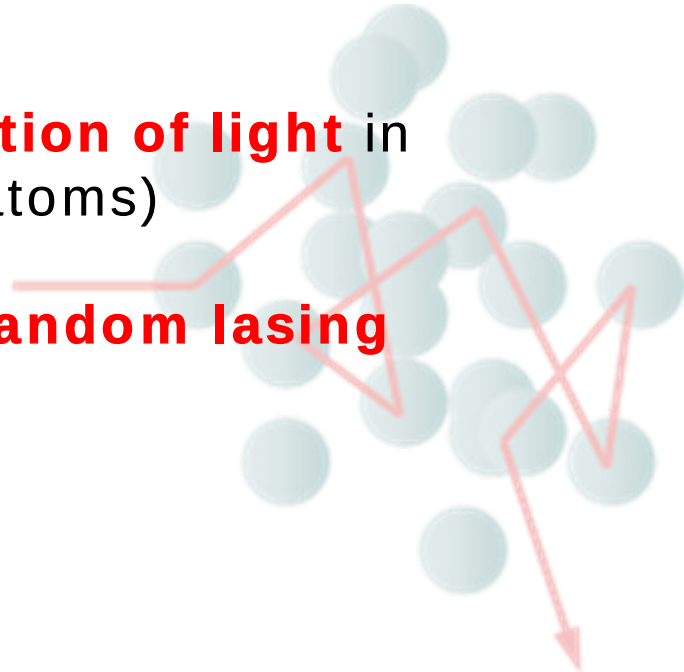
A. Goetschy & S.E. Skipetrov, EPL **96**, 34005 (2011)

Conclusions

Random Green's matrix approach is an efficient way to study collective phenomena in multiple scattering of waves.

We successfully applied it to demonstrate:

- The difference between scalar and vector models in the problem of **spontaneous emission of light** by large atomic clouds

- The **absence of Anderson localization of light** in a 3D ensemble of point scatterers (atoms)

- A simple **threshold condition for random lasing**