The Abdus Salam International Centre for Theoretical Physics

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Advanced Workshop on Nonlinear Photonics, Disorder and Wave Turbulence

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Transport in potentials random in space and time: From Anderson localization to super-ballistic motion

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Transport in potentials random in space and time: From Anderson localization to super-ballistic motion

- Yevgeny Krivolapov, Michael Wilkinson, SF
- Liad Levi, Moti Segev

Outline

- Experimental Motivation
- Theory for particles in 1D
- Universality Classes
- Chirikov resonances
- Fokker—Planck Equation
- Theory for particles for d>1
- Short scales and uniform acceleration
- Comparison between spreading of particles and waves

Experimental motivation

Optics : Transverse Localization Scheme

Suggested by De Raedt, Lagengijk & de Vries (PRL 1987)

Wave Equation

$$\nabla^2 \vec{E} - \mu_0 \varepsilon(\vec{r}) \frac{\partial^2}{\partial t^2} \vec{E} = 0$$

+

- **Scalar and Time harmonic**
- **slow variations of index of refraction**
- **slowly varying amplitude solution**

$$\lambda \frac{\partial^2 A}{\partial z^2} << \frac{\partial A}{\partial z}$$

$$\vec{E}(x,y,z) = A(x,y,z)e^{i(kz-wt)}\hat{x}$$

Optics

$$i\frac{\partial A}{\partial z} = -\frac{1}{2k}\nabla_\perp^2 A - \frac{k}{n_0}\Delta n(x,y,z)A$$

Solid State Physics

$\hbar = 1$

$$i\frac{\partial \Psi}{\partial t} = -\frac{1}{2m}\nabla^2 \Psi + V(\vec{r},t)\Psi$$

$$A \leftrightarrow \Psi$$

$$z \leftrightarrow t$$

$$k\,\Delta n / n_0 \leftrightarrow -V$$

$$k_\perp, \lambda_\perp \leftrightarrow p, \lambda_{de-broglie}$$

LETTERS

Transport and Anderson localization in disordered two-dimensional photonic lattices

Tal Schwartz[1], Guy Bartal[1], Shmuel Fishman[1] & Mordechai Segev[1]

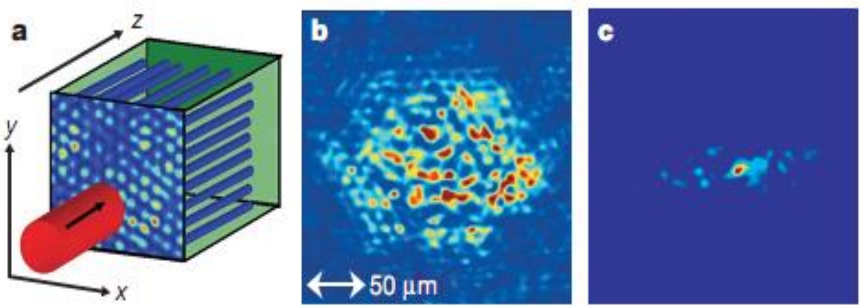
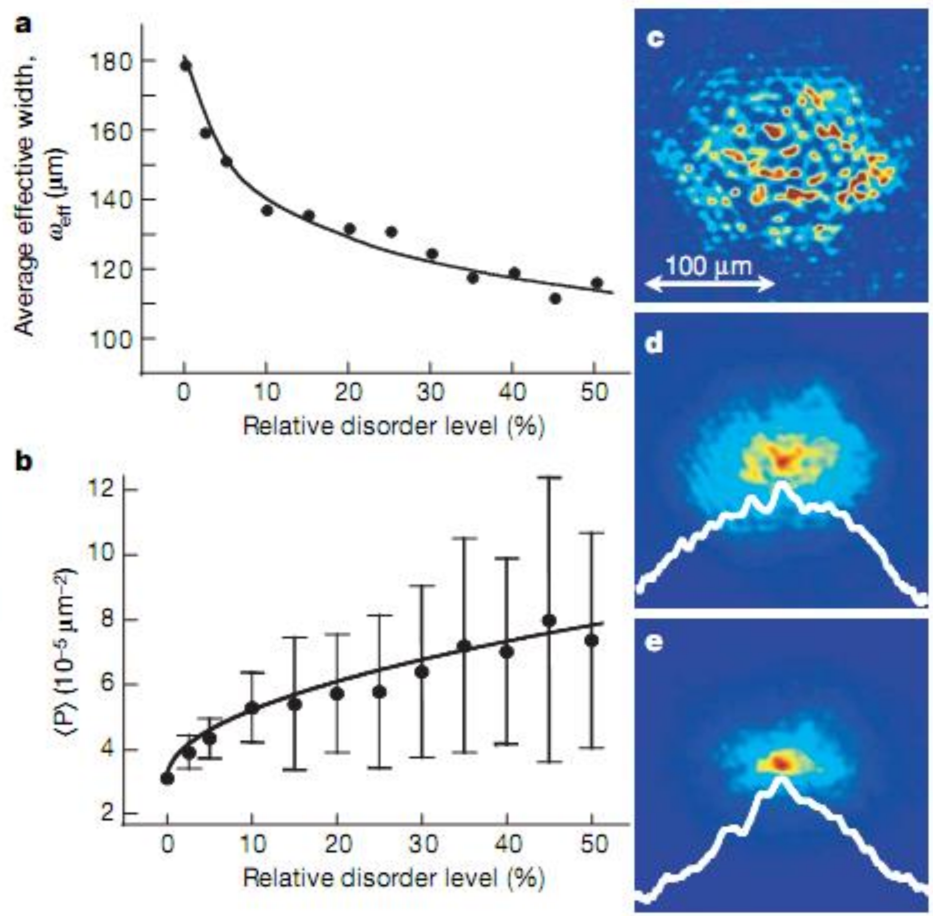


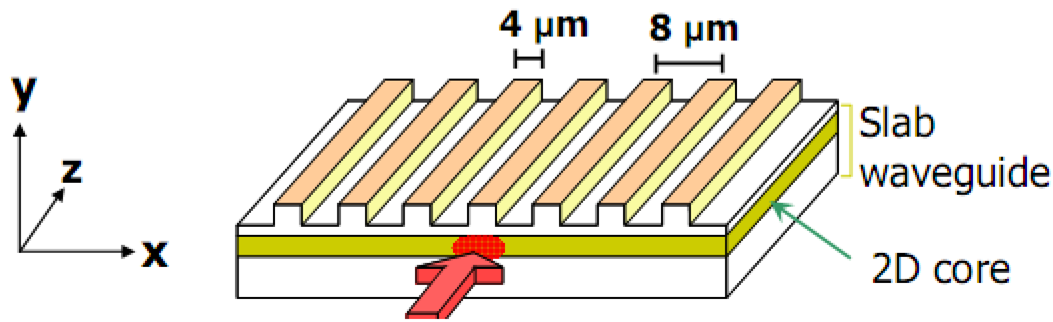
Figure 1 | Transverse localization scheme. a, A probe beam entering a disordered lattice, which is periodic in the two transverse dimensions (*x* and *y*) but invariant in the propagation direction (*z*). In the experiment described here, we use a triangular (hexagonal) photonic lattice with a periodicity of 11.2 µm and a refractive-index contrast of ~5.3 × 10^{-4}. The lattice is induced optically, by transforming the interference pattern among three plane waves

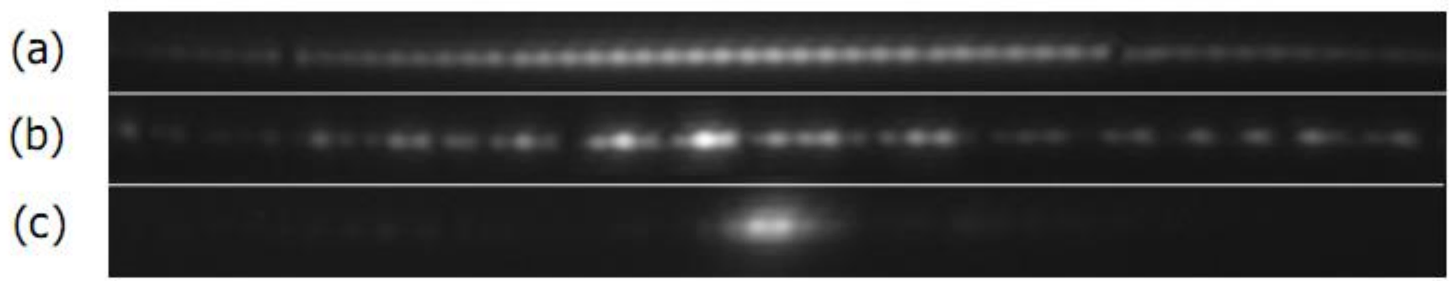


PRL 100, 013906 (2008) PHYSICAL REVIEW LETTERS week ending 11 JANUARY 2008

Anderson Localization and Nonlinearity in One-Dimensional Disordered Photonic Lattices

Yoav Lahini,[1,*] Assaf Avidan,[1] Francesca Pozzi,[2] Marc Sorel,[2] Roberto Morandotti,[3]
Demetrios N. Christodoulides,[4] and Yaron Silberberg[1]





