The Abdus Salam
International Centre
for Theoretical Physics

2472-1

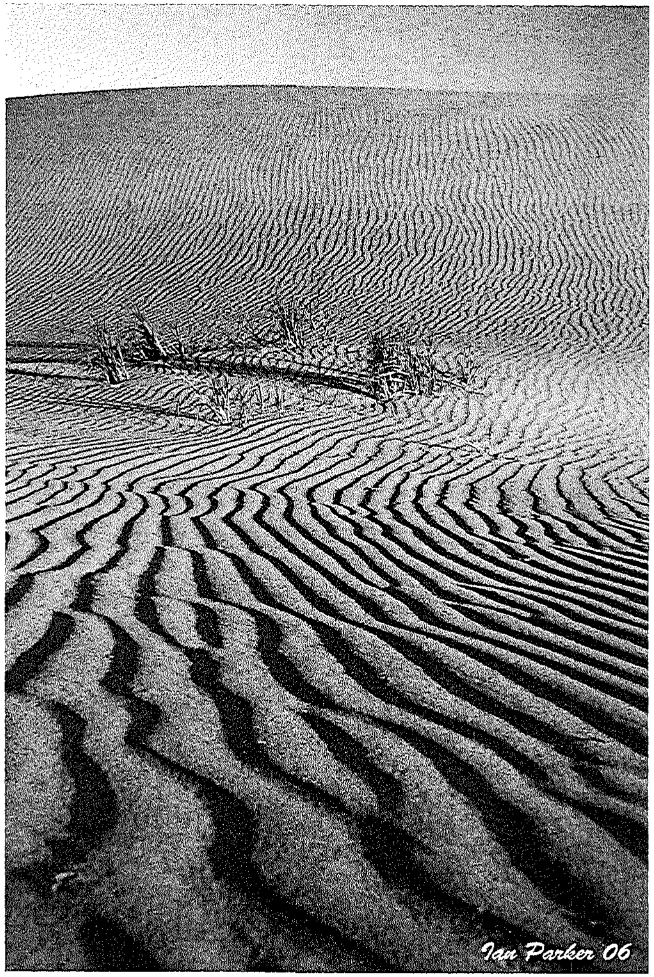
Advanced Workshop on Nonlinear Photonics, Disorder and Wave Turbulence

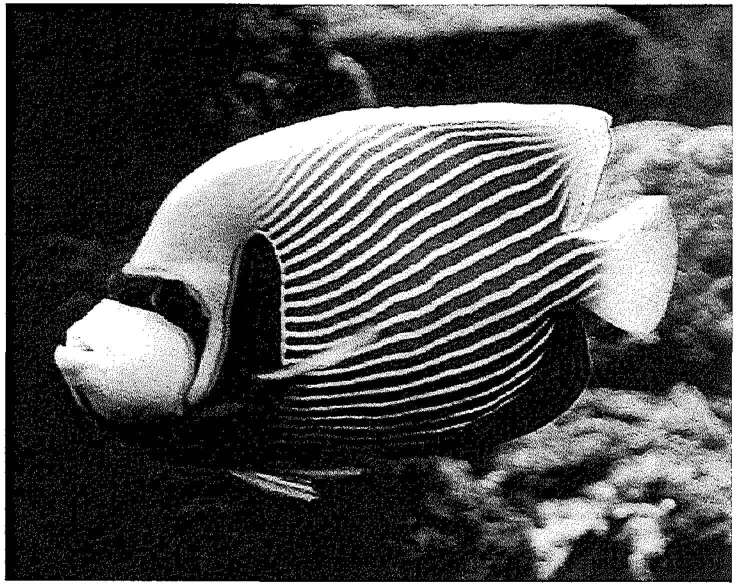
15 - 19 July 2013

And now for something completely different ... Fibonacci and Optics?

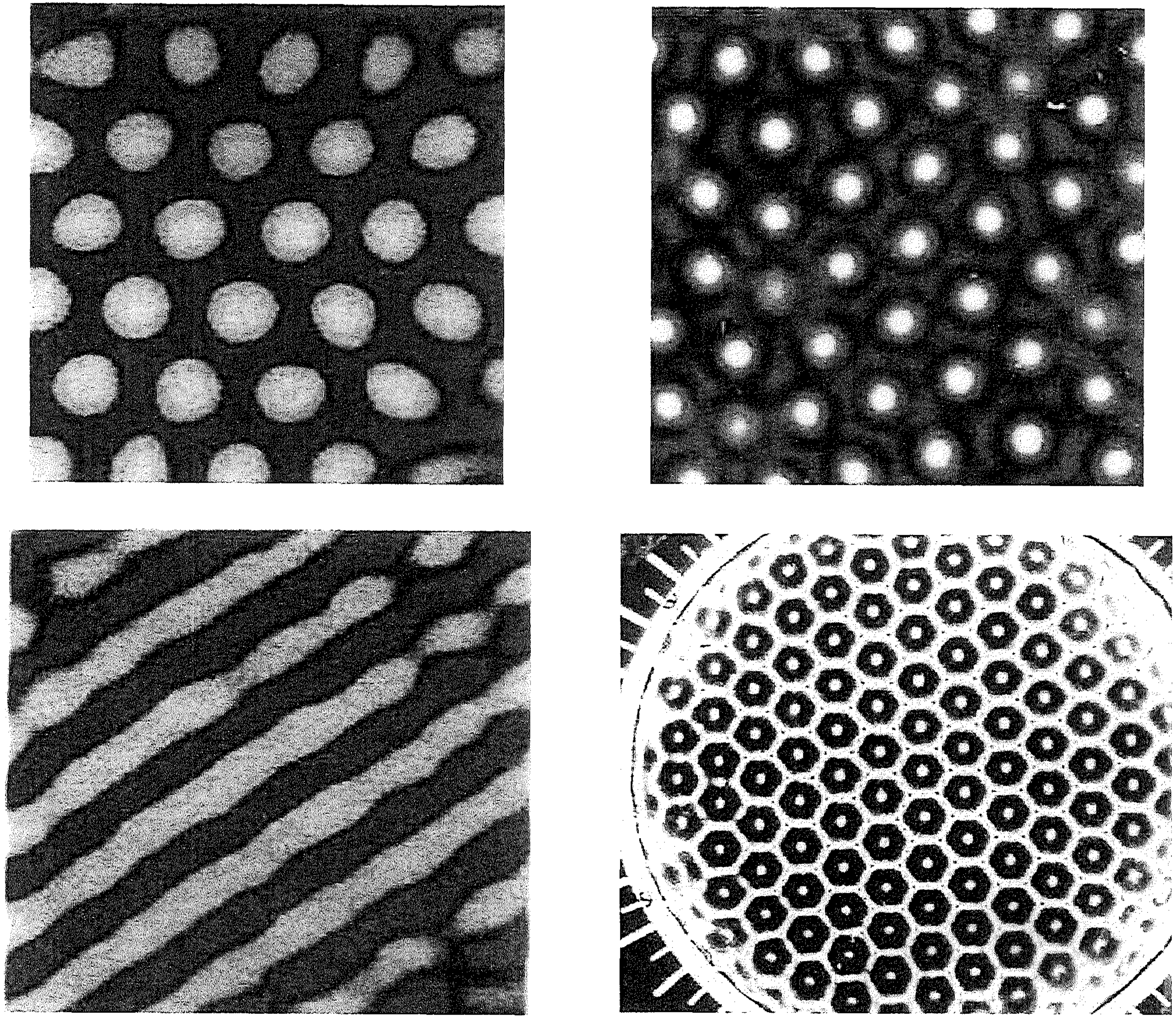
A. Newell
University of Arizona, Tucson
USA

Stripe Patterns

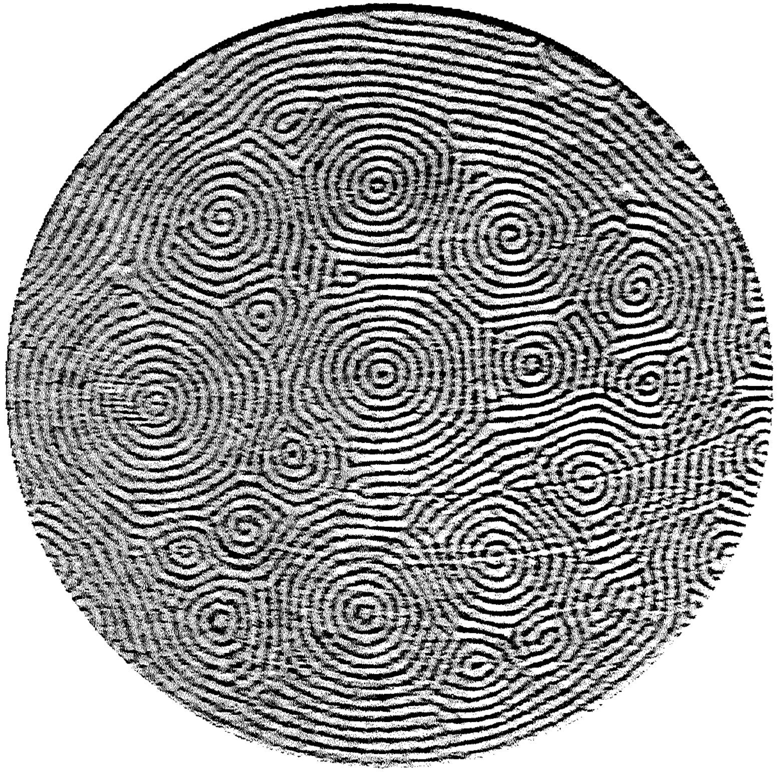




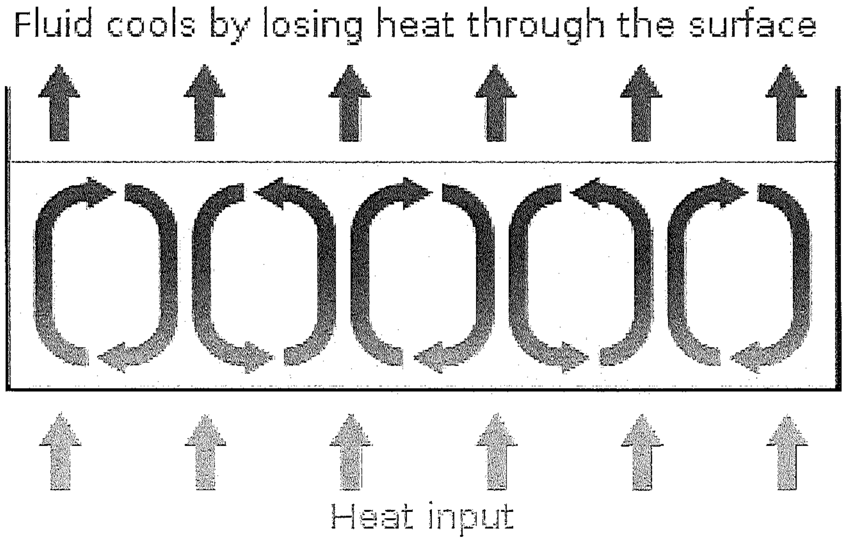
Hexagons (and Stripes)

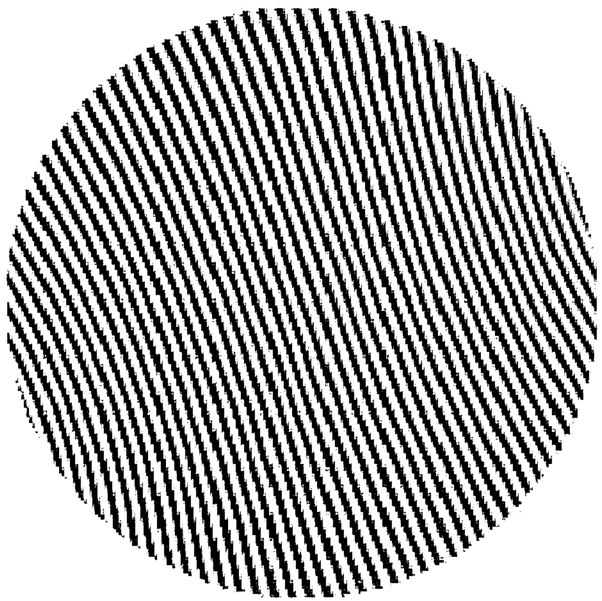


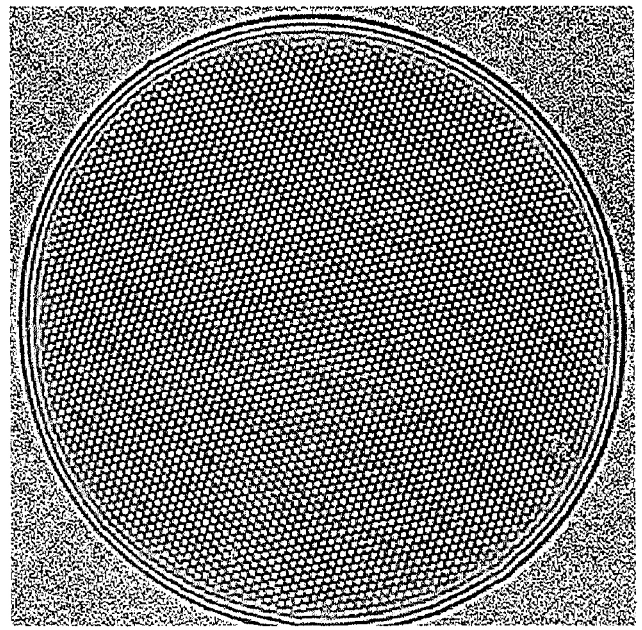
Convection and Megalithic Art



Convection Patterns

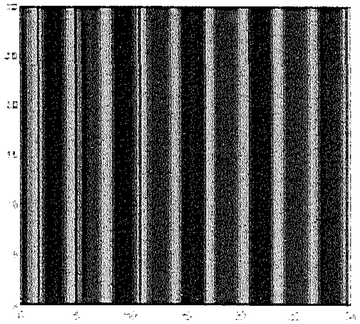




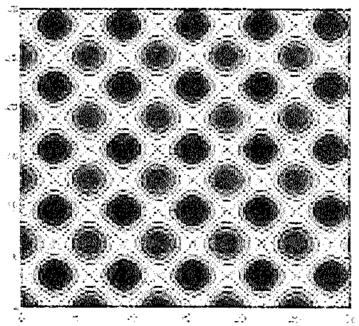


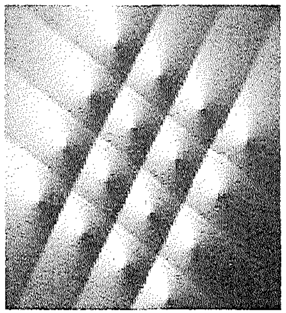
Competitors: $\{ e^{i\vec{k_j}\cdot\vec{x}} \}$, $|\vec{k_j}| = k_0$

Rolls/ Stripes $w = cos(x)$ $(k_0 = 1)$

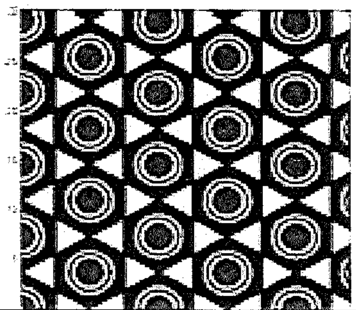


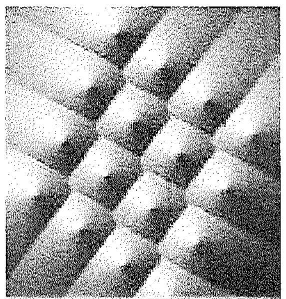
Squares $w = cos(x) + cos(y)$





Hexagons $w = cos(x) + cos(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y) + cos(-\frac{1}{2}x - \frac{\sqrt{3}}{2}y)$





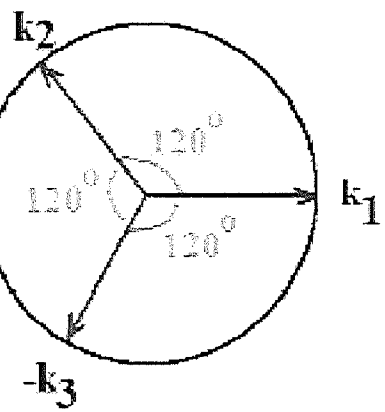
Why hexagons?

