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From dawn to decadence: Creation and decay of coherent pulses in one dimensional wave turbulence

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From dawn to decadence: Creation and decay of coherent pulses in one dimensional wave turbulence

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- The Majda-McLaughlin-Tabak model
- Instability of wave turbulence
- Energy transfer by radiating pulses

Majda-McLaughlin-Tabak (MMT) model for weakly nonlinear waves

$$\left(i\frac{\partial}{\partial t} - \mathcal{L}\right)\psi(x,t) \quad = \quad \lambda\psi(x,t)|\psi(x,t)|^2$$

- complex wave amplitude $\psi(x,t)$
- linear operator $\mathcal{L}\exp(ikx) = \omega_k \exp(ikx)$,
- dispersion $\omega_k = \sqrt{|k|}$.

(Fourier modes $a_k = \int_{-L/2}^{L/2} \psi(x,t)\exp(-ikx)dx/\sqrt{2\pi}$
Large system size L with periodic boundary conditions)

A.J. Majda, D.W. McLaughlin, E.G. Tabak, J. Nonlinear Sci. 6, 9 (1997),

Conserved quantities of the MMT equation

- Hamiltonian or 'energy'

$$E = \sum_k \omega_k |a_k|^2 + (\lambda/2) \int_{-L/2}^{L/2} |\psi|^4 dx$$

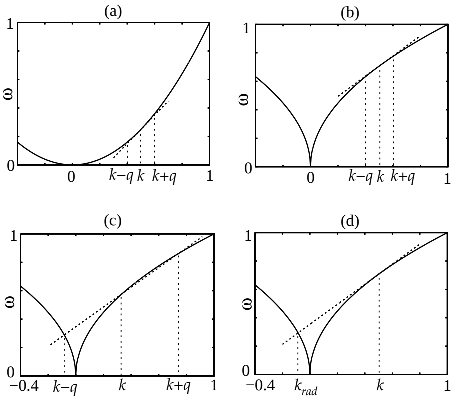
- waveaction

$$N = \sum_k |a_k|^2$$

- momentum

$$P = \sum_k k |a_k|^2$$

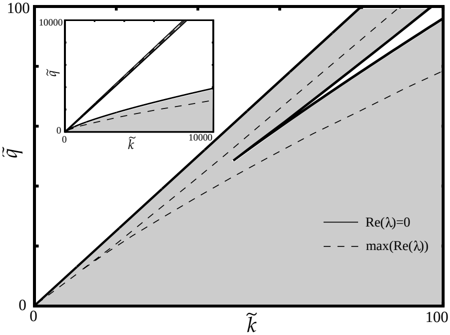
Dispersion $\omega = k^2$ vs. $\omega = \sqrt{|k|}$



(a) modulational instability for the nonlinear Schrödinger equation
(b) modulational instability for MMT with $k \gg q$
(c) instability for MMT with $k \sim q$
(d) radiating quasisolitons for MMT

Instability by short modulations for $\lambda = 1$

- Monochromatic wave $\psi = (\psi_0 + \delta a)\exp(ikx)$
- Modulation $\delta a = \delta a_+ \exp(iqx) + \delta a_- \exp(-iqx)$
- Instability near $q \approx 5k/4$.

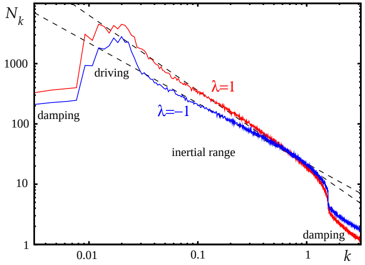


Rescaled wavenumbers:

- carrier wave $\tilde{k} = k/A^4$,
- modulation $\tilde{q} = q/A^4$

Damped and driven MMT equation

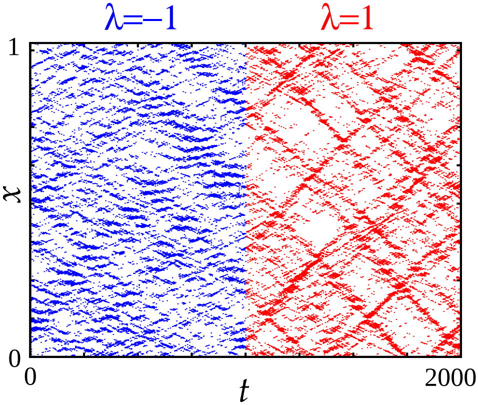
- Driving force at moderate wavenumbers
- Damping at high and at low wavenumbers
- $N_k = \langle |a_k|^2 \rangle$