(a) Periodic array – *expansion*

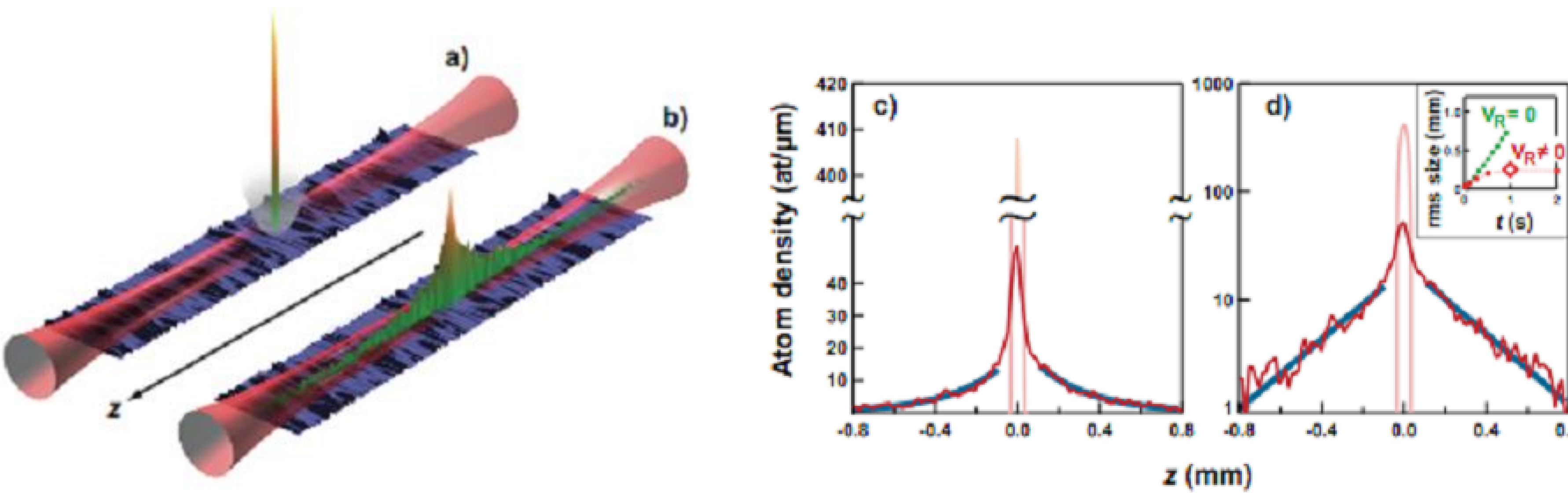
(b) Disordered array - *expansion*

(c) Disordered array - *localization*

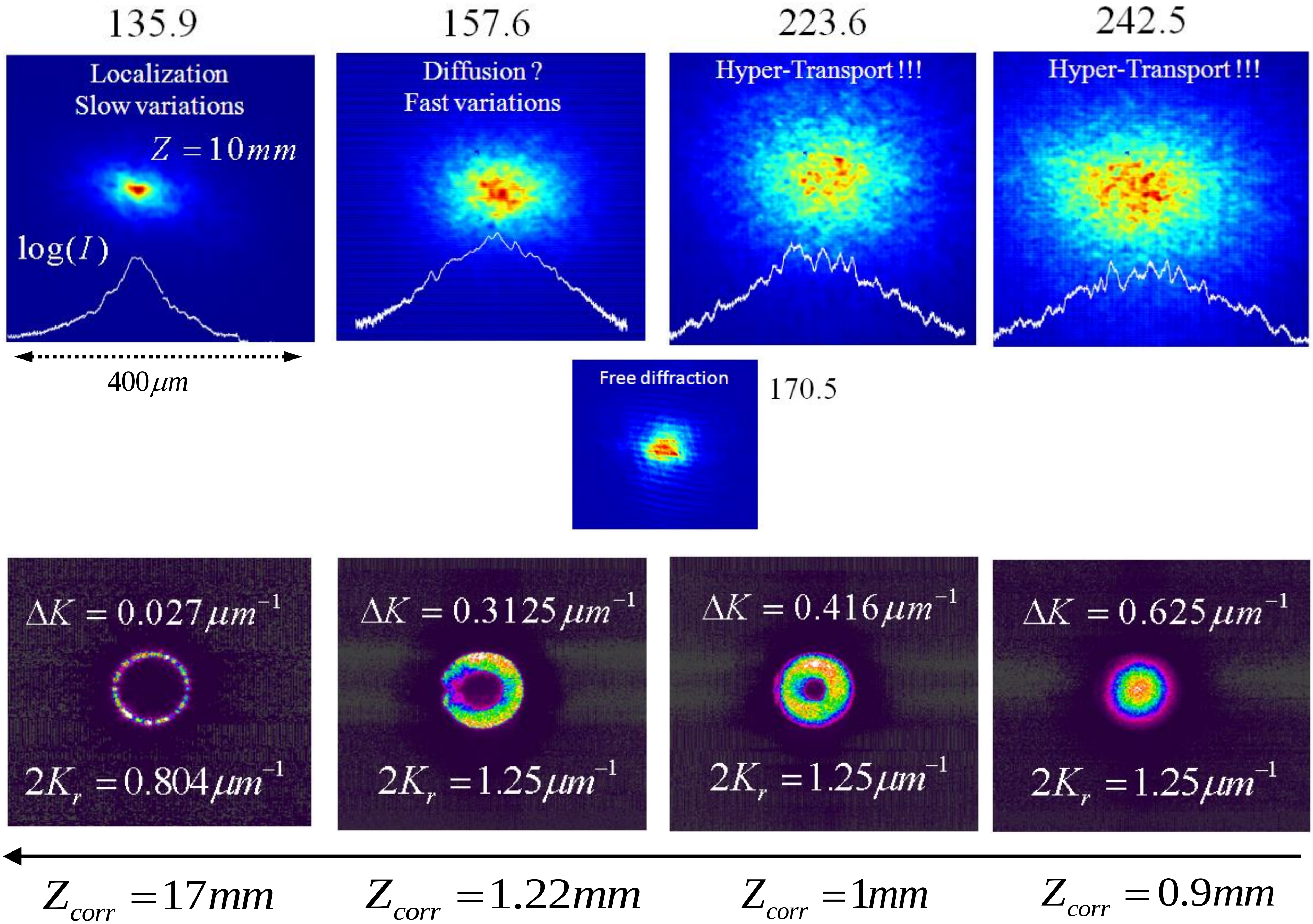
Direct observation of Anderson localization of matter-waves in a controlled disorder

Juliette Billy[1], Vincent Josse[1], Zhanchun Zuo[1], Alain Bernard[1], Ben Hambrecht[1], Pierre Lugan[1], David Clément[1], Laurent Sanchez-Palencia[1], Philippe Bouyer[1] & Alain Aspect[1]

[1]*Laboratoire Charles Fabry de l'Institut d'Optique, CNRS and Univ. Paris-Sud, Campus Polytechnique, RD 128, F-91127 Palaiseau cedex, France*



Hyper Transport of Light (experimental results)



Theory

How are the details and structure of noise reflected in transport?

Interest in spreading to high momentum

Is spreading of waves similar to the one of particles?

Theory for Particles in 1D

Interest in spreading to high momentum

mass=1

Wave nature is probably not important.

$$H = \frac{v^2}{2} + V(x,t)$$

Natural to use Fourier expansion in optics and in atom optics

$$V(x,t) = \frac{1}{\sqrt{N}} \sum_{m=-N}^{N} A_m e^{\left[i\left(k_m x - \omega_m t\right)\right]}$$

$$A_{-m} = A_m^*$$

$$\langle A_m \rangle = \langle A_m A_n \rangle = 0$$

$$\langle A_m A_n^* \rangle = \sigma^2 \delta_{n,m}$$

A_m **Independent random variables**

Chirikov Resonances

$$V(x,t) = \frac{1}{\sqrt{N}} \sum_{m=-N}^{N} A_m e^{\left[i\left(k_m x - \omega_m t\right)\right]}$$

Dominant energy transfer at resonance, points where phase is statinary

$$v_m^{res} = \frac{\omega_m}{k_m}$$

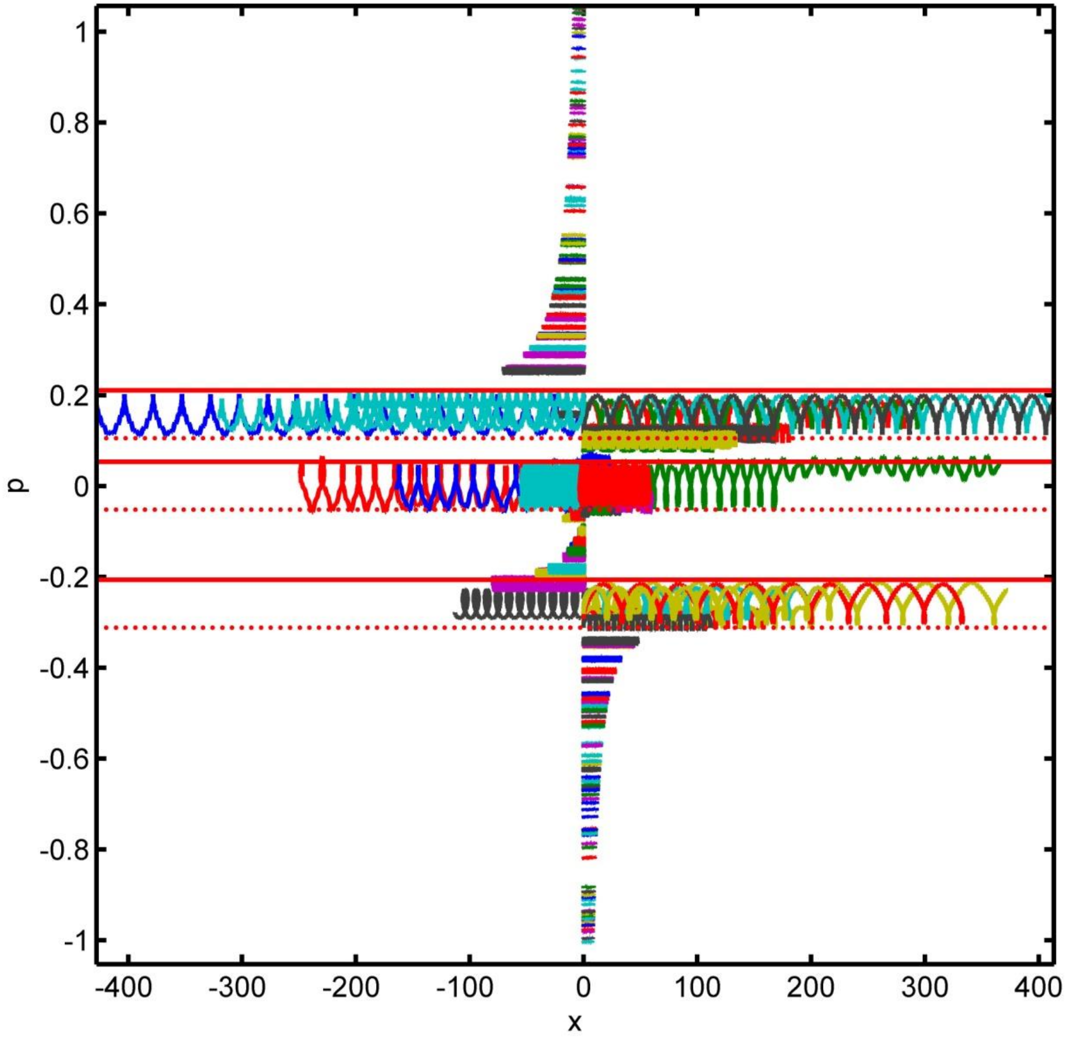
Effective pendulum

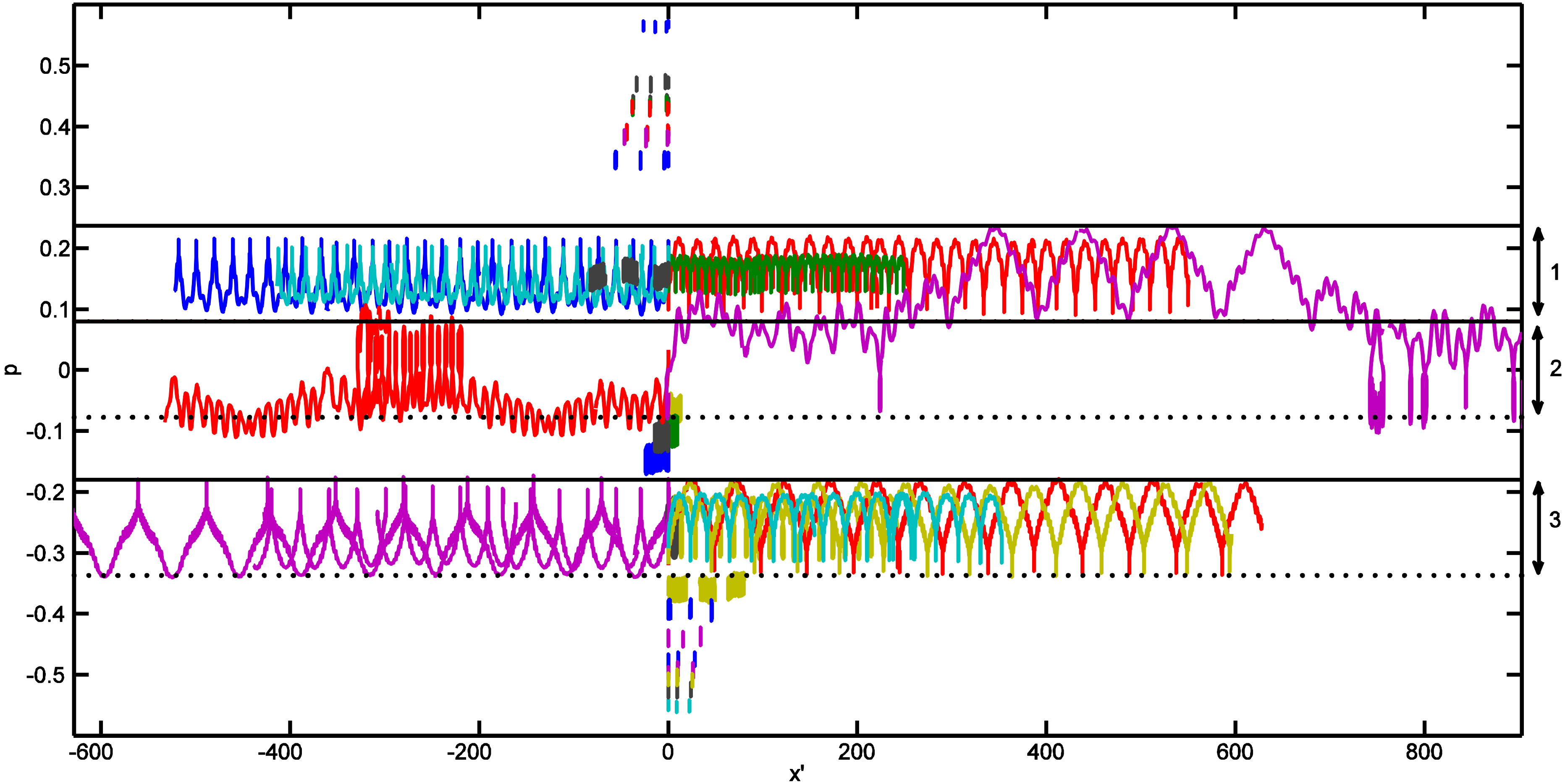
$$H^{res} = \frac{v^2}{2} + \frac{2A_m}{\sqrt{N}} \cos k_m x$$

