A unified approach to the problems of outliers and related eigenvectors for spiked additive or multiplicative deformations of classical random matrices as well as Information-Plus-Noise type models, through free subordination properties.

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Notation:
For any $N \times N$ hermitian matrix X,
$\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_N(X)$ eigenvalues of X.

$$\mu_X := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(X)}$$

A) Seminal works on spiked models dealing with finite rank perturbations

I) The BBP phase transition

$L.U.E$ matrix

Definition

$$X_N = \frac{1}{p} B_N B_N^*$$

B_N is a $N \times p(N)$ matrix,

$$(B_N)_{u,v} = Z_{u,v} + iY_{u,v}$$

$Z_{u,v}$, $Y_{u,v}$, $u = 1, \ldots, N$, $v = 1, \ldots, p(N)$ are independent Gaussian variables $\mathcal{N}(0, \frac{1}{2})$

Convergence of the spectral measure:

Theorem

Marchenko-Pastur *(1967):*

If $c_N := \frac{N}{p} \to c \in]0;1]$ *when* $N \to \infty$,

$$\mu_{\frac{B_N B_N^*}{p}} := \frac{1}{N}\sum_{i=1}^{N} \lambda_i(\frac{B_N B_N^*}{p}) \to \mu_c \quad a.s \text{ when } \; N \to +\infty$$

$$\frac{d\mu_c}{dx}(x) = \frac{1}{2\pi c x}\sqrt{(b-x)(x-a)}\,1_{[a,b]}(x)$$

$a = (1-\sqrt{c})^2$, $b = (1+\sqrt{c})^2$,

Convergence of the largest eigenvalue

Theorem

(Geman 1980) (Bai-Yin-Krishnaiah 1988) (Bai-Silverstein-Yin 1988)

$$\lambda_1\left(\frac{B_N B_N^*}{p(N)}\right) \to (1+\sqrt{c})^2 \text{ a.s when } N \to +\infty.$$

$$\lambda_N\left(\frac{B_N B_N^*}{p(N)}\right) \to (1-\sqrt{c})^2 \text{ a.s when } N \to +\infty.$$

$$M_N = \frac{1}{p}\Sigma^{1/2} B_N B_N^* \Sigma^{1/2}$$

$$\Sigma = \text{diag}\,(\underbrace{1,\ldots,1}_{N-r \text{ times}}, \pi_1, \ldots, \pi_r)$$

r: fixed, independent of N.
$\pi_1 \geq \pi_2 \geq \pi_r > 0$ fixed, independent of N;
$\forall i \in \{1, \ldots, r\},\ \pi_i \neq 1$. (spikes)

Σ is a finite rank perturbation of I_N
$\Longrightarrow \mu_{M_N} := \frac{1}{N}\sum_{i=1}^{N} \lambda_i(M_N) \to \mu_c$ a.s when $N \to +\infty$

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Baik-Ben Arous-Péché (2005) (BBP phase transition)

π_1: the largest eigenvalue of Σ distinct from 1.

$\omega_c = 1 + \sqrt{c}$,

- If $\pi_1 > \omega_c$, a.s when $N \to +\infty$

$$\lambda_1\left(\frac{1}{p}\Sigma^{1/2}B_N B_N^* \Sigma^{1/2}\right) \to \sigma^2 \pi_1 \left(1 + \frac{c}{(\pi_1 - 1)}\right) > (1+\sqrt{c})^2.$$

  Therefore the largest eigenvalue of $\frac{1}{p}\Sigma^{1/2}B_N B_N^*\Sigma^{1/2}$ is an "outlier" since it converges outside the support of the limiting empirical spectral distribution μ_c and then does not stick to the bulk.

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result extended by Baik-Silversten (2006) when B_N has i.i.d entries which are not necessarily Gaussian

II) Spiked multiplicative

finite rank deformation

of a unitarily invariant matrix

$$M_N = (I_N + P_N)^{1/2} U_N B_N U_N^* (I_N + P_N)^{1/2},$$

• B_N is a deterministic $N \times N$ Hermitian non negative definite matrix such that:

- $\mu_{B_N} := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(B_N)}$ weakly converges to some probability measure μ with compact support $[a; b]$.
- the smallest and largest eigenvalue of B_N converge to a and b.

• U_N is a random $N \times N$ unitary matrix distributed according to Haar measure.

• P_N is a deterministic Hermitian matrix having r non-zero eigenvalues $\gamma_1 \geq \cdots \geq \gamma_s > 0 > \gamma_{s+1} \geq \cdots \geq \gamma_r > -1$, $r, \gamma_i, i = 1, \ldots r$, fixed independent of N .