$\{ e^{i\vec{k}_j \cdot \vec{x}} \}$ compete

Quadratic interaction (strongest near onset)

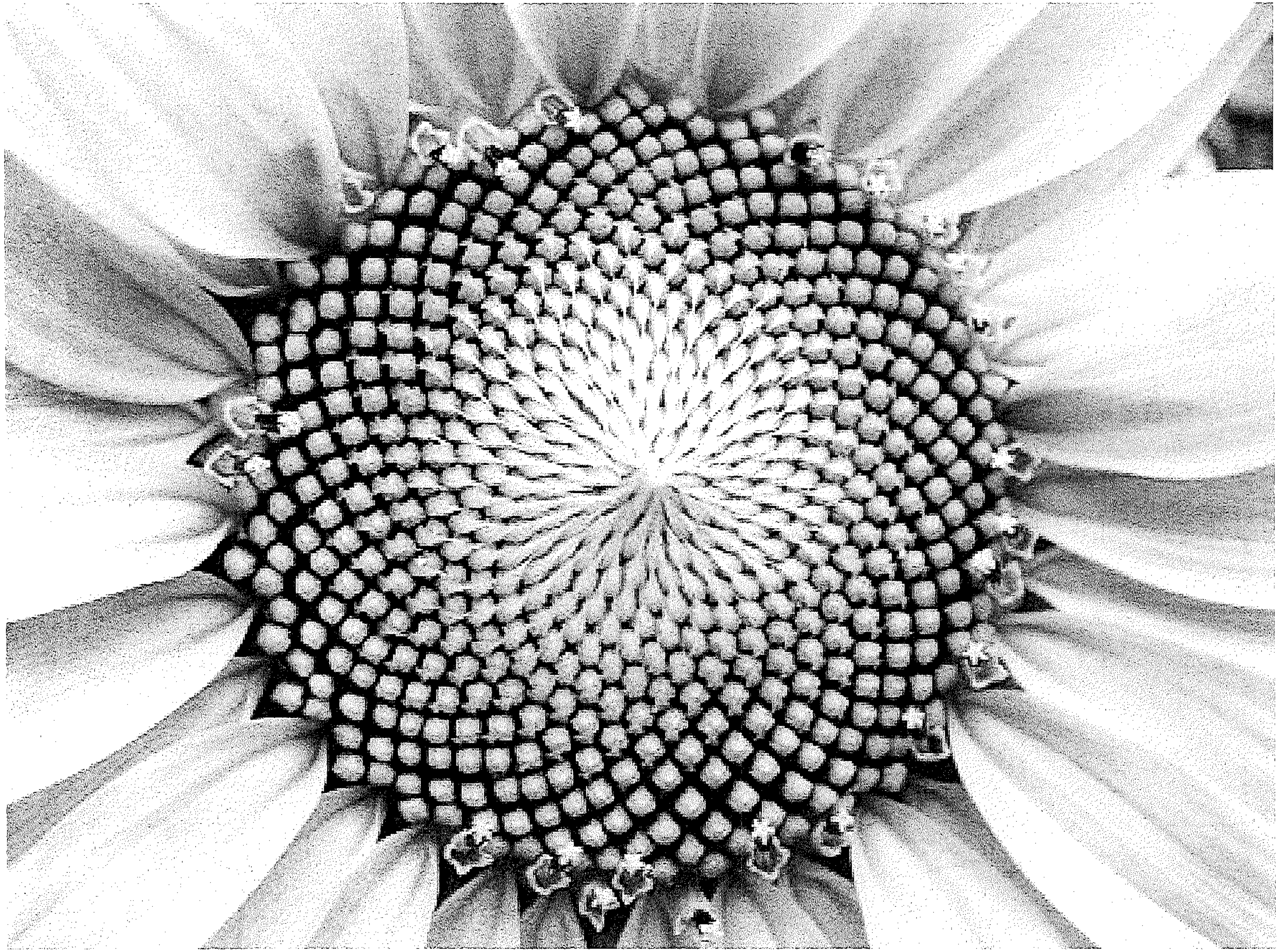
$$e^{i\vec{k}_1 \cdot \vec{x}} \cdot e^{i\vec{k}_2 \cdot \vec{x}} = e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{x}}$$

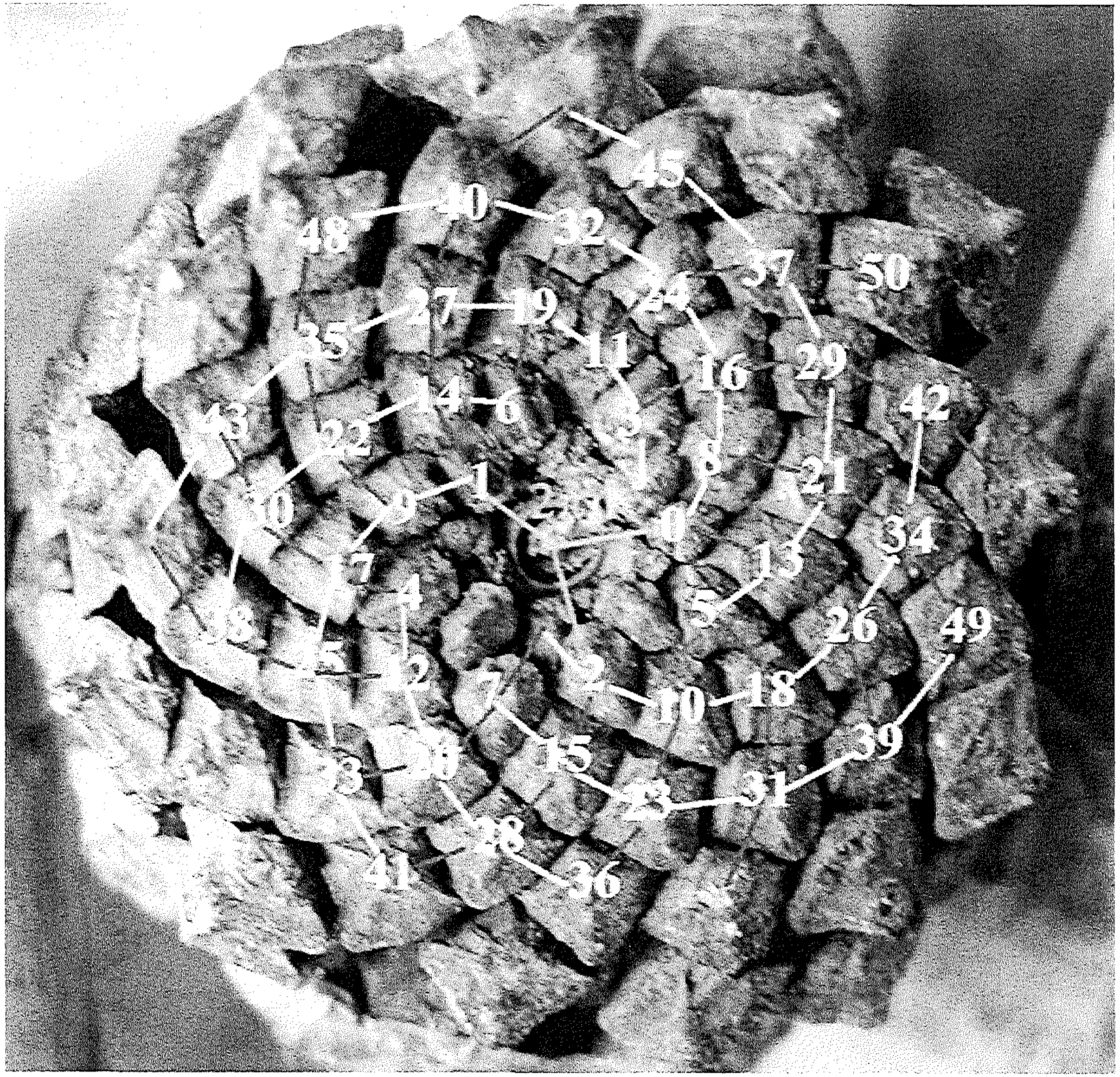
When do $\vec{k}_1$, $\vec{k}_2$, $\vec{k}_3 = \vec{k}_1 + \vec{k}_2$ all have length k_0?



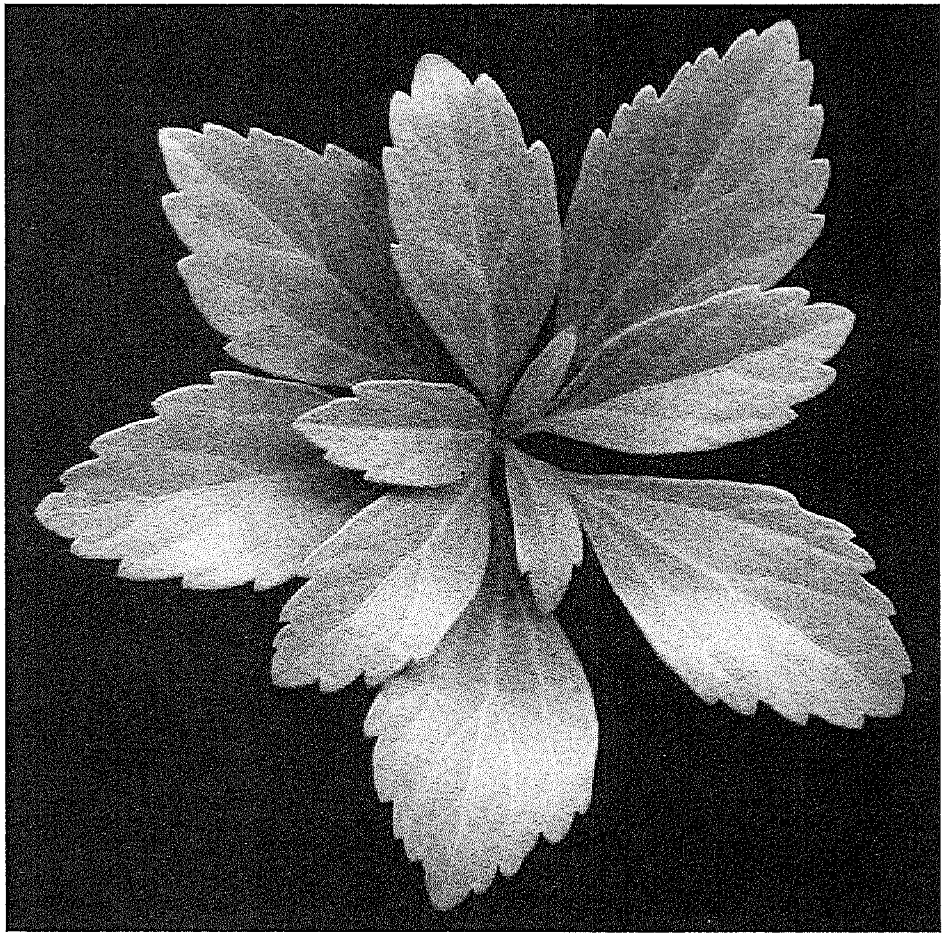
Quadratic interactions favor hexagons

Plant Patterns (Fibonacci)

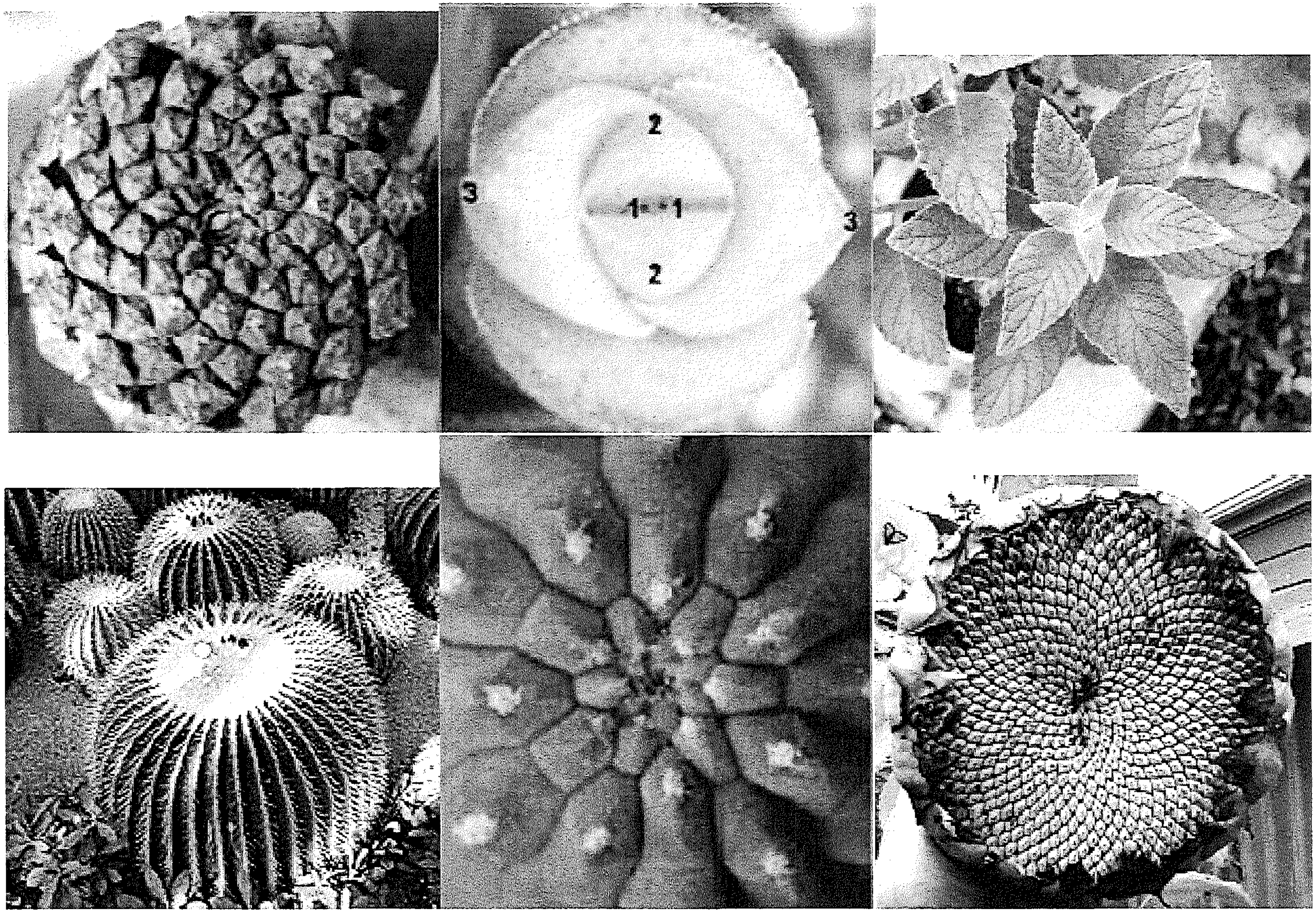



45
48
40
32
37
50
27
19
24
35
11
16
29
14
6
3
42
43
22
8
21
1
9
30
0
34
17
13
4
5
26
38
49
25
12
2
7
10
18
39
15
33
20
31
23
28
41
36

What is Phyllotaxis?