Width

$$\sqrt{\frac{8\left|A_m\right|}{\sqrt{N}}} = \Delta_N$$

No overlap





A phase-space, x'=x-p , for two overlapping resonances. The double arrows indicate the region of the resonances. Resonances (1) and (2) overlap. Different colors (shades) describe different trajectories. The horizontal black solid and dashed lines, designate the edges of the resonance chains. The initial conditions were uniformly distributed on the p axis (x(t=0)=0). The system was integrated up-to t=10^4 (dimensionless variables).

Random walk between resonances

Chaos, correlations decay

Diffusion in momentum modeled by a Fokker-Planck Equation

Decay of Correlations and the Fokker-Planck Equation

$$\delta v = \int_{o}^{\delta t} dt F\left(x(t),t\right)$$

Force $F = -\dfrac{\partial V}{\partial x}$

Correlation Function $K(x_1,t_1;x_{2,}t_2) = \left\langle F\left(x_1,t_1\right) F\left(x_2,t_2\right)\right\rangle$

Stationary and translation invariant

Continuum limit

$$S\left(k,\omega\right)$$

Fourier Transform of potential correlation function=Power spectral density

$$K(x_1 - x_2, t_1 - t_2) = \iint d\omega dk k^2 S\left(k,\omega\right) e^{i\left[k\left(x_1 - x_2\right) - \omega\left(t_1 - t_2\right)\right]}$$

Assume rapid decay of correlations.

In the regime where correlation important, acceleration negligible

Therefore in correlation functions $$x_1 - x_2 = v\left(t_1 - t_2\right)$$

Diffusion coefficient

$$D\left(v\right) = \frac{1}{2}\int_{-\infty}^{\infty} d\tau \iint dk\,d\omega\, k^2 S\left(k,\omega\right) e^{i\left(kv-\omega\right)\tau}$$

$$D\left(v\right) = \pi \int k k^2 S\left(k, kv\right)$$

Fokker Planck equation $$\frac{\partial P\left(v,t\right)}{\partial t} = \frac{\partial}{\partial v} D\left(v\right) \frac{\partial}{\partial v} P\left(v,t\right)$$

Universality Classes

First $d=1$

$$D(v)=\pi\int dk\,k^2 S(k,kv)$$

What happens in the asymptotic limit of large velocity?

1. Main class:

define $k'=kv$

for $v\to\infty$

$$D(v)=\frac{D_0}{v^3}$$

$$D_0=\int_{-\infty}^{\infty}\mathrm{d}k'\,k'^2 S(0,k')$$

$$\frac{\partial P}{\partial t} = \left(\frac{\partial}{\partial v} \frac{D_0}{v^3} \frac{\partial}{\partial v} \right) P$$

Scaling solution

$$P(v,t) = \frac{1}{t^{d/5}} g\left(\frac{v^5}{t} \right)$$

$$\langle v^2 \rangle \sim t^{2/5}$$

Typically for large t

$$v \sim t^{1/5}$$

$$x \sim t^{6/5}$$

Hyper-transport

For white noise $v \sim t^{1/2}$

2. Class 2 Sum for potential

$$V(x,t) = \frac{1}{\sqrt{N}} \sum_{m=-N}^{N} A_m e^{i\left[i\left(k_m x - \omega_m t\right)\right]}$$

truncated as is the case in many experiments in optics and atom optics

$$v_m^{res} = \frac{\omega_m}{k_m} \leq v^{nax}$$

$$S\left(k,\omega\right) = 0 \qquad \omega / k > v_{\max}$$

$$D\left(v\right) = \pi \int \mathrm{d}k\, k^2 S\left(k, kv\right)$$

$$D\left(v\right) = 0 \qquad v > v_{\max}$$ **ballistic**

Note, the correlation function may be infinitely differentiable.

3. Class 3 Singular correlations

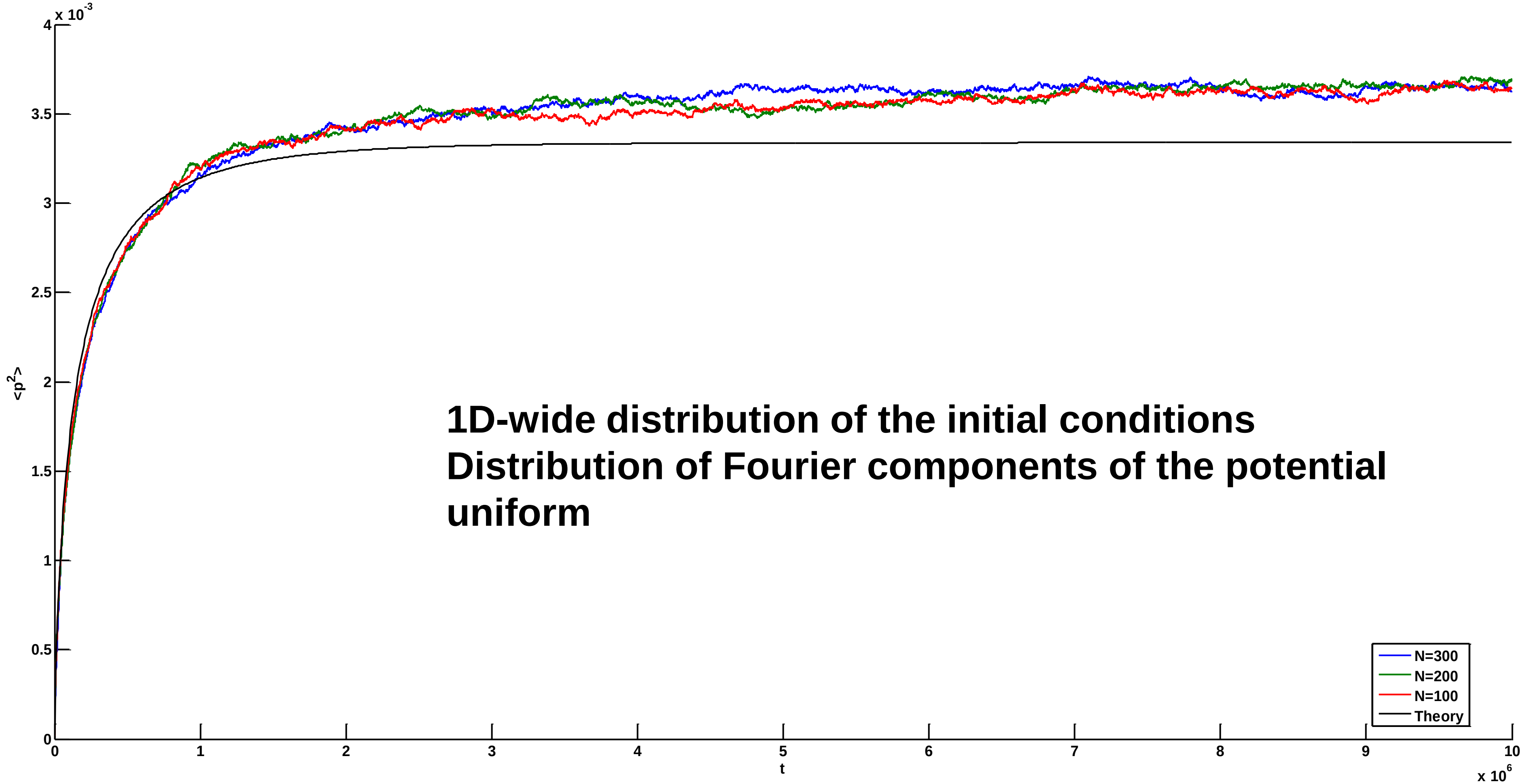
$$D(v) = \pi \int \mathrm{d}k\, k^2 S(k, kv)$$

$$S(k,\omega) = h\left(k, \frac{k}{\omega}\right)$$

Examples:

$$S(k,\omega) = h(k,x) \sim e^{-1/x} \qquad x = \frac{1}{v}$$

Also possible, Taylor expansion starts from $\frac{1}{v^n}$ $\quad n > 3$



$\langle p^2 \rangle$ as a function of t . Dashed gray line are numerical solution of the equation of motion for H for 1000 initial conditions taken from a Gaussian distribution and averaged over 10 realizations of the potential, V . Different shades and styles of lines designate different N (see legend). The solid thick line is the numerical solution of the Fokker-Planck equation.

Dispersion relations

In some cases there is a relation $\omega(k)$

Often $\omega(k) = \frac{k^2}{2}$

$$\frac{\omega}{k} = \frac{k}{2}$$

Optics potentials

$$E(\mathbf{x},t) = \int \mathrm{d}\mathbf{k}\, \hat{E}(\mathbf{k}) \exp i\left(\mathbf{k}\cdot\mathbf{x} - \omega(k)t\right)$$

$$\left\langle \hat{E}(\mathbf{k}) \right\rangle = 0$$

$$\left\langle \hat{E}(\mathbf{k}_1)\hat{E}^*(\mathbf{k}_2) \right\rangle = I_0 f(\mathbf{k}_1)\delta(\mathbf{k}_1 - \mathbf{k}_2)$$

$$\left\langle \hat{E}(\mathbf{k}_1)\hat{E}(\mathbf{k}_2) \right\rangle = 0$$

$$V(\mathbf{x},t) = \left|E(\mathbf{x},t)\right|^2 =$$

$$\int \mathrm{d}\mathbf{k}_1 \mathrm{d}\mathbf{k}_2\, \hat{E}(\mathbf{k}_1)\hat{E}^*(\mathbf{k}_2) e^{i\left[(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x} - \left(\omega(k_1)-\omega(k_2)\right)t\right]}$$

Higher dimensions $d > 1$

Chirikov resonance condition is satisfied even if bounds on ω **And** k

Since $v \bullet k = \omega$

Diffusion tensor

$$D_{i,j}(v) = \frac{1}{2}\int_{-\infty}^{\infty} d\tau \iint d\mathbf{k}\, d\omega\, k_i k_j S(\mathbf{k},\omega) e^{i(\mathbf{k}\bullet\mathbf{v}-\omega)\tau}$$