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μ_{M_N} converges to μ.

$$T_\mu : \mathbb{C} \setminus \mathrm{supp}(\mu) \to \mathbb{C}, \quad T_\mu(z) = \int_{\mathbb{R}} \frac{t d\mu(t)}{z-t}.$$

Theorem (Benaych-Georges-Rao (2010))

Denote by $\lambda_1(M_N) \geq \cdots \geq \lambda_N(M_N)$ the ordered eigenvalues of $(I_N + P_N)^{1/2} U_N B_N U_N^*(I_N + P_N)^{1/2}$. Then, we have for each $1 \leq i \leq s$, almost surely,

$$\lambda_i(M_N) \to_{N\to+\infty} \begin{cases} T_\mu^{-1}(1/\gamma_i) & \text{if } \gamma_i > 1/\lim_{z\downarrow b} T_\mu(z), \\ b & \text{otherwise.} \end{cases}$$

Similarly, for the smallest eigenvalues, we have for each $0 \leq j < r - s$, a.s,

$$\lambda_{N-j}(M_N) \to_{N\to+\infty} \begin{cases} T_\mu^{-1}(1/\gamma_{r-j}) & \text{if } \gamma_{r-j} < 1/\lim_{z\uparrow a} T_\mu(z), \\ a & \text{otherwise.} \end{cases}$$

III) Additive finite rank deformation of a Wigner matrix

Definition

A G.U.E (N, σ^2) matrix W_N is a $N \times N$ Hermitian matrix such that :

$(W_N)_{ii}$, $\sqrt{2}\Re e((W_N)_{ij})_{i<j}$, $\sqrt{2}\Im m((W_N)_{ij})_{i<j}$ are independent gaussian $\mathcal{N}(0, \sigma^2)$ random variables.

Theorem

Convergence of the spectral measure: *Wigner (50')*

$$\mu_{\frac{W_N}{\sqrt{N}}} := \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i\left(\frac{W_N}{\sqrt{N}}\right)} \rightarrow \mu_\sigma \quad a.s \text{ when } \quad N \rightarrow +\infty$$

$$\frac{d\mu_\sigma}{dx}(x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}\, 1_{[-2\sigma,2\sigma]}(x)$$

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Theorem

Convergence of the extremal eigenvalues *(Bai-Yin 1988):*

$$\lambda_1(\frac{W_N}{\sqrt{N}}) \to 2\sigma \text{ and } \lambda_N(\frac{W_N}{\sqrt{N}}) \to -2\sigma \text{ a.s when } N \to +\infty.$$

Spiked finite rank deformation : $M_N = \frac{1}{\sqrt{N}} W_N + A_N$

$$A_N = \mathrm{diag}\,(\underbrace{0, \ldots, 0}_{N-r \text{ times}}, \gamma_1, \ldots, \gamma_r)$$

r: fixed, independent of N.

A_N : a deterministic Hermitian matrix of fixed finite rank r with r non-null eigenvalues (spikes) $\gamma_1 \geq \cdots \geq \gamma_r$ independent of N,

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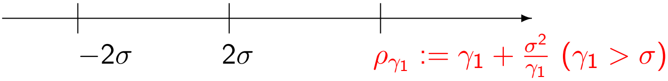
A_N : a deterministic Hermitian matrix of fixed finite rank r with r non-null eigenvalues (spikes) $\gamma_1 \geq \cdots \geq \gamma_r$ independent of N,

$\Longrightarrow$ Convergence of the spectral measure $\mu_{M_N} := \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i(M_N)}$ towards the semi-circular distribution μ_σ.

Theorem (Péché 2006)

- *If* $\gamma_1 \leq \sigma$, $\lambda_1(M_N) \to 2\sigma$
- *If* $\gamma_1 > \sigma$, $\lambda_1(M_N) \to \rho_{\theta_1}$ *with* $\rho_{\gamma_1} := \gamma_1 + \frac{\sigma^2}{\gamma_1}$.



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result extended to Wigner matrices whose entries are not necessarily Gaussian by Féral-Péché (2007), Capitaine-Donati-Martin-Féral (2009)

IV) Spiked additive

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- B_N is a deterministic $N \times N$ Hermitian matrix such that:
  - $\mu_{B_N} := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i(B_N)}$ weakly converges to some probability measure μ with compact support $[a; b]$.
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- U_N is a random $N \times N$ unitary matrix distributed according to Haar measure.

- A_N is a deterministic Hermitian matrix having r non-zero eigenvalues $\gamma_1 \geq \cdots \geq \gamma_s > 0 > \gamma_{s+1} \geq \cdots \geq \gamma_r$, r, $\gamma_i, i = 1, \ldots r$, fixed independent of N (spikes).