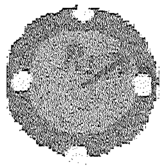


More examples of patterns on plants

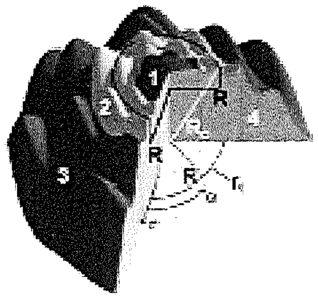


How do leaves/flowers form on a plant?

Top view

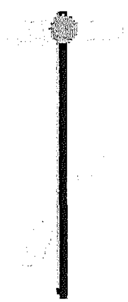


Schematic of SAM

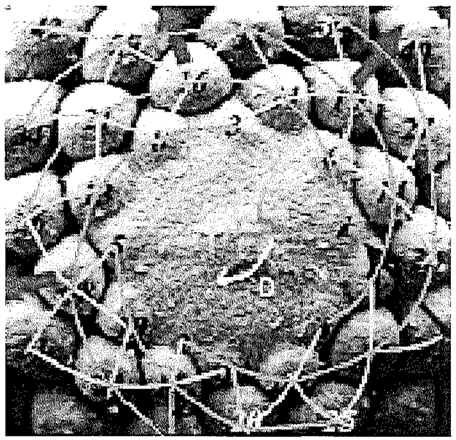


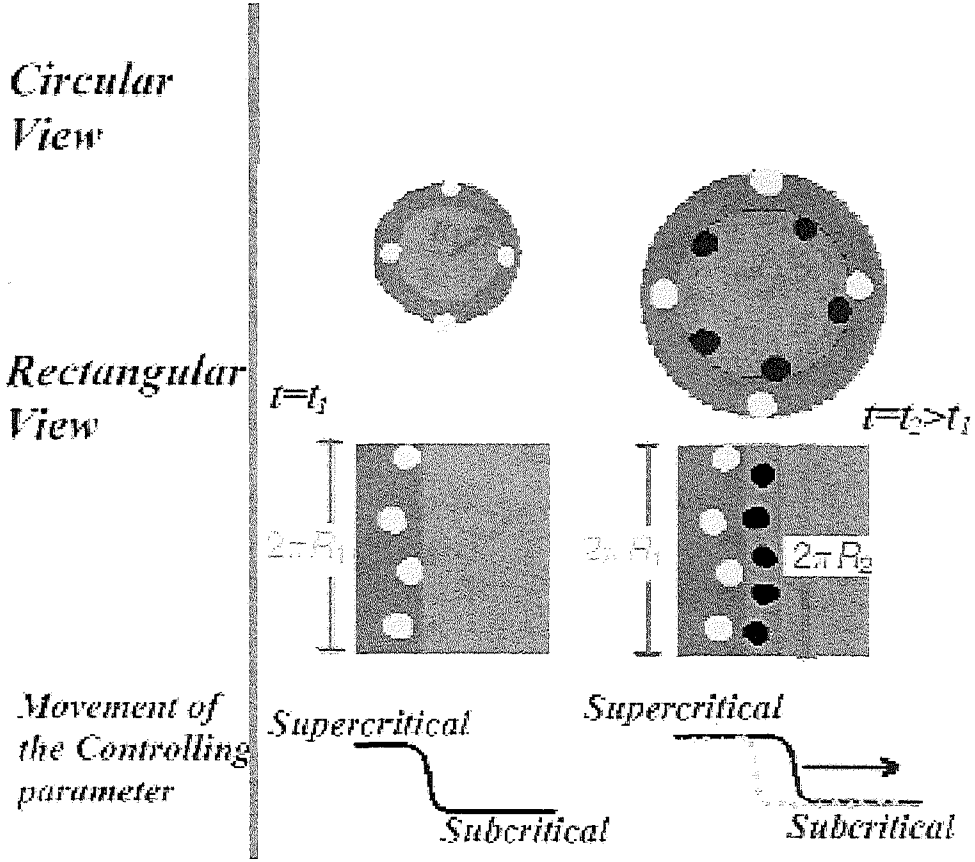
Side view

SAM

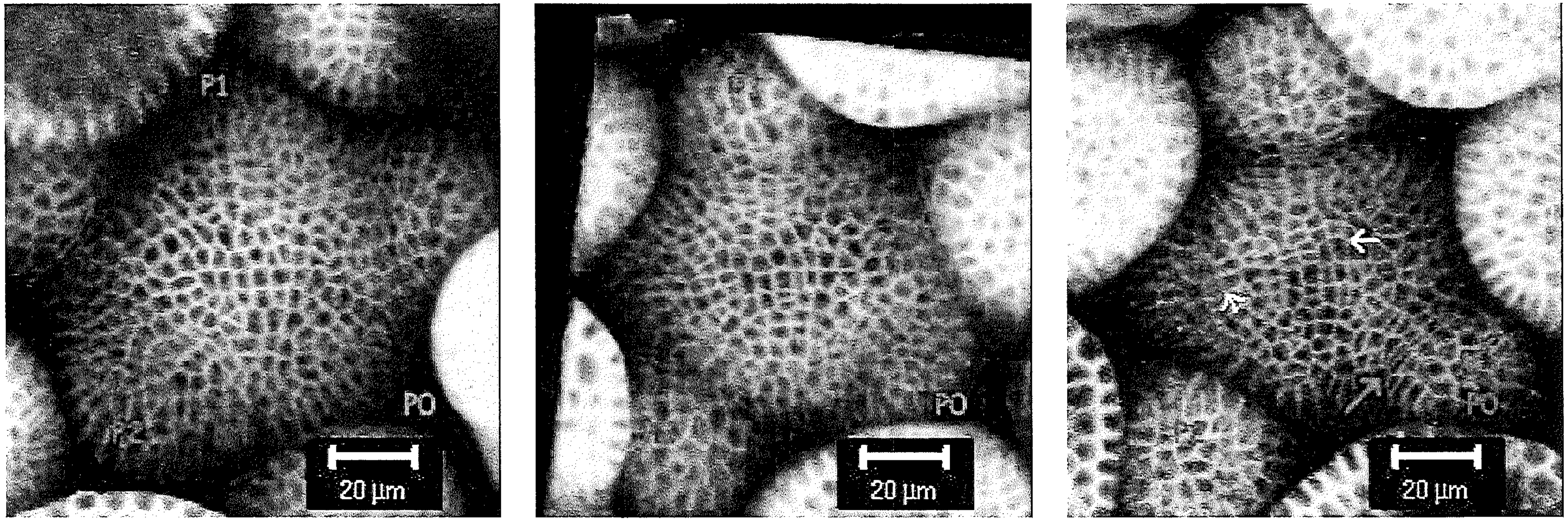


Divergence angle
plastochrone ratio

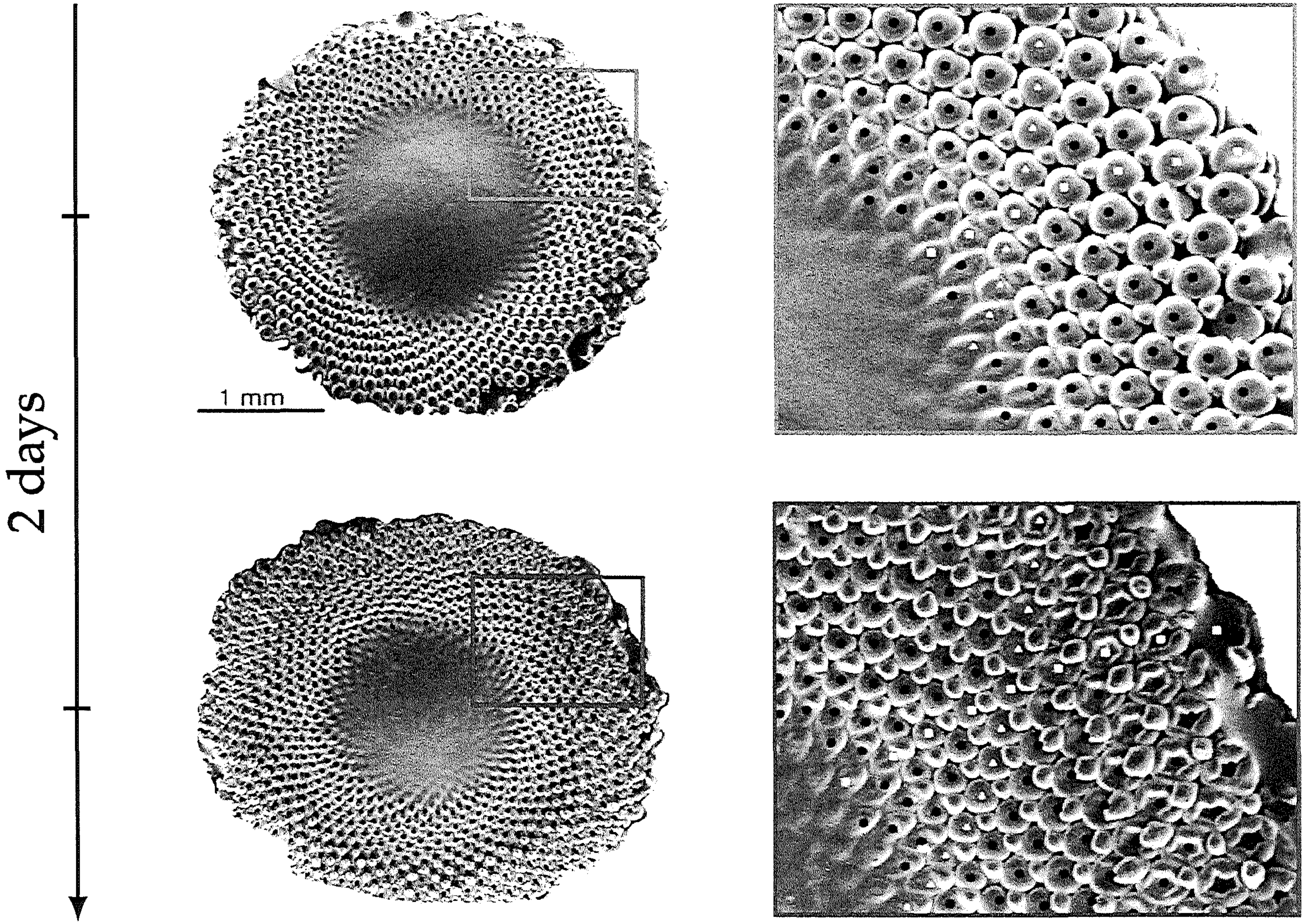




Microscopic Formation



Microscopic Formation of Phylla



From (Hotton et al. 2006)

Explanations

Teleological, the "whys", final cause or primary reason.
X is so in order that Y ...
Tigers have stripes to better camouflage.
Phylla are positioned to optimize access to nutrients, light ...

Mechanistic, the "hows", cause or secondary reason.
Physical (differential growth induced compressive stresses) and biochemical (uniform auxin concentration instability) processes leading to patterned states ...

D'Arcy Wentworth Thompson (1860-1942)

Polymath, son of a Professor of Classics at Galway, grandson of a highwayman, an immigrant to Botany Bay, and author of 'On Growth and Form' one goal of which was

... 'to show that a certain mathematical aspect of morphology ... is helpful, nay critical to the proper study and comprehension of growth and form.'

He also has a jaundiced eye for and takes aim at excessive reliance on teleological explanations ...

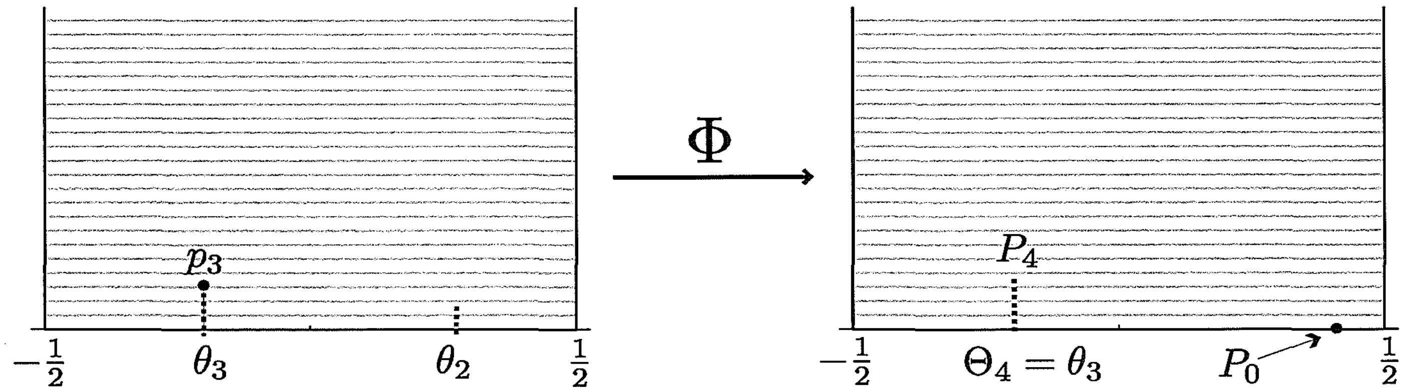
Teleological

Hofmeister Rules (1868)

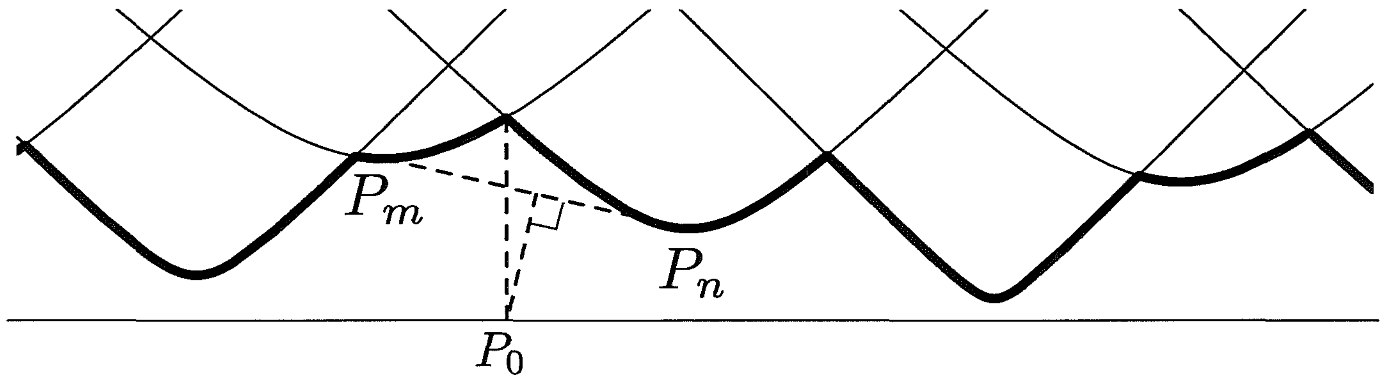
1. The meristem is axisymmetric.
2. Primordia form in a generative annulus on the periphery of the apex.
3. New primordia form at regular time intervals.
4. Primordia move radially away from the apex.
5. Each new primordium forms in the least crowded spot left by the existing primordia.