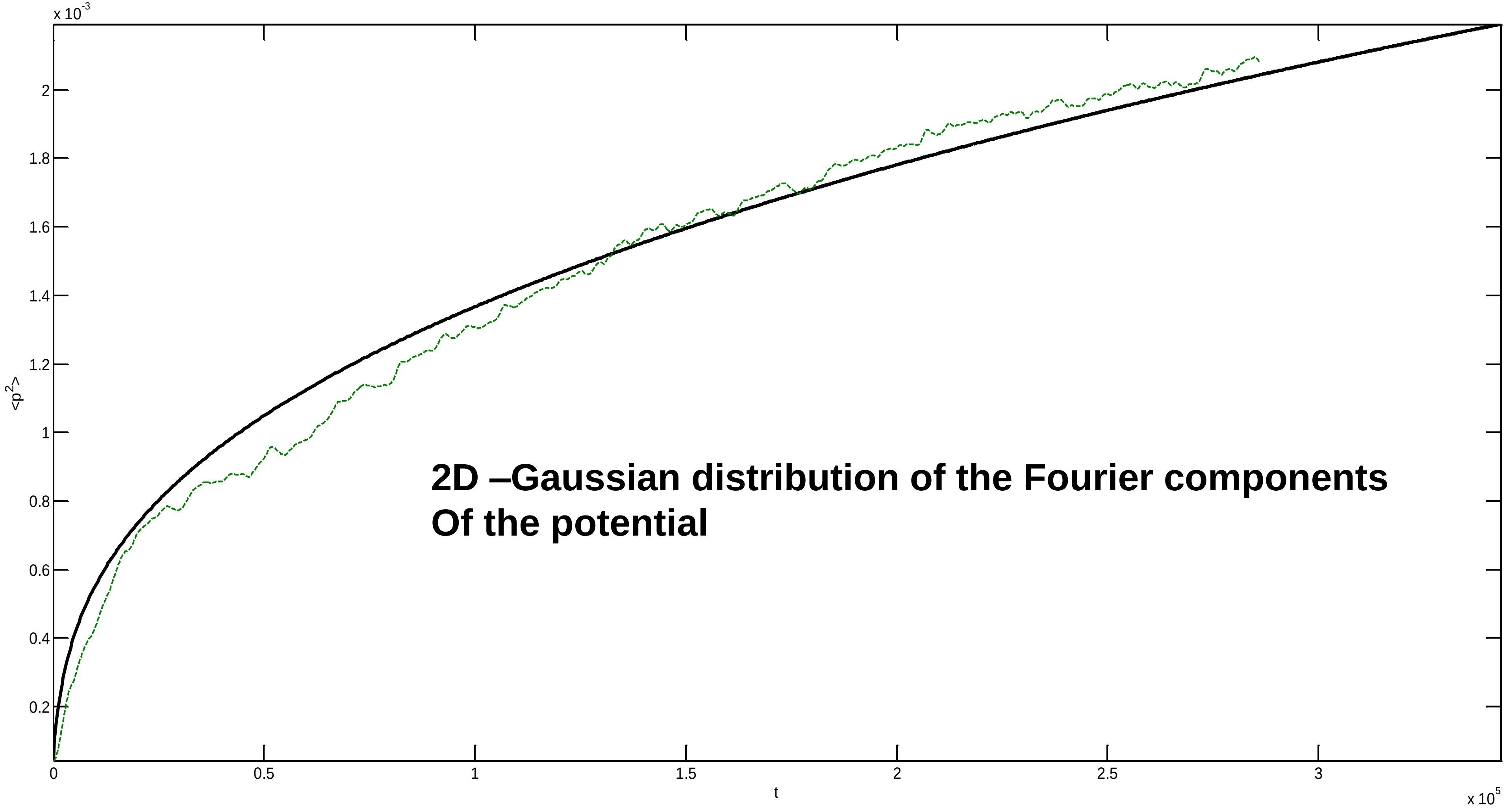
$$D(v) = \pi \int d\mathbf{k} (\mathbf{k}\cdot\hat{v})^2 S(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})$$

$\langle v^2 \rangle \sim t^{2/5}$ **Spectral spread**

$$\frac{\partial P}{\partial t} = \left(v^{-(d-1)} \frac{\partial}{\partial v} v^{(d-1)} \frac{D_0}{v^3} \frac{\partial}{\partial v} \right) P$$

$\langle x^2 \rangle \sim t^2$ **ballistic**

Main universality class much more robust than for $d = 1$



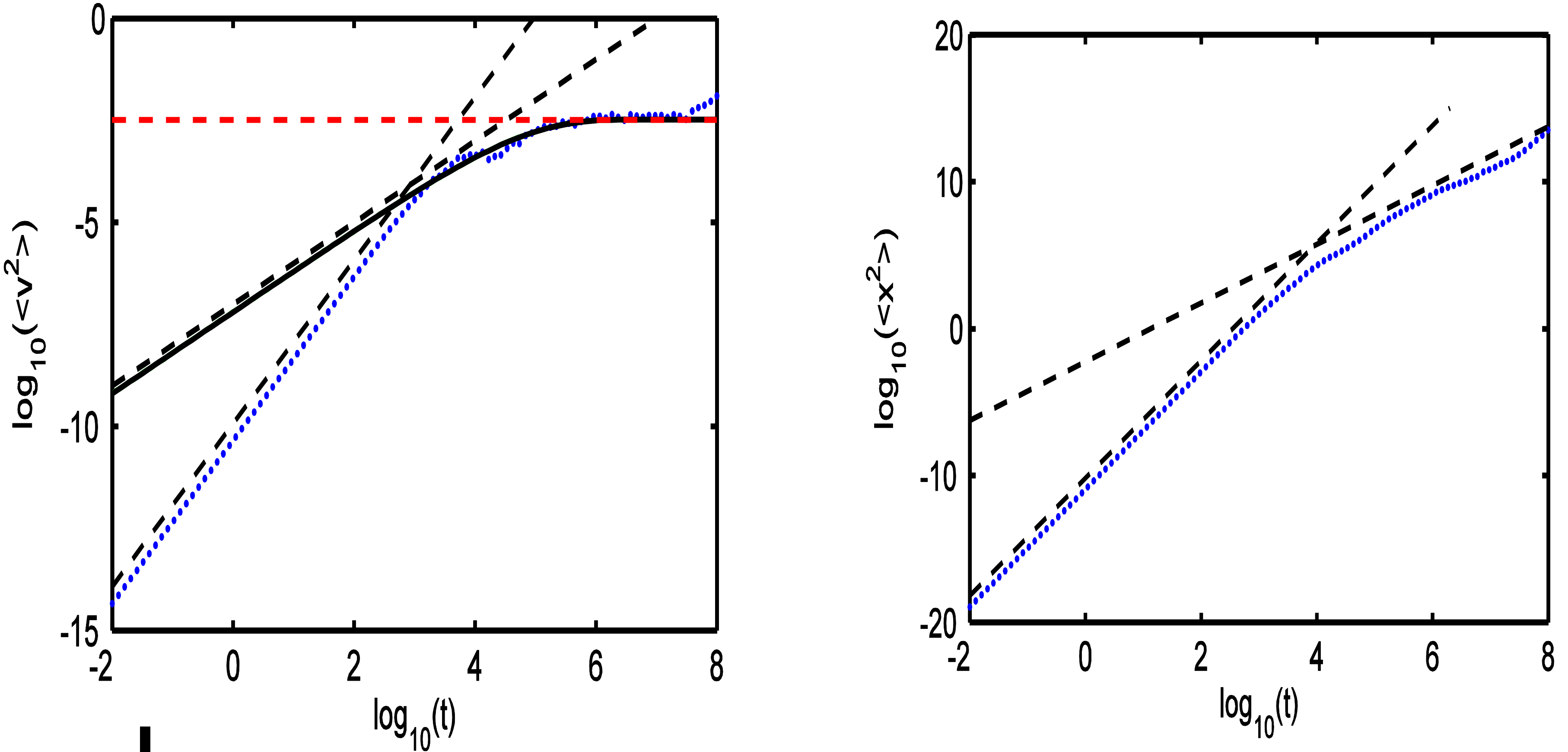
Same as previous figure, but for a two-dimensional system, for N=800 and parameters commonly used in the experiment.

When does the onset of fluctuations of the potential take place?