- $\lambda = -1$: Kolmogorov-Zakharov spectrum $N_k \sim k^{-1}$ $\rightarrow$ wave turbulence
- $\lambda = 1$: Steeper spectrum $N_k \sim k^{-1.25}$ $\rightarrow$ unknown mechanism of turbulence

Contour plot of regions with high amplitudes:

Switching the sign $\lambda = -1$ from to $\lambda = 1$



Wave turbulence - Coherent structures

Envelope equation for wave turbulence

- Ensemble average $\langle u(\mathbf{x},t)u^*(\mathbf{x}+\mathbf{r},t)\rangle$ depends on $\mathbf{x}$
- Slow spatial variations of the waveaction

$$N(\mathbf{k},\mathbf{x},t) = \int \langle u(\mathbf{x},t)u^*(\mathbf{x}+\mathbf{r},t)\rangle \exp(i\mathbf{k}\cdot\mathbf{r})d\mathbf{r}$$

- Kinetic equation is extended by a Vlasov term

$$\frac{\partial N}{\partial t} + \frac{\partial \tilde{\omega}}{\partial \mathbf{k}}\frac{\partial N}{\partial \mathbf{x}} - \frac{\partial \tilde{\omega}}{\partial \mathbf{x}}\frac{\partial N}{\partial \mathbf{k}} = T_4[N]$$

- Nonlinear by the renormalized frequency

$$\tilde{\omega}(k,\mathbf{x},t) = \omega_k + 2\lambda \int N(\mathbf{p},\mathbf{x},t)d\mathbf{p}$$

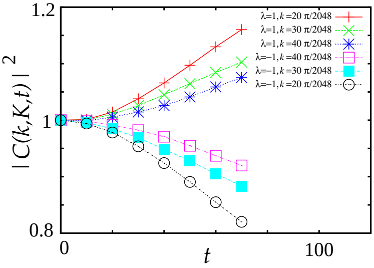
Breaking of the spatial homogeneity symmetry of wave turbulence

- Linearization $N(\mathbf{k}, \mathbf{x}, t) = N_0(k) + \Delta N(\mathbf{k}, \mathbf{x}, t)$
- Kolmogorov-Zakharov spectrum $N_0(k)$
- Modulation $\Delta N(\mathbf{k}, \mathbf{x}, t) = a(\mathbf{k}) \exp(i\mathbf{K} \cdot \mathbf{x} - i\Omega t)$

Stability:

- $\lambda > 0$:
  - unstable in one dimension $\sim$ negative Landau damping
  - no instability in two dimensions
- $\lambda < 0$:
  - no instability

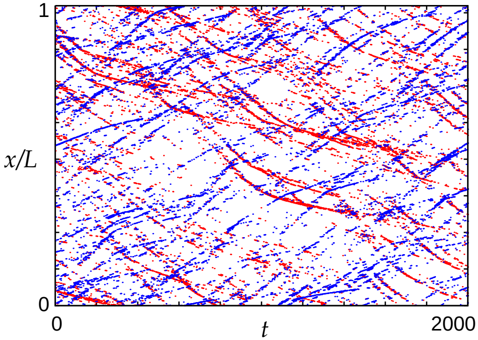
Growth of correlations for $\lambda = 1$; decay for $\lambda = -1$



- time evolution of the correlation $|C(k,K,t)|^2 = |\langle A_k(t) A^*_{k+K}(t)\rangle|^2 / |\langle A_k(0) A^*_{k+K}(0)\rangle|^2$ with $K = 6\pi/2048$ for an ensemble of 400,000 trajectories
- initial conditions contain a small correlation on top of a KZ spectrum

Formation of coherent structures for $\lambda = 1$:

