

Nuclear Structure (II) Collective models

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TALENT school

TALENT (Training in Advanced Low-Energy Nuclear Theory, see <http://www.nucleartalent.org>).

Course 5: Theory for exploring nuclear structure experiments.

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Audience: Master of Science and PhD students, and early post-doctoral students.

Place: GANIL. Dates: 11–29 August 2014.

Collective nuclear models

Rotational models

Vibrational models

Liquid-drop model of vibrations and rotations

Interacting boson model

Rotation of a symmetric top

Energy spectrum:

$$E_{\text{rot}}(I) = \frac{\hbar^2}{2\mathfrak{I}} I(I+1)$$

$$\equiv A I(I+1), \quad I = 0, 2, 4, \dots$$

$$E(I) - E(I-2)$$

$$6^+ \quad \underline{\quad 42A \quad}$$

Large deformation \Rightarrow

large $\mathfrak{I} \Rightarrow$ low $E_x(2^+)$.

$$22A$$

$$4^+ \quad \underline{\quad 20A \quad}$$

R_{42} energy ratio:

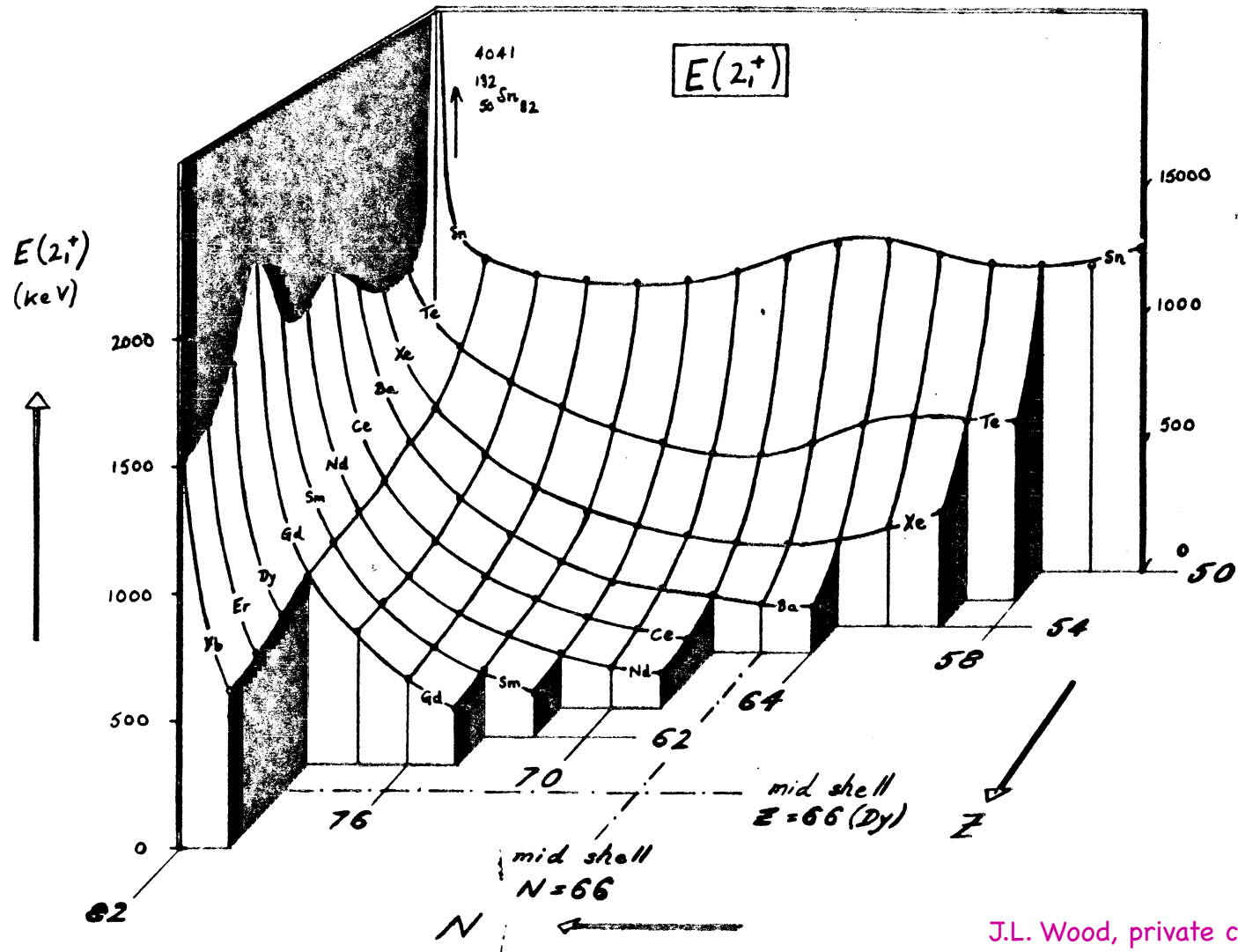
$$14A$$

$$E_{\text{rot}}(4^+) / E_{\text{rot}}(2^+) = 3.333\dots$$

$$\begin{array}{c} 2^+ \quad \underline{\quad 6A \quad} \\ 0^+ \quad \underline{\quad 0 \quad} \end{array}$$

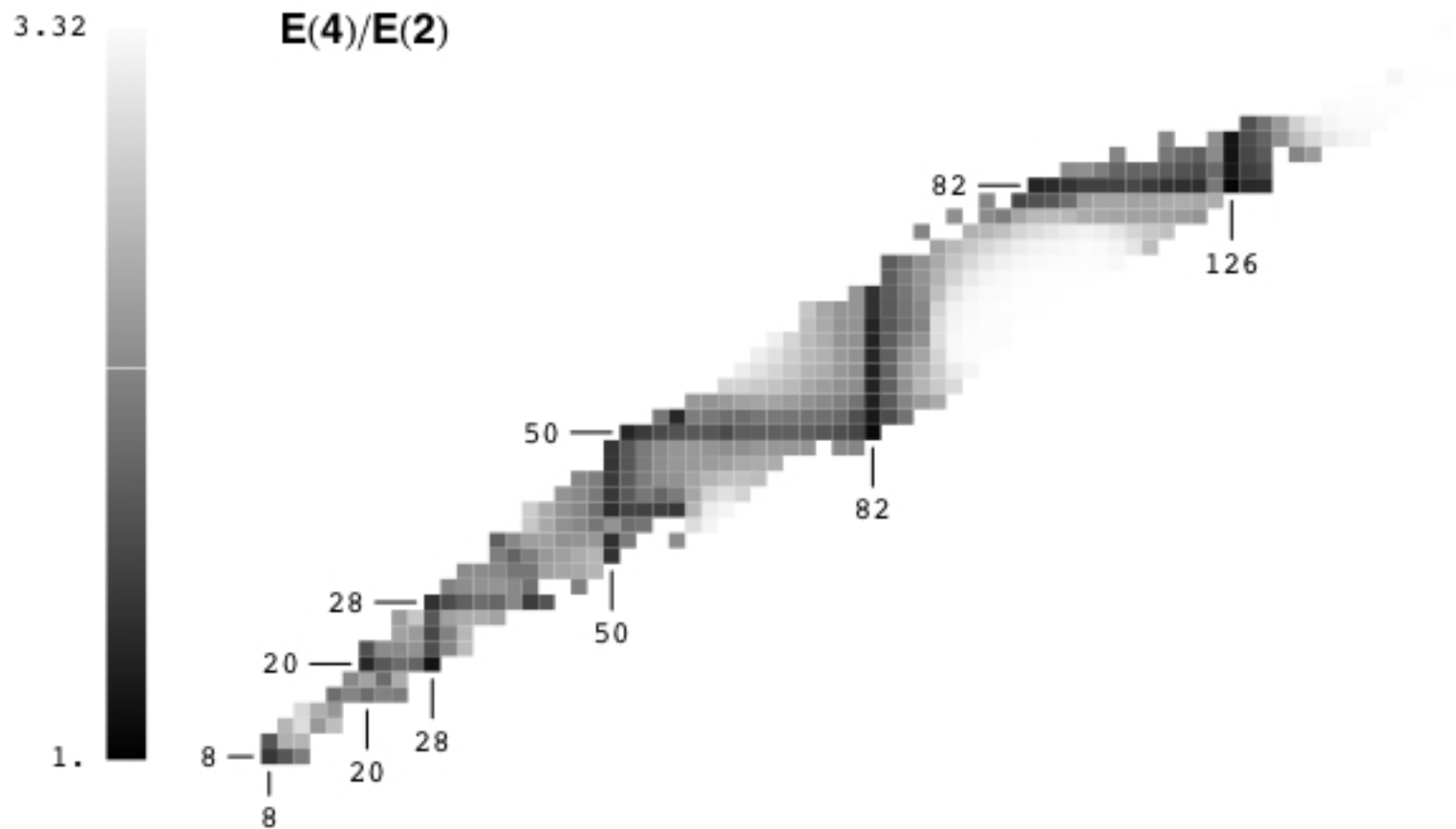
$$6A$$

Evolution of $E_x(2^+)$



J.L. Wood, private communication

The ratio R_{42}



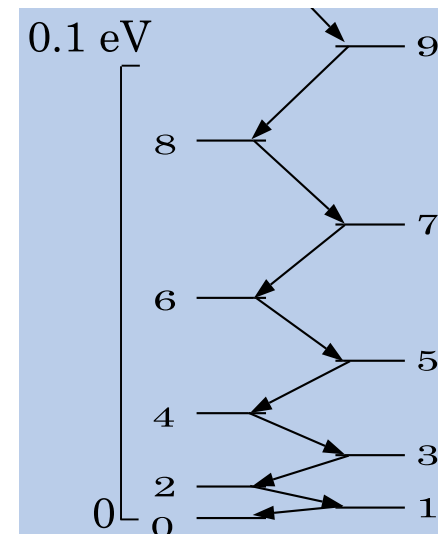
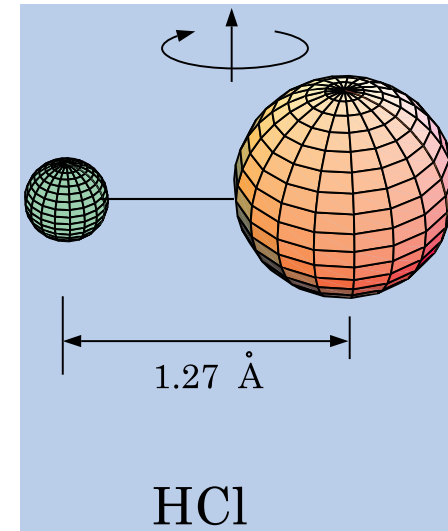
Rotation of an asymmetric top

Energy spectrum:

$$E_{\text{rot}}(I^\pi) = \frac{\hbar^2}{2\mathfrak{I}} I(I+1)$$

$$I^\pi = 0^+, 1^-, 2^+, 3^-, 4^+, \dots$$

Reflection symmetry
only allows even I with
positive parity π .



Nuclear shapes

Shapes are characterized by the variables $\alpha_{\lambda\mu}$ in the surface parameterization:

$$R(\theta, \varphi) = R_0 \left(1 + \sum_{\lambda} \sum_{\mu=-\lambda}^{+\lambda} \alpha_{\lambda\mu} Y_{\lambda\mu}^*(\theta, \varphi) \right)$$

$\lambda=0$: *compression (high energy)*

$\lambda=1$: *translation (not an intrinsic deformation)*

$\lambda=2$: *quadrupole deformation*

$\lambda=3$: *octupole deformation*

Quadrupole shapes

Since the surface $R(\theta, \varphi)$ is real: $(\alpha_{\lambda\mu})^* = (-1)^\mu \alpha_{\lambda-\mu}$

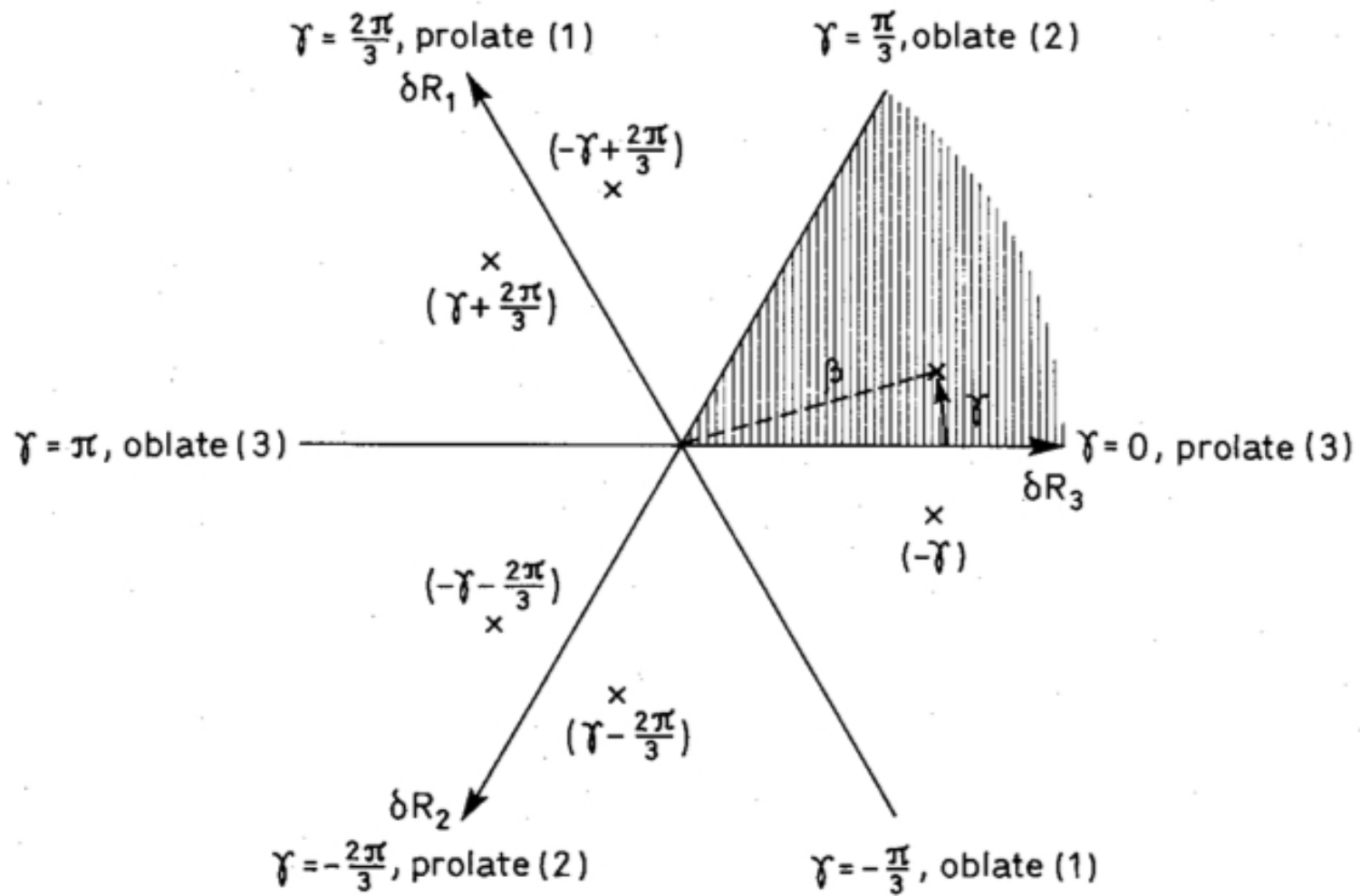
→ Five independent quadrupole variables ($\lambda=2$).

Equivalent to three Euler angles and two intrinsic variables β and γ :

$$\alpha_{2\mu} = \sum_{\nu} a_{2\nu} D_{\mu\nu}^2(\Omega), \quad a_{21} = a_{2-1} = 0, \quad a_{22} = a_{2-2}$$

$$a_{20} = \beta \cos \gamma, \quad a_{22} = \frac{1}{\sqrt{2}} \beta \sin \gamma$$

The (β, γ) plane



Modes of nuclear vibration

Nucleus is considered as a droplet of nuclear matter with an equilibrium shape. Vibrations are modes of excitation around that shape.