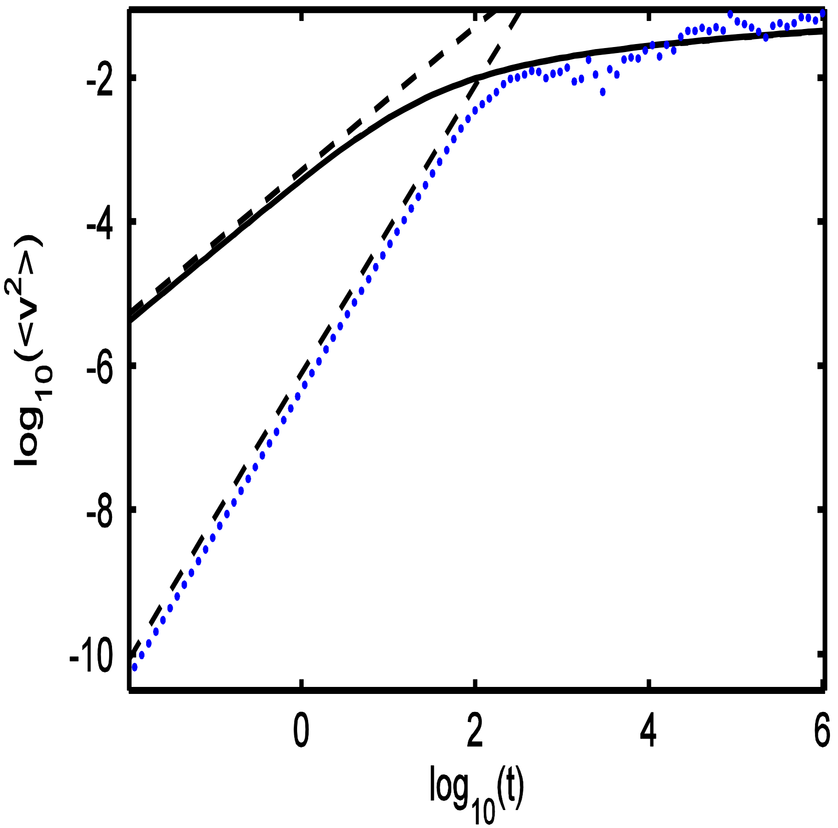
On short distances l_x

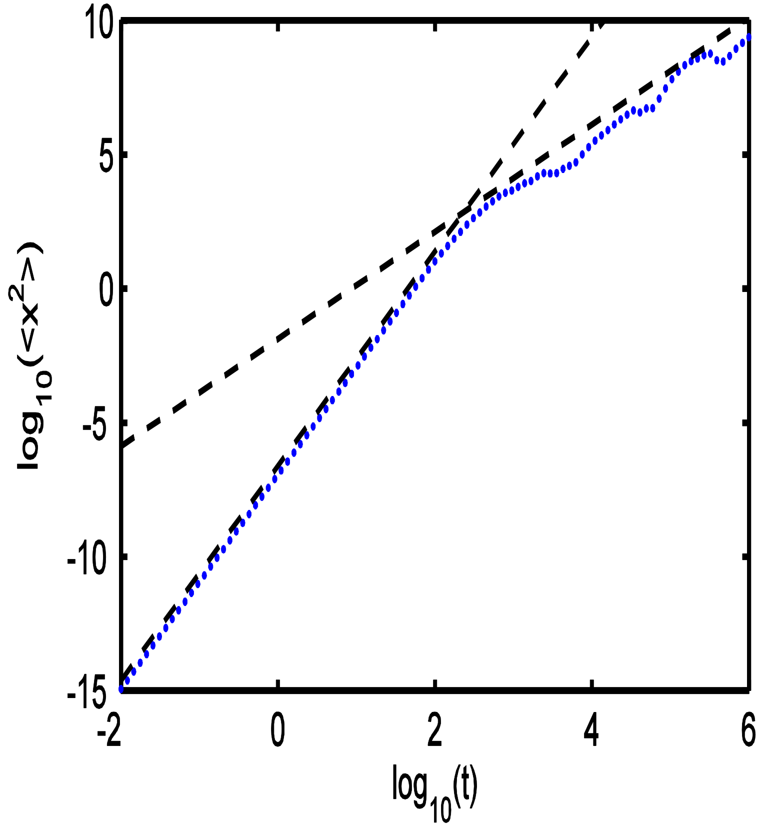
And short time scales l_t

Fluctuations not important and motion is uniform acceleration

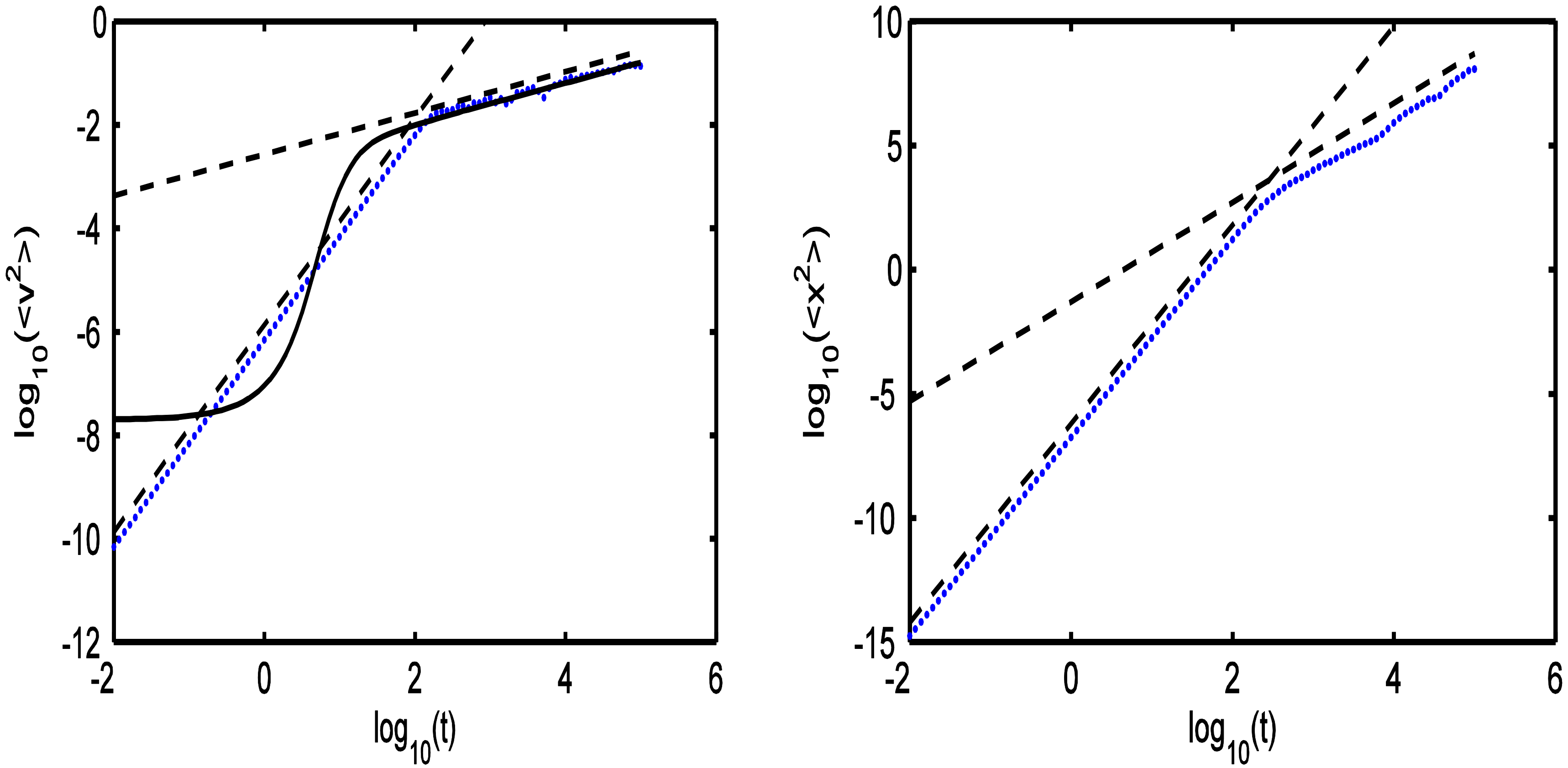


A log-log plot of average squared velocity as a function of time for one dimensional system with an optical potential and **a uniform distribution of wave-numbers over a segment**. The blue dots represent the result of the Monte-Carlo solution averaged over 20 realizations and the black solid line is the numerical solution of the Fokker-Planck equation for the velocity. The dashed black and red lines are guides for the eye with the corresponding slopes of 2 , 1 and 0. The initial condition was a narrow distribution of velocities for the Fokker-Planck and x=v=0 for the Monte-Carlo calculation. The parameters used for this simulation are, V0=1e-4 , kr=0.1 .

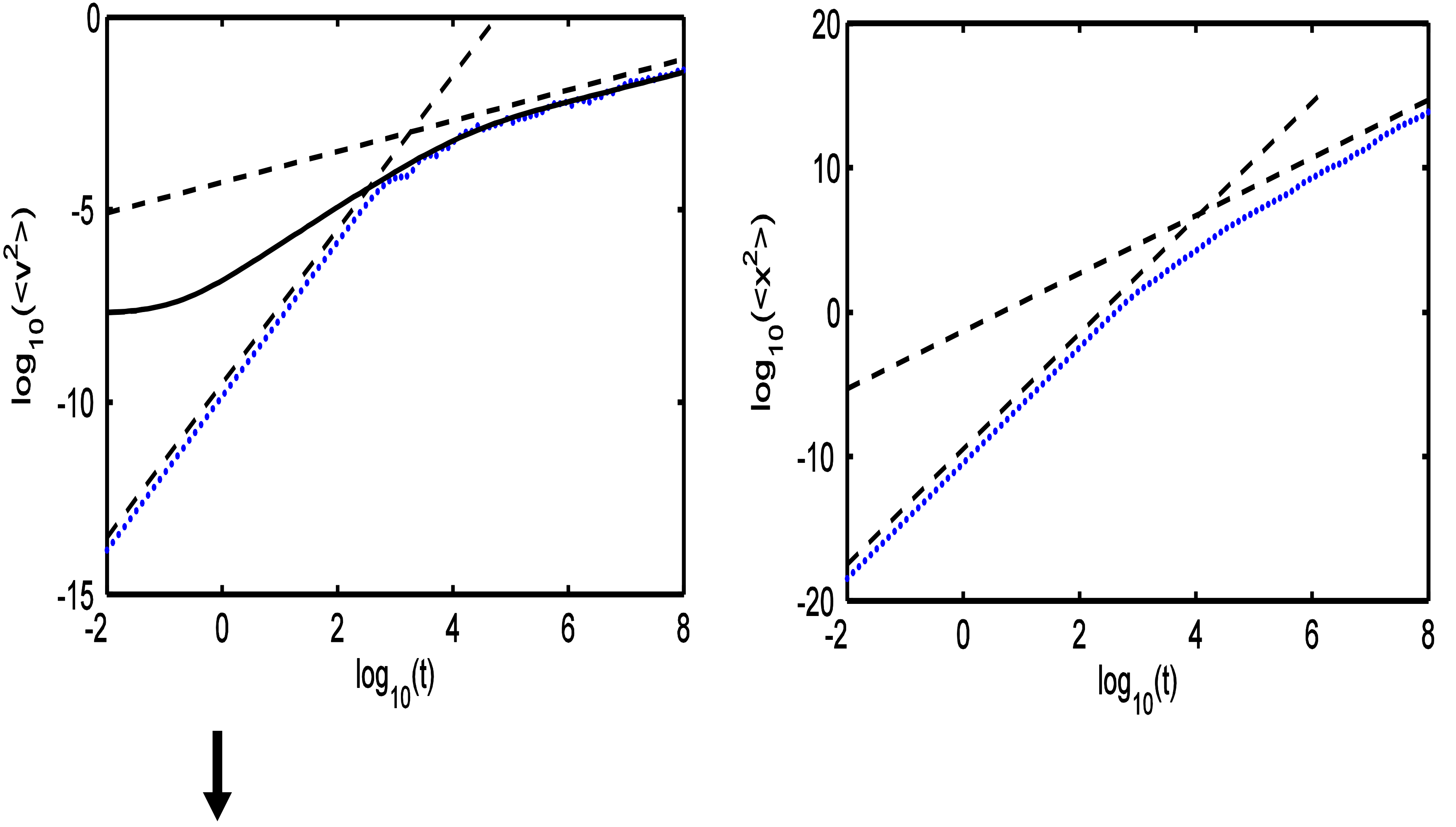




A log-log plot of average squared velocity as a function of time for one dimensional system with an optical potential and a **Gaussian distribution** of wave-numbers over a disk. The blue dots represent the result of the Monte-Carlo solution averaged over 20 realizations and the black solid line is the numerical solution of the Fokker-Planck equation for the velocity. The dashed black ines are guides for the eye with the corresponding slopes of 2 ,1. The initial condition was a narrow distribution of velocities for the Fokker-Planck and x=v=0 for the Monte-Carlo calculation. The parameters used for this simulation are, V0=1e-2 , kr=0.1 .



A log-log plot of average squared velocity as a function of time for a two dimensional system with a cosine potential and a **uniform distribution** of wave-numbers over a disk. The blue dots represent the result of the Monte-Carlo solution averaged over 20 realizations and the black solid line is the numerical solution of the Fokker-Planck equation for the velocity. The dashed black lines are guides for the eye with the corresponding slopes of 2 ,2/5. The initial condition was a narrow distribution of velocities for the Fokker-Planck and x=v=0 for the Monte-Carlo calculation. The parameters used for this simulation are, V0=1e-2 , kr=0.1 .



A log-log plot of average squared velocity as a function of time for a two dimensional system with **an optical potential** and a **uniform distribution** of wave-numbers over a disk. The blue dots represent the result of the Monte-Carlo solution averaged over 20 realizations and the black solid line is the numerical solution of the Fokker-Planck equation for the velocity. The dashed black ines are guides for the eye with the corresponding slopes of 2 ,2/5. The initial condition was a narrow distribution of velocities for the Fokker-Planck and x=v=0 for the Monte-Carlo calculation. The parameters used for this simulation are, V0=1e-2 , kr=0.1 .