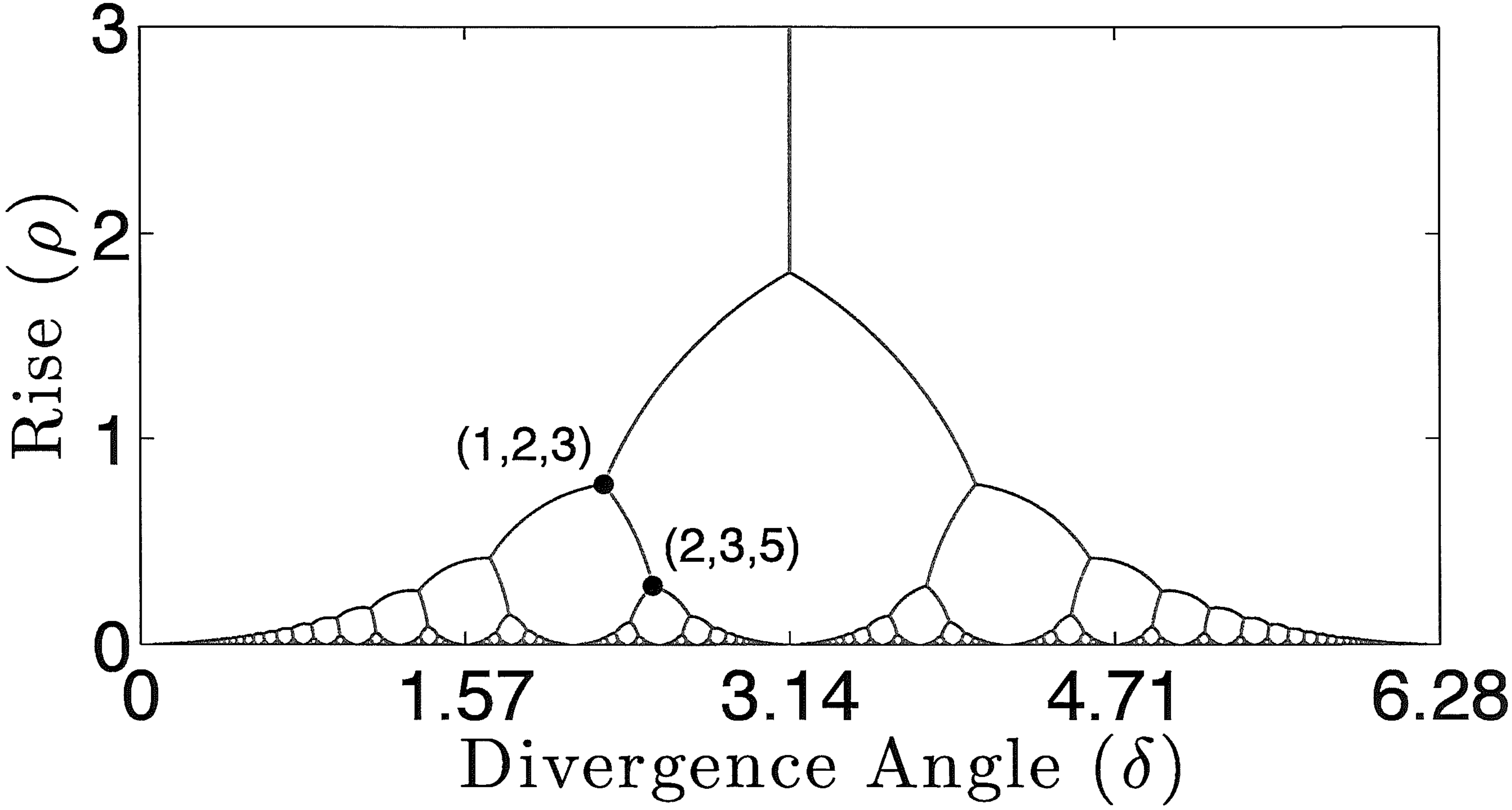
Van Iterson (1907)
Snow and Snow (1952)
Levitov (1991)
Douady and Couder (1996)
Atela, Golé and Hotton (2003)

Atela, Golé and Hotton

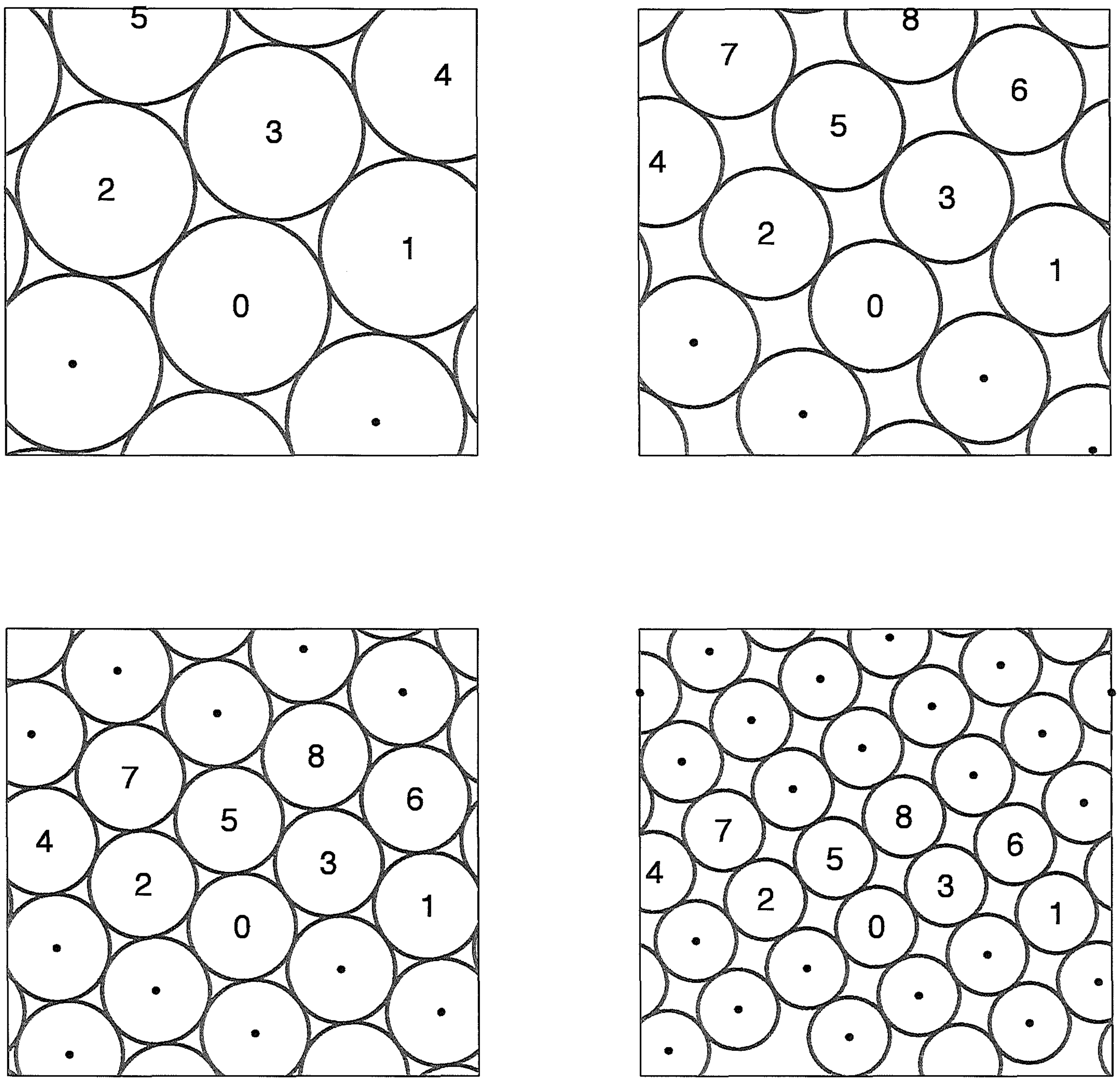


$$P_0 = \operatorname{argmax} \min_{n>0} \|P_n - P_0\|$$





3
2
1
0
Rise (ρ)
(1,2,3)
(2,3,5)
0
1.57
3.14
4.71
6.28
Divergence Angle (δ)


5
4
3
2
1
0
8
7
6
5
4
3
2
1
0
7
8
6
5
4
3
2
0
1
8
7
6
5
4
3
2
1
0

Patterns on Plants

It is our suggestion that patterns on plants near the growth shoots have much in common with patterns in planar geometries.

They arise as instabilities.

They have a preferred length scale.

Like hexagons in planar geometries, quadratic interactions reflecting the breaking of 'up-down' symmetries play important roles.

But, there are also important differences arising from the way in which they are made, annulus by annulus, in a generative region surrounding the meristem. This leads to planforms which, as a function of radius, alternate between rhombi and hexagons and whose maxima lie on families of spirals enumerated by Fibonacci sequences.

The Mechanistic Approach

What are the underlying physical processes?

Biomechanics: Green (1998) suggests that the formation of phylla may be due to a bucking instability caused by internal compressive stress on the surface of the plant.

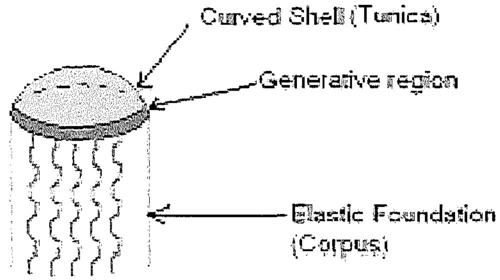
Biochemistry: Meyerowitz (2006) and Traas (2003) suggest that phylla are induced by the plant hormone auxin, whose transport is controlled by a PIN1 protein.

Near the onset of instability, the governing equations of both processes take the same form!

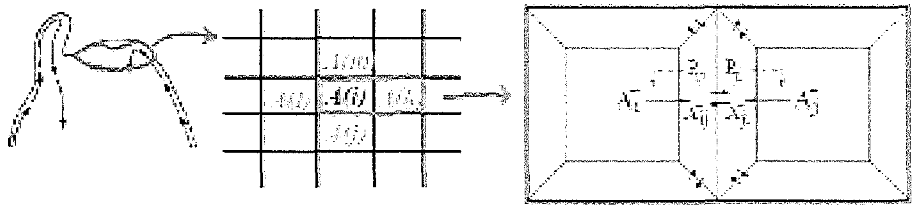
$$\frac{\partial u}{\partial t} = \mu u - (\nabla^2 + 1)^2 u - \frac{\beta}{3}\left(|\nabla u|^2 + 2u\nabla^2 u\right) - u^3$$

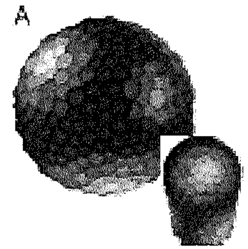
Mechanisms and Models

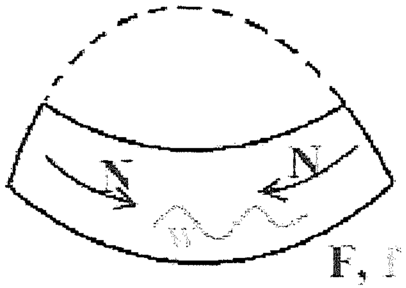
$w(r,\alpha,t)$ Surface normal deformation, $f(r,\alpha,t)$ Airy Stress tensor fluctuation
$g(r,\alpha,t)$ Growth/auxin concentration fluctuation (continuum limit)

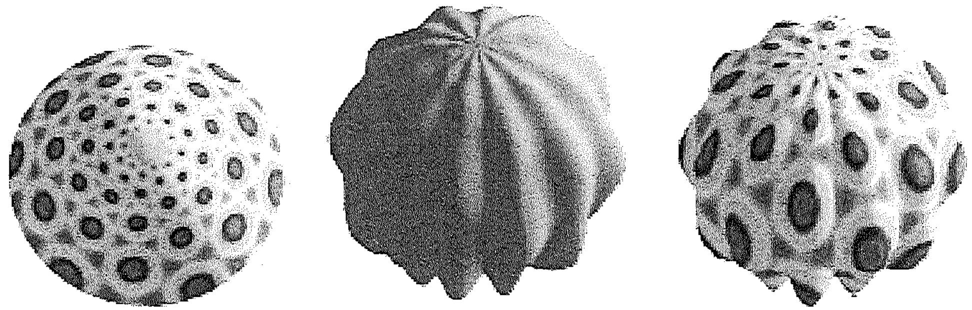


[Reinhardt, Meyerowitz, Traas et al]