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$\Longrightarrow$ Convergence of the spectral measure μ_{M_N} towards μ.

$$g_\mu : \mathbb{C} \setminus \mathrm{supp}(\nu) \to \mathbb{C}, \quad g_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{z-t}.$$

Theorem (Benaych-Georges-Rao (2010))

Denote by $\lambda_1(M_N) \geq \cdots \geq \lambda_N(M_N)$ the ordered eigenvalues of $M_N = U_N B_N U_N^* + A_N$. Then, we have for each $1 \leq i \leq s$, almost surely,

$$\lambda_i(M_N) \to_{N\to+\infty} \begin{cases} g_\mu^{-1}(1/\gamma_i) & \text{if } \gamma_i > 1/\lim_{z\downarrow b} g_\mu(z), \\ b & \text{otherwise.} \end{cases}$$

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V) Spiked Information-plus-noise

(finite rank deformation case)

Gaussian "Information plus noise" type model

$$M_N = (\sigma \frac{X_N}{\sqrt{N}} + A_N)(\sigma \frac{X_N}{\sqrt{N}} + A_N)^*$$

$\sigma > 0$. $n \in \mathbb{N}$, $N \in \mathbb{N}$, $n \leq N$.
X_N: a $n \times N$ matrix such that $(X_N)_{ij} = X_{ij}$. $\{X_{ij}, i \in \mathbb{N}^*, j \in \mathbb{N}^*\}$ independent random standard complex Gaussian variables.

$$A_N = \begin{pmatrix} a_1 & & & & & & (0) \\ & (0) & \ddots & & (0) & & \\ & & & a_r & & & \\ & & & & 0 & & \\ & & (0) & & & \ddots & (0) \\ & & & & & 0 & (0) \end{pmatrix} \text{ deterministic.}$$

$A_N A_N^*$ has a **finite** number r of **fixed** eigenvalues (independent of N) **(spikes)** $\theta_1 > \ldots > \theta_r > 0$, $\theta_i = |a_i|^2$).

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$A_N A_N^*$ has **a finite** number r of **fixed** eigenvalues (independent of N) **(spikes)** $\theta_1 > \ldots > \theta_r > 0$, $\theta_i = |a_i|^2$).
$\mu_{M_N} \longrightarrow \sigma^2 \mu_c$, when $N \to +\infty$ and $n/N \to c \in]0;1]$.

Theorem (Benaych-Georges-Rao; Loubaton-Vallet (2010))

Denote by $\lambda_1(M_N) \geq \cdots \geq \lambda_N(M_N)$ the ordered eigenvalues of $M_N = (\sigma \frac{X_N}{\sqrt{N}} + A_N)(\sigma \frac{X_N}{\sqrt{N}} + A_N)^*$. Then, we have for each $1 \leq i \leq r$, almost surely,

$$\lambda_i(M_N) \underset{\substack{N \to +\infty \\ n/N \to c \in]0;1]}}{\longrightarrow} \begin{cases} \frac{(\sigma^2+\theta_i)(\sigma^2 c+\theta_i)}{\theta_i} & \text{if } \theta_i > \sigma^2\sqrt{c}, \\ \sigma^2(1+\sqrt{c})^2 & \text{otherwise.} \end{cases}$$

B) How free probability may shed light on these phenomena and provide a unified understanding, allowing to extend them to non-finite rank deformations ?

$\mathcal{M}$: the set of probability measures supported on the real line
$\mathcal{M}^+$: the set of probability measures supported on $[0;+\infty[$.
Free probability theory defines:

- a binary operation on $\mathcal{M}$:the free additive convolution $\mu \boxplus \nu$ for μ and ν in $\mathcal{M}$,
- binary operations on $\mathcal{M}^+$: the free multiplicative convolution $\mu \boxtimes \nu$ and the free rectangular convolution with ratio $c \in]0;1]$ $\mu \boxplus_c \nu$, for μ and ν in $\mathcal{M}^+$,

(cf Voiculescu and Benaych-Georges)

For several matricial models where A_N and B_N are independent $N \times N$ Hermitian random matrices, free probability provides a good understanding of the asymptotic global behaviour of the spectrum of $A_N + B_N$ and $A_N^{\frac{1}{2}} B_N A_N^{\frac{1}{2}}$

$$\mu_{A_N+B_N} \rightarrow_{N\rightarrow+\infty} \mu_a \boxplus \mu_b$$

$$\mu_{A_N^{\frac{1}{2}} B_N A_N^{\frac{1}{2}}} \rightarrow_{N\rightarrow+\infty} \mu_a \boxtimes \mu_b$$

where $\mu_{A_N} \rightarrow_{N\rightarrow+\infty} \mu_a$ and $\mu_{B_N} \rightarrow_{N\rightarrow+\infty} \mu_b$.

