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Entanglement Entropy and Spacetime

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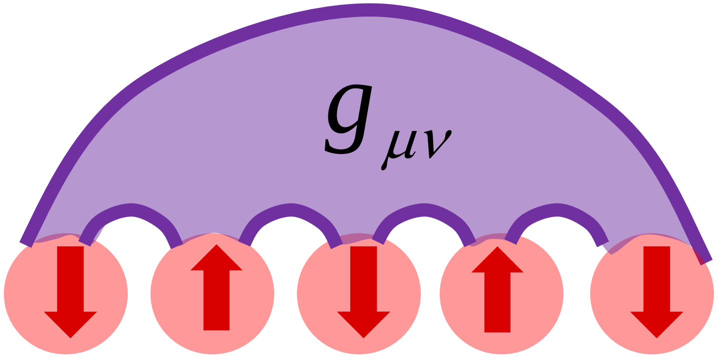


References (Review Articles)

(i) EE in QFTs and Quantum Many-body Systems

Calabrese-Cardy, arXiv:0905.4013, J.Phys.A42:504005,2009. **[2d CFT]**

Casini-Huerta, arXiv:0903.5284, J.Phys.A42:504007,2009. **[Free CFT]**

J. Eisert, M. Cramer and M. B. Plenio, arXiv:0808.3773, Rev. Mod. Phys. 82 (2010) 277. **[Quant-ph/cond-mat view, Area laws]**

(ii) Holographic EE

Nishioka-Ryu-TT, arXiv:0905.0932, J.Phys.A42:504008,2009.

TT, arXiv:1204.2450, Class.Quant.Grav. 29 (2012) 153001.

(ii) EE and Black holes

Solodukhin, arXiv:1104.3712, Living Rev. Relativity 14, (2011), 8.

① Introduction

(1-1) What is Entanglement Entropy ?

What is quantum entanglement ?

In quantum mechanics, a physical state is described by a vector in Hilbert space.

If we consider **a spin of an electron** (= two dimensional Hilbert space), for example, a state is generally described by **a linear combination:**

$$\left|\Psi\right\rangle = a\left|\uparrow\right\rangle + b\left|\downarrow\right\rangle, \qquad |a|^2 + |b|^2 = 1.$$

Consider **two spin systems**.

We can think of the following states:

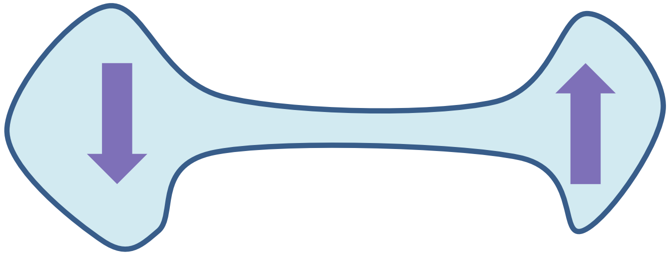
(i) **A direct product state** (unentangled state)

$$|\Psi\rangle = \frac{1}{2}\left[|\uparrow\rangle_A + |\downarrow\rangle_A\right] \otimes \left[|\uparrow\rangle_B + |\downarrow\rangle_B\right].$$

Independent

(ii) **An entangled state**

$$|\Psi\rangle = \left[|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B\right] /\sqrt{2}.$$



One determines the other !

∃ Non-local correlation

Density matrix formalism

For a **pure state**, using the wave function $|\Psi\rangle$,
the density matrix is given by $\rho_{tot} = |\Psi\rangle\langle\Psi|$.

We can express physical expectation values as

$$\langle O \rangle = \mathrm{Tr}[O \cdot \rho_{tot}]. \qquad (\mathrm{Tr}[\rho_{tot}] = 1)$$

In a generic quantum system such as the one at finite temperature, it is not a pure state, but is a **mixed state**.

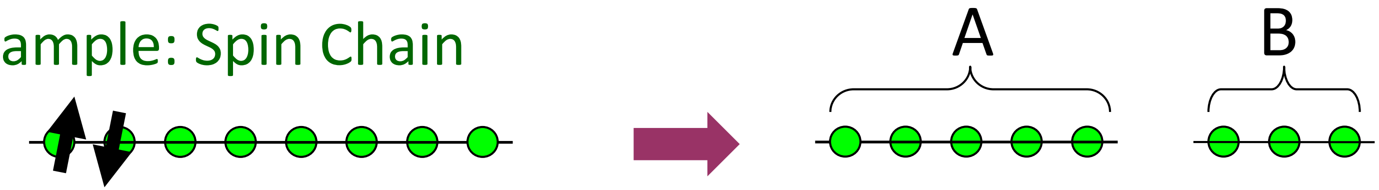
e.g. $\rho_{tot} = \dfrac{e^{-\beta H}}{\mathrm{Tr}[e^{-\beta H}]}$ for the canonical ensemble.