Character of vibrations depends on symmetry of equilibrium shape. Two important cases in nuclei:

Spherical equilibrium shape

Spheroidal equilibrium shape

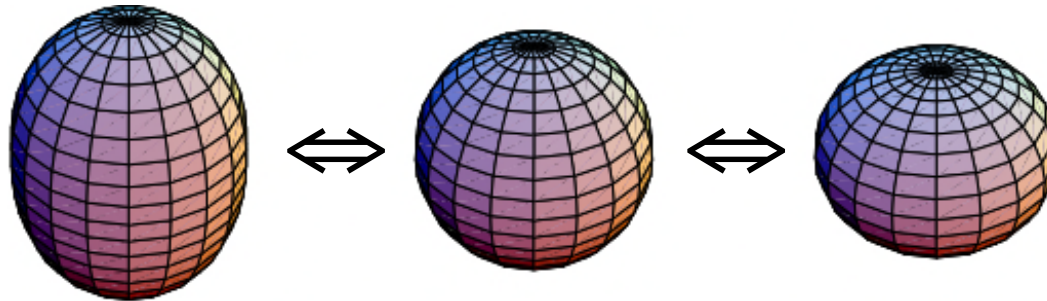
Vibrations about a spherical shape

Vibrations are characterized by λ in the surface parameterization:

$\lambda=0$: *compression (high energy)*

$\lambda=1$: *translation (not an intrinsic excitation)*

$\lambda=2$: *quadrupole vibration*



$\lambda=3$: *octupole vibration*

Spherical quadrupole vibrations

Energy spectrum:

$$E_{\text{vib}}(n) = \left(n + \frac{5}{2}\right)\hbar\omega, n = 0, 1, \dots$$

$$\begin{array}{c} 3 \\ \hline 6^+ 4^+ 3^+ 2^+ 0^+ \end{array}$$

R_{42} energy ratio:

$$E_{\text{vib}}(4^+) / E_{\text{vib}}(2^+) = 2$$

$$\begin{array}{c} 2 \\ \hline 4^+ 2^+ 0^+ \end{array}$$

E2 transitions:

$$B(\text{E}2; 2_1^+ \rightarrow 0_1^+) = \alpha^2$$

$$B(\text{E}2; 2_2^+ \rightarrow 0_1^+) = 0$$

$$\begin{array}{c} 1 \\ \hline 2^+ \end{array}$$

$$B(\text{E}2; n = 2 \rightarrow n = 1) = 2\alpha^2$$

$$\begin{array}{c} 0 \\ \hline 0^+ \end{array}$$

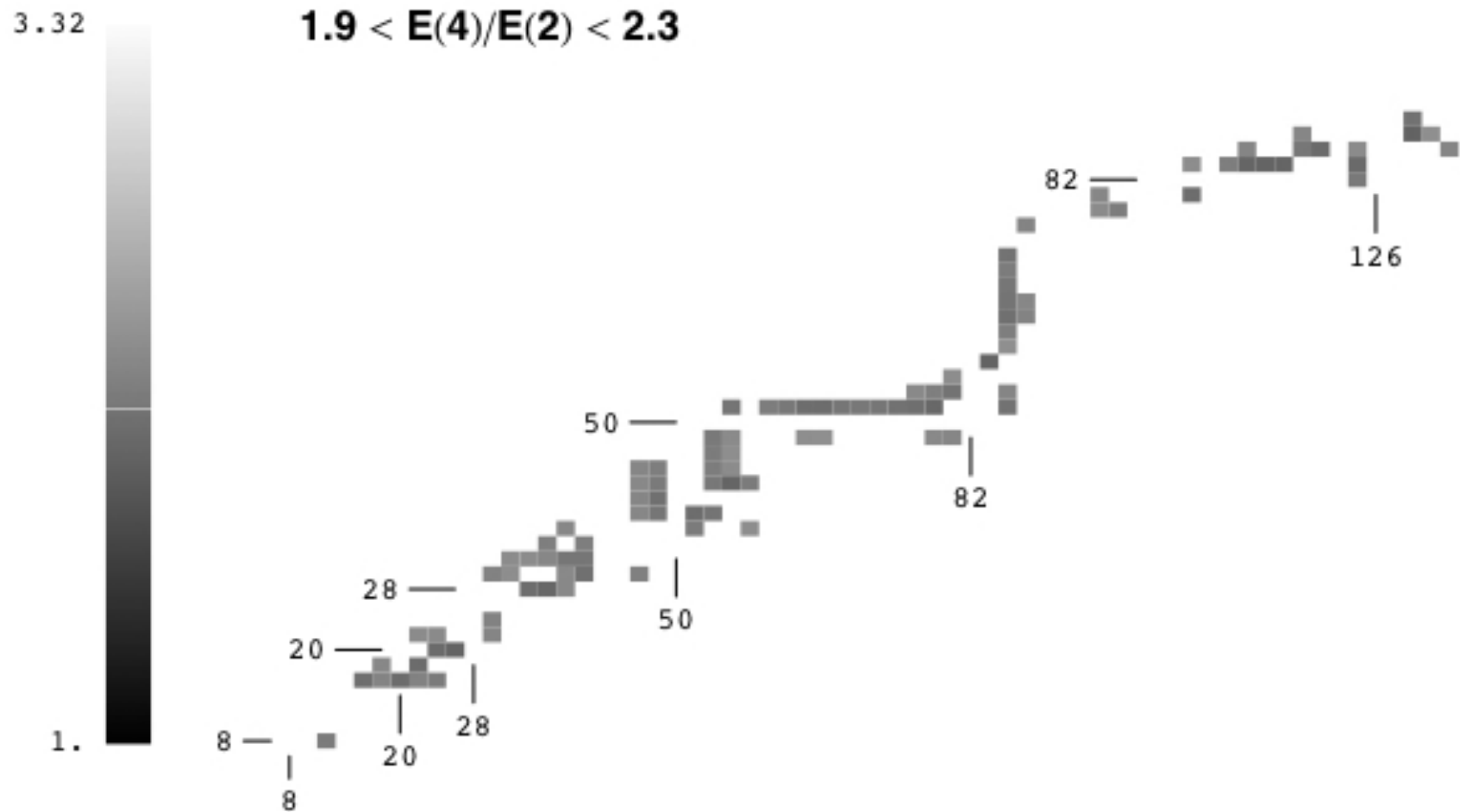
^{112}Cd

Y - BAND G - BAND I - BAND QUASI - γ BAND

I - BAND

QUASI - γ BAND

Possible vibrational nuclei from R_{42}



Spheroidal quadrupole vibrations

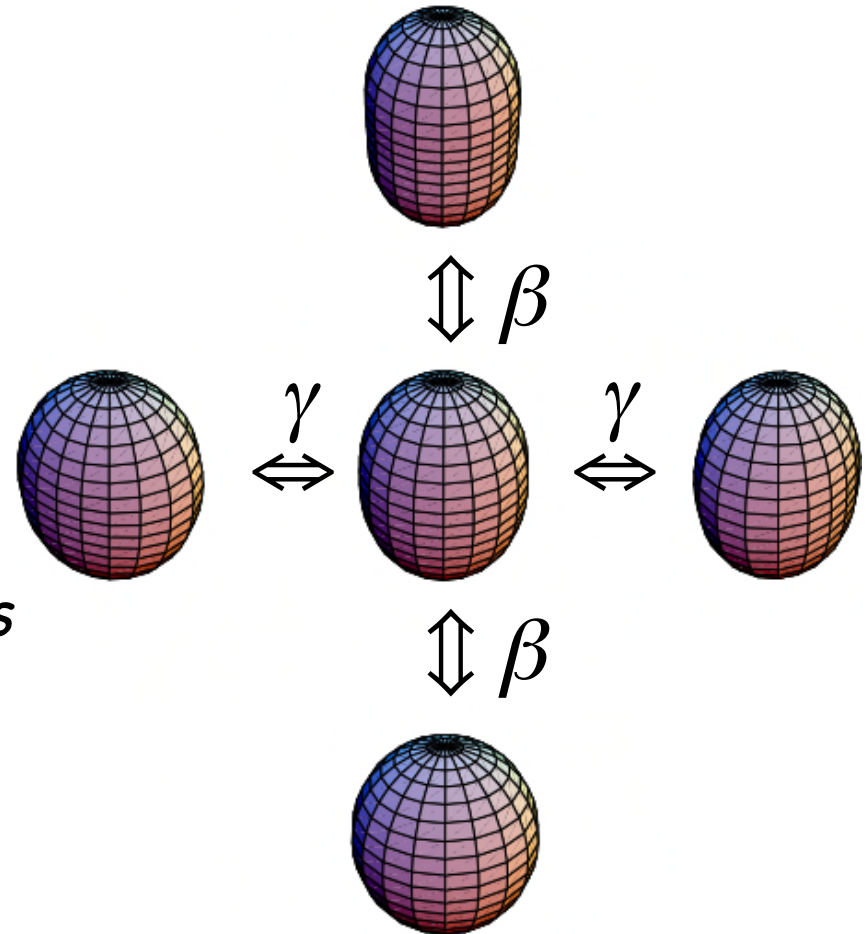
The vibration of a shape with axial symmetry is characterized by $a_{\lambda\nu}$.

Quadrupole oscillations:

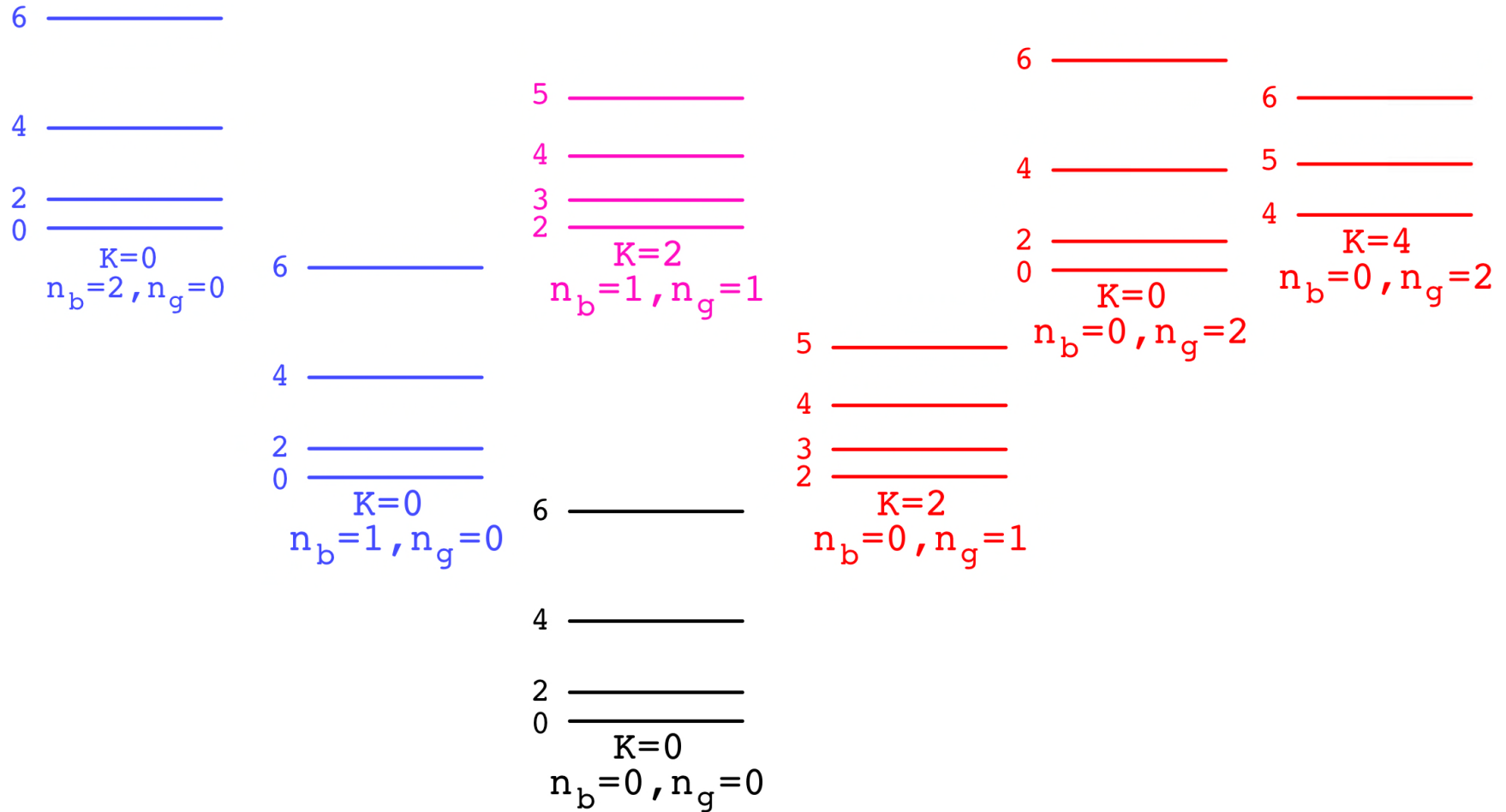
$\nu=0$: along the axis of symmetry (β)

$\nu=\pm 1$: spurious rotation

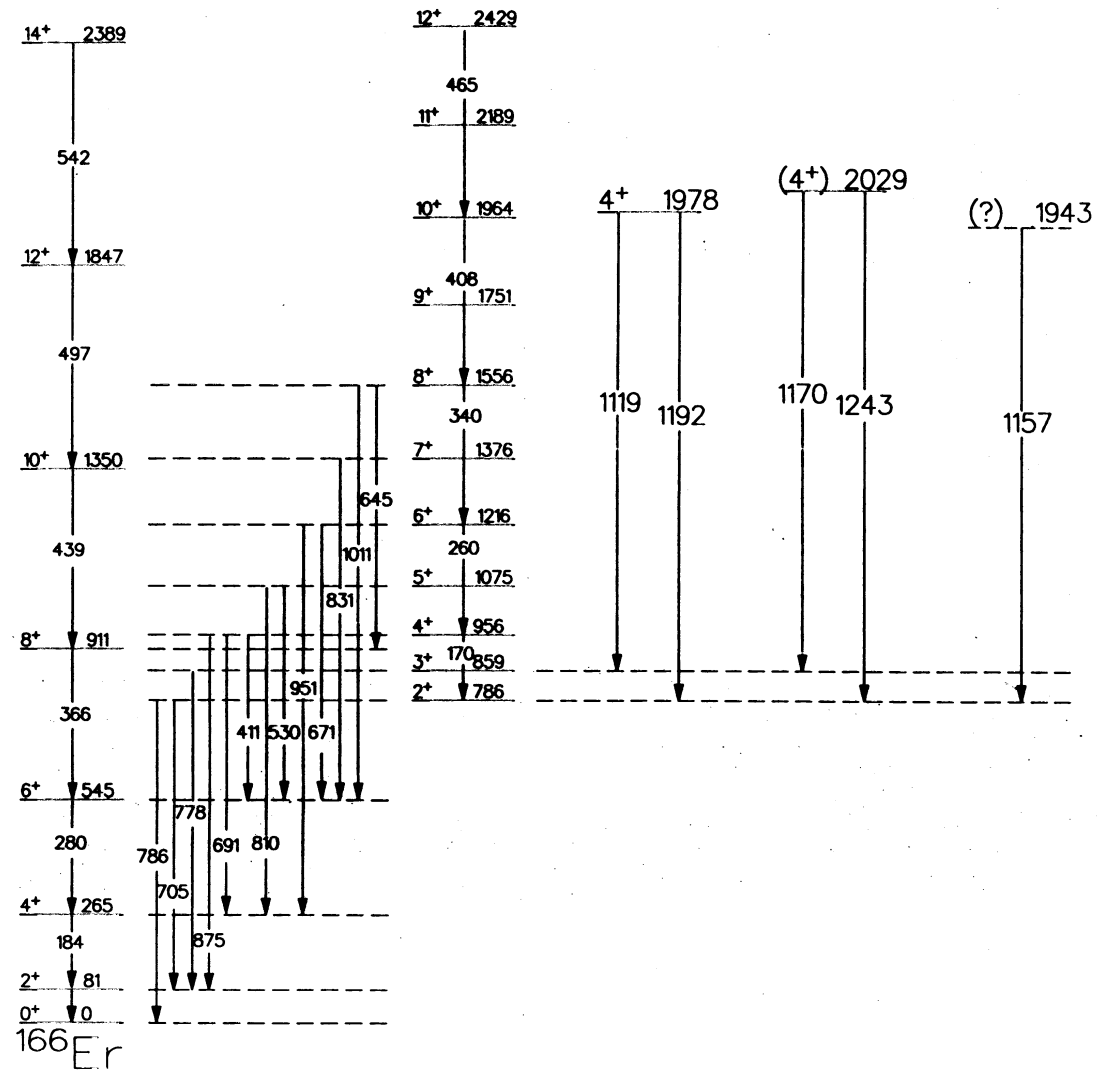
$\nu=\pm 2$: perpendicular to axis of symmetry (γ)



Spectrum of spheroidal vibrations



Example of ^{166}Er

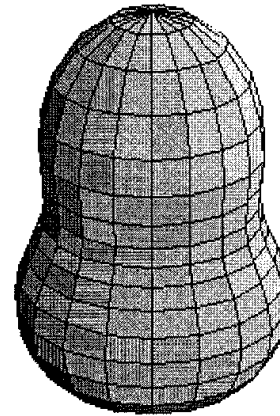


Quadrupole-octupole shapes

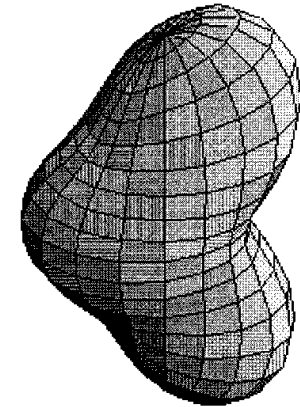
It is difficult to define an intrinsic frame for a pure octupole shape.