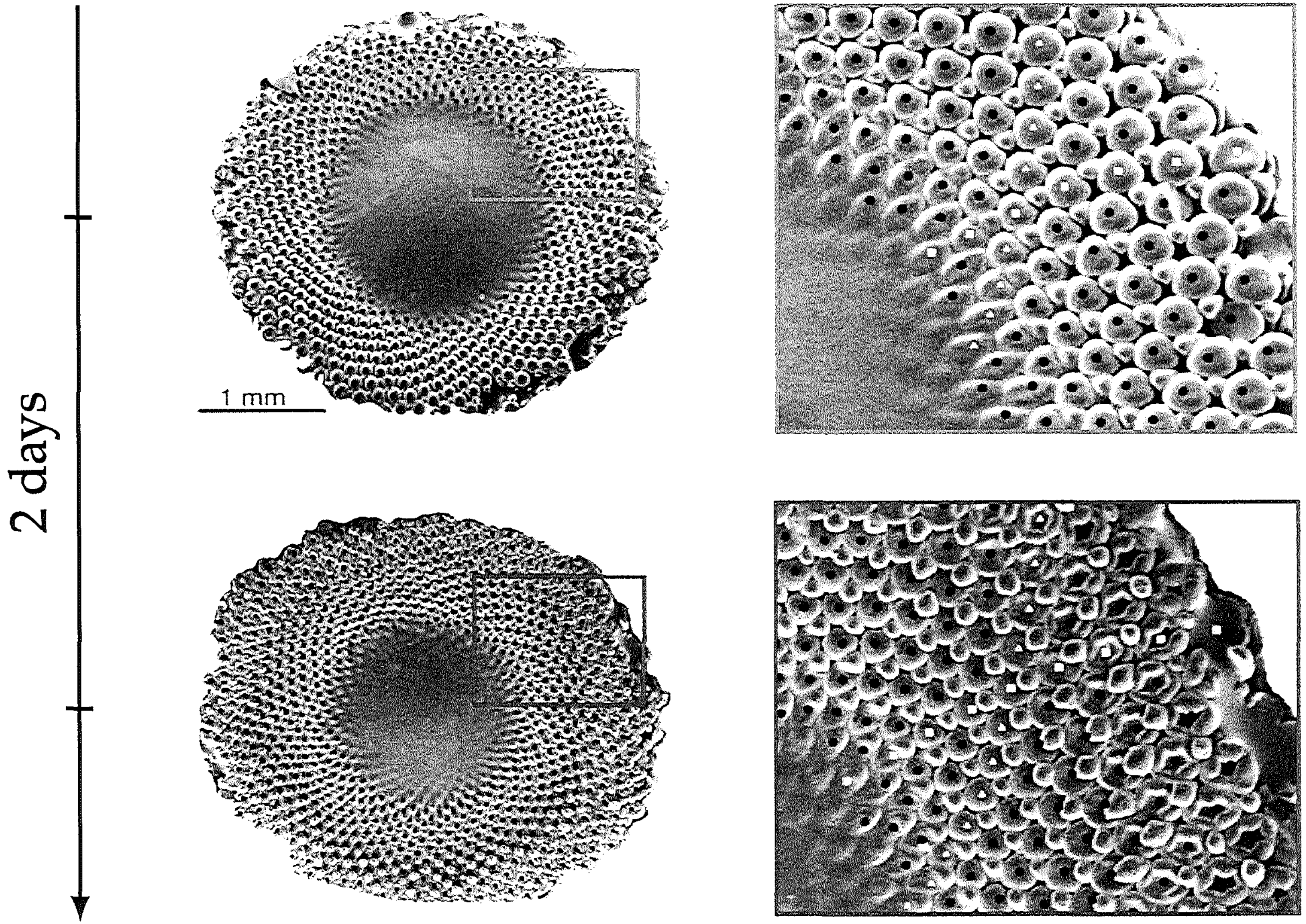




$$\zeta w_t + D\Delta^2 w + N\Delta_\chi w + \frac{1}{R_\alpha}\Delta_\rho f - [f,w] + \kappa w + \gamma w^3 = 0$$
$$\frac{1}{Eh}\Delta^2 f - \frac{1}{R_\alpha}\Delta_\rho w + \frac{1}{2}[w,w] + \Delta g = 0$$
$$g_t = -Lg - H\nabla^2 g - \nabla^4 g - \bar{\kappa}_1 \nabla(g\nabla g) - \bar{\kappa}_2\nabla(\nabla g \nabla^2 g) - \delta g^3 + \beta\Delta f$$

Canonical form: $u_t = -(\nabla^2+1)^2 u + \epsilon u + \text{Quad} - u^3$

Microscopic Formation of Phylla



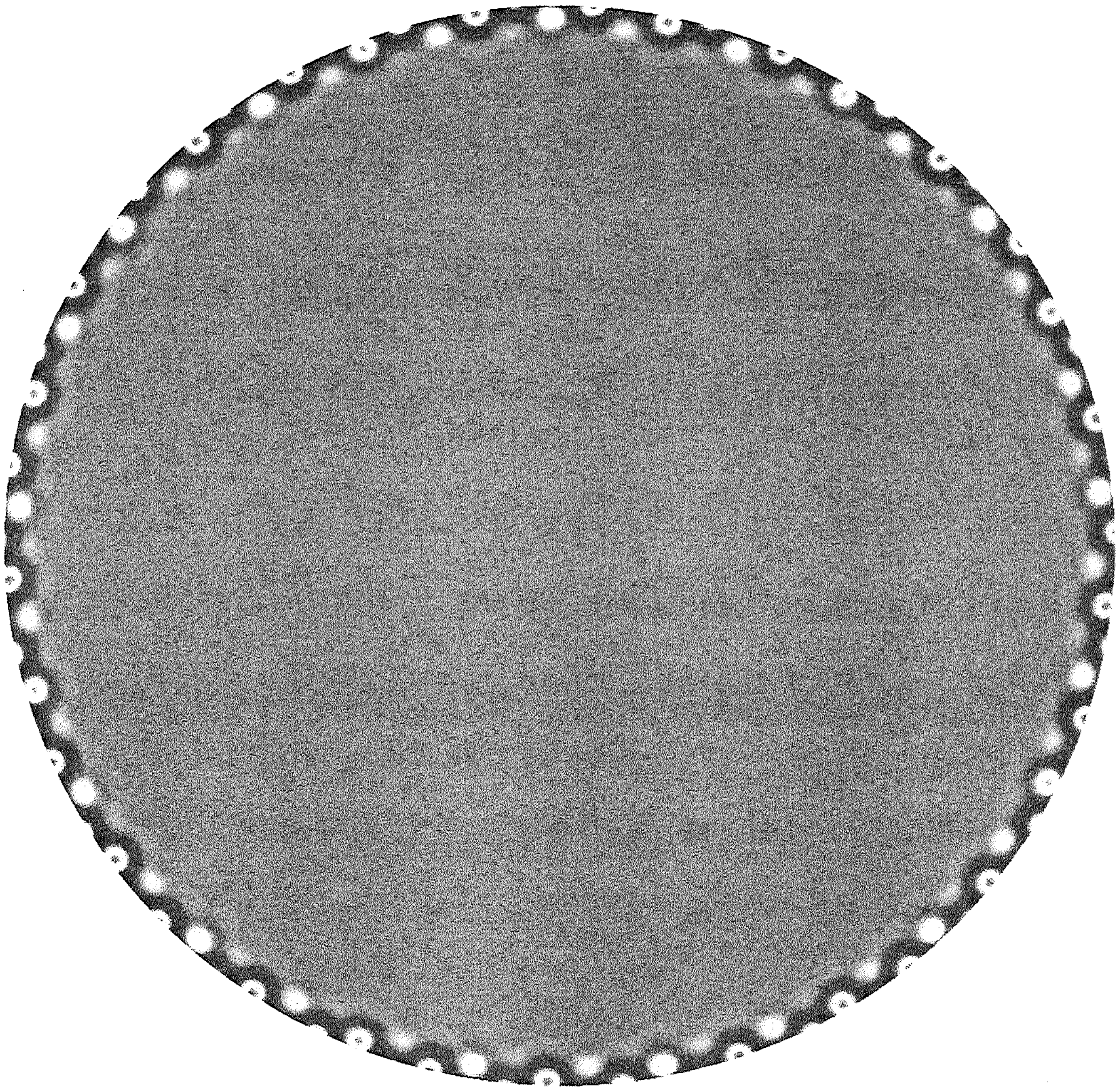
From (Hotton et al. 2006)

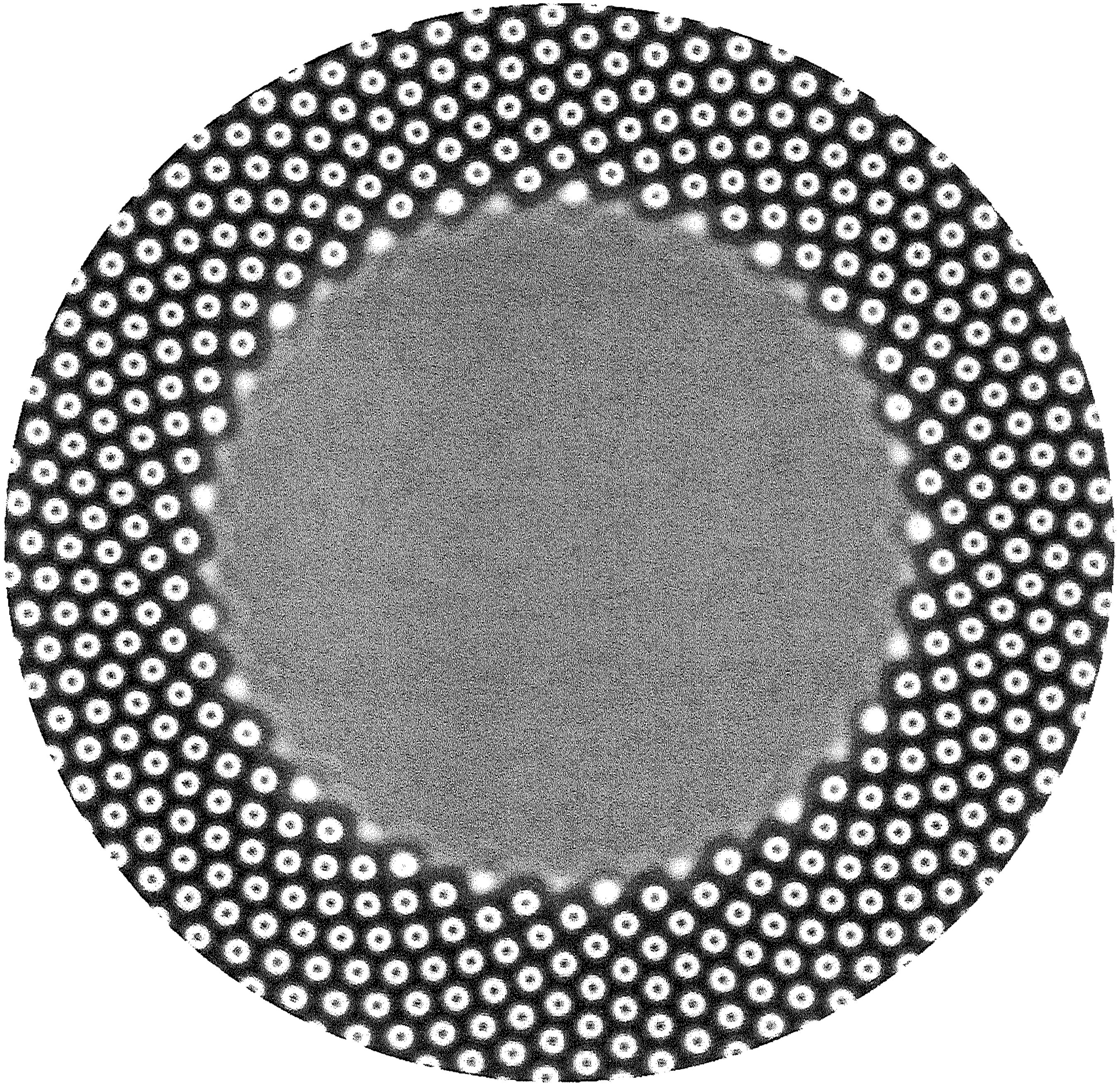
What We Monitor?

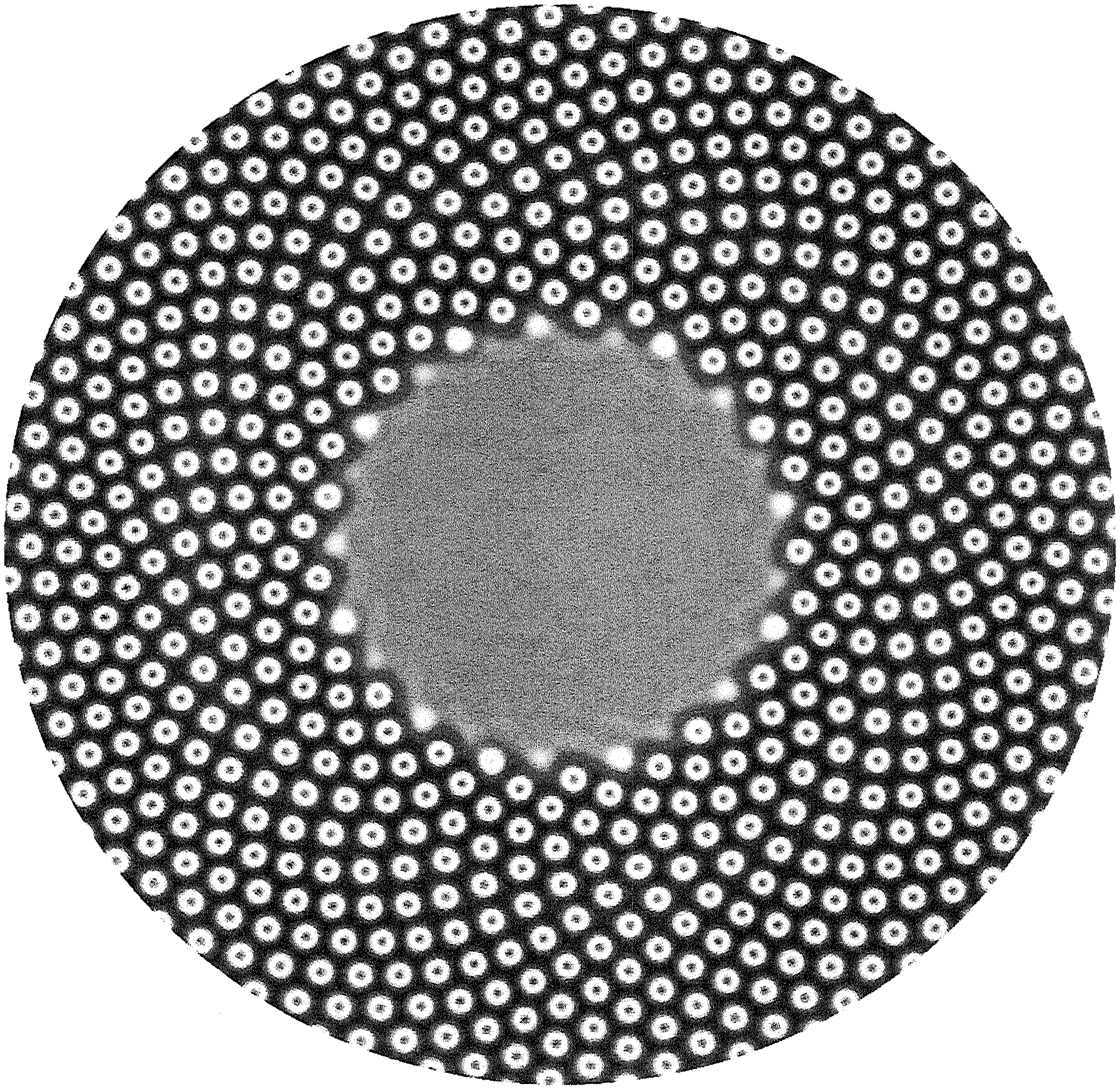
$$u(r,\theta,t) = \sum_{m\in\mathbb{Z}} u_m(r,t)\exp(-im\theta)$$