Pionnering work 90' of D. Voiculescu extended by several authors

For several matricial models where A_N and B_N are independent rectangular $n \times N$ random matrices such that $n/N \to c \in]0;1]$, rectangular free convolution provides a good understanding of the asymptotic global behaviour of the singular values of $A_N + B_N$:

$$\frac{1}{n} \sum_{s \text{ sing. val. of } A_N+B_N} \delta_s \to \nu_a \boxplus_c \nu_b.$$

(where $\frac{1}{n} \sum_{s \text{ sing. val. of } A_N} \delta_s \to \nu_a$, $\frac{1}{n} \sum_{s \text{ sing. val. of } B_N} \delta_s \to \nu_b$)

(cf work of Benaych-Georges when A_N or B_N is invariant, in law, under multiplication, on the right and on the left, by any unitary matrix)

Does the spectrum of

- $A_N + B_N$
- $A_N^{\frac{1}{2}} B_N A_N^{\frac{1}{2}}$
- $(A_N + B_N)(A_N + B_N)^*$

have outliers? i.e

For large N, are there eigenvalues of $A_N + B_N$, $A_N^{\frac{1}{2}} B_N A_N^{\frac{1}{2}}$ and $[(A_N + B_N)(A_N + B_N)^*]^{\frac{1}{2}}$ outside the support of the respective limiting empirical spectral distributions $\mu_a \boxplus \mu_b$, $\mu_a \boxtimes \mu_b$ and $\nu_a \boxplus_c \nu_b$?

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Actually free probability will shed light on this question through free subordination property

I) Additive Spiked deformations

$$M_N = X_N + A_N$$

General framework

- X_N is a $N \times N$ random Hermitian matrix such that almost surely:

$$\mu_{X_N} \to \mu \quad \text{weakly }, \mu \text{ compactly supported},$$

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- A_N is a deterministic Hermitian matrix,

$$\mu_{A_N} \to \nu \quad \text{weakly}\ , \nu \text{ compactly supported}.$$

The eigenvalues of A_N:

- $N - r$ (r fixed) eigenvalues $\beta_i(N)$ such that

$$\max_{i=1}^{N-r} \operatorname{dist}(\beta_i(N), \operatorname{supp}(\nu)) \to_{N \to \infty} 0$$

- a **finite** number J of **fixed** (independent of N) eigenvalues **(SPIKES)** $\theta_1 > \ldots > \theta_J$, $\forall i = 1, \ldots, J$, $\theta_i \notin \operatorname{supp}(\nu)$, each θ_j having a fixed multiplicity k_j, $\sum_j k_j = r$.

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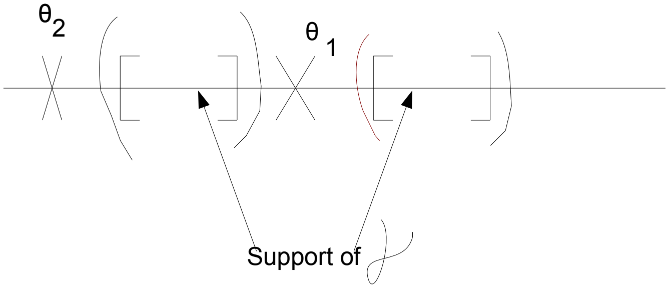
- Almost surely $\mu_{X_N + A_N} \to_{N \to +\infty} \mu \boxplus \nu$, weakly.

For large N, the $\beta_i(N)$ are inside an ε neighborhood of the support of



SPECTRUM OF A_N

Additive free subordination property

For a probability measure τ on $\mathbb{R}$, $z \in \mathbb{C}\backslash\mathbb{R}$, $g_\tau(z) = \int_{\mathbb{R}} \frac{d\tau(x)}{z-x}$.

Theorem (D.Voiculescu, P. Biane)

Let μ and ν be two probability measures on $\mathbb{R}$, there exists a unique analytic map $\omega_{\mu,\nu} : \mathbb{C}^+ \to \mathbb{C}^+$ such that

$$\forall z \in \mathbb{C}^+, g_{\mu\boxplus\nu}(z) = g_\nu(\omega_{\mu,\nu}(z)),$$

$\forall z \in \mathbb{C}^+, \Im\omega_{\mu,\nu}(z) \geq \Im z$ *and* $\lim_{y\uparrow+\infty} \frac{\omega_{\mu,\nu}(iy)}{iy} = 1$.
$\omega_{\mu,\nu}$ *is called the subordination map of* $\mu \boxplus \nu$ *with respect to* ν.

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Conjecture

Conjecture: for large N, the θ_i's such that the equation

$$\omega_{\mu,\nu}(\rho) = \theta_i$$

has solutions ρ outside $\mathrm{support}\ \mu \boxplus \nu$ generate k_i eigenvalues of M_N in a neighborhood of each of these ρ...

Definition

For each $j \in \{1, \ldots, J\}$, define O_j the set of solutions ρ in $\mathbb{R} \setminus \mathrm{supp}(\mu \boxplus \nu)$ of the equation

$$\omega_{\mu,\nu}(\rho) = \theta_j,$$

and

$$O = \bigcup_{1 \le j \le J} O_j.$$

The sets O_j defined above may be empty, finite, or countably infinite.