Quadrupole-octupole:
use quadrupole frame
→ two quadrupole and
seven octupole intrinsic
variables $\alpha_{3\mu}$.

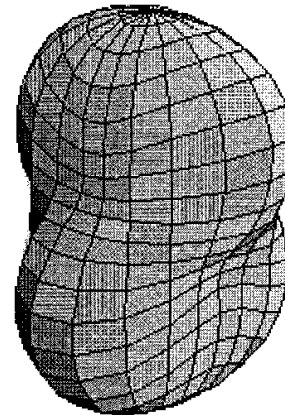
Most important case:
 $\beta_3 = \alpha_{30}$ (axial symmetry).



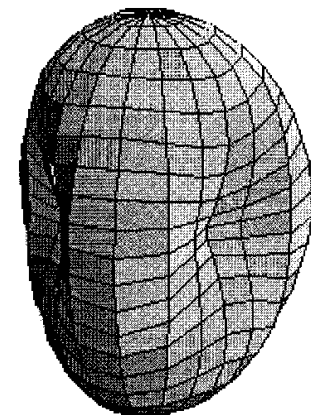
$\mu=0$



$\mu=1$

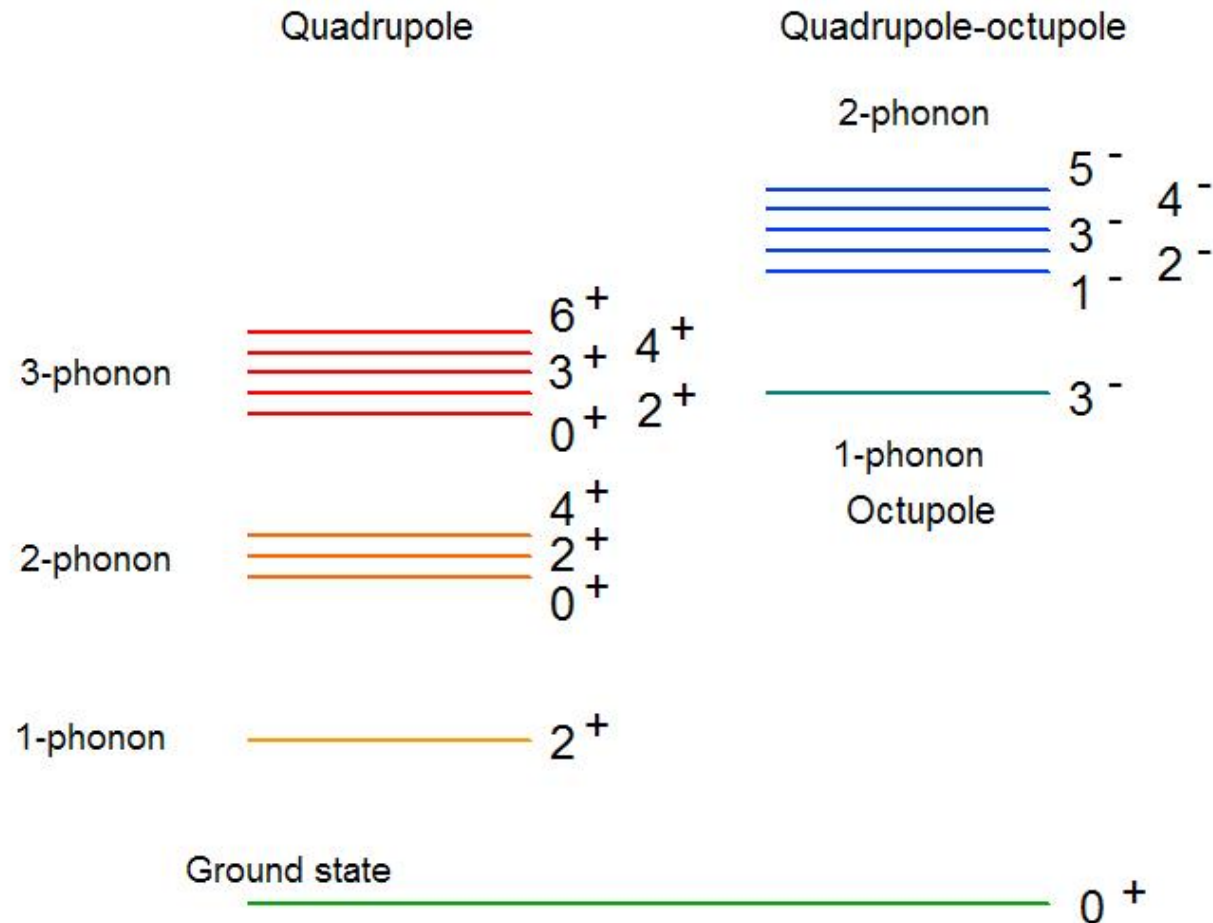


$\mu=2$

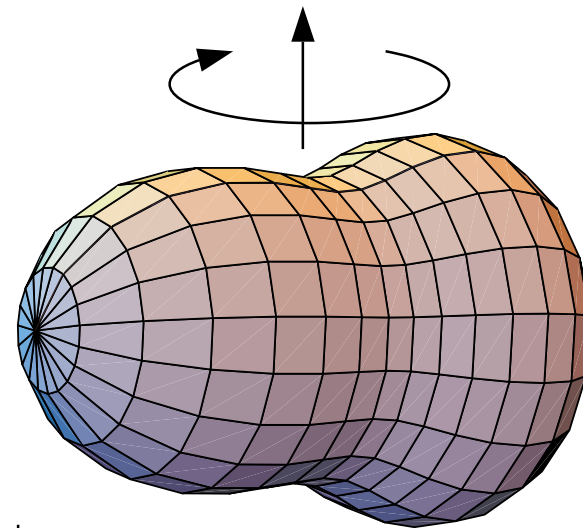
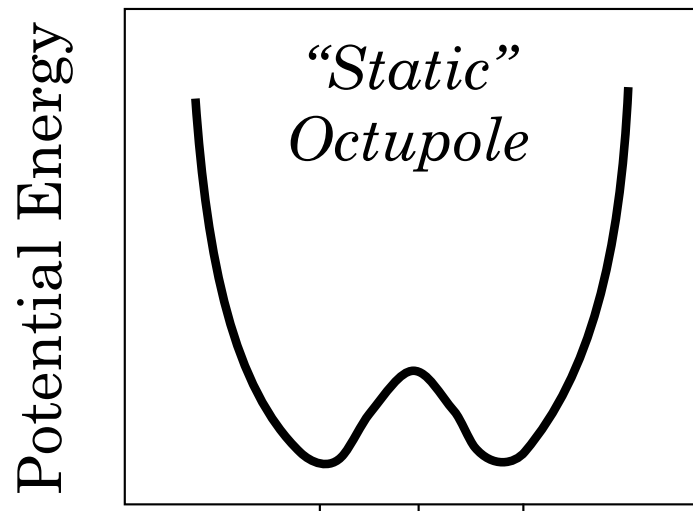
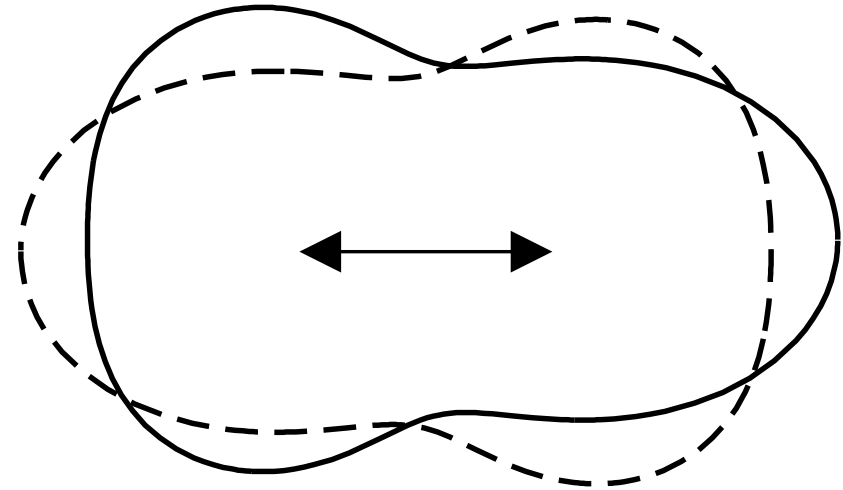
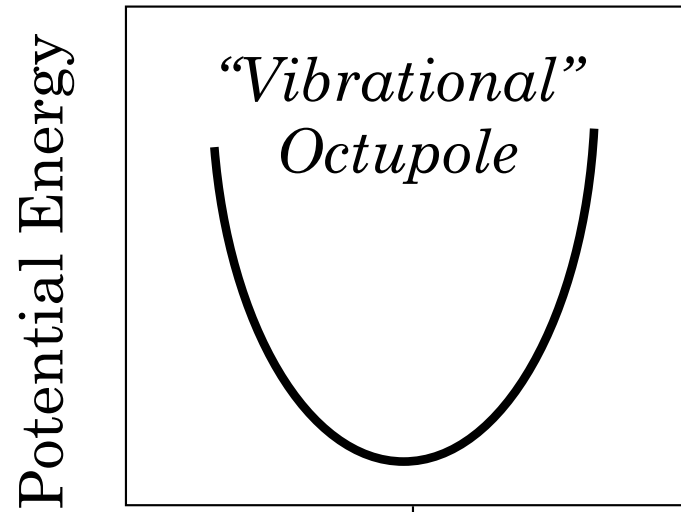


$\mu=3$

Quadrupole-octupole vibrations

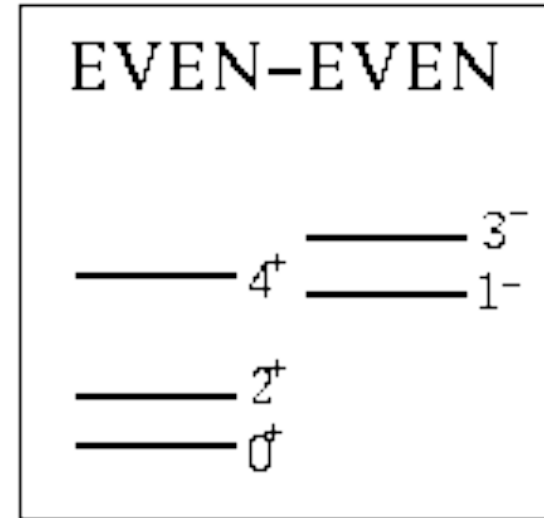
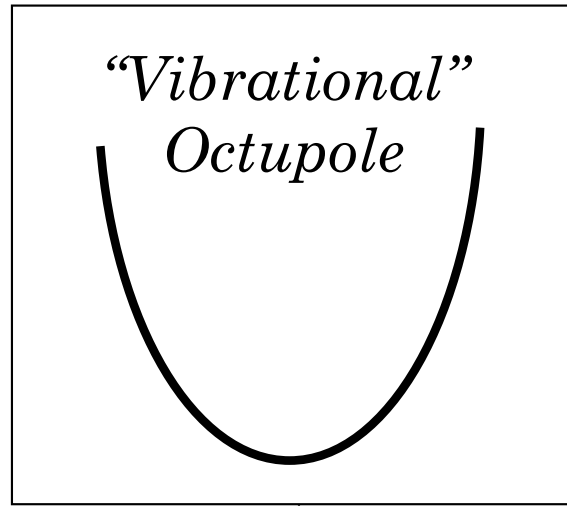


Octupole rotation-vibrations

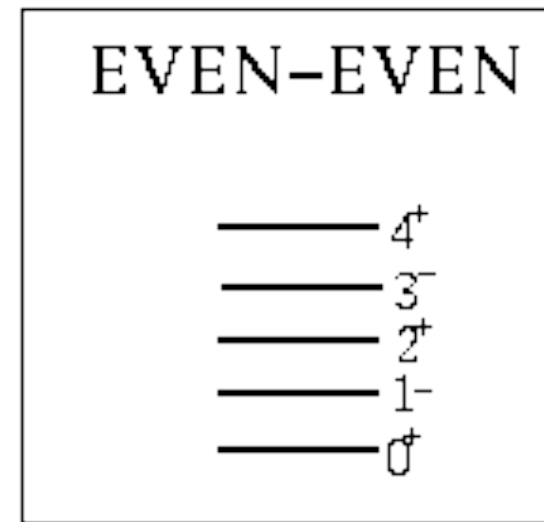
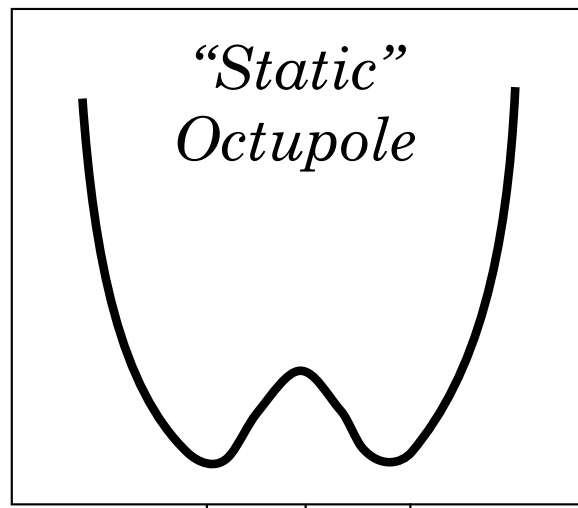


Octupole rotation-vibrations

Potential Energy

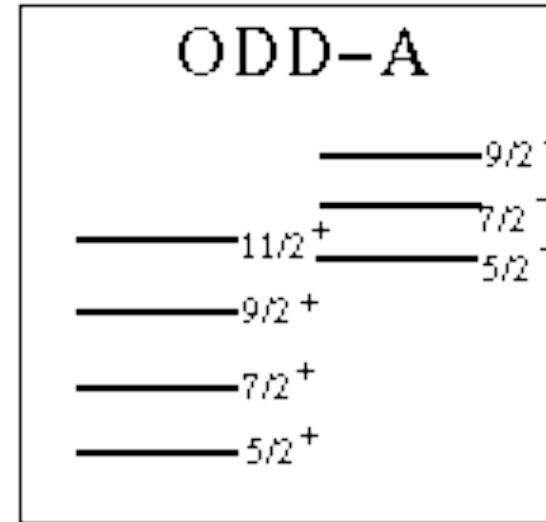
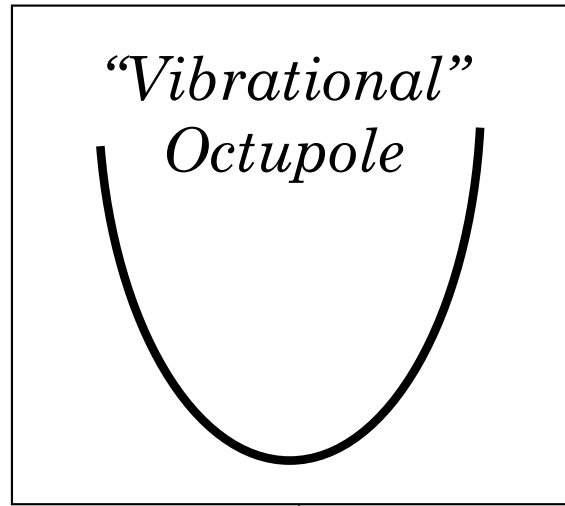