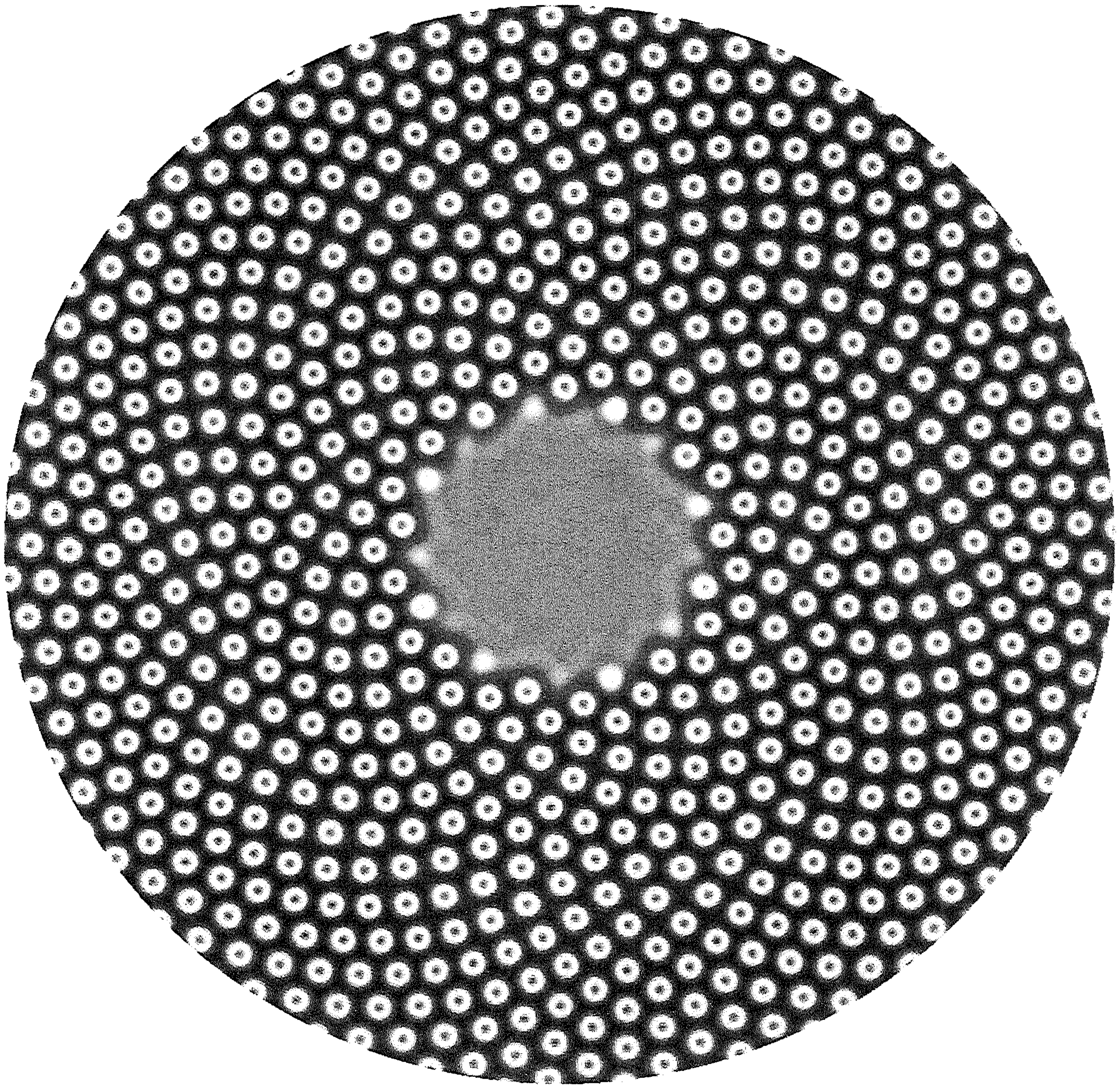
$$u_m(r,t) = a_m \exp(i\phi_m)$$

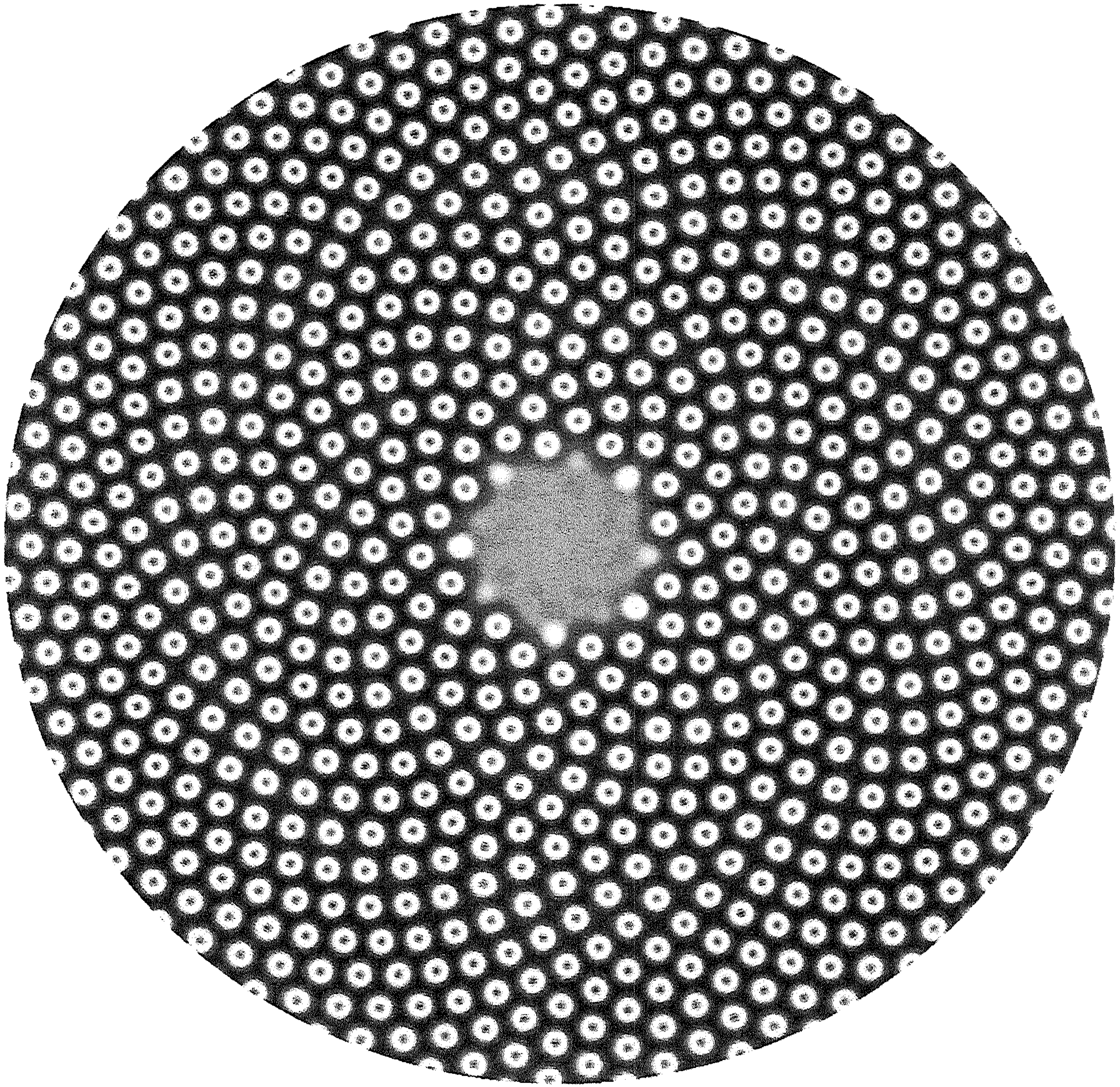
- $a_m(r)$ – amplitude of mode m.
- $\ell_m(r)$ – local radial wavenumber.
- $\epsilon(r)$ – local energy density.
- $\eta(r)$ – local packing efficiency.
- $\nu(r)$ – local front speed.