Conjecture proved when X_N is a Wigner matrix (with technical conditions) by Capitaine-Donati-Martin-Féral-Février (2011) or when the distribution of X_N is invariant under conjugation by any unitary matrix by Belinschi-Bercovici-Capitaine-Février (2012).

Theorem

Denote by $\mathrm{sp}(M_N)$ the spectrum of $M_N = X_N + A_N$ The following results hold almost surely:

- *for each $\rho \in O_j$, for all small enough $\varepsilon > 0$, for all large enough N,*

$$\mathrm{card}\{\mathrm{sp}(M_N) \bigcap]\rho - \epsilon; \rho + \epsilon[\} = k_j;$$

- *for all $\varepsilon > 0$, for large enough N,*

$$\mathrm{sp}(M_N) \subset \{x \in \mathbb{R} \mid d(x, \mathrm{supp}(\mu \boxplus \nu) < \epsilon\} \bigcup_{\rho \in O}]\rho - \epsilon; \rho + \epsilon[.$$

The restriction to the real line of some subordination map may be many-to-one so that for one θ_j, there may exists several distinct ρ solving the equation

$$\omega_{\mu,\nu}(\rho) = \theta_j.$$

$\Longrightarrow$ A single spiked eigenvalue of A_N may generate a finite or countably infinite set of outliers of X_N.

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$$\forall z \in \mathbb{C}^+, \omega_{\mu_\sigma,\nu}(z) = z - \sigma^2 g_{\mu_\sigma \boxplus \nu}(z)$$

$\omega_{\mu_\sigma,\nu} : \mathbb{C}^+ \cup \mathbb{R} \to \omega_{\mu_\sigma,\nu}(\mathbb{C}^+ \cup \mathbb{R})$ is a homeomorphism with inverse

$$H_{\sigma,\nu} : z \mapsto z + \sigma^2 g_\nu(z).$$

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$$O = \{\rho_{\theta_j} = H_{\sigma,\nu}(\theta_j), \text{for } \theta_j \text{ such that } H'_{\sigma,\nu}(\theta_j) > 0\}$$

remark

If $\mathrm{supp}(\mu) = [\mathrm{a};\mathrm{b}]$ *and* $\nu = \delta_0$,

$$\omega_{\mu,\delta_0}(z) = \frac{1}{g_\mu(z)}$$

Hence, $\omega_{\mu,\delta_0}(z) = \theta_j$ *has a solution on* $[b;+\infty[$ *if and only if* $\theta_j > 1/\lim_{z\downarrow b} g_\mu(z)$ *and then the solution is equal to* $g_\mu^{-1}(1/\theta_j)$ *so that we recover the results of Benaych-Georges-Rao and Péché, Baik-Silverstein.*

II) Multiplicative Spiked perturbations

$$M_N = \Sigma^{1/2} X_N \Sigma^{1/2}$$

General framework

- X_N is a $N \times N$ random nonnegative matrix such that a.s.:

$$\mu_{X_N} \to \mu \quad \text{weakly }, \mu \text{ compactly supported on } [0; +\infty[$$

$$\max_{1 \leq j \leq N} \operatorname{dist}(\lambda_j^{(N)}(X_N), \operatorname{supp}(\mu)) \to_{N \to \infty} 0.$$

- Σ_N is a deterministic definite positive Hermitian matrix,

$$\mu_{\Sigma_N} \to \nu \quad \text{weakly }, \nu \text{ compactly supported on } [0; +\infty[$$

The eigenvalues of Σ_N:

- $N - r$ (r fixed) eigenvalues $\beta_i(N)$ such that

$$\max_{i=1}^{N-r} \operatorname{dist}(\beta_i(N), \operatorname{supp}(\nu)) \to_{N \to \infty} 0$$

- a **finite** number J of **fixed** (independent of N) eigenvalues **(SPIKES)** $\theta_1 > \ldots > \theta_J > 0$, $\forall i = 1, \ldots, J$, $\theta_i \notin \operatorname{supp}(\nu)$, each θ_j having a fixed multiplicity k_j, $\sum_j k_j = r$.

- Almost surely $\mu_{\Sigma^{1/2} X_N \Sigma^{1/2}} \to_{N \to +\infty} \mu \boxtimes \nu$, weakly.

Multiplicative free subordination property

$$\Psi_\tau(z) = \int \frac{tz}{1-tz} d\tau(t) = \frac{1}{z} g_\tau(\frac{1}{z}) - 1,$$

for complex values of z such that $\frac{1}{z}$ is not in the support of τ.