Potential Energy

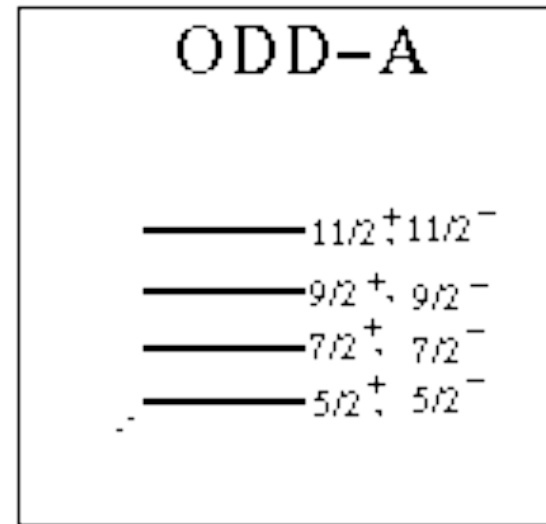
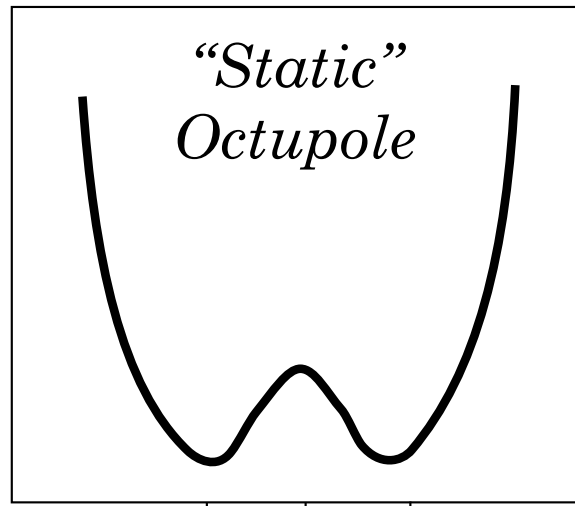


Octupole rotation-vibrations

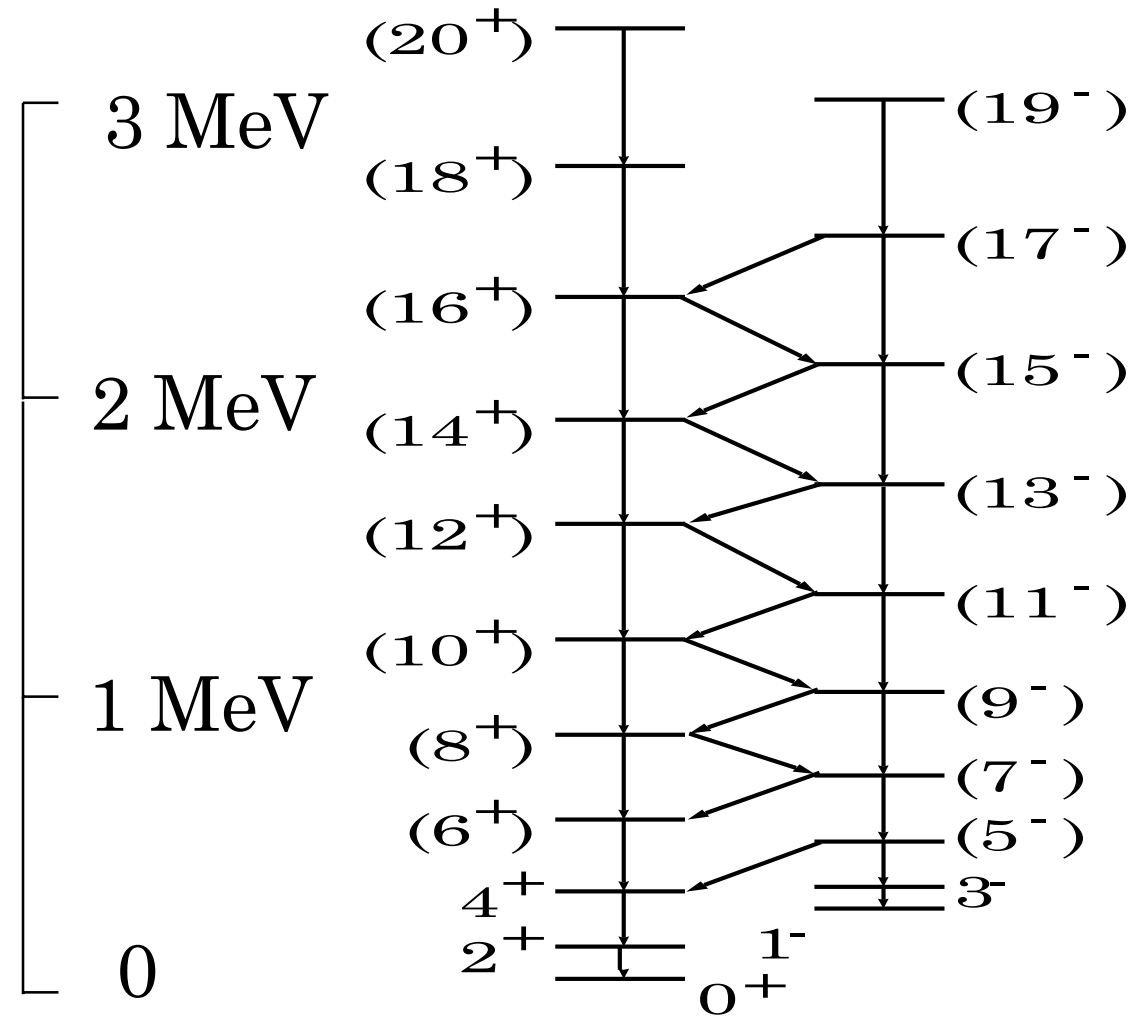
Potential Energy



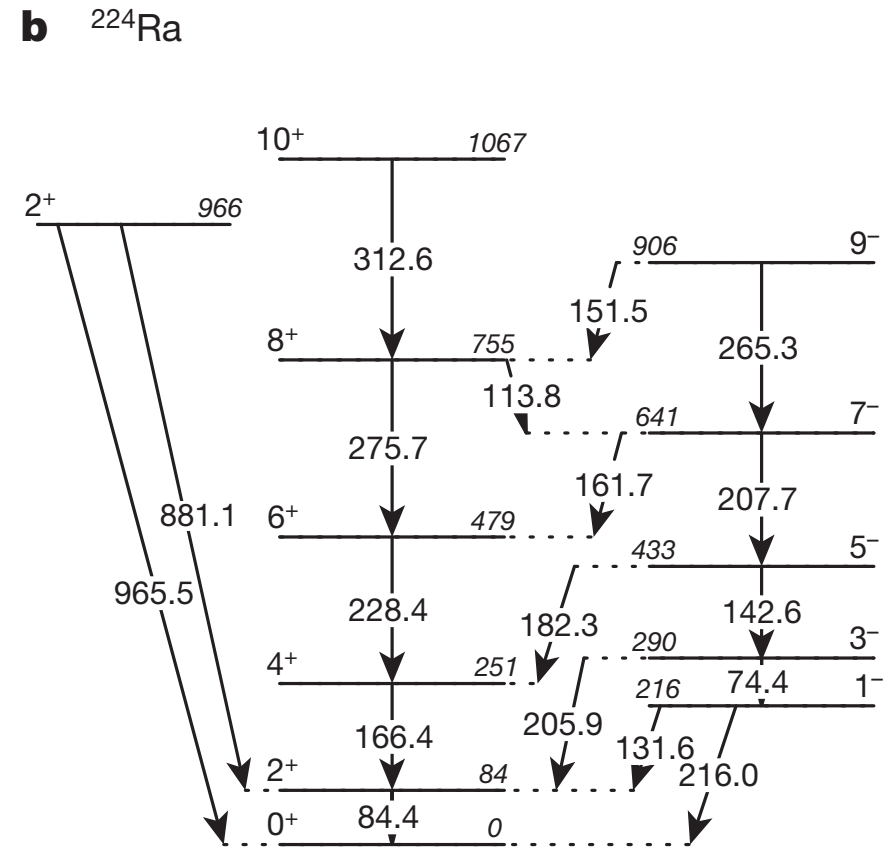
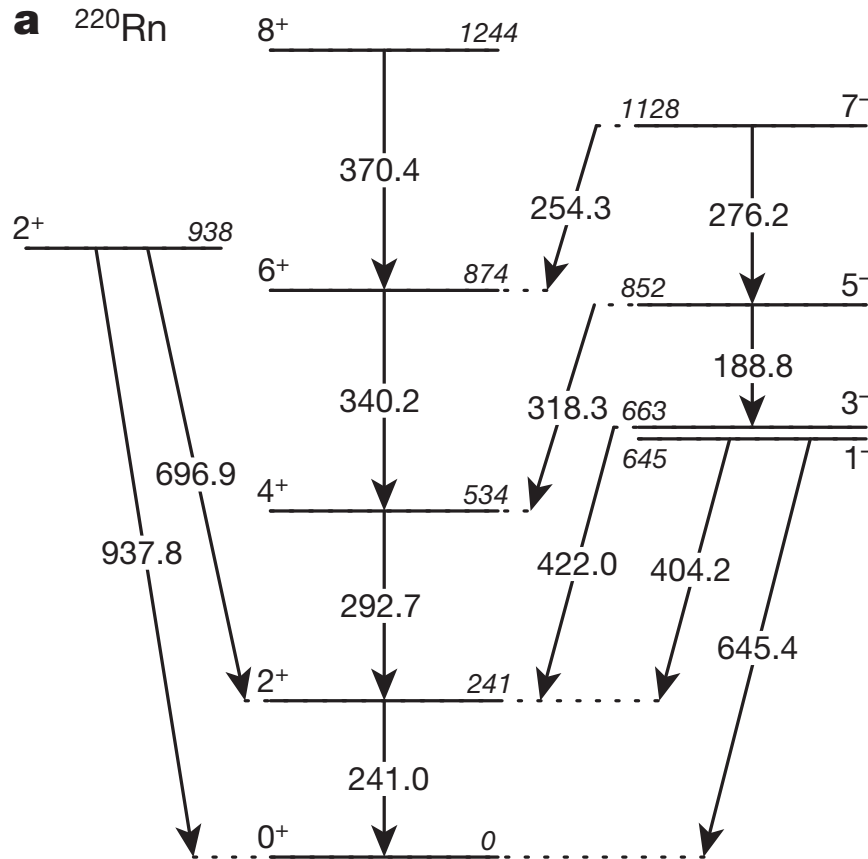
Potential Energy



Example: ^{222}Ra



Example: ^{220}Rn and ^{224}Ra



Discrete nuclear symmetries

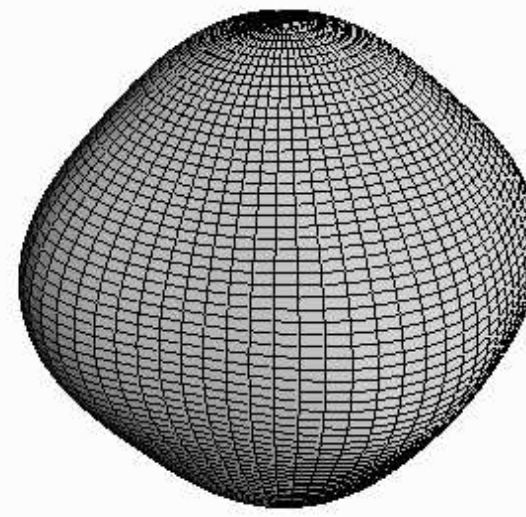
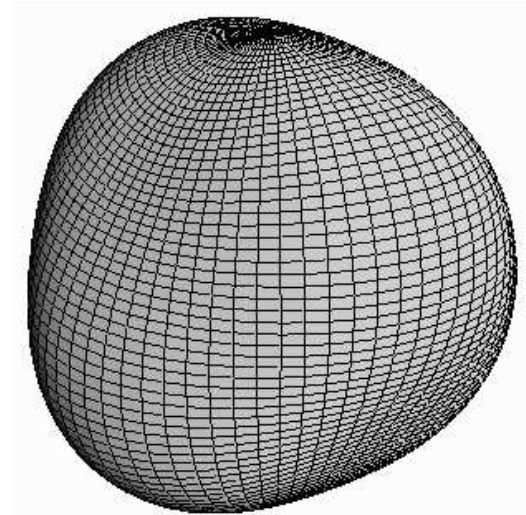
Tetrahedral symmetry:

$$\alpha_{3\pm 2} \neq 0$$

Octahedral symmetry:

$$\alpha_{40} = \sqrt{\frac{14}{5}} \alpha_{4\pm 2} \neq 0$$

Experimental evidence?



Rigid rotor model

Hamiltonian of quantum-mechanical rotor in terms of 'rotational' angular momentum \mathbf{R} :

$$\hat{H}_{\text{rot}} = \frac{\hbar^2}{2} \left[\frac{R_1^2}{\mathfrak{I}_1} + \frac{R_2^2}{\mathfrak{I}_2} + \frac{R_3^2}{\mathfrak{I}_3} \right] = \frac{\hbar^2}{2} \sum_{i=1}^3 \frac{R_i^2}{\mathfrak{I}_i}$$

Nuclei have an additional intrinsic part H_{intr} with 'intrinsic' angular momentum \mathbf{J} .

The total angular momentum is $\mathbf{I} = \mathbf{R} + \mathbf{J}$.

Rigid axially symmetric rotor

For $\mathfrak{I}_1 = \mathfrak{I}_2 = \mathfrak{I} \neq \mathfrak{I}_3$ the rotor hamiltonian is

$$\hat{H}_{\text{rot}} = \sum_{i=1}^3 \frac{\hbar^2}{2\mathfrak{I}_i} I_i^2 = \frac{\hbar^2}{2\mathfrak{I}} \sum_{i=1}^3 I_i^2 + \frac{\hbar^2}{2} \left(\frac{1}{\mathfrak{I}_3} - \frac{1}{\mathfrak{I}} \right) I_3^2$$

Eigenvalues of H_{rot} :

$$E_{KI} = \frac{\hbar^2}{2\mathfrak{I}} I(I+1) + \frac{\hbar^2}{2} \left(\frac{1}{\mathfrak{I}_3} - \frac{1}{\mathfrak{I}} \right) K^2$$

Eigenvectors $|KIM\rangle$ of H_{rot} satisfy:

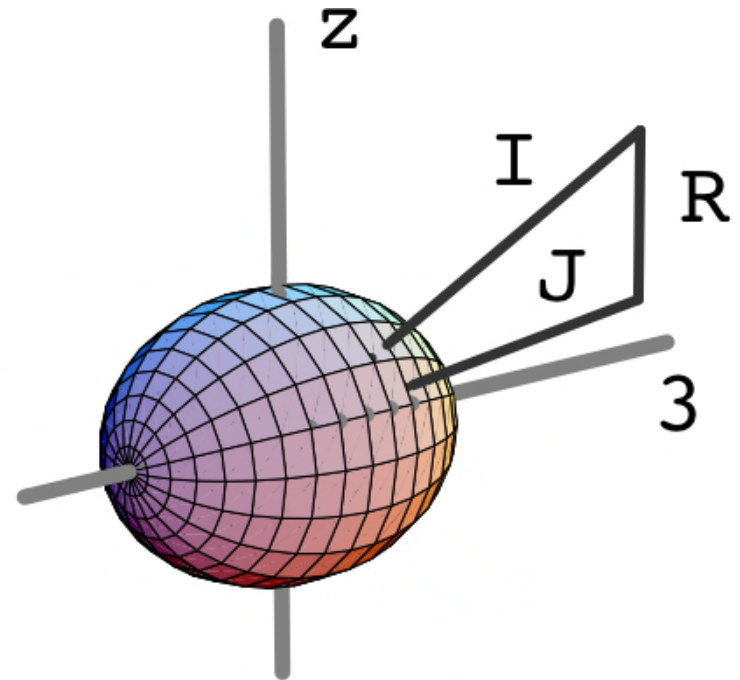
$$I^2 |KIM\rangle = I(I+1) |KIM\rangle,$$

$$I_z |KIM\rangle = M |KIM\rangle, \quad I_3 |KIM\rangle = K |KIM\rangle$$

Ground-state band of axial rotor

The ground-state spin of even-even nuclei is $I=0$. Hence $K=0$ for ground-state band:

$$E_I = \frac{\hbar^2}{2\mathfrak{I}} I(I+1)$$



E2 properties of rotational nuclei

Intra-band E2 transitions:

$$B(\text{E2}; KI_i \rightarrow KI_f) = \frac{5}{16\pi} \langle I_i K \ 20 | I_f K \rangle^2 e^2 Q_0(K)^2$$