For particles for spectrally bounded potentials:

no hyper-transport for d=1

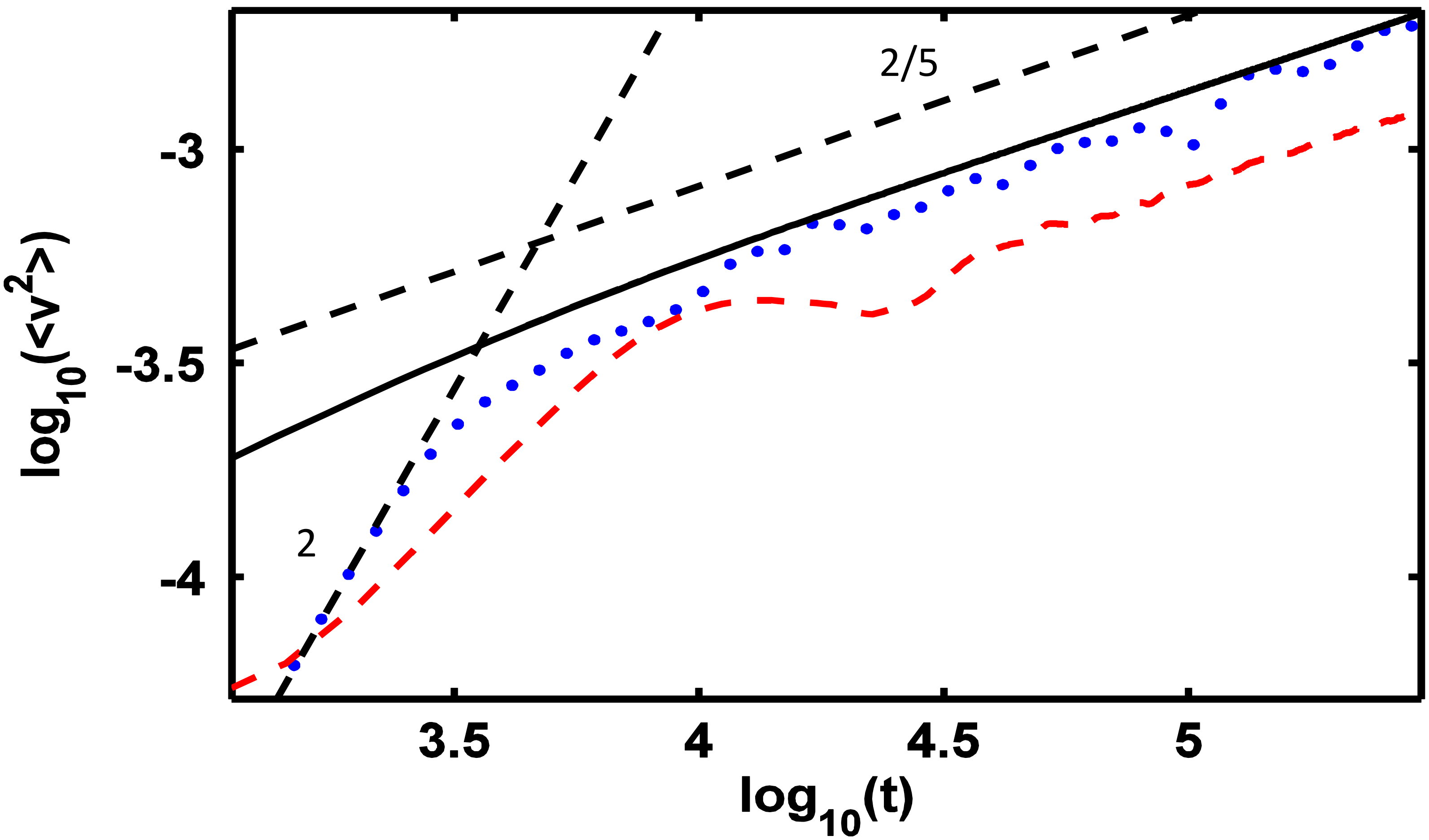
spectral hyper-transport d>1

no hyper-transport in space for d>1

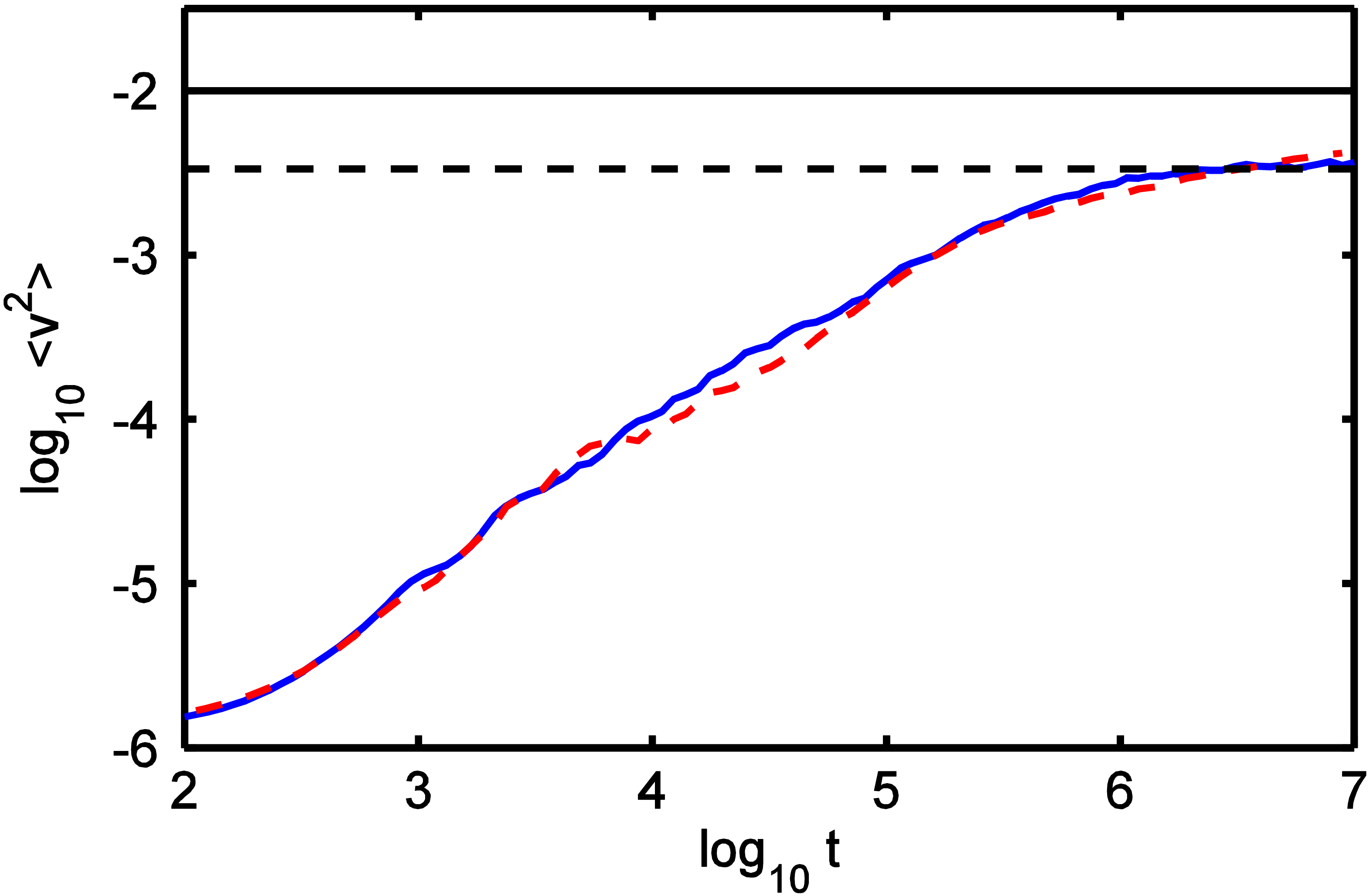
Comparison Between Particles and Waves

$$d = 1$$

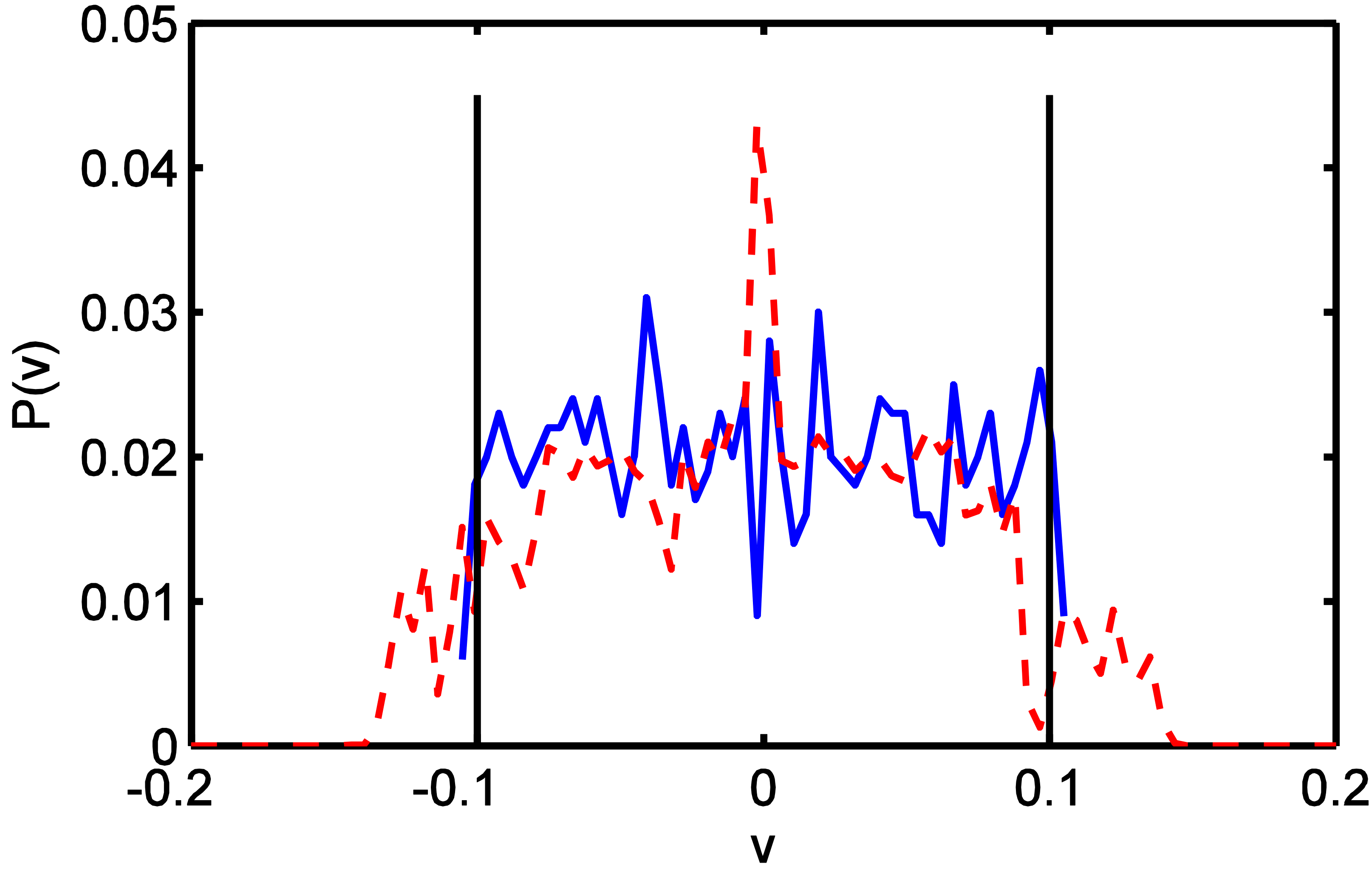
$$|A_m| = A$$



Log-log plots of the velocity and the position second moments vs. time in dimensionless units. Solid black line is the Fokker-Planck approximation, blue circles are Monte-Carlo of particles. Red solid line is a simulations for waves. Black dashed lines are guides for the eye. The parameters are the same as used in the experiment.