Theorem (Biane; Belinschi-Bercovici)

Let $\mu \neq \delta_0$ and $\nu \neq \delta_0$ be two probability measures on $[0;+\infty[$. There exists a unique analytic map $F_{\mu,\nu}$ defined on $\mathbb{C} \setminus [0;+\infty[$ such that

$$\forall\, z \in \mathbb{C} \setminus [0;+\infty[,\ \Psi_{\mu \boxtimes \nu}(z) = \Psi_\nu(F_{\mu,\nu}(z))$$

and $\forall\, z \in \mathbb{C}^+$,

$$F_{\mu,\nu}(z) \in \mathbb{C}^+,\ F_{\mu,\nu}(\bar{z}) = \overline{F_{\mu,\nu}(z)},\ \arg(F_{\mu,\nu}(z)) \geq \arg(z).$$

Conjecture

Conjecture: for large N, the θ_i's such that the equation

$$\frac{1}{F_{\mu,\nu}(\frac{1}{\rho})} = \theta_i$$

has solutions ρ outside support $\mu \boxtimes \nu$ *generate k_i eigenvalues of M_N in a neighborhood of these ρ...*

This conjecture is true when X_N is a Wishart matrix : the results of R. Rao, J. Silverstein and Z.D Bai, J. Yao can be described in terms of the subordination function related to the free multiplicative convolution by the Marchenko-Pastur distribution.

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This conjecture is still true when the distribution of X_N is invariant under unitary conjugation (work in progress with Belinschi, Bercovici, Février).

III) Spiked Information-Plus-Noise type matrices

$$M_N = (\sigma \frac{X_N}{\sqrt{N}} + A_N)(\sigma \frac{X_N}{\sqrt{N}} + A_N)^*$$

General framework

- X_N is a $n \times N$ random matrix such that almost surely:

$$\mu_{X_N X_N^*} \to \mu \quad \text{weakly}, \mu \text{ compactly supported on } [0; +\infty[$$

$$\max_{1 \leq j \leq N} \mathrm{dist}(\lambda_j^{(N)}(X_N X_N^*), \mathrm{supp}(\mu)) \to_{N \to \infty} 0.$$

- A_N is a deterministic $n \times N$ matrix,

$$\mu_{A_N A_N} \to \nu \quad \text{weakly}, \nu \text{ compactly supported on } [0; +\infty[$$

The eigenvalues of $A_N A_N^*$:

- $n - r$ (r fixed) eigenvalues $\beta_i(N)$ such that $\max_{i=1}^{n-r} \mathrm{dist}(\beta_i(N), \mathrm{supp}(\nu)) \to_{N \to \infty} 0$
- a **finite** number J of **fixed** (independent of N) eigenvalues **(SPIKES)** $\theta_1 > \ldots > \theta_J > 0$, $\forall i = 1, \ldots, J$, $\theta_i \notin \mathrm{supp}(\nu)$, each θ_j having a fixed multiplicity k_j, $\sum_j k_j = r$.

- Almost surely

$$\mu_{(\sigma \frac{X_N}{\sqrt{N}} + A_N)(\sigma \frac{X_N}{\sqrt{N}} + A_N)^*} \to_{\substack{N \to +\infty \\ n/N \to c}} (\sqrt{\mu} \boxplus_c \sqrt{\nu})^2, \text{ weakly.}$$

(F. Benaych-Georges):rectangular R-transform with ratio c

τ probability measure on $\mathbb{R}^+$; $c \in]0;1]$.

$$M_\tau(z) = \int_{\mathbb{R}^+} \frac{t^2 z}{1 - t^2 z} d\tau(t).$$

$$H_\tau^{(c)}(z) := z\left(cM_\tau(z) + 1\right)\left(cM_\tau(z) + 1\right).$$

$$C_\tau^{(c)}(z) = T^{(c)-1}\left(\frac{z}{{H_\tau^{(c)}}^{-1}(z)}\right), \quad \text{for} \quad z \neq 0;$$

$$C_\tau^{(c)}(0) = 0.$$

$$T^{(c)}(z) = (cz+1)(z+1).$$

Rectangular subordination

Theorem (Belinschi-Benaych-Georges-Guionnet)

Assume that the rectangular R-transform $C_\tau^{(c)}$ of τ extends analytically to $\mathbb{C} \setminus \mathbb{R}^+$; this happens for example if τ is $\boxplus_c$ infinitely divisible. Then there exist two unique meromorphic functions Ω_1, Ω_2 on $\mathbb{C} \setminus \mathbb{R}^+$ so that

$$H_\tau^{(c)}(\Omega_1(z)) = H_\nu^{(c)}(\Omega_2(z)) = H_{\tau \boxplus_c \nu}^{(c)}(z),$$

$\Omega_j(\bar{z}) = \overline{\Omega_j(z)}$ *and* $\lim_{x \uparrow 0} \Omega_j(x) = 0$, $j \in \{1; 2\}$.