A gas of solitary waves



Pattern of solitary waves ('pulses') with high positive or negative momenta

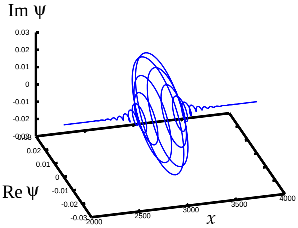
Quasisolitons for $q \ll k_m$

- slow modulation by the envelope $\phi^{(sol)}(x,t)$

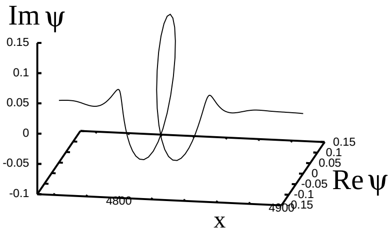
$$\psi(x,t) = \phi^{(sol)}(x,t)\exp(ik_m x - i\omega_m t)$$

- soliton solution

$$\phi^{(sol)}(x,t) = q\sqrt{-\omega''_m}\exp(i\omega''_m q^2 t/2)\mathrm{sech}(q(x-\omega'_m t))$$



Narrow pulse with $q \sim k_m$

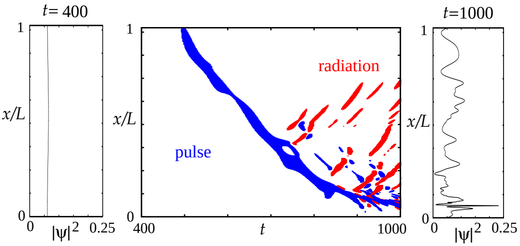


- Shape of a pulse

$$\psi^{(f)} = q\sqrt{\omega_m}{k_m}^{-1} f(\theta)\exp(i\alpha)\exp(i\Omega t)$$

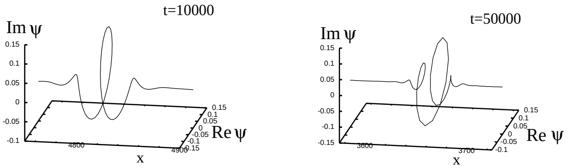
$$\theta_x = q,\ \theta_t = -qv,$$
$$\alpha_x = k_m,\ \alpha_t = -k_m v$$

Pulses emerge from the instability at $q \sim k_m$



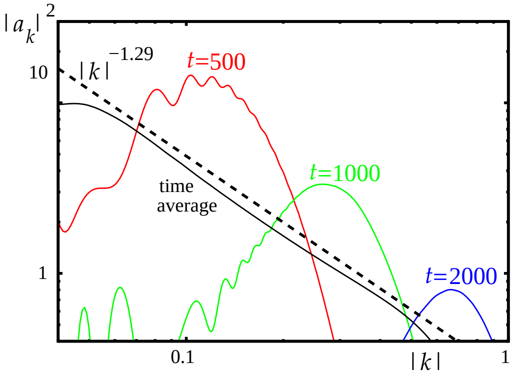
- The pulse-speed decays
- The pulse emits radiation

A radiating pulse evolves in time



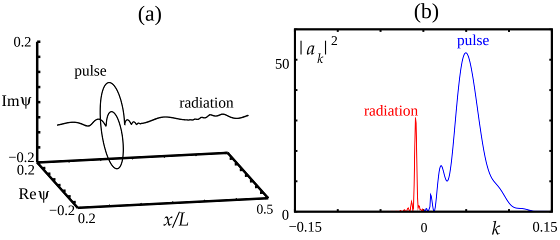
- The pulse narrows in real space - q increases
- The number of loops increases - k_m increases

Time-average of an evolving pulse is MMT-like



- Pulse in k-space at three different times
- Time-average spectrum $\langle |a_k^{(f)}|^2 \rangle \sim k^{-1.29}$

Radiation: Resonant driving of linear waves



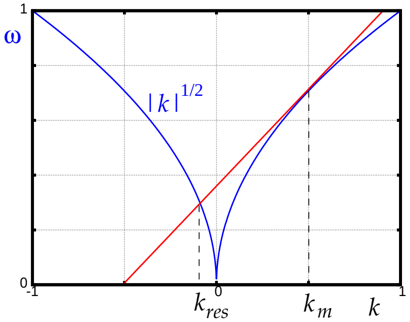
linear wave: driving force by the pulse:

$$i\dot{a}_k - \omega_k a_k \quad = \quad |T_k| \exp(-i\Lambda_k t)$$

with

$$T_k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi^{(f)} |\psi^{(f)}|^2 \exp(-ikx) dx$$