E2 moments:

$$Q(KI) = \frac{3K^2 - I(I+1)}{(I+1)(2I+3)} Q_0(K)$$

$Q_0(K)$ is the 'intrinsic' quadrupole moment:

$$e\hat{Q}_0 \equiv \int \rho(\mathbf{r}') r'^2 (3\cos^2 \theta' - 1) d\mathbf{r}', \quad Q_0(K) = \langle K | \hat{Q}_0 | K \rangle$$

E2 properties of gs bands

For the ground state ($K=I$):

$$Q(K=I) = \frac{I(2I-1)}{(I+1)(2I+3)} Q_0(K)$$

For the gsb in even-even nuclei ($K=0$):

$$B(E2; I \rightarrow I-2) = \frac{15}{32\pi} \frac{I(I-1)}{(2I-1)(2I+1)} e^2 Q_0^2$$

$$Q(I) = -\frac{I}{2I+3} Q_0$$

$$\Rightarrow |eQ(2_1^+)| = \frac{2}{7} \sqrt{16\pi \cdot B(E2; 2_1^+ \rightarrow 0_1^+)}$$

Generalized intensity relations

Mixing of K arises from

Dependence of Q_0 on I (stretching)

Coriolis interaction

Triaxiality

Generalized *intra-* and *inter-*band matrix elements
(eg E2):

$$\frac{\sqrt{B(E2; K_i I_i \rightarrow K_f I_f)}}{\left| \langle I_i K_i \ 2K_f - K_i | I_f K_f \rangle \right|} = M_0 + M_1 \Delta + M_2 \Delta^2 + \dots$$

$$\text{with } \Delta = I_f(I_f + 1) - I_i(I_i + 1)$$

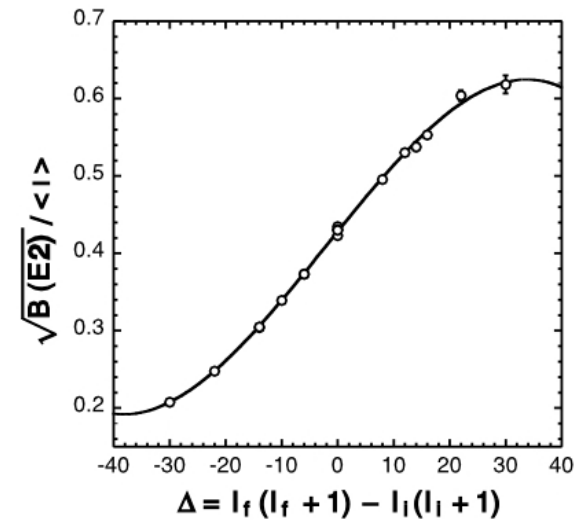
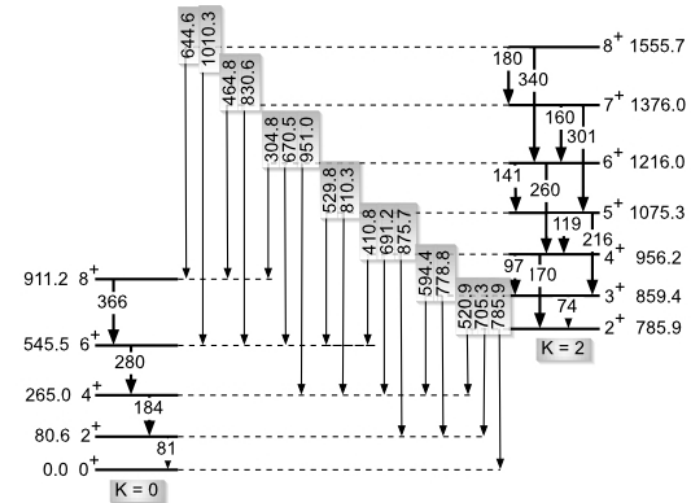
Inter-band E2 transitions

Example of $\gamma \rightarrow g$ transitions in ^{166}Er :

$$\frac{\sqrt{B(E2; I_\gamma \rightarrow I_g)}}{\left| \langle I_\gamma 2 2 - 2 | I_g 0 \rangle \right|}$$

$$= M_0 + M_1 \Delta + M_2 \Delta^2 + \dots$$

$$\Delta = I_g(I_g + 1) - I_\gamma(I_\gamma + 1)$$



Rigid triaxial rotor

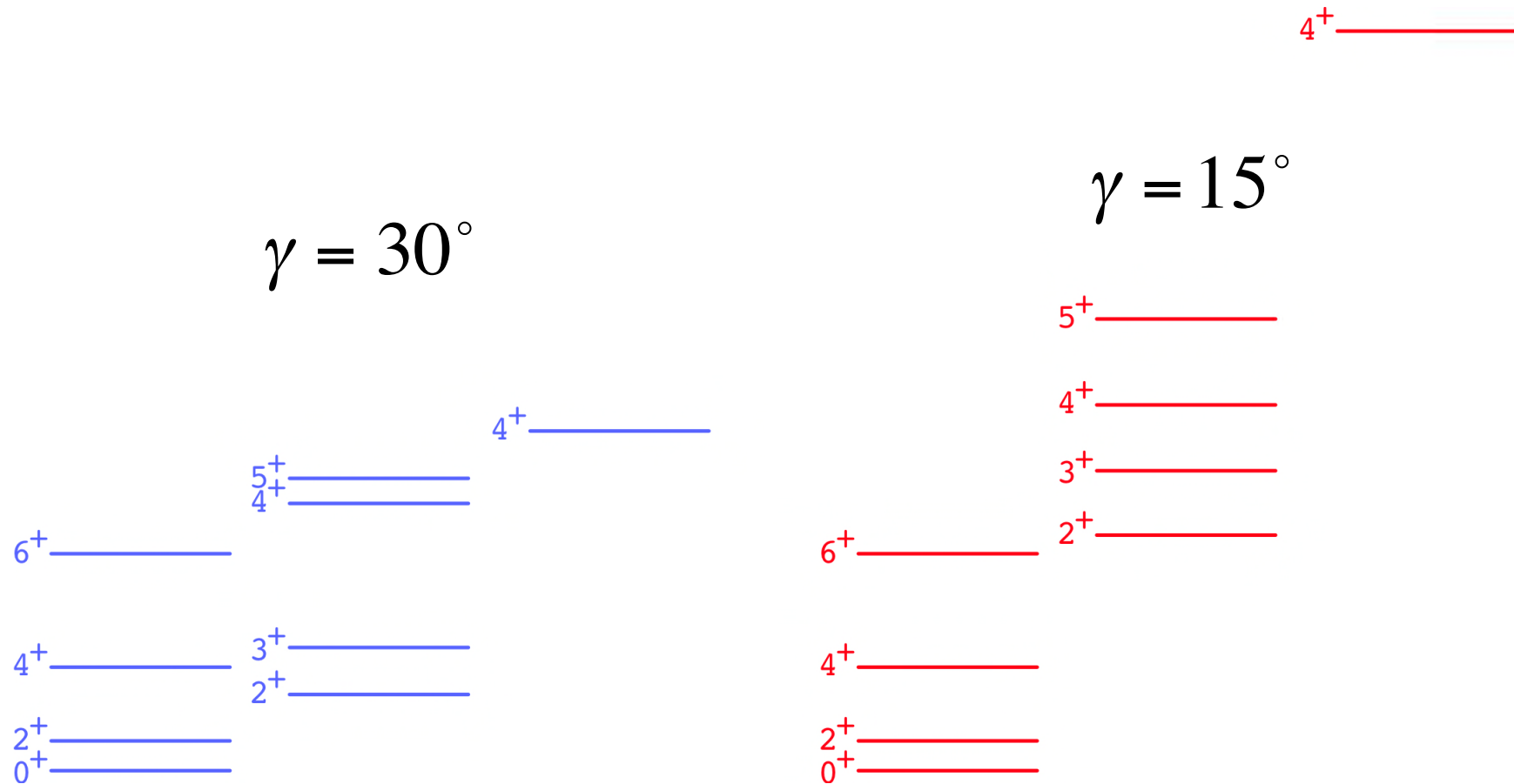
Triaxial rotor hamiltonian $\mathfrak{I}_1 \neq \mathfrak{I}_2 \neq \mathfrak{I}_3$:

$$\hat{H}'_{\text{rot}} = \sum_{i=1}^3 \frac{\hbar^2}{2\mathfrak{I}_i} I_i^2 = \underbrace{\frac{\hbar^2}{2\mathfrak{I}} I^2 + \frac{\hbar^2}{2\mathfrak{I}_f} I_3^2}_{\hat{H}'_{\text{axial}}} + \underbrace{\frac{\hbar^2}{2\mathfrak{I}_g} (I_+^2 + I_-^2)}_{\hat{H}'_{\text{mix}}}$$

$$\frac{1}{\mathfrak{I}} = \frac{1}{2} \left(\frac{1}{\mathfrak{I}_1} + \frac{1}{\mathfrak{I}_2} \right), \quad \frac{1}{\mathfrak{I}_f} = \frac{1}{\mathfrak{I}_3} - \frac{1}{\mathfrak{I}}, \quad \frac{1}{\mathfrak{I}_g} = \frac{1}{4} \left(\frac{1}{\mathfrak{I}_1} - \frac{1}{\mathfrak{I}_2} \right)$$

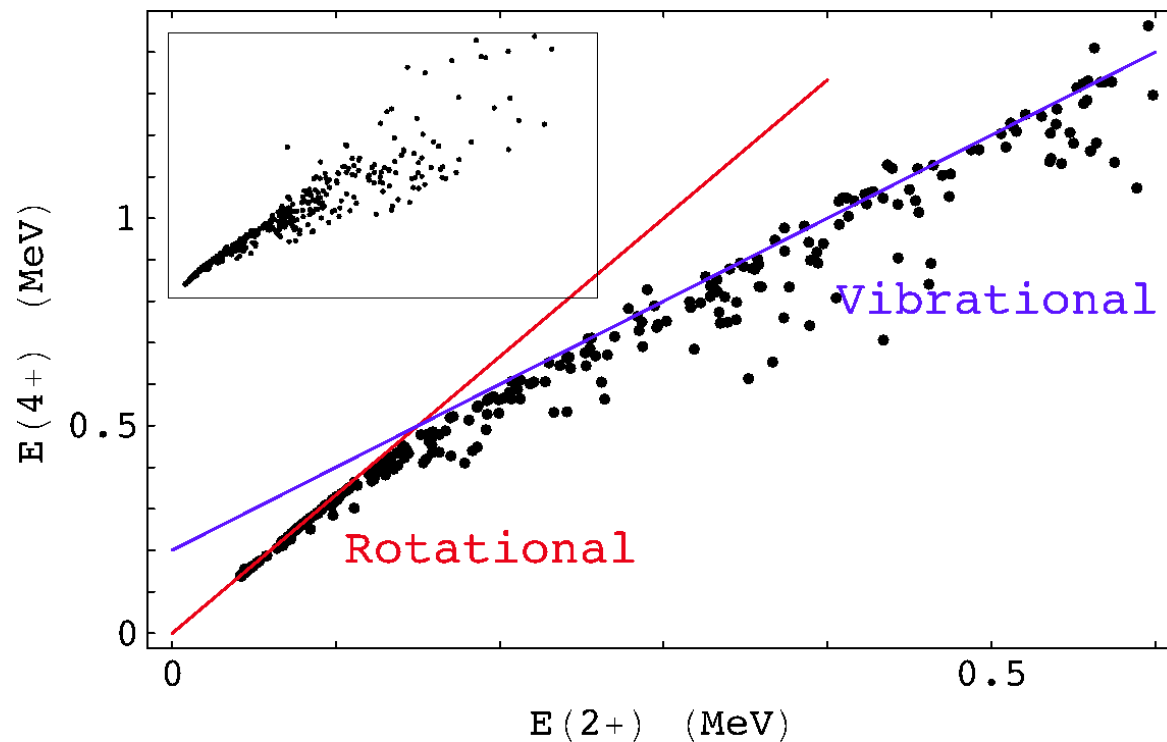
H'_{mix} non-diagonal in axial basis $|KIM\rangle \Rightarrow K$ is *not* a conserved quantum number

Rigid triaxial rotor spectra



Tri-partite classification of nuclei

Empirical evidence for seniority-type, vibrational- and rotational-like nuclei.



Interacting boson model

Describe the nucleus as a system of N interacting s and d bosons. Hamiltonian:

$$\hat{H}_{\text{IBM}} = \sum_{i=1}^6 \varepsilon_i \hat{b}_i^+ \hat{b}_i + \sum_{i_1 i_2 i_3 i_4=1}^6 v_{i_1 i_2 i_3 i_4} \hat{b}_{i_1}^+ \hat{b}_{i_2}^+ \hat{b}_{i_3} \hat{b}_{i_4}$$

Justification from

Shell model: s and d bosons are associated with S and D fermion (Cooper) pairs.

Geometric model: for large boson number the IBM reduces to a liquid-drop hamiltonian.

Dimensions

Assume Ω available 1-fermion states. Number of n -fermion states is $\binom{\Omega}{n} = \frac{\Omega!}{n!(\Omega - n)!}$

Assume Ω available 1-boson states. Number of n -boson states is $\binom{\Omega + n - 1}{n} = \frac{(\Omega + n - 1)!}{n!(\Omega - 1)!}$

Example: $^{162}\text{Dy}_{96}$ with 14 neutrons ($\Omega=44$) and 16 protons ($\Omega=32$) ($^{132}\text{Sn}_{82}$ inert core).

SM dimension: $7 \cdot 10^{19}$

IBM dimension: 15504

Dynamical symmetries

Boson hamiltonian is of the form

$$\hat{H}_{\text{IBM}} = \sum_{i=1}^6 \varepsilon_i \hat{b}_i^+ \hat{b}_i + \sum_{i_1 i_2 i_3 i_4 = 1}^6 v_{i_1 i_2 i_3 i_4} \hat{b}_{i_1}^+ \hat{b}_{i_2}^+ \hat{b}_{i_3} \hat{b}_{i_4}$$

In general not solvable analytically.