$\mu_{X_N X_N^*} \to \mu$, $\mu_{A_N A_N^*} \to \nu$,

$$\mu_{(\sigma \frac{X_N}{\sqrt{N}}+A_N)(\sigma \frac{X_N}{\sqrt{N}}+A_N)^*} \underset{\substack{N \to +\infty \\ n/N \to c}}{\longrightarrow} (\sqrt{\mu} \boxplus_c \sqrt{\nu})^2, \text{weakly.}$$

$$H^{(c)}_{\sqrt{\nu}}(\Omega_{\mu,\nu}(z)) = H^{(c)}_{\sqrt{\mu}\boxplus_c\sqrt{\nu}}(z)$$

$$H^{(c)}_{\sqrt{\tau}} = \frac{c}{z} g_\tau(\frac{1}{z})^2 + (1-c) g_\tau(\frac{1}{z})$$

$\mu_{X_N X_N^*} \to \mu$, $\mu_{A_N A_N^*} \to \nu$,

$$\mu_{(\sigma \frac{X_N}{\sqrt{N}} + A_N)(\sigma \frac{X_N}{\sqrt{N}} + A_N)^*} \underset{\substack{N \to +\infty \\ n/N \to c}}{\longrightarrow} (\sqrt{\mu} \boxplus_c \sqrt{\nu})^2, \text{weakly.}$$

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Conjecture

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$$\frac{1}{\Omega_{\mu,\nu}(\frac{1}{\rho})} = \theta_i$$

has solutions ρ outside support $(\sqrt{\mu} \boxplus_c \sqrt{\nu})^2$ *generate k_i eigenvalues of $M_N = (\sigma \frac{X_N}{\sqrt{N}} + A_N)(\sigma \frac{X_N}{\sqrt{N}} + A_N)^*$ in a neighborhood of these ρ...*

This conjecture is proved in [C. 2013] when $X_N = (X_{ij})$ where $\{X_{ij}, i \in \mathbb{N}, j \in \mathbb{N}\}$ is an infinite set of i.i.d standardized complex random variables $(\mathbb{E}(X_{ij}) = 0, \mathbb{E}(|X_{ij}|^2) = 1)$ with finite fourth moment,

$$A_N = \begin{pmatrix} a_1(N) & & & (0) \\ & & (0) & \\ & \ddots & & (0) \\ (0) & & & \\ & & a_n(N) & (0) \end{pmatrix}$$

$\sup_N \|A_N\| < +\infty$, $\mu_{A_N A_N^*} = \frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i(A_N A_N^*)} \rightarrow_{N\rightarrow+\infty} \nu$ weakly, the support of ν is compact and has a finite number of connected components.

In the previous case (X_N has i.i.d entries), the spikes θ which generate outliers are explicitely those which satisfy

$$\Phi'_{\sigma,\nu,c}(\theta) > 0, g_\nu(\theta) > -\frac{1}{\sigma^2 c},$$

where $g_\nu(z) = \int_{\mathbb{R}} \frac{d\nu(x)}{z-x}$,

$$\Phi_{\sigma,\nu,c} : \begin{array}{l} \mathbb{R} \setminus \operatorname{supp}(\nu) \to \mathbb{R} \\ x \mapsto x(1 + c\sigma^2 g_\nu(x))^2 + \sigma^2(1-c)(1 + c\sigma^2 g_\nu(x)) \end{array},$$

and the corresponding limiting outliers are equal to $\rho_\theta = \Phi_{\sigma,\nu,c}(\theta)$.

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and the corresponding limiting outliers are equal to $\rho_\theta = \Phi_{\sigma,\nu,c}(\theta)$.

Open question: Proof of the conjecture when X_N is invariant, in law, under multiplication, on the right and on the left, by any unitary matrix, and $\nu \neq \delta_0$.

Conclusion

Solving the problem of outliers consists in solving an equation involving the free subordination function and the spikes of the perturbation

$M_N = A_N + B_N$ $\mu_{A_N} \to_{N\to+\infty} \nu$ $\mu_{B_N} \to_{N\to+\infty} \mu$	$M_N = A_N^{1/2} B_N A_N^{1/2}$ $\mu_{A_N A_N^*} \to_{N\to+\infty} \nu$ $\mu_{B_N B_N^*} \to_{N\to+\infty} \mu$	$M_N = (A_N + B_N)(A_N + B_N)^*$ $\mu_{A_N A_N^*} \to_{N\to+\infty} \nu$ $\mu_{B_N B_N^*} \to_{N\to+\infty} \mu$
$\mu_{M_N} \to_{N\to+\infty} \mu \boxplus \nu$	$\mu_{M_N} \to_{N\to+\infty} \mu \boxtimes \nu$	$\mu_{M_N} \to_{N\to+\infty} (\sqrt{\mu} \boxplus_c \sqrt{\nu})^2$
$g_\tau(z) = \int_{\mathbb{R}} \frac{d\tau(x)}{z-x}$	$\Psi_\tau(z) = \frac{1}{z} g_\tau(\frac{1}{z}) - 1$	$H^{(c)}_{\sqrt{\tau}} = \frac{c}{z} g_\tau(\frac{1}{z})^2 + (1-c) g_\tau(\frac{1}{z}$
$g_{\mu\boxplus\nu}(z) = g_\nu(\omega_{\mu,\nu}(z))$	$\Psi_{\mu\boxtimes\nu}(z) = \Psi_\nu(F_{\mu,\nu}(z))$	$H^{(c)}_{\sqrt{\mu}\boxplus_c\sqrt{\nu}}(z) = H^{(c)}_{\sqrt{\nu}}(\Omega_{\mu,\nu}(z)$
$\omega_{\mu,\nu}(\rho) = \theta$	$\frac{1}{F_{\mu,\nu}(1/\rho)} = \theta$	$\frac{1}{\Omega_{\mu,\nu}(1/\rho)} = \theta$