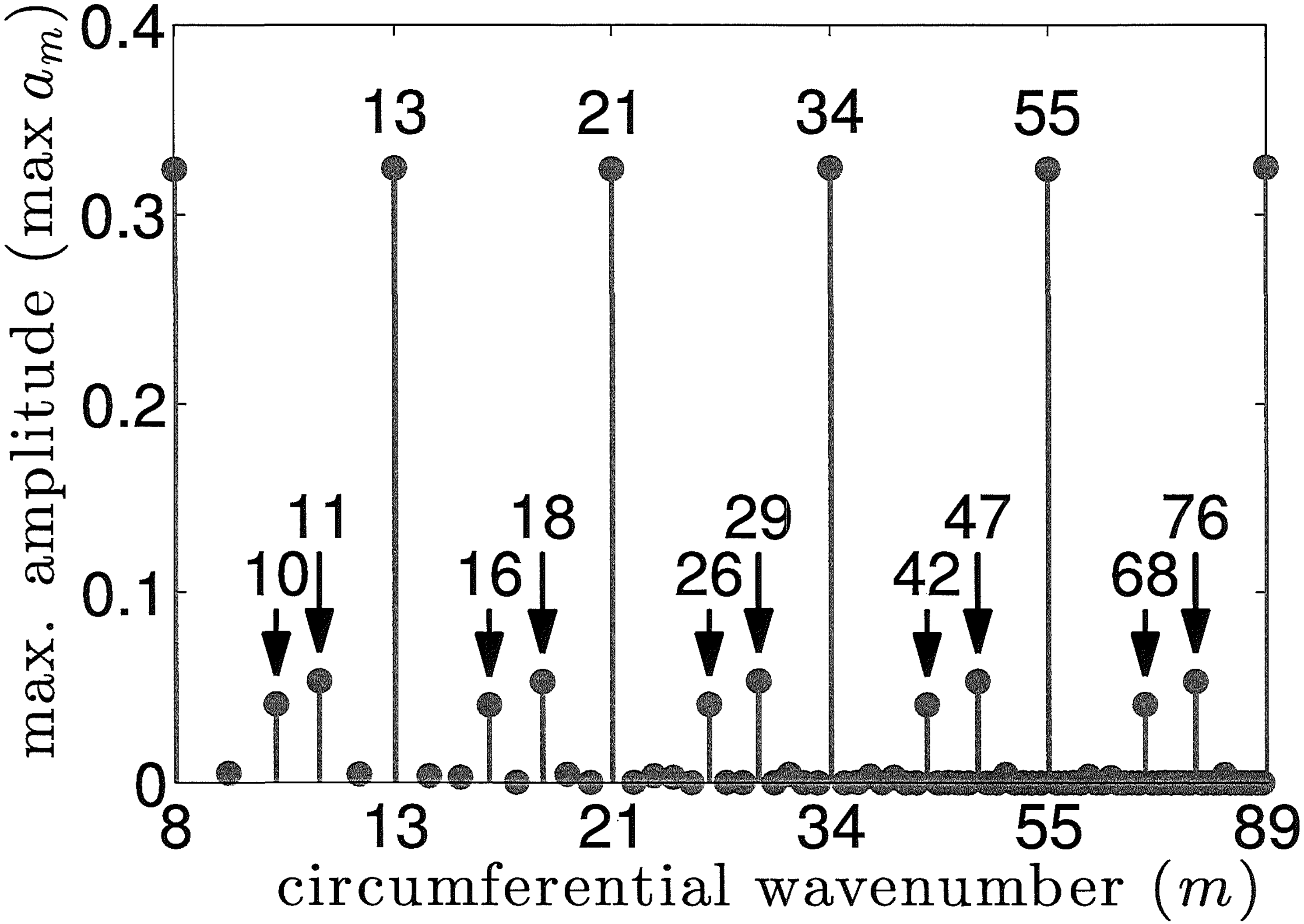








Maximum Amplitudes



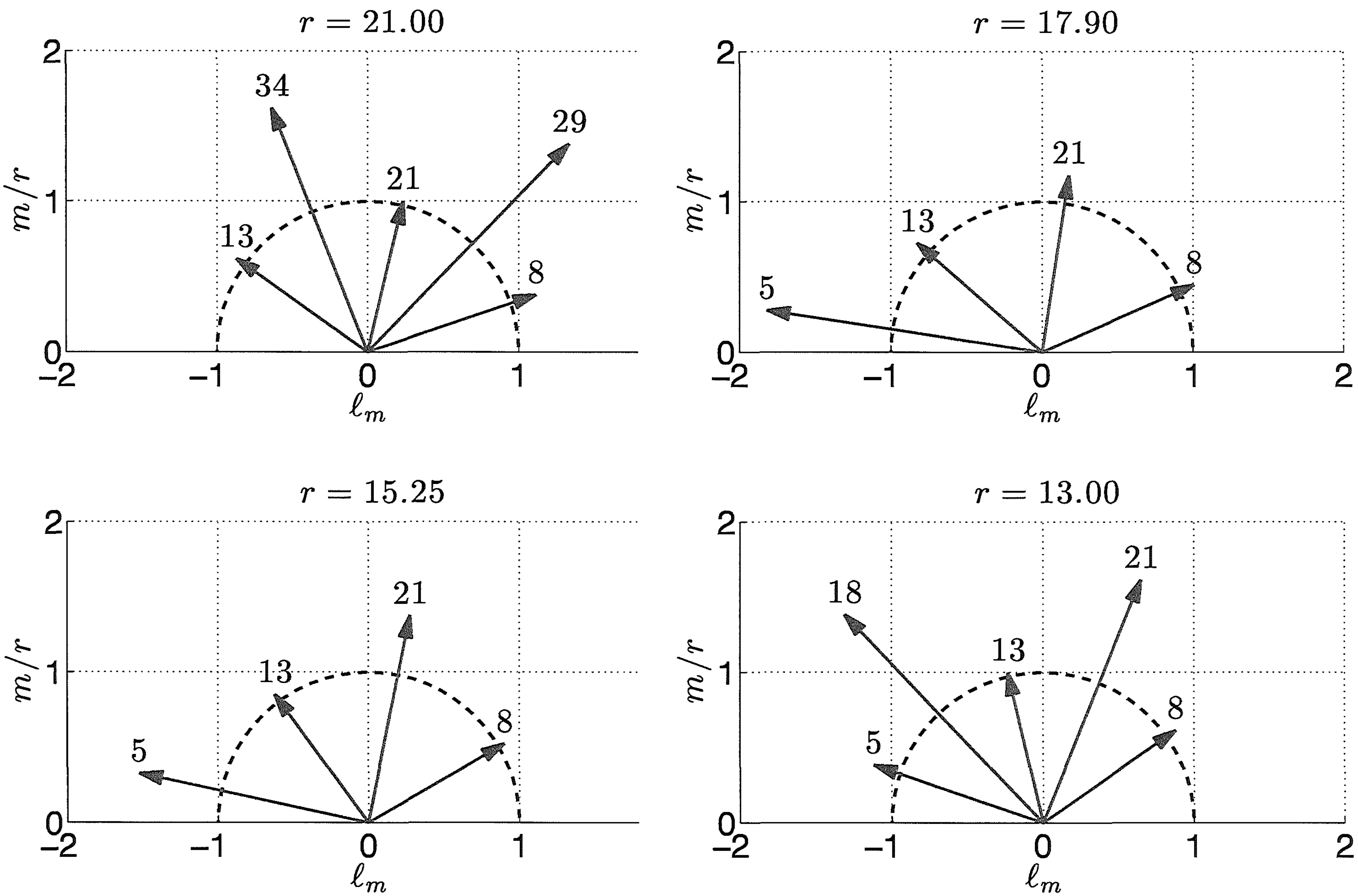
Self-Similarity

For $m_j \in \mathcal{F}$, we find for all r that

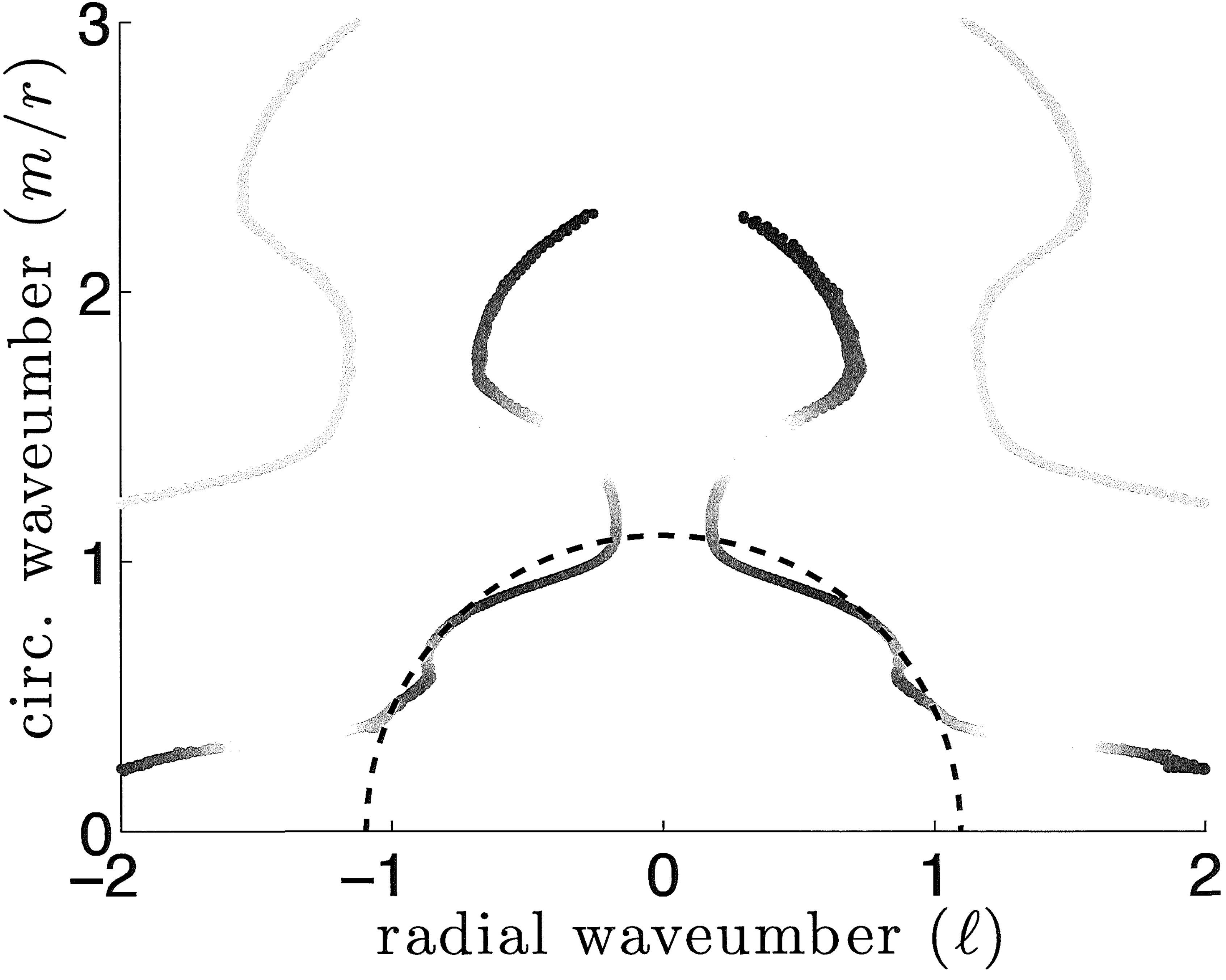
$$a_{m_j}(r) = a_{m_{j+1}}(r\varphi)$$

$$\ell_{m_j}(r) = -\ell_{m_{j+1}}(r\varphi)$$

where $\varphi = \lim m_{j+1}/m_j = (\sqrt{5}+1)/2$.

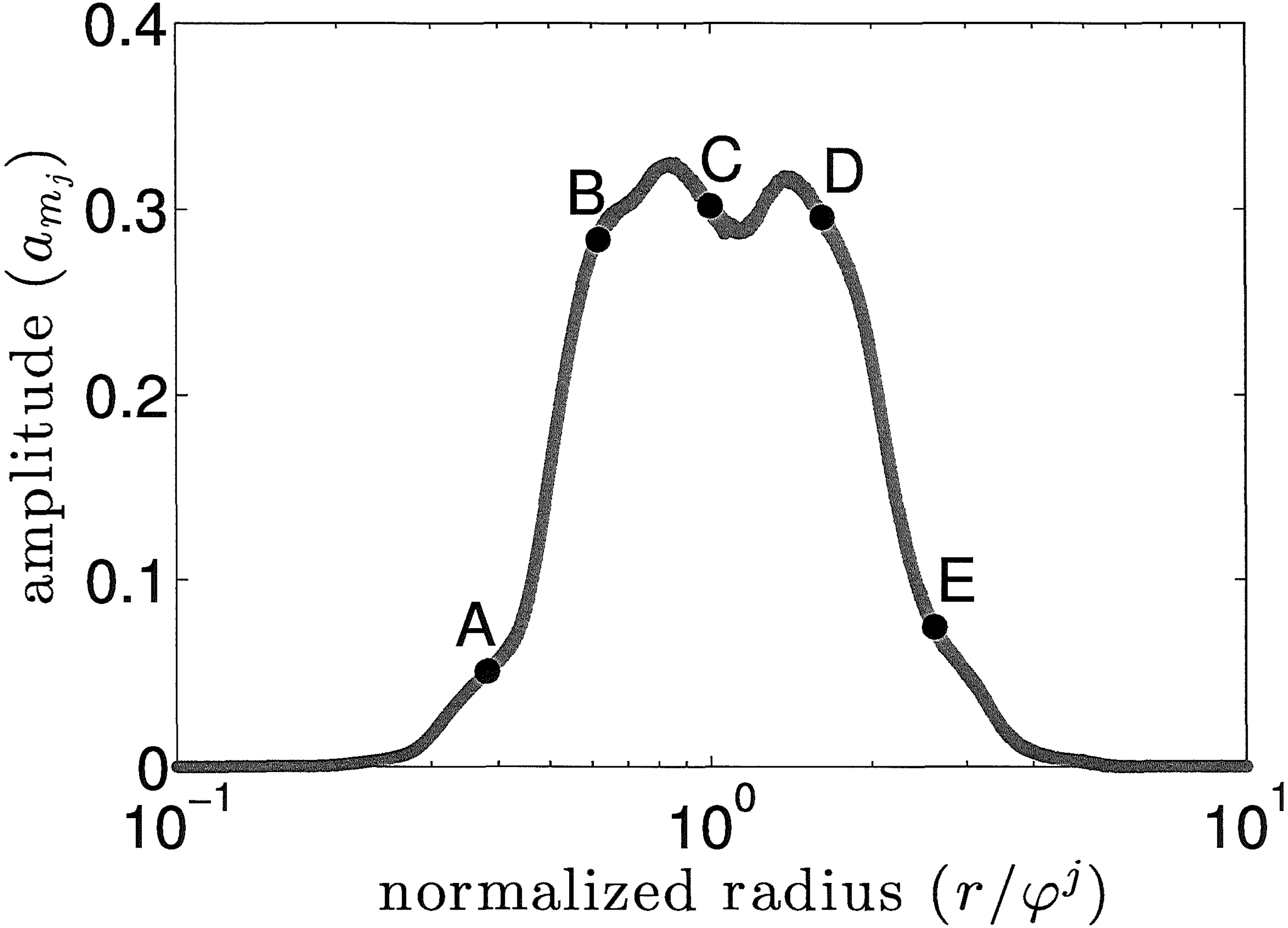


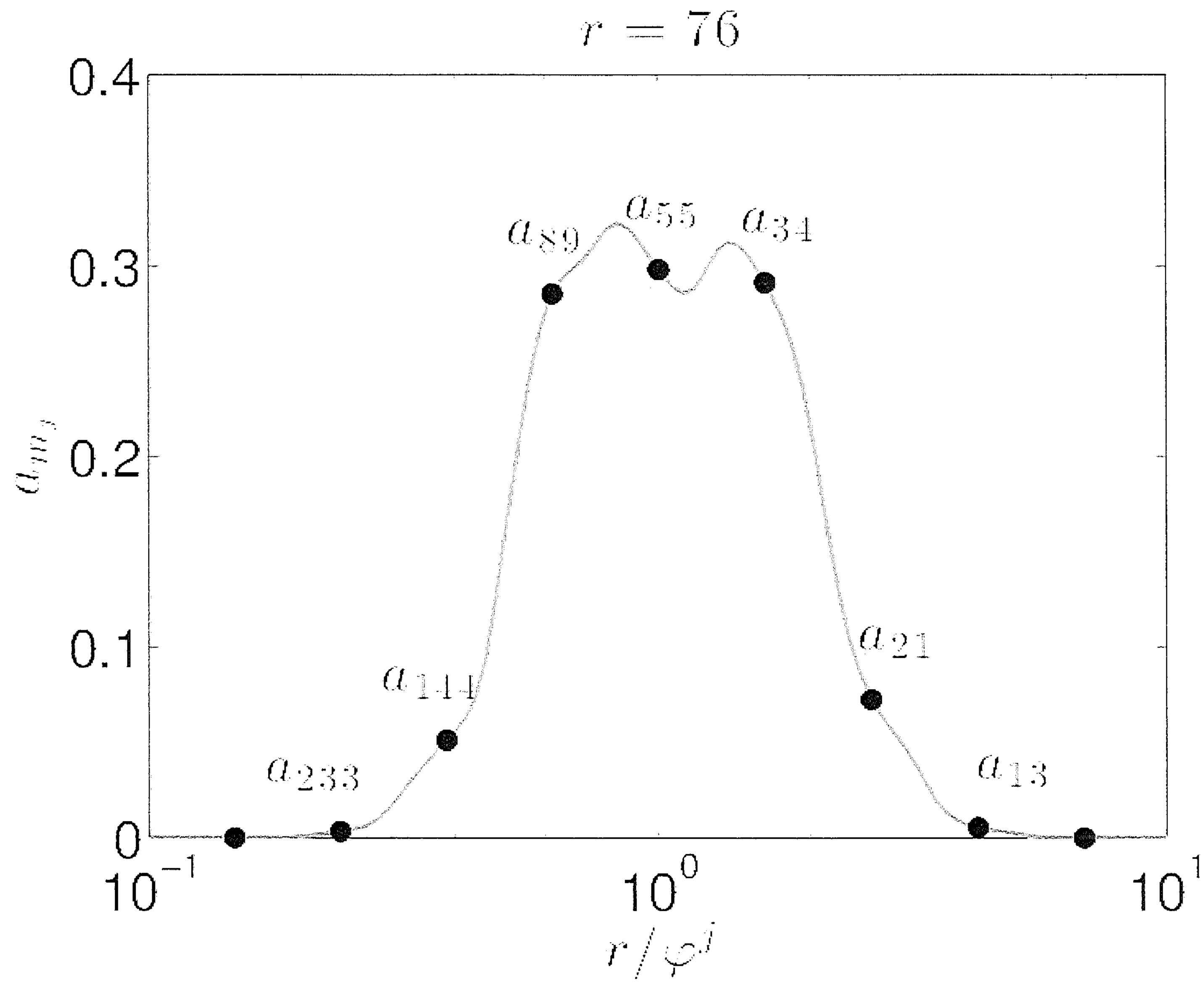
r = 21.00
r = 17.90
r = 15.25
r = 13.00
m/r
ℓ_m
34
29
21
13
8
5
18