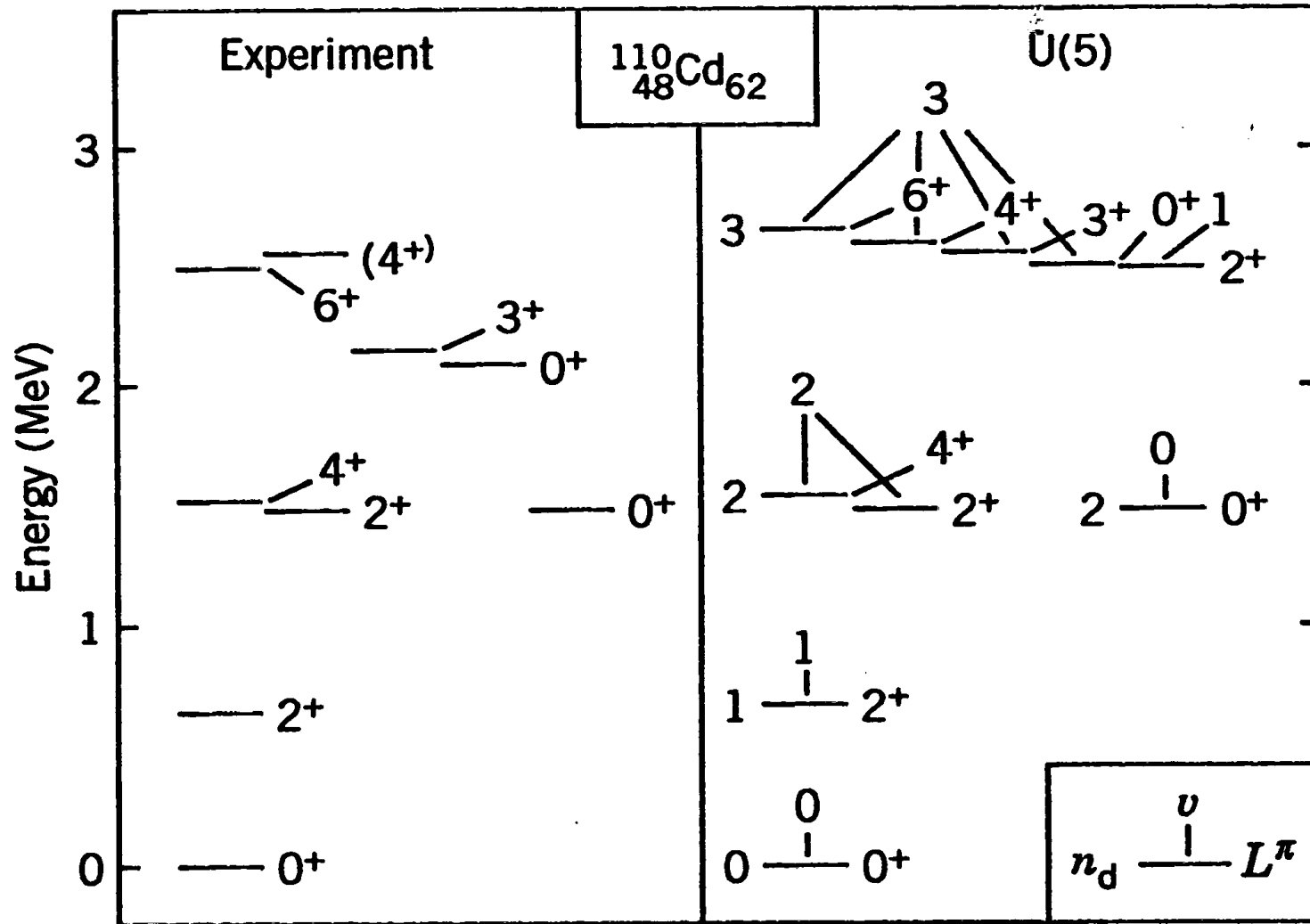
Three solvable cases with $SO(3)$ symmetry:

$$U(6) \supset U(5) \supset SO(5) \supset SO(3)$$

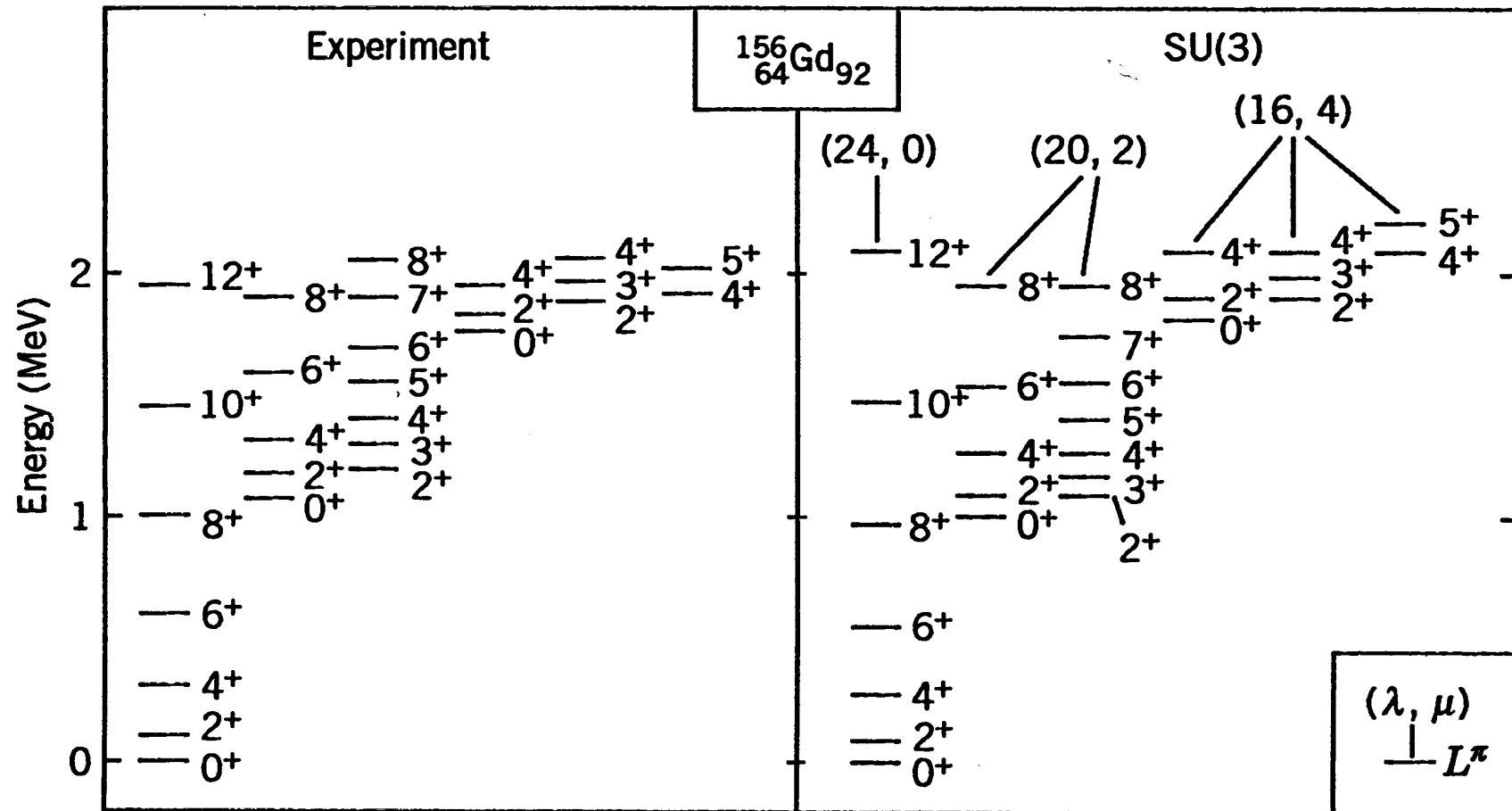
$$U(6) \supset SU(3) \supset SO(3)$$

$$U(6) \supset SO(6) \supset SO(5) \supset SO(3)$$

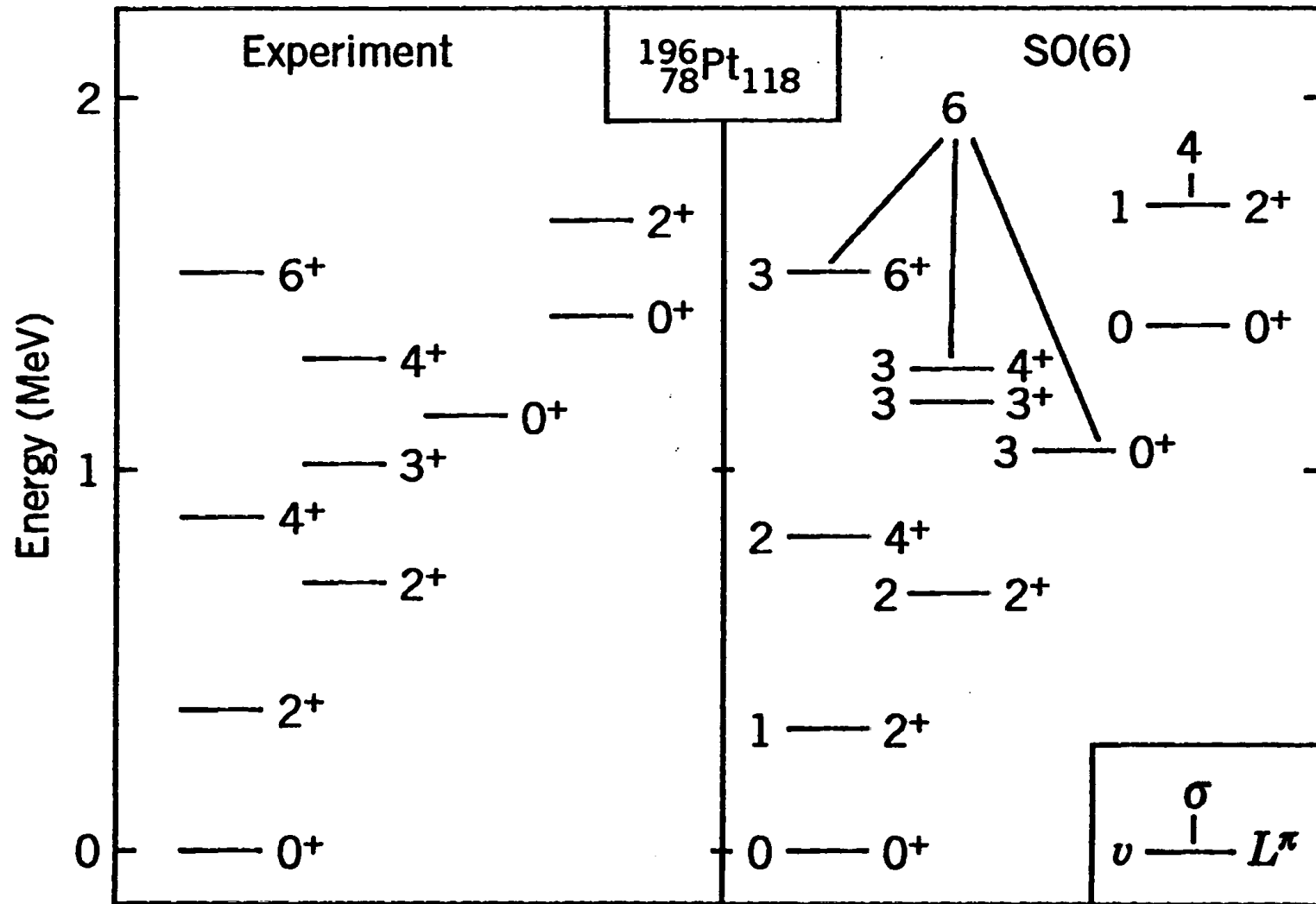
U(5) vibrational limit: $^{110}\text{Cd}_{62}$



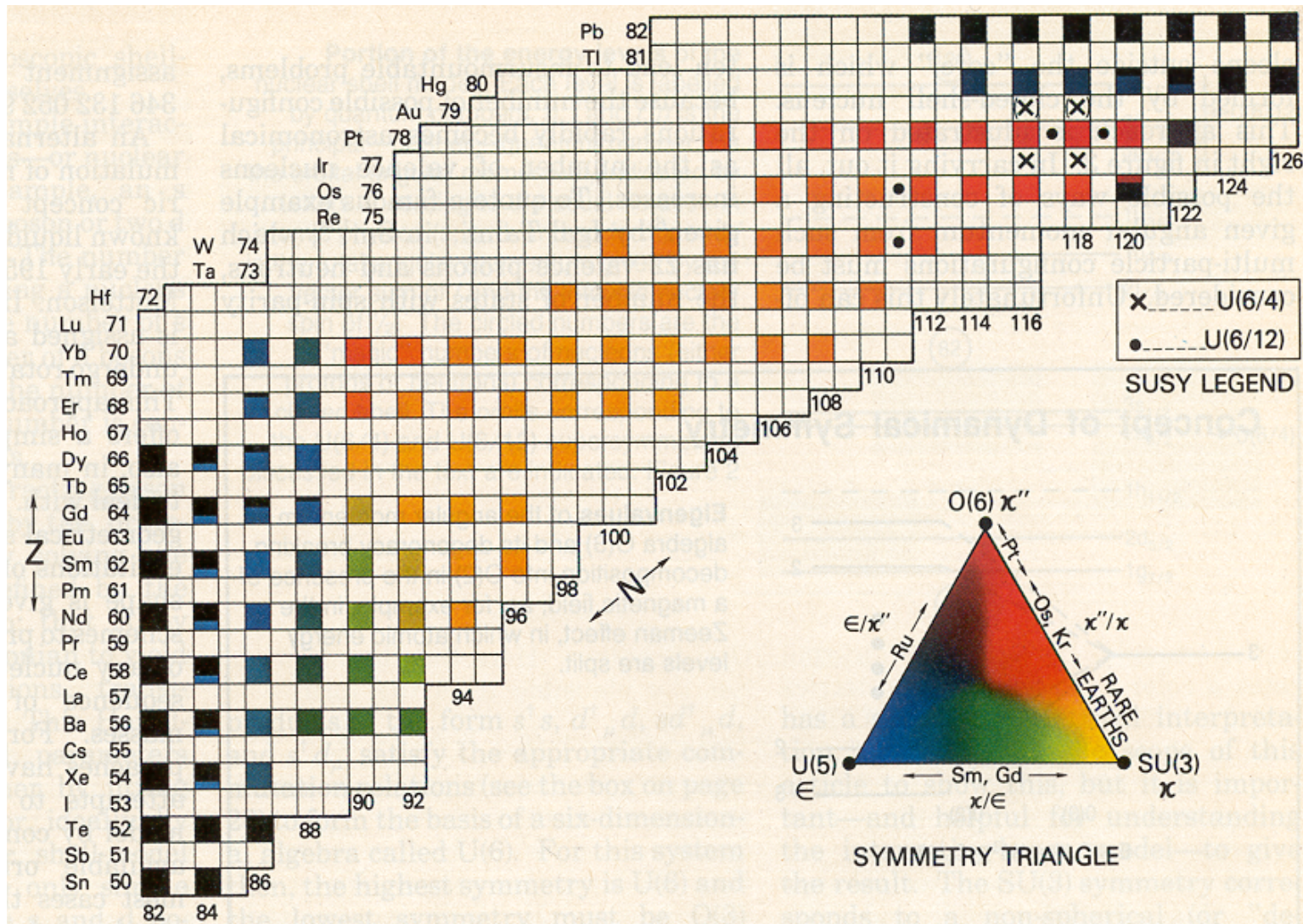
SU(3) rotational limit: $^{156}\text{Gd}_{92}$



SO(6) γ -unstable limit: $^{196}\text{Pt}_{118}$



Applications of IBM



Classical limit of IBM

For large boson number N , a *coherent* (or *intrinsic*) state is an approximate eigenstate,

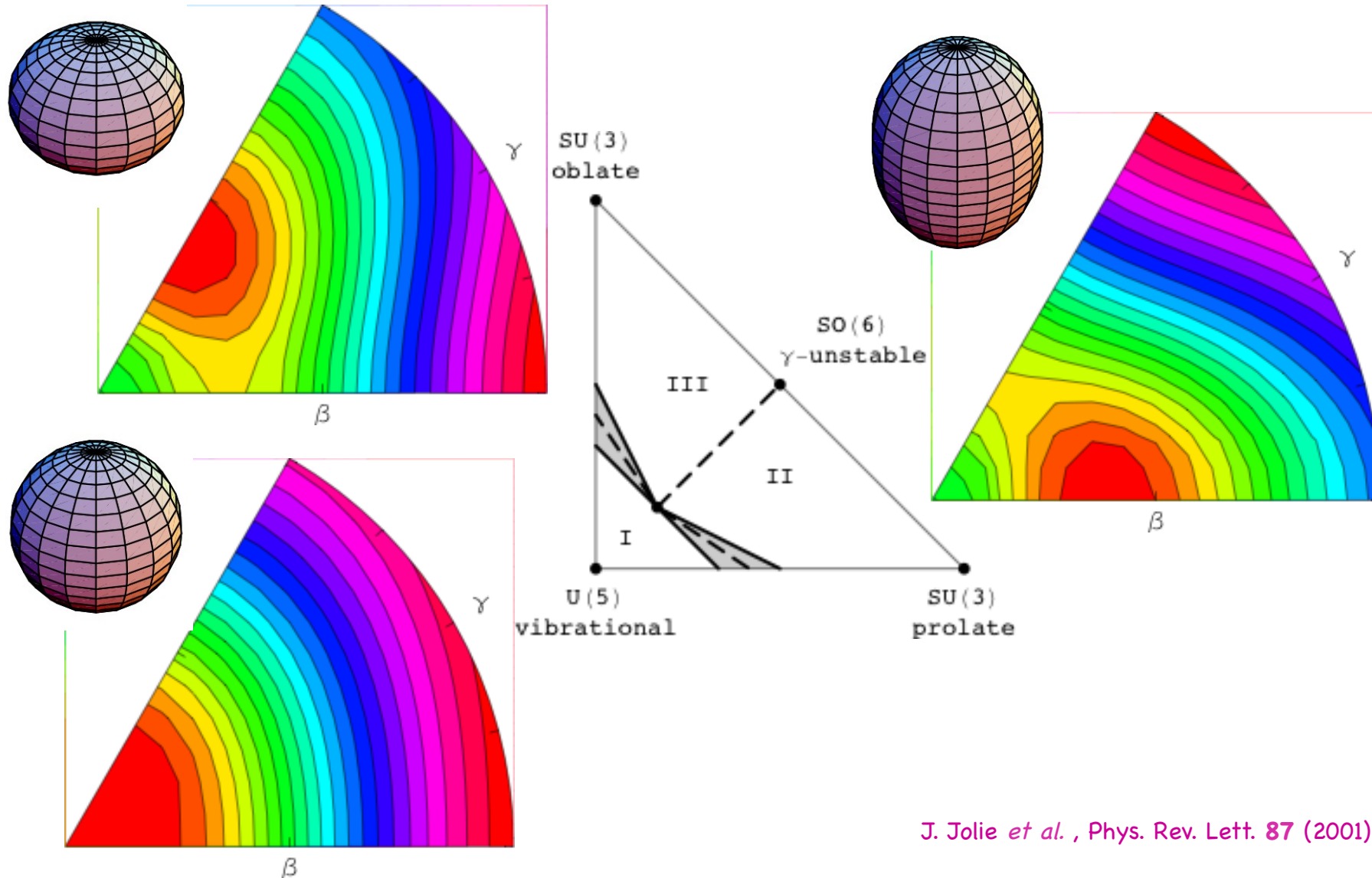
$$\hat{H}_{\text{IBM}}|N;\alpha_\mu\rangle \approx E|N;\alpha_\mu\rangle, \quad |N;\alpha_\mu\rangle \propto \left(s^+ + \sum_\mu \alpha_\mu d_\mu^+\right)^N |0\rangle$$

The real parameters α_μ are related to the three Euler angles and shape variables β and γ .