$$A = 10^{-4}, N = 100, v_{\max} = 0.1, \Delta_x = 0.1.$$



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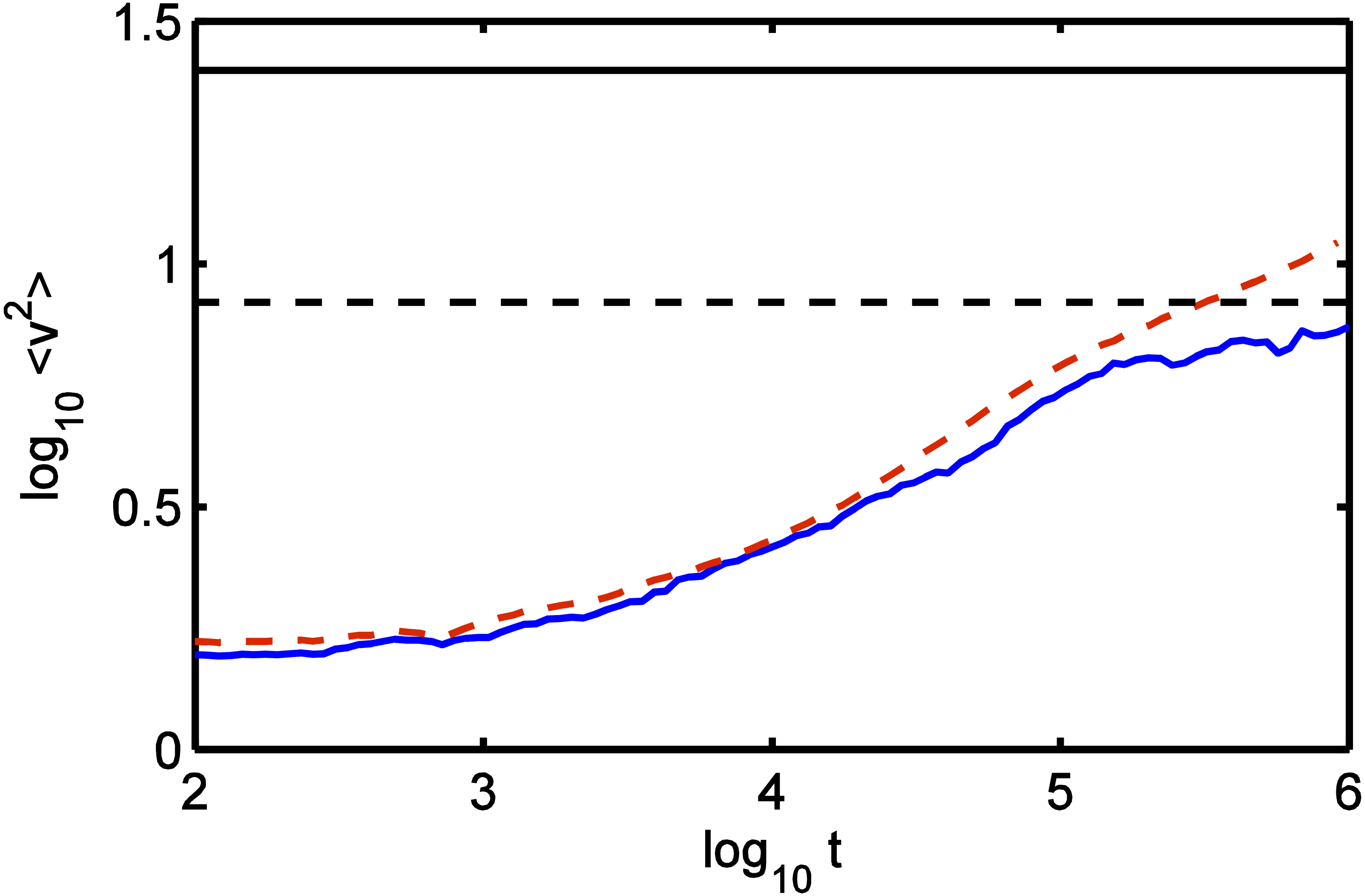
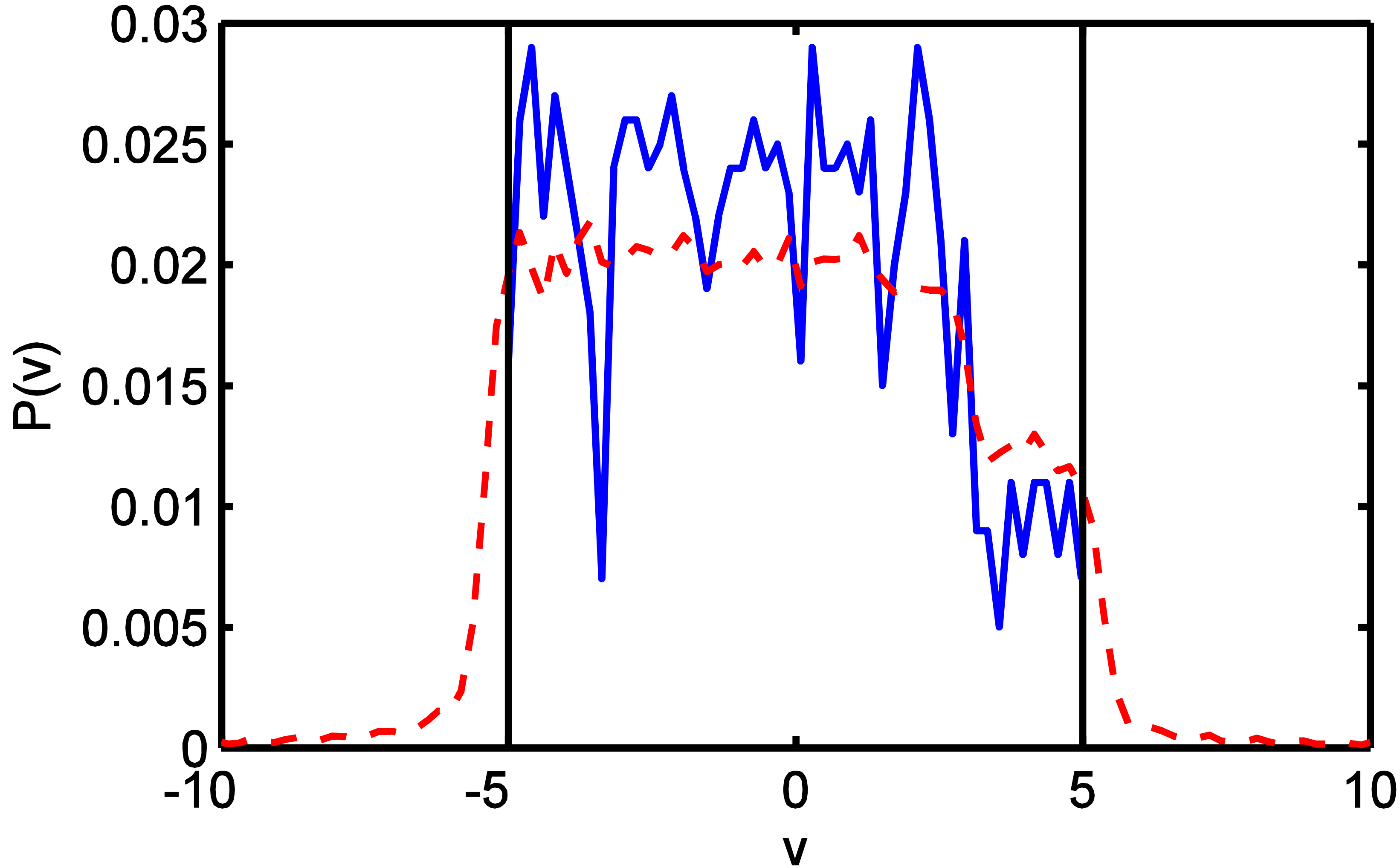
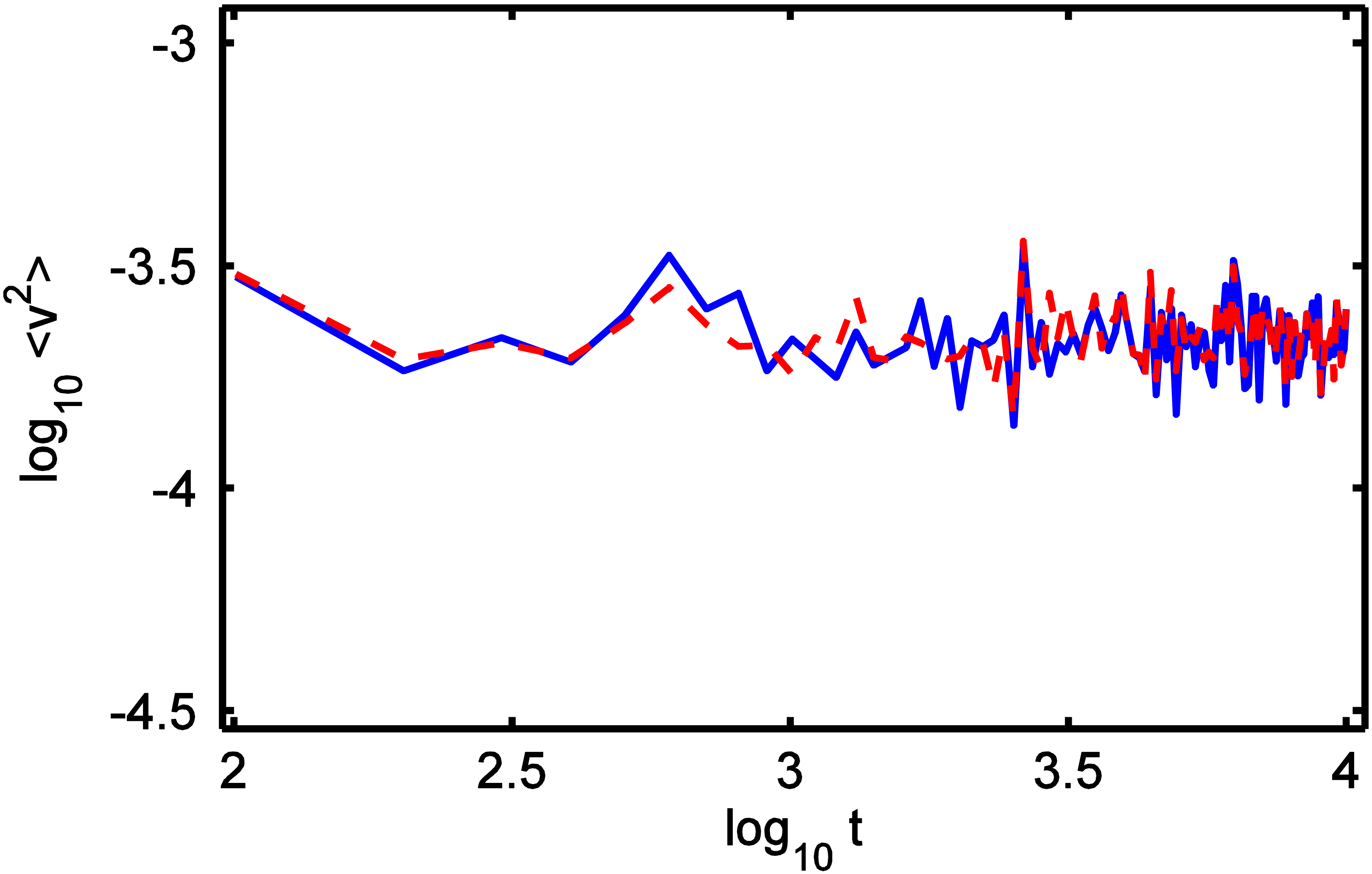