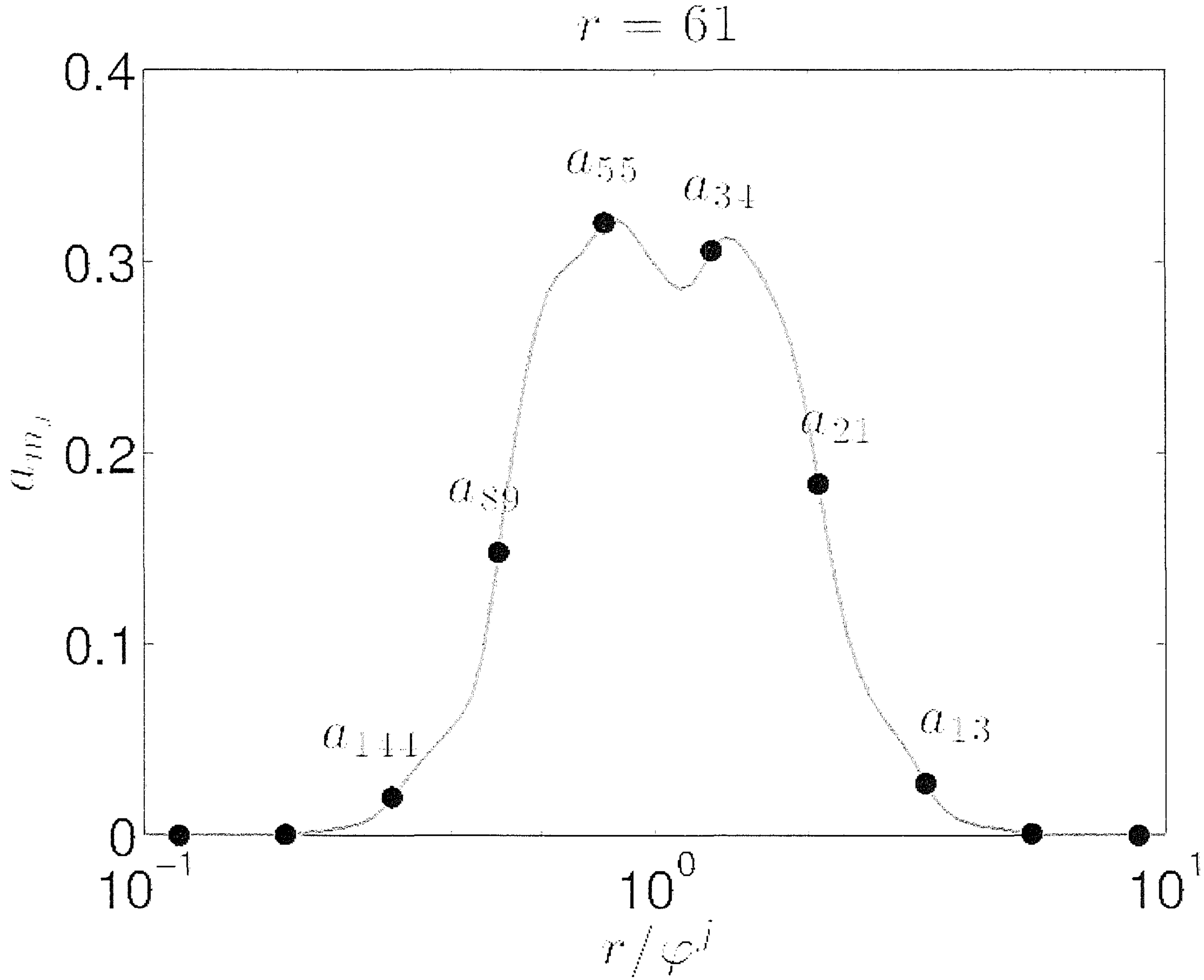
3
2
1
0
−2
−1
0
1
2
circ. waveumber (m/r)
radial waveumber (ℓ)

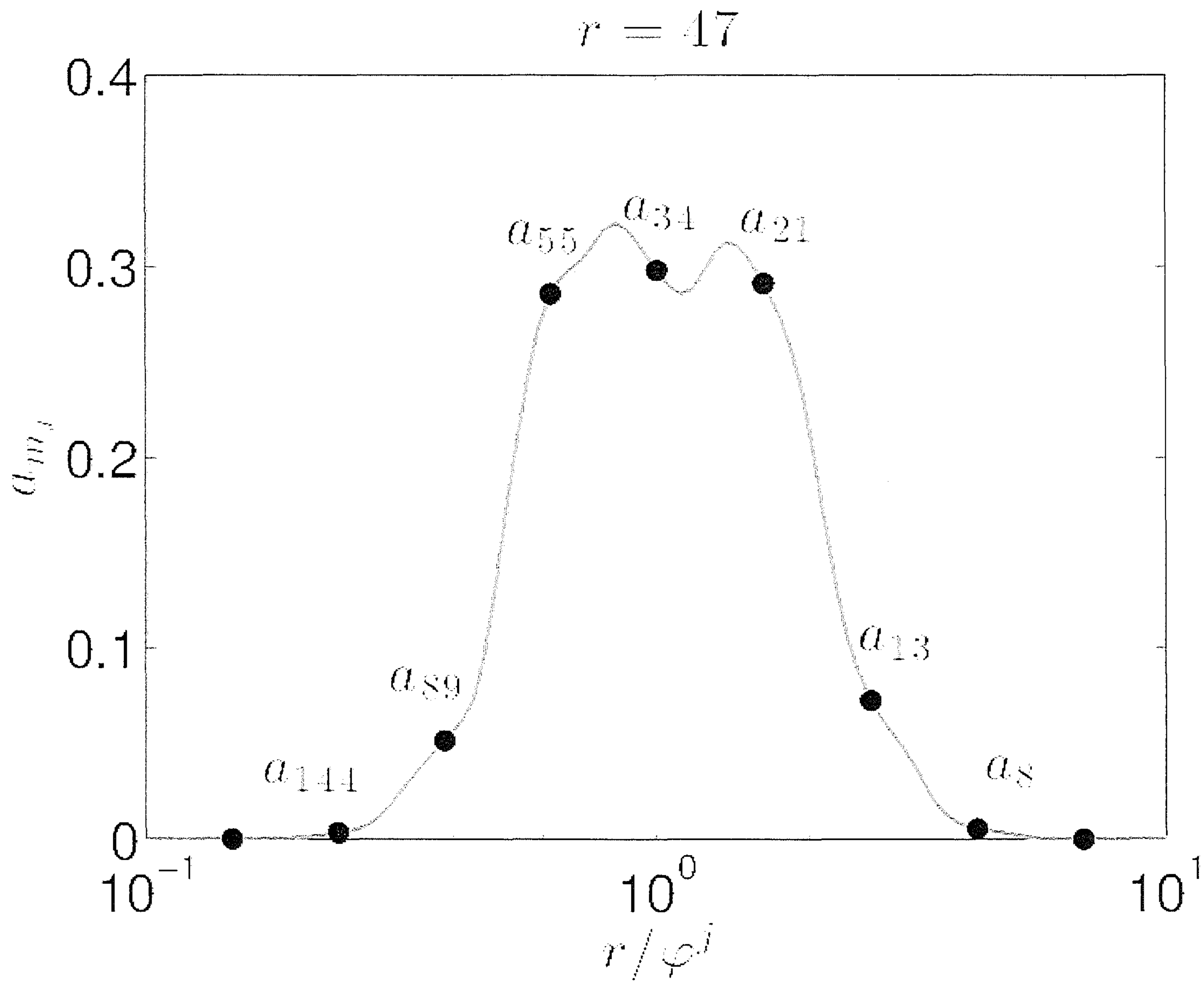
The Amplitude Invariant Curve





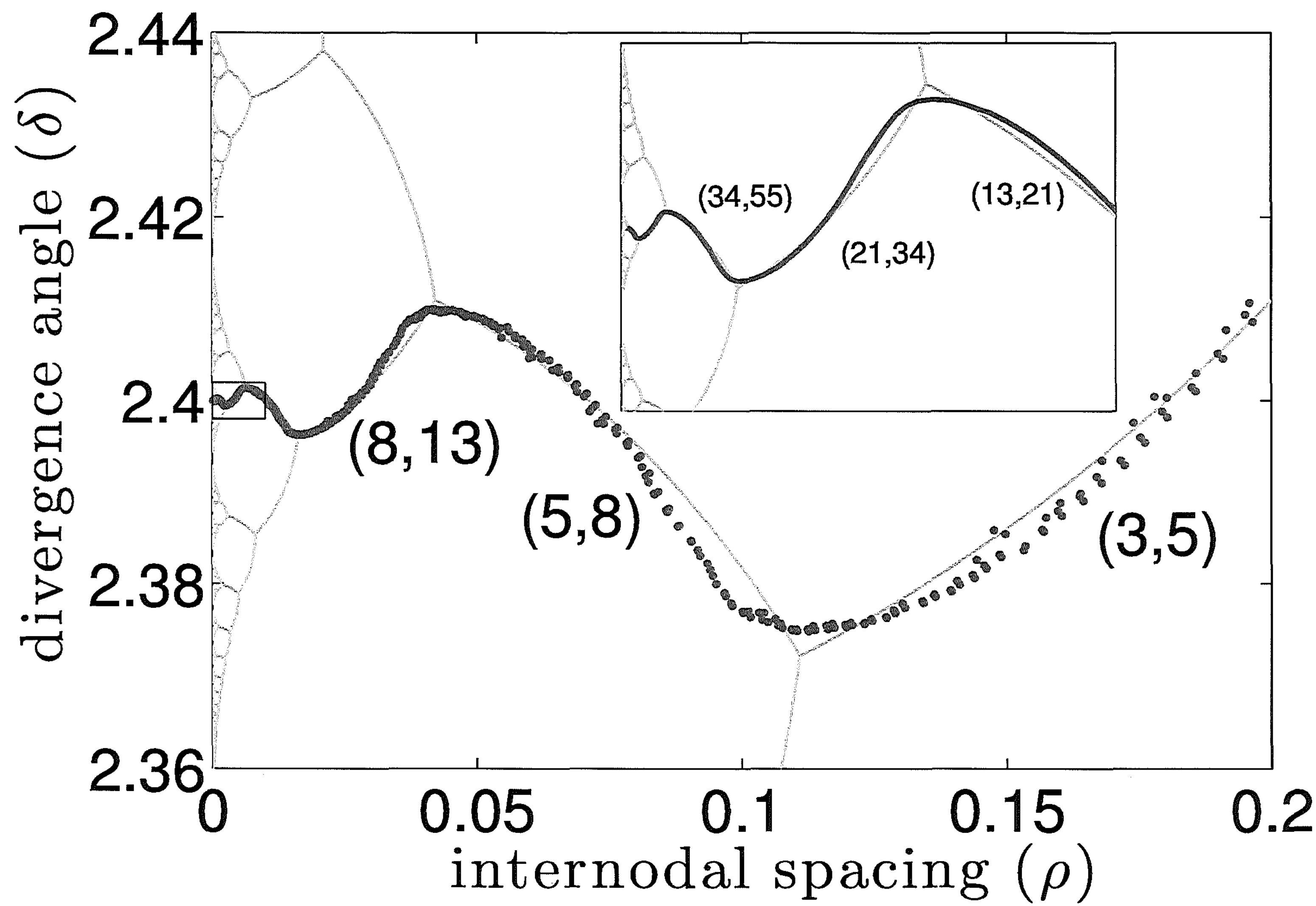
$r = 76$
0.4
0.3
0.2
0.1
0
a_{m_j}
a_{233}
a_{144}
a_{89}
a_{55}
a_{34}
a_{21}
a_{13}
10^{-1}
10^{0}
10^{1}
r/φ^j

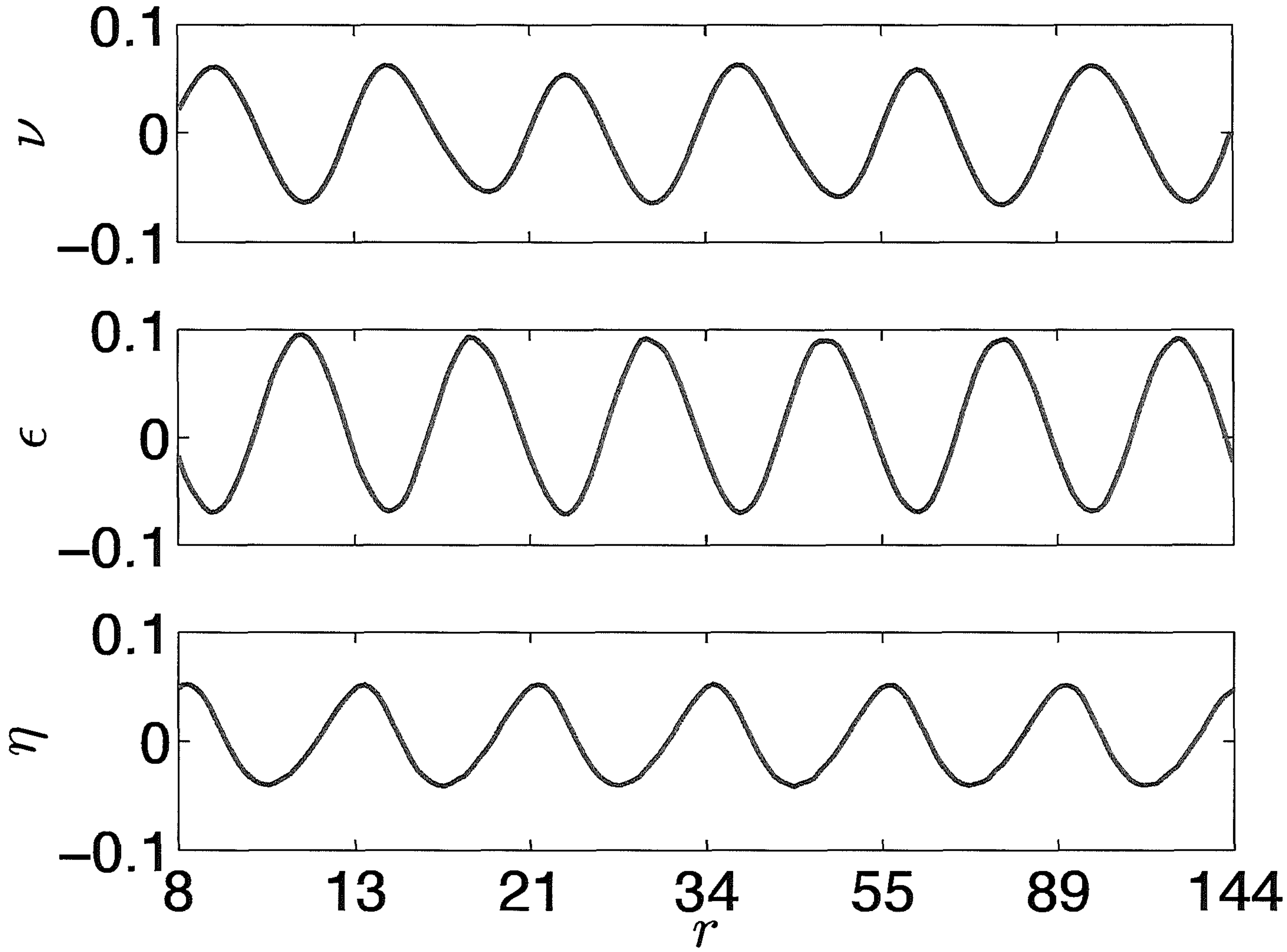

r = 61
0.4
0.3
0.2
0.1
0
a_{m_j}
a_{55}
a_{34}
a_{21}
a_{89}
a_{144}
a_{13}
10^{-1}
10^{0}
10^{1}
r/φ^j



$r = 47$
0.4
0.3
0.2
0.1
0
a_{m_j}
a_{55}
a_{34}
a_{21}
a_{13}
a_{89}
a_{144}
a_8
10^{-1}
10^0
10^1
r/φ^j

Lattice Geometry





0.1
0
−0.1
ν
0.1
0
−0.1
ϵ
0.1
0
−0.1
η
8
13
21
34
55
89
144
r

Open Challenges and Opportunities

1. Pushed pattern forming fronts and optimal strategies …
2. Invariants and self-similarities: Quo Venitis? …
3. Universality of Fibonacci patterns …
4. Other plant patterns? …
5. “Fibonacci rules” in 3D shell geometries …