Any IBM hamiltonian yields energy surface:

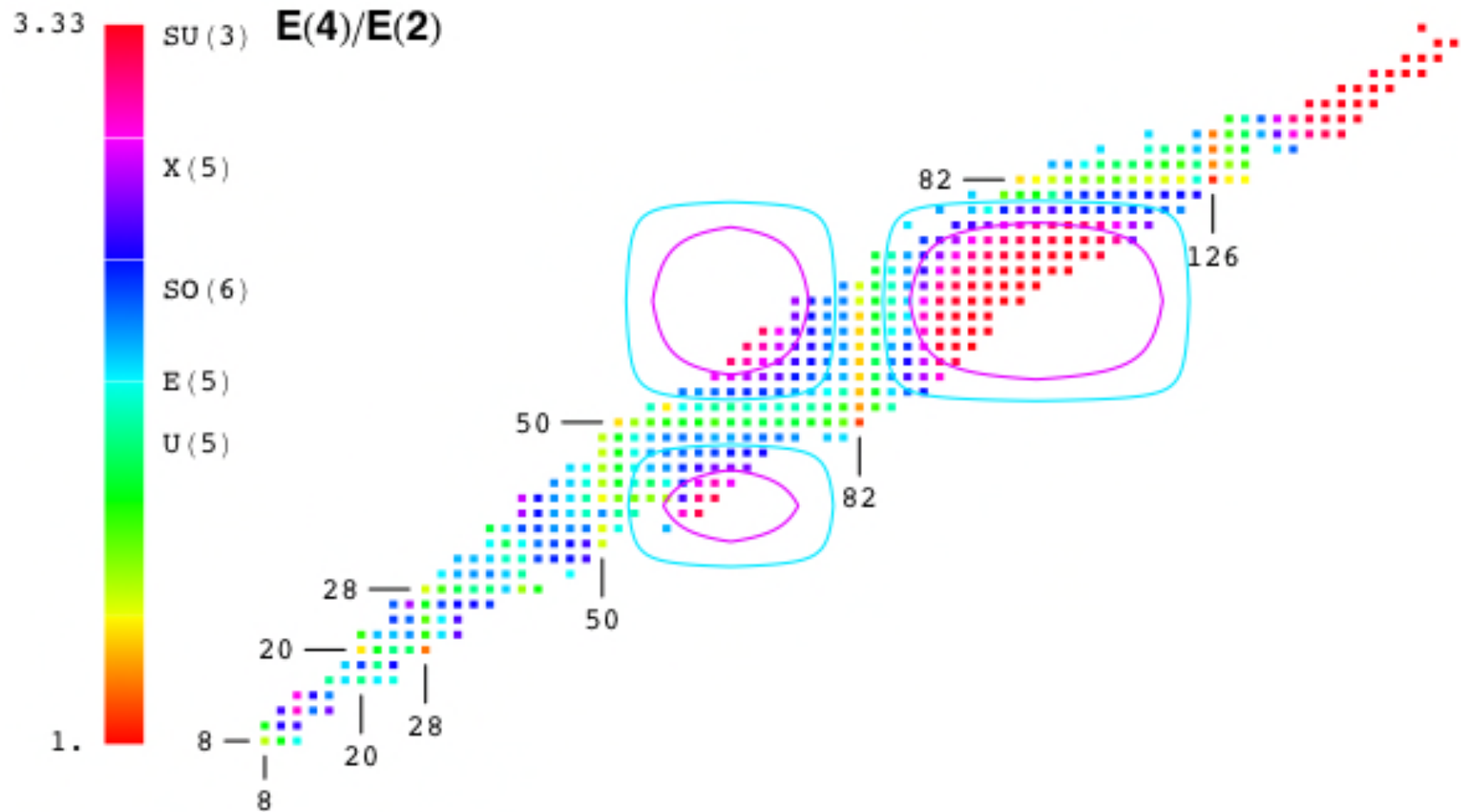
$$\langle N;\alpha_\mu | \hat{H}_{\text{IBM}} | N;\alpha_\mu \rangle = \langle N;\beta\gamma | \hat{H}_{\text{IBM}} | N;\beta\gamma \rangle \equiv V(\beta,\gamma)$$

Phase diagram of IBM



J. Jolie *et al.*, Phys. Rev. Lett. **87** (2001) 162501

The ratio R_{42}



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Electric (quadrupole) properties

Partial γ -ray half-life:

$$T_{1/2}^{\gamma}(\text{E}\lambda) = \ln 2 \left\{ \frac{8\pi}{\hbar} \frac{\lambda + 1}{\lambda [(2\lambda + 1)!!]^2} \left(\frac{E_{\gamma}}{\hbar c} \right)^{2\lambda + 1} B(\text{E}\lambda) \right\}^{-1}$$

Electric quadrupole transitions:

$$B(\text{E}2; I_i \rightarrow I_f) = \frac{1}{2I_i + 1} \sum_{M_i} \sum_{M_f \mu} \left| \langle I_f M_f | \sum_{k=1}^A e_k r_k^2 Y_{2\mu}(\theta_k, \varphi_k) | I_i M_i \rangle \right|^2$$

Electric quadrupole moments:

$$eQ(I) = \langle IM = I | \sqrt{\frac{16\pi}{5}} \sum_{k=1}^A e_k r_k^2 Y_{20}(\theta_k, \varphi_k) | IM = I \rangle$$

Magnetic (dipole) properties

Partial γ -ray half-life:

$$T_{1/2}^{\gamma}(\text{M}\lambda) = \ln 2 \left\{ \frac{8\pi}{\hbar} \frac{\lambda + 1}{\lambda [(2\lambda + 1)!!]^2} \left(\frac{E_{\gamma}}{\hbar c} \right)^{2\lambda + 1} B(\text{M}\lambda) \right\}^{-1}$$

Magnetic dipole transitions:

$$B(\text{M}1; I_i \rightarrow I_f) = \frac{1}{2I_i + 1} \sum_{M_i} \sum_{M_f \mu} \left| \langle I_f M_f | \sum_{k=1}^A (g_k^l l_{k,\mu} + g_k^s s_{k,\mu}) | I_i M_i \rangle \right|^2$$

Magnetic dipole moments:

$$\mu(I) = \langle IM = I | \sum_{k=1}^A (g_k^l l_{k,z} + g_k^s s_{k,z}) | IM = I \rangle$$

Extensions of IBM

Neutron and proton degrees freedom (IBM-2):

F-spin multiplets ($N_v + N_\pi = \text{constant}$)

Scissors excitations

Fermion degrees of freedom (IBFM):

Odd-mass nuclei

Supersymmetry (doublets & quartets)

Other boson degrees of freedom:

Isospin $T=0$ & $T=1$ pairs (IBM-3 & IBM-4)

Higher multipole (g,...) pairs

Scissors mode

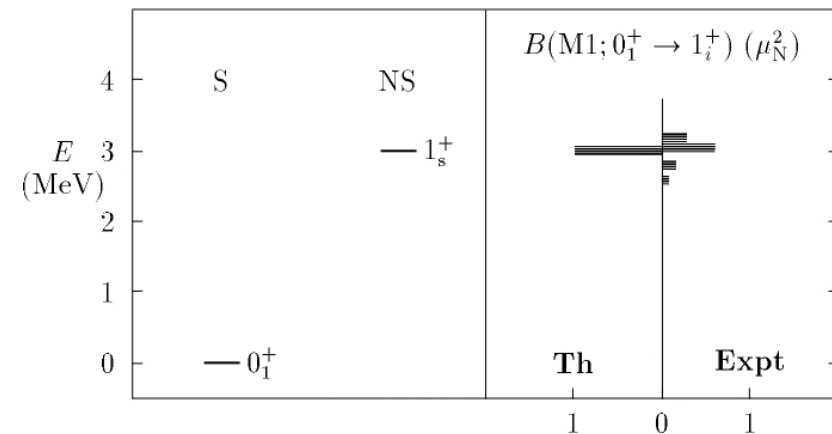
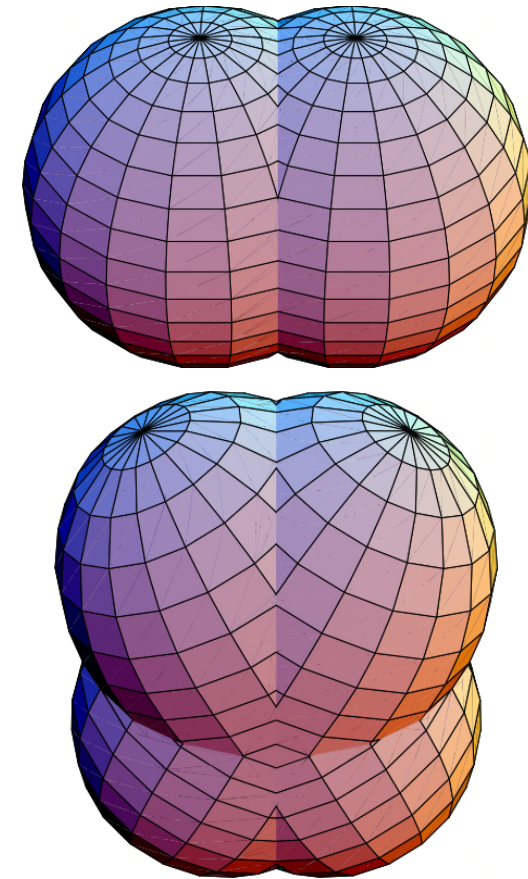
Collective displacement
modes between neutrons
and protons:

*Linear displacement
(giant dipole resonance):*

$R_v - R_\pi \Rightarrow E1$ excitation.

*Angular displacement
(scissors resonance):*

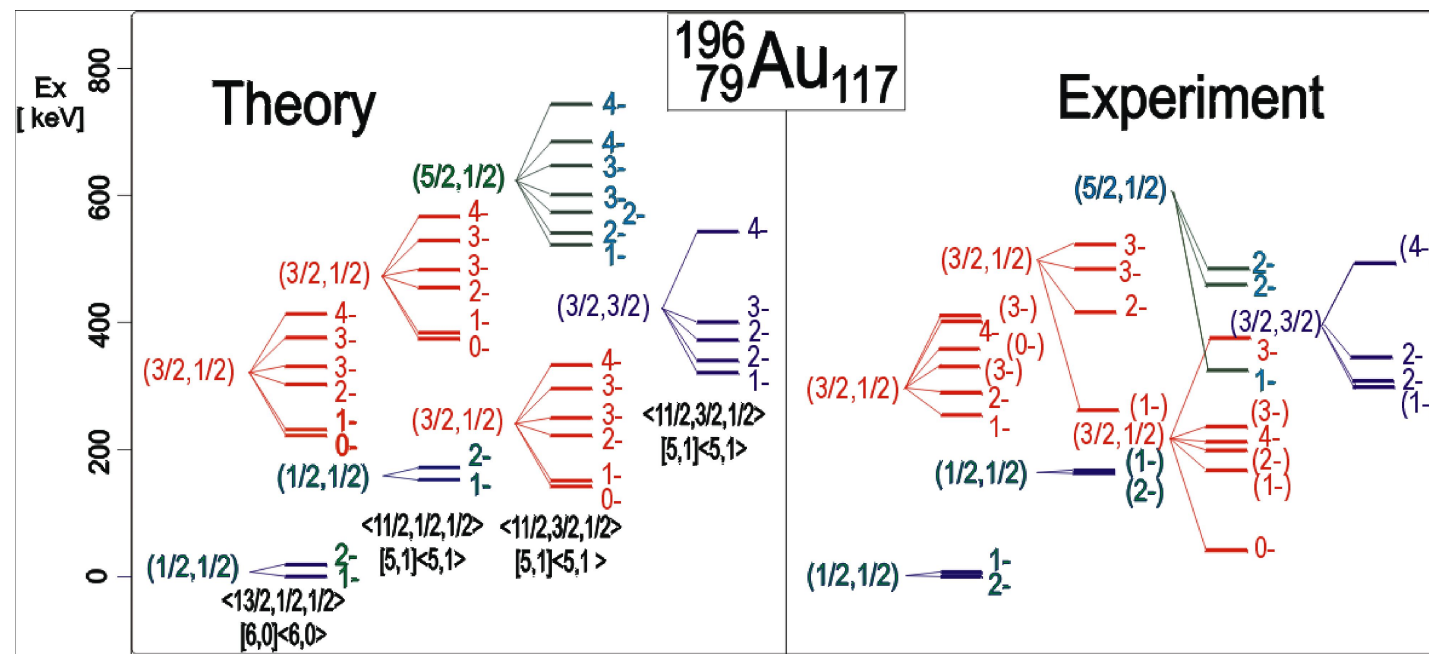
$L_v - L_\pi \Rightarrow M1$ excitation.



Supersymmetry

A simultaneous description of even- and odd-mass nuclei (doublets) or of even-even, even-odd, odd-even and odd-odd nuclei (quartets).

Example of ^{194}Pt , ^{195}Pt , ^{195}Au & ^{196}Au :



Bosons + fermions

Odd-mass nuclei are fermions.