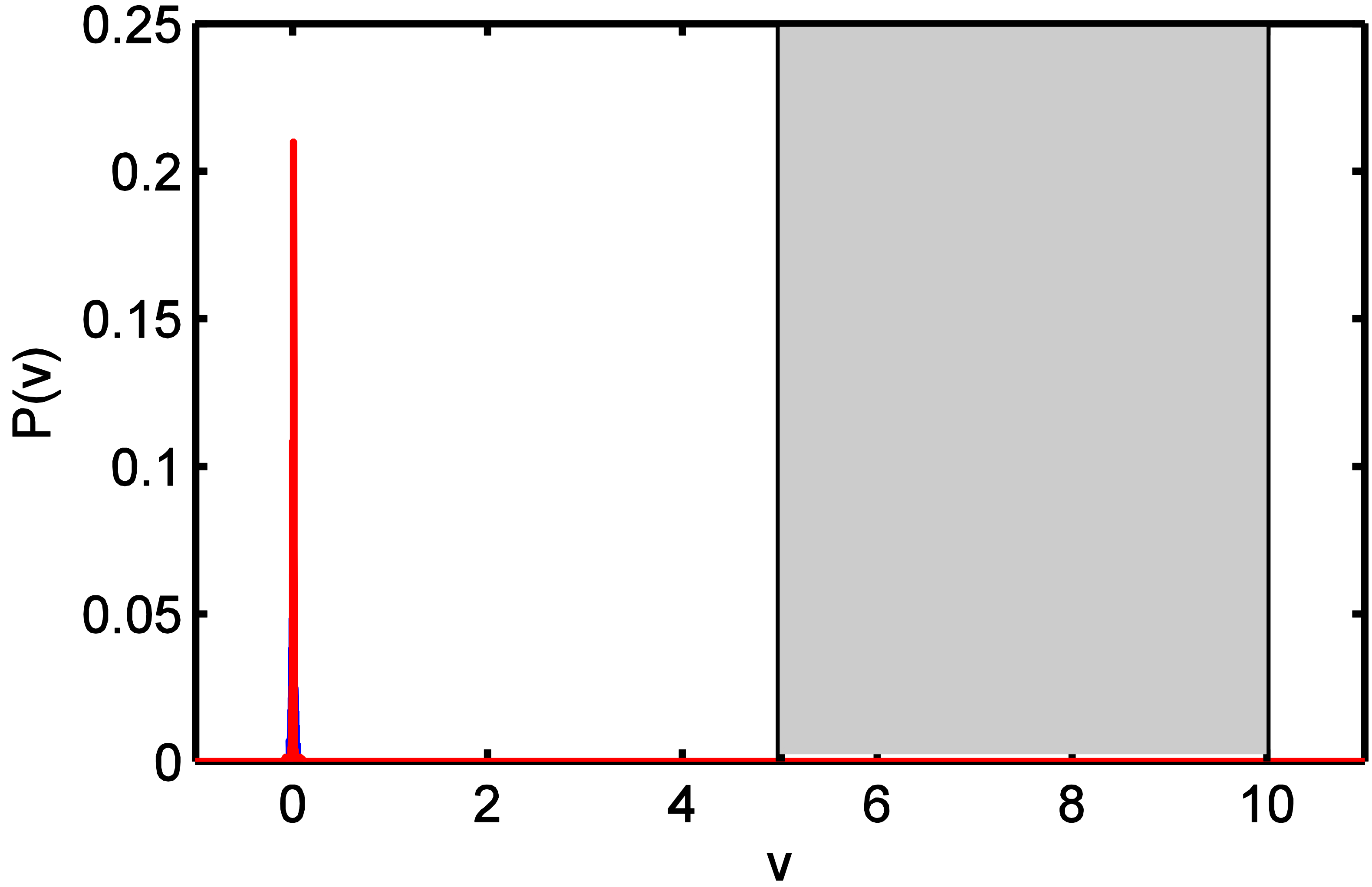
$$A = 10^{-1}, N = 100, v_{\max} = 5, \Delta_x = 0.1.$$



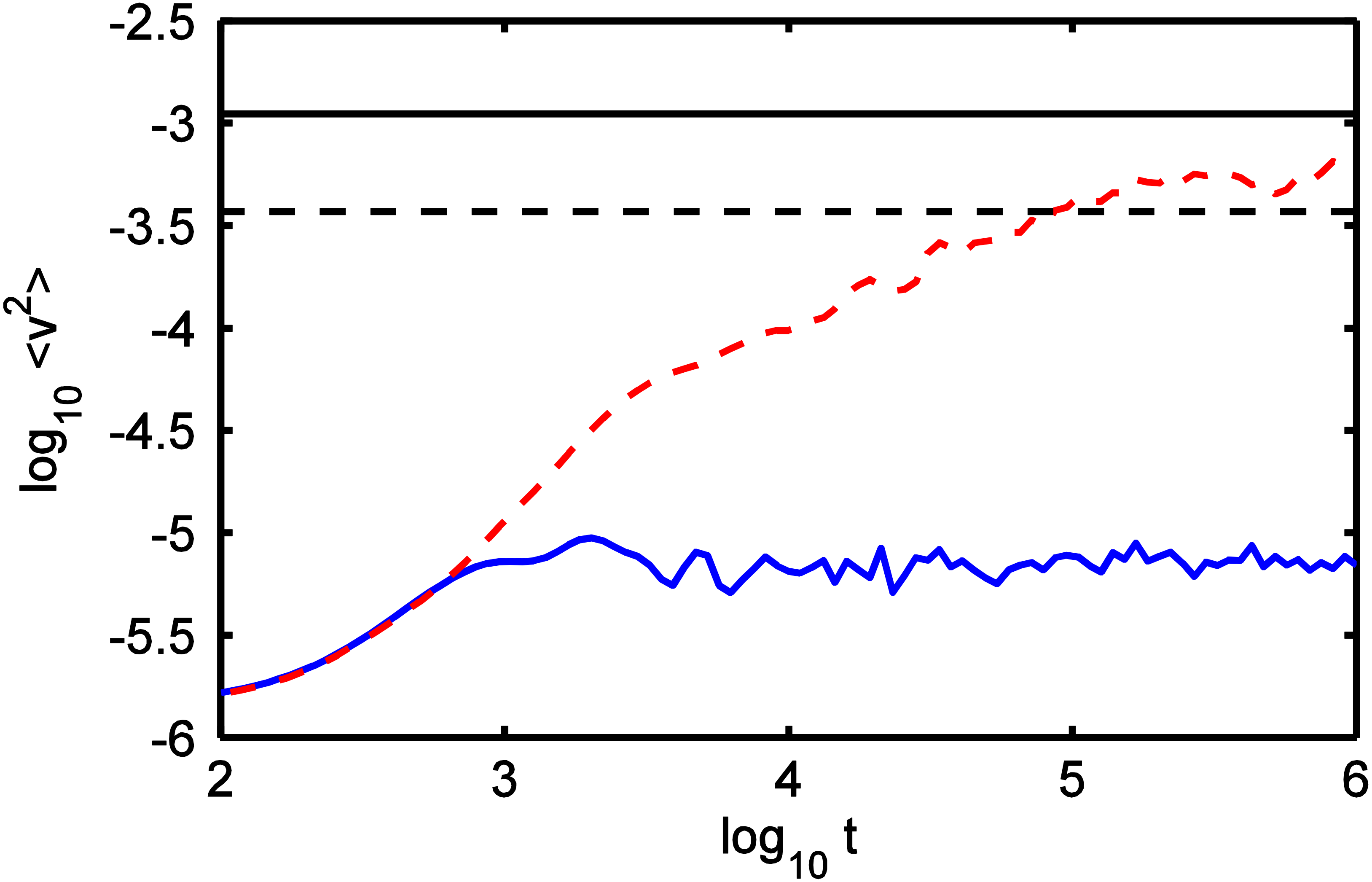
$$A = 10^{-1}, N = 100, v_{\max} = 5, \Delta_x = 0.1.$$



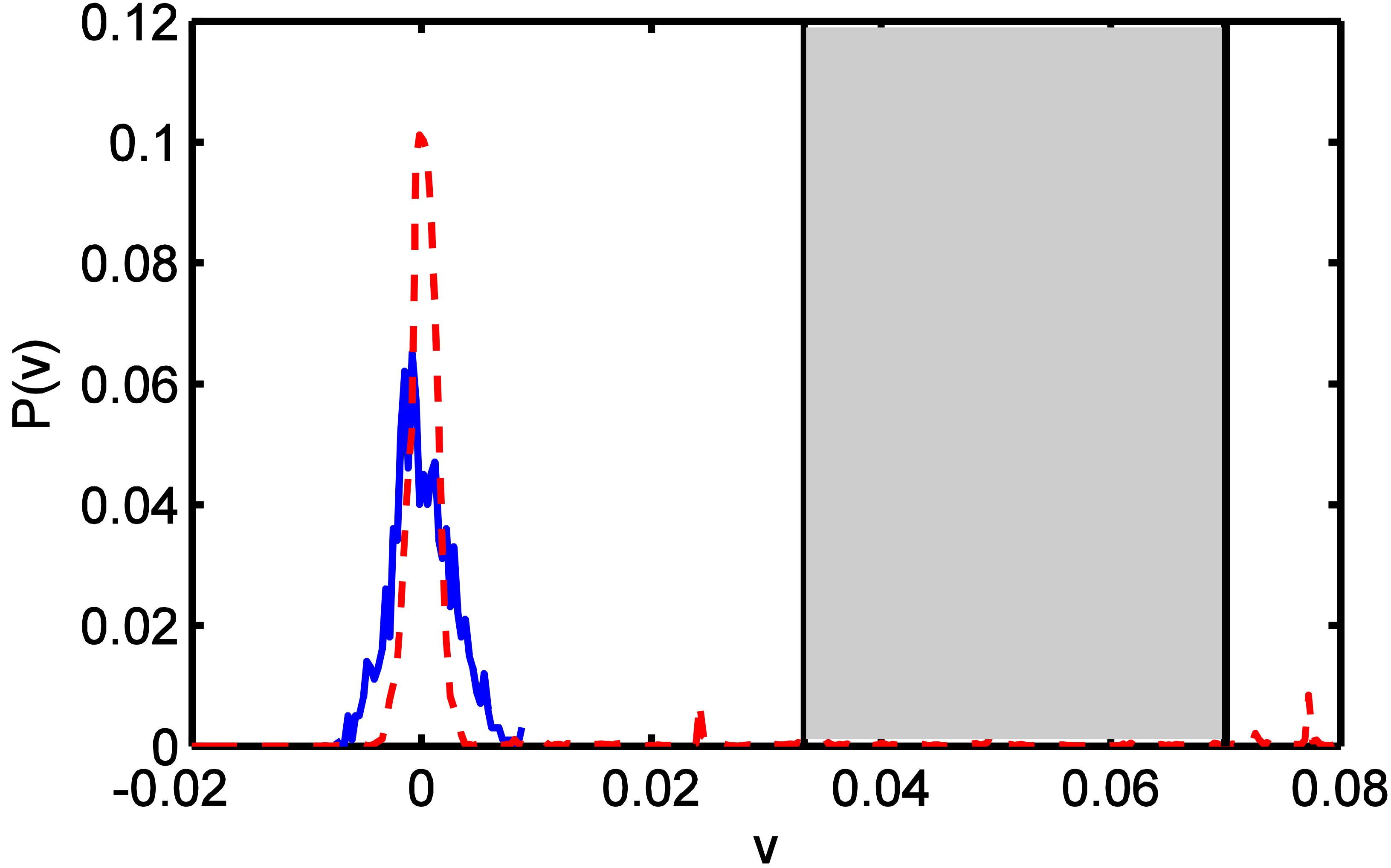
$$A = 0.1, 5 \le v_i \le 10, \Delta_x = 0.1,$$



$$A = 0.1, 5 \le v_i \le 10, \Delta_x = 0.1,$$



$$A = 10^{-4}, 0.0228 \le v_i \le 0.07, \Delta_x = 0.1$$



$$A = 10^{-4}, 0.0228 \leq v_i \leq 0.07, \Delta_x = 0.1$$

summary

1. A formula for the diffusion coefficient in momentum in terms of the distribution function of the Fourier components of the potential was developed. Natural for potentials used in optics and atom optics
2. Classification into Universality classes and Identification of new classes.
3. Identification of the uniform acceleration regime for short time.
4. Asymptotically particles agree with waves in present calculations (d=1)

Spreading

For particles for spectrally bounded potentials:

no hyper-transport for $d=1$

spectral hyper-transport $d>1$

no hyper-transport in space for $d>1$

Open problems

1. Identification of regimes that are not uniform acceleration where Fokker—Planck fails
2. An analytic theory for spreading in coordinate space.
3. Is there a regime of high momentum where waves spread in a way fundamentally different from particles?