A measure of quantum entanglement is known as **entanglement entropy** (EE), defined as follows:

Divide a quantum system into two parts A and B. The total Hilbert space becomes factorized:

$$H_{tot} = H_A \otimes H_B \quad .$$

Example: Spin Chain



Define the reduced density matrix ρ_A for A by

$$\rho_A = \mathrm{Tr}_B \rho_{tot} \quad , \quad (\text{for pure state}: \ \rho_{tot} = |\Psi\rangle\langle\Psi|).$$

Finally, the entanglement entropy (EE) S_A is defined by

$$\boxed{S_A = -\mathrm{Tr}_A\, \rho_A \log \rho_A \quad .}$$

(von-Neumann entropy)

The Simplest Example: two spins (2 qubits)

$$\text{(i)}\quad |\Psi\rangle = \frac{1}{2}\left[|\uparrow\rangle_A + |\downarrow\rangle_A\right]\otimes\left[|\uparrow\rangle_B + |\downarrow\rangle_B\ \right]$$

$$\Rightarrow \rho_{\mathrm{A}} = \mathrm{Tr}_{\mathrm{B}}\left[|\Psi\rangle\langle\Psi|\right] = \frac{1}{2}\left[|\uparrow\rangle_A + |\downarrow\rangle_A\right]\cdot\left[\langle\uparrow|_A + \langle\downarrow|_A\right] \cong \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

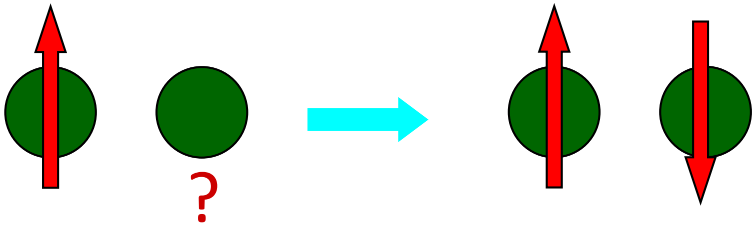
Not Entangled

$$\boxed{\mathrm{S}_{\mathrm{A}} = 0.}$$

$$\text{(ii)}\quad |\Psi\rangle = \left[|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_{\mathrm{B}}\ \right]\ /\sqrt{2}$$

$$\Rightarrow \rho_A = \mathrm{Tr}_{\mathrm{B}}\left[|\Psi\rangle\langle\Psi|\right] = \frac{1}{2}\left[|\uparrow\rangle_A\langle\uparrow|_A + |\downarrow\rangle_A\langle\downarrow|_A\right] \cong \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Entangled



$$\boxed{\mathrm{S}_{\mathrm{A}} = -2\cdot\frac{1}{2}\cdot\log\frac{1}{2} = \log 2.}$$

Note: The standard thermal entropy is obtained as a particular case of EE: i.e. A=total space.

$$\rho = \frac{e^{-\beta H}}{Z}, \quad Z = \mathrm{Tr}[e^{-\beta H}].$$

$$\Rightarrow S = -\frac{\partial}{\partial n}\log[\mathrm{Tr}[\rho^n]]\Big|_{n\to 1} = -\frac{\partial}{\partial n}\left(\log[\mathrm{Tr}[e^{-\beta n H}]] - n\cdot \log Z\right)$$

$$= \beta\langle H\rangle + \log Z = \beta(E-F) = S_{thermal}.$$

A Generalization of EE: Renyi entropy (REE)

Entanglement (n-th) Renyi entropy is defined by

$$S_A^{(n)} = \frac{\log \mathrm{Tr}[(\rho_A)^n]}{1-n}.$$

This is related to EE in the limit $\lim_{n\to 1} S_A^{(n)} = S_A$.

If we know $S_A^{(n)}$ for all n, we can obtain all eigenvalues of ρ_A . They are called the entanglement spectrum.

Entanglement entropy (EE)

= A measure how much a given quantum state is quantum mechanically entangled (or complicated).

~ `active' degrees of freedom (or its information)

Why interesting and useful ?

It seems still difficult to observe EE in real experiments (→ a developing subject).

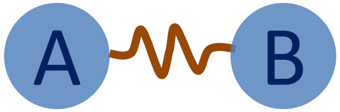
But,

(i) recently it is very common to calculate EE in **`numerical experiments'** of cond-mat systems.

(ii) It offers us a **geometrical way to understand QFTs**.

(cf. conventional approaches in QFT: local and algebraic)

Ex. EE in two interacting harmonic oscillators



Consider the system with the Hamiltonian:

$$H = a^{+}a + b^{+}b + \lambda(a^{+}b^{+} + ab).$$

The creation and annihilation operators satisfy

$$[a, a^{+}] = [b, b^{+}] = 1.$$

We can diagonalize H by the Bogoliubov transformation

$$\begin{cases} \tilde{a} = \cosh\theta \cdot a + \sinh\theta \cdot b^{+}, \\ \tilde{b} = \sinh\theta \cdot a^{+} + \cosh\theta \cdot b. \end{cases} \qquad \left(\lambda \equiv \frac{2\sinh\theta\cosh\theta}{1+2\sinh^{2}\theta} \right)$$

$$\Rightarrow H = \frac{1}{1+2\sinh^{2}\theta}(\tilde{a}^{+}\tilde{a} + \tilde{b}^{+}\tilde{b}).$$

The ground state of H:

$$\begin{cases} \tilde{a}\left|\Psi\right\rangle = 0 \quad \Leftrightarrow \quad (\cosh\theta\cdot a + \sinh\theta\cdot b^{+})\left|\Psi\right\rangle = 0, \\ \tilde{b}\left|\Psi\right\rangle = 0 \quad \Leftrightarrow \quad (\sinh\theta\cdot a^{+} + \cosh\theta\cdot b)\left|\Psi\right\rangle = 0. \end{cases}$$

$$\Rightarrow \left|\Psi\right\rangle = \left|0\right\rangle_{\tilde{a}} \otimes \left|0\right\rangle_{\tilde{b}} = \frac{1}{\cosh\theta}\cdot e^{-\tanh\theta\cdot a^{+}b^{+}}\left|0\right\rangle_{a} \otimes \left|0\right\rangle_{b}.$$

Now we define the subsystems A and B as follows:

$$H_{tot} = \underbrace{H_A}_{\text{Generated by}\{a^{+n}\ \left|0\right\rangle_a\}} \otimes \underbrace{H_B}_{\text{Generated by}\{b^{+n}\ \left|0\right\rangle_b\}}$$

The reduced density matrix is computed as

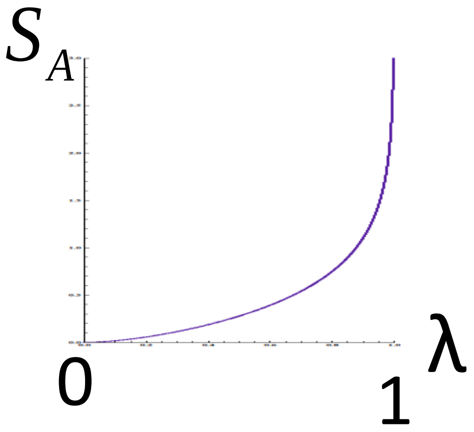
$$\rho_A = \frac{1}{\cosh^2\theta}\sum_{n=0}^{\infty}\frac{(\tanh 2\theta)^{2n}}{n!}(a^+)^n|0\rangle_a\langle 0|_a a^n$$

$$\longrightarrow e^{-n/T}$$

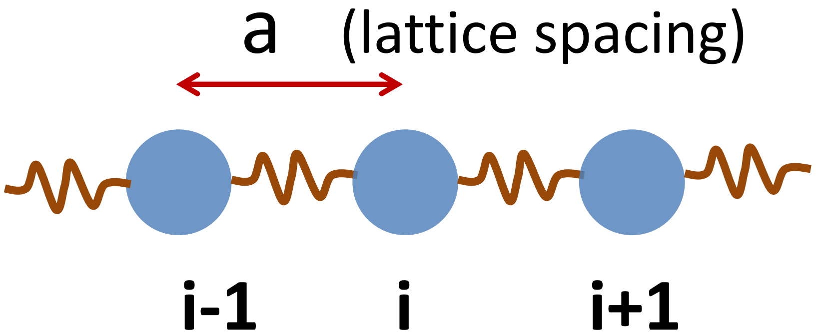
$$= \frac{1}{\cosh^2\theta}\sum_{n=0}^{\infty}(\tanh 2\theta)^{2n}|n\rangle_a\langle n|_a.$$

$$\left(\text{The number state: } \quad |n\rangle_a \equiv \frac{1}{\sqrt{n!}}(a^+)^n|0\rangle_a\right)$$

Thus we find $S_A = \cosh\theta\cdot\log\cosh^2\theta - \sinh\theta\cdot\log\sinh^2\theta.$



From quantum many-body systems to QFTs



$$H = \sum_i \left[\frac{1}{2}\left(\frac{d\phi_i}{dt}\right)^2 + \frac{(\phi_i)^2 - \phi_{i+1}\phi_i}{a^2} \right]$$

$$= \sum_i \left[\frac{1}{2}\left(\frac{d\phi_i}{dt}\right)^2 + \frac{(\phi_{i+1} - \phi_i)^2}{2a^2} \right]$$

Continuum Limit

x=i▪a, a→0

$$H = \frac{1}{2}\int dx \left[\left(\frac{\partial\phi(t,x)}{\partial t}\right)^2 + \left(\frac{\partial\phi(t,x)}{\partial x}\right)^2 \right]$$

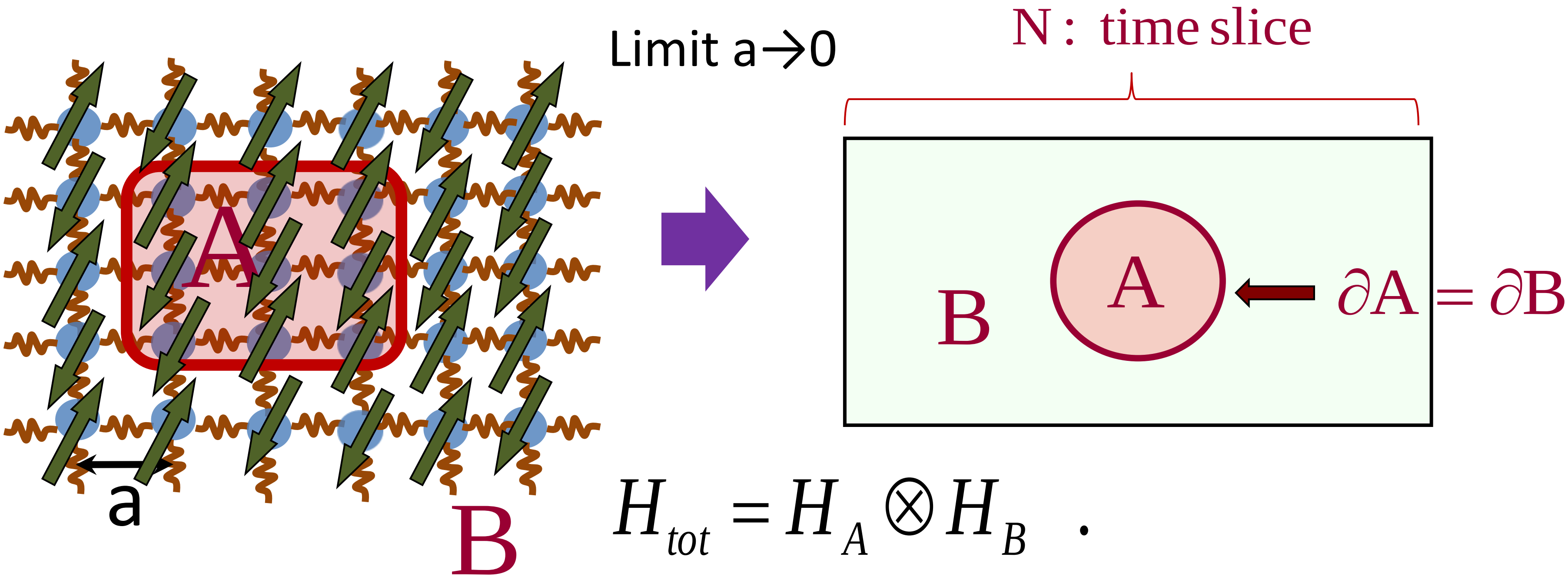
EE in Quantum Many-body Systems and QFTs

The EE is defined geometrically

(sometime called geometric entropy).

Continuum

Limit a→0



$$H_{tot} = H_A \otimes H_B \quad .$$

Quantum Many-body Systems **Quantum Field Theories (QFTs)**

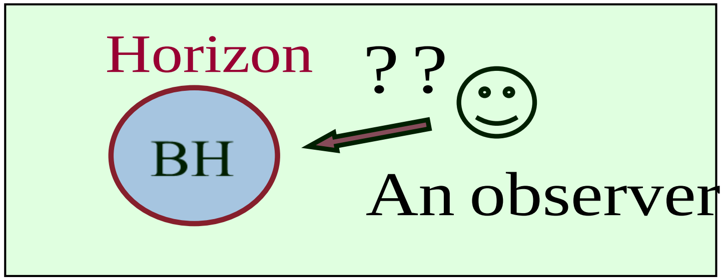
Historical origin: an analogy with black hole entropy

['t Hooft 85, Bombelli-Koul-Lee-Sorkin 86, Srednicki 93, …]

As EE is defined by smearing out the Hilbert space for B,

E.E. ~ `Lost Information' hidden in B