Describe an odd-mass nucleus as N bosons + 1 fermion mutually interacting. Hamiltonian:

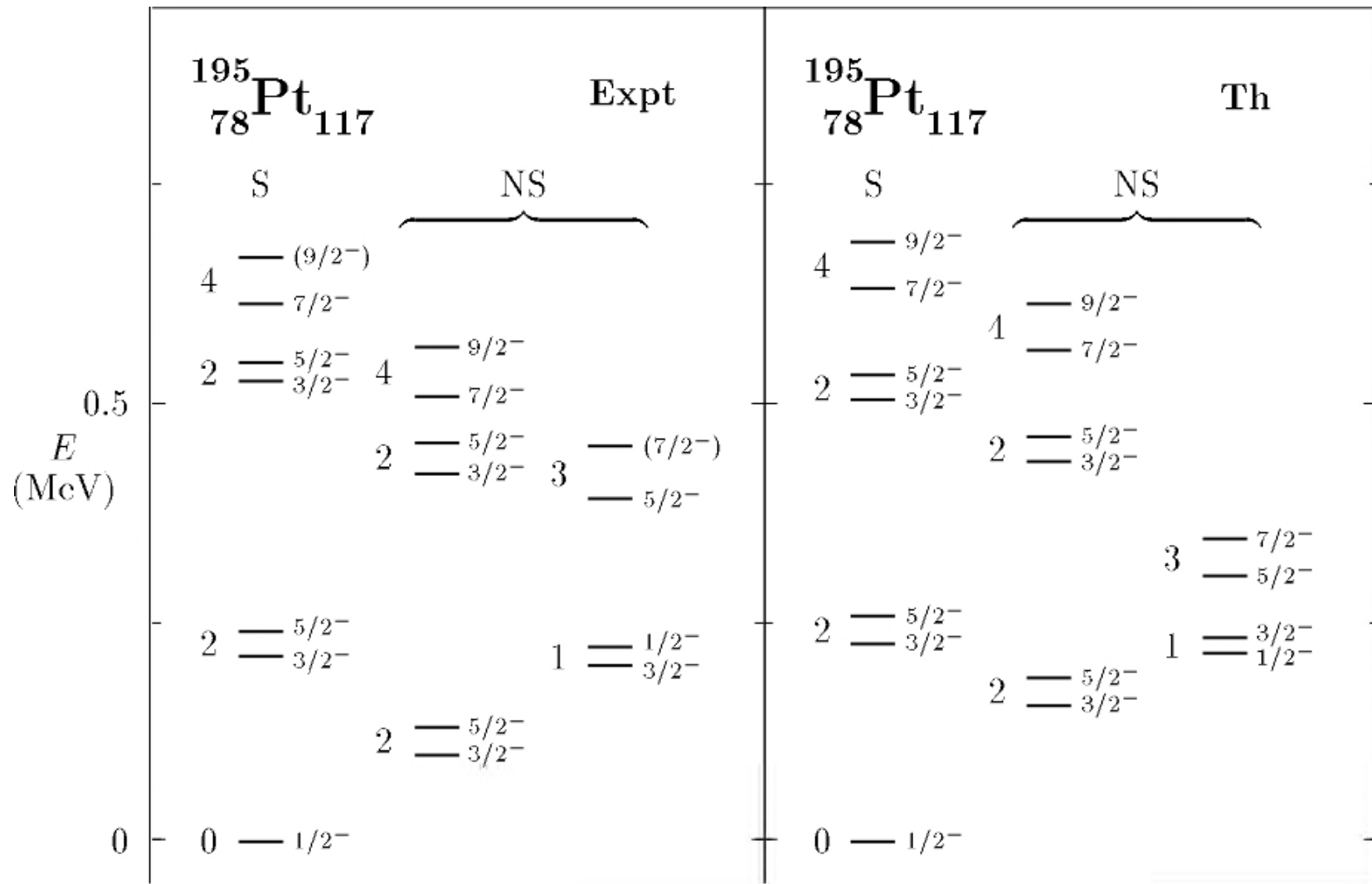
$$\hat{H}_{\text{IBFM}} = \hat{H}_{\text{IBM}} + \sum_{j=1}^{\Omega} \bar{\varepsilon}_j \hat{a}_j^+ \hat{a}_j + \sum_{i_1 i_2=1}^6 \sum_{j_1 j_2=1}^{\Omega} \bar{v}_{i_1 j_1 i_2 j_2} \hat{b}_{i_1}^+ \hat{a}_{j_1}^+ \hat{b}_{i_2} \hat{a}_{j_2}$$

Algebra:

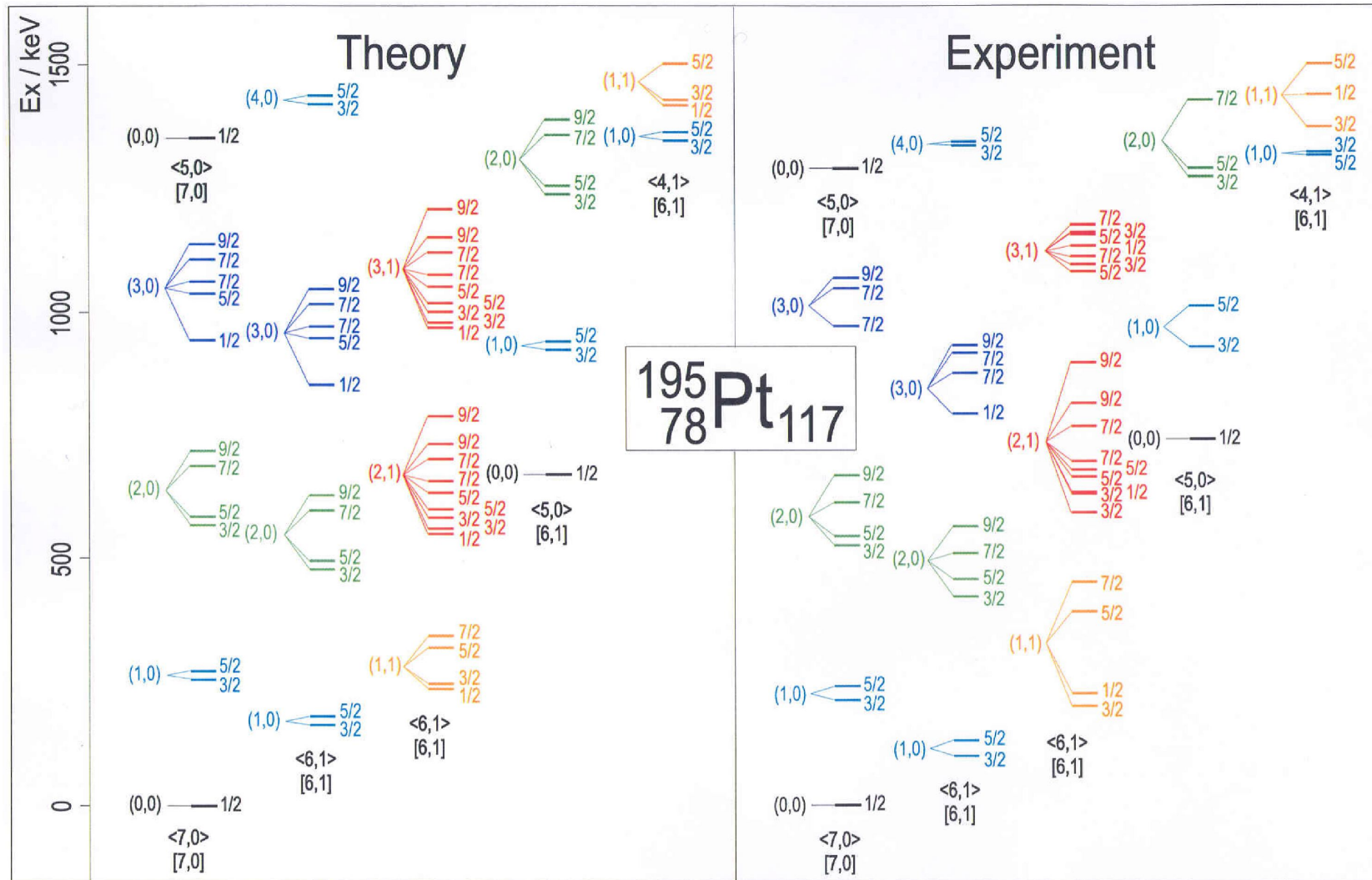
$$\text{U}(6) \oplus \text{U}(\Omega) = \left\{ \begin{array}{c} \hat{b}_{i_1}^+ \hat{b}_{i_2} \\ \hat{a}_{j_1}^+ \hat{a}_{j_2} \end{array} \right\}$$

Many-body problem is solved analytically for certain energies ε and interactions v .

Example: $^{195}\text{Pt}_{117}$



Example: $^{195}\text{Pt}_{117}$ (new data)



Nuclear supersymmetry

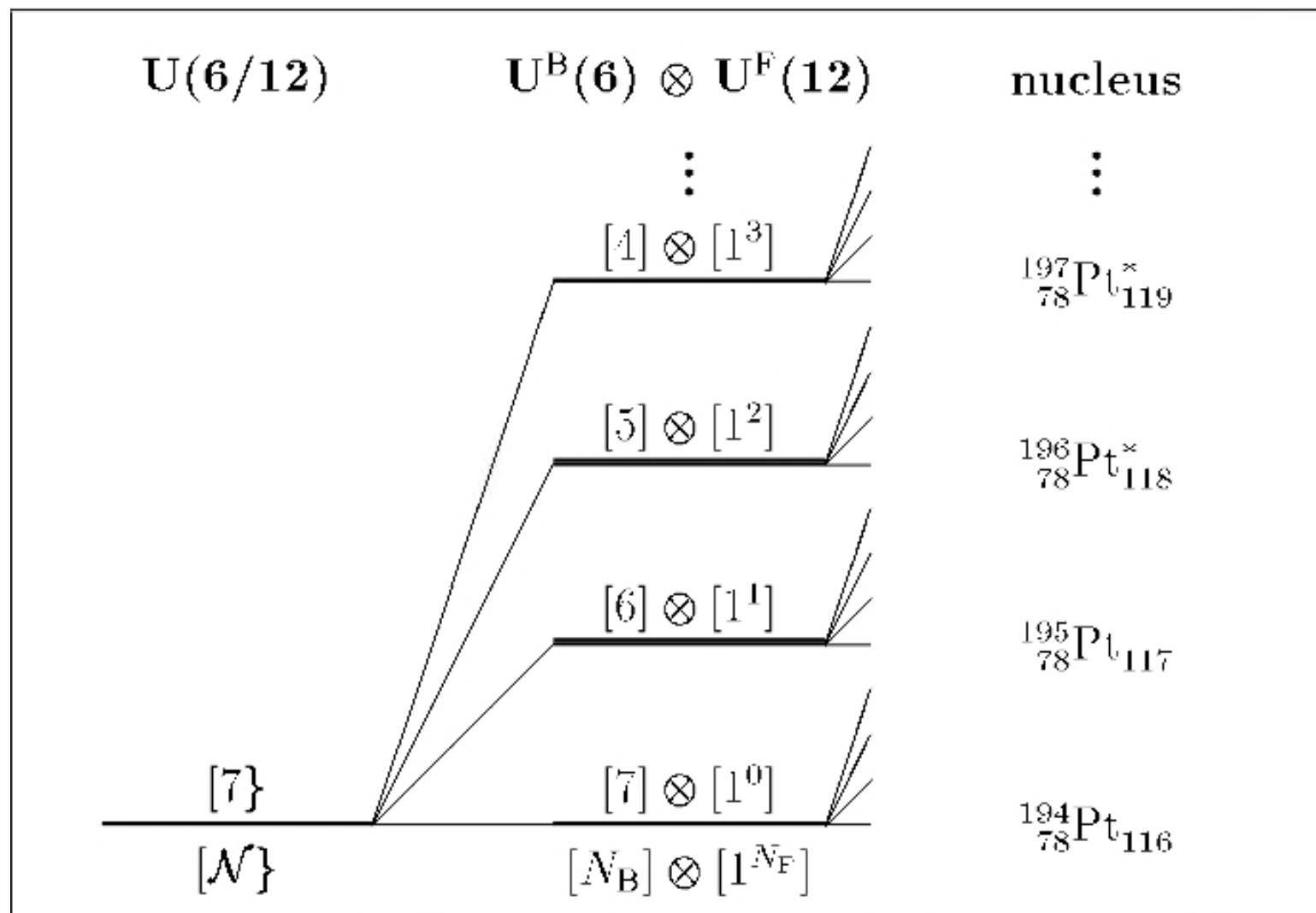
Up to now: separate description of even-even and odd-mass nuclei with the algebra

$$U(6) \oplus U(\Omega) = \left\{ \begin{array}{c} \hat{b}_{i_1}^+ \hat{b}_{i_2} \\ \hat{a}_{j_1}^+ \hat{a}_{j_2} \end{array} \right\}$$

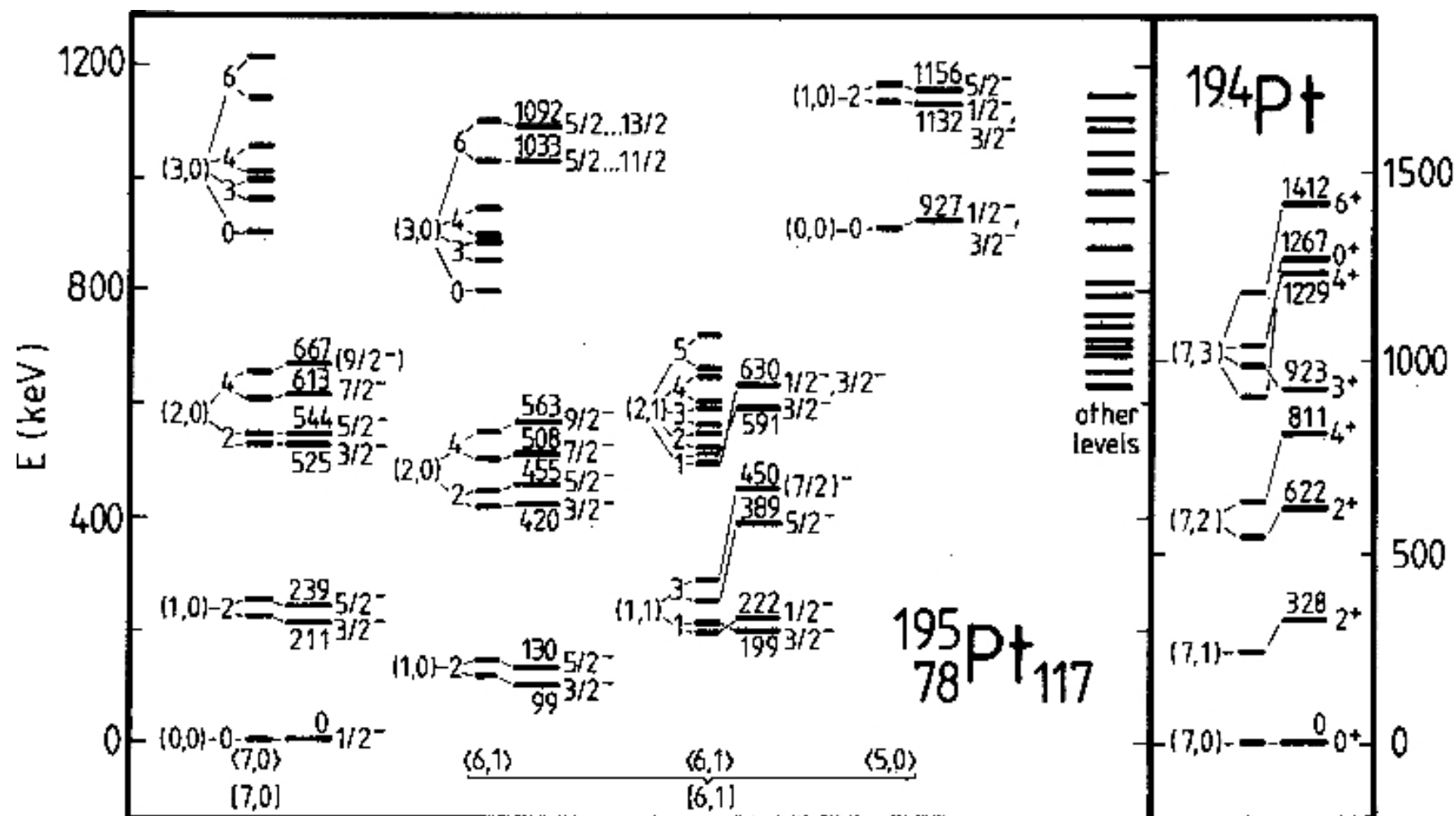
Simultaneous description of even-even and odd-mass nuclei with the superalgebra

$$U(6/\Omega) = \left\{ \begin{array}{cc} \hat{b}_{i_1}^+ \hat{b}_{i_2} & \hat{b}_{i_1}^+ \hat{a}_{j_2} \\ \hat{a}_{j_1}^+ \hat{b}_{i_2} & \hat{a}_{j_1}^+ \hat{a}_{j_2} \end{array} \right\}$$

U(6/12) supermultiplet



Example: $^{194}\text{Pt}_{116}$ & $^{195}\text{Pt}_{117}$



Example: $^{196}\text{Au}_{117}$