C) **When some eigenvalues deviate from the bulk, how do the corresponding eigenvectors project onto those of the perturbation?**

Conjecture

ρ: a solution of the equation involving the subordination map and θ_i
Almost surely, for δ small enough:

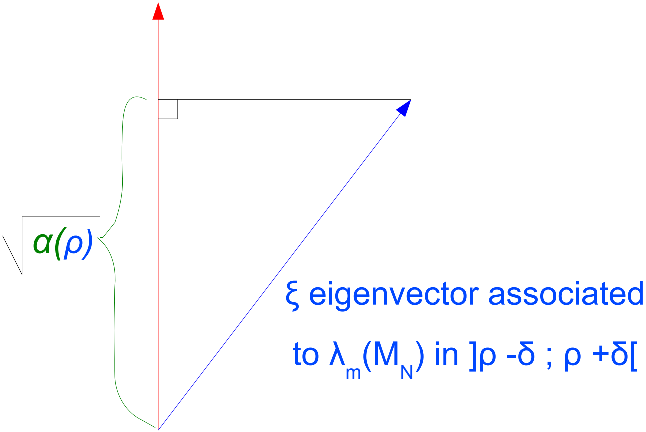
- *for all large N*, $\mathrm{card}\,\{\mathrm{spect} M_N \cap]\rho-\delta, \rho+\delta[\} = k_i$.
- *for any $\zeta > 0$, for N large enough, for any orthonormal system $(\xi_1(\rho), \cdots, \xi_{k_i}(\rho))$ of eigenvectors associated to the k_i eigenvalues of X_N in $]\rho-\delta, \rho+\delta[$, for any $n = 1, \ldots, k_i$, for any $l = 1, \ldots, J$,*

$$\left| \quad \left\| P_{Ker\,(\theta_l I_N - A_N)} \xi_n(\rho) \right\|_2^2 - \delta_{l,i}\alpha(\rho) \right| < \zeta.$$

$$\alpha(\rho) = \begin{cases} \frac{1}{\omega'_{\mu,\nu}(\rho)} & \text{if } M_N = X_N + A_N \\ \\ \frac{\rho F_{\mu,\nu}(1/\rho)}{F'_{\mu,\nu}(1/\rho)} & \text{if } M_N = A_N^{1/2} X_N A_N^{1/2} \end{cases}$$

$k_j=1$ When N goes to infinity,

V eigenvector associated to $\theta_j=\lambda_p(A_N)$



This conjecture was proved

• for $X_N + A_N$ when X_N is a Wigner matrix [C. 2011] and when the distribution of X_N is unitarily invariant [Belinschi, Bercovici, C., Février work in progress]

First results of Benaych-Georges-Rao (2010) when the distribution of X_N is unitarily invariant and dealing with finite rank perturbations.

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First results of Benaych-Georges-Rao (2010) when the distribution of X_N is unitarily invariant and dealing with finite rank perturbations.

Note that we have explicitely for deformed Wigner matrices with standardized entries $\omega(\rho) = \theta \Leftrightarrow \rho = \theta + g_\nu(\theta)$ and $\alpha(\rho) = 1 - \int \frac{1}{(\theta - x)^2} d\nu(x)$

• for $A_N^{1/2} X_N A_N^{1/2}$ when X_N is a Wishart matrix [C. 2011] and when the distribution of X_N is unitarily invariant [Belinschi, Bercovici, C. , Février work in progress]

First results of Benaych-Georges-Rao (2010) when the distribution of X_N is unitarily invariant and dealing with finite rank perturbations of the Identity matrix.

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Note that when X_N is a Wishart matrix, we have

$$\alpha(\rho) = \frac{1 - c \int \frac{x^2}{(\theta - x)^2} d\nu(x)}{1 + c \int \frac{x}{(\theta - x)} d\nu(x)}.$$

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Work in progress for information-plus-noise type models.
First results of Benaych-Georges-Rao (2012) dealing with finite rank perturbations