Doppler-shifted phase frequency $\Lambda_k = \Omega + kv$



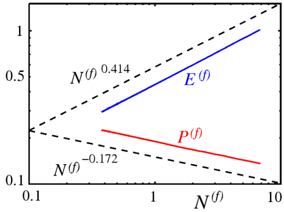
Resonance $\Lambda_k = \omega_k$
at $k_{res} \approx -(\sqrt{2}-1)^2 k_m$

Evolution of energy and momentum

- Balance of $dE^{(rad)}$ and $dP^{(rad)}$ for the radiation:
  $dP^{(rad)} = k_{res} dN^{(rad)}$
  $dE^{(rad)} = \sqrt{|k_{res}|} dN^{(rad)}$
- Balance of energy $dE^{(f)} = -dE^{(rad)}$ and momentum $dP^{(f)} = -dP^{(rad)}$ of the pulse:

$$\left(\frac{dE^{(f)}(N^{(f)}, P^{(f)}(N^{(f)}))}{dN^{(f)}}\right)^2 = -\frac{dP^{(f)}(N^{(f)})}{dN^{(f)}}$$

- $N^{(f)}$ decays in time
- $E^{(f)} \sim {N^{(f)}}^{\sqrt{2}-1}$ decays
- $P^{(f)} \sim {N^{(f)}}^{\sqrt{8}-3}$ increases



- Radiation driven by a pulse:

$$i\dot{a}_k - \omega_k a_k = |T_k| \exp(-i\Lambda t)$$

- Time-dependent pulse frequency with linear chirp approximation $\Lambda(t) \approx \omega_k + \dot{\Lambda} t$
- Amplitude of radiation

$$|a_k|^2 \sim T_k^2 \sqrt{|k_m|}/\dot{k}_m$$

after the driving frequency $\Lambda(t)$ has moved through resonance

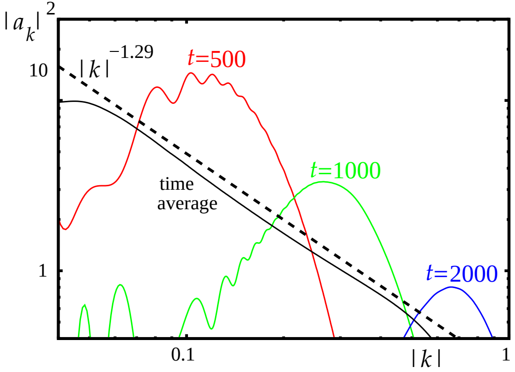
The spectrum of the pulses

- Driving force $T_k \sim q^2 k_m^{-9/4}$
- Speed of a pulse in k-space: $\dot{k}_m \sim q^3 k_m^{-3/2}$
- Wave action of a pulse $N^{(f)} \sim q k_m^{-3/2}$
- Spectrum:

$$
\begin{aligned}
\langle |a^{(f)}_{k=k_m}|^2 \rangle &\sim N^{(f)}/\dot{k}_m \\
&\sim q^{-2} \\
&\sim \sim k_m^{(3\sqrt{2}-5)/(2-\sqrt{2})} \sim k_m^{-1.29}
\end{aligned}
$$

- Solve coupled equations for pulse and radiation
- Time-average of the pulse yields the spectrum

$$\langle |a_k^{(pulse)}|^2 \rangle_{time} \sim k^{-2+1/\sqrt{2}} \sim k^{-1.29}$$



Conclusions

- Spatial homogeneity of wave turbulence spontaneously broken
- Transfer of energy by radiating pulses

B.R., A.C. Newell, V.E. Zakharov, PRL 103, 074502 (2009);
A.C. Newell, B.R., V.E. Zakharov, PRL 108, 194502 (2012);
B.R., A.C. Newell, PLA 377, 1260 (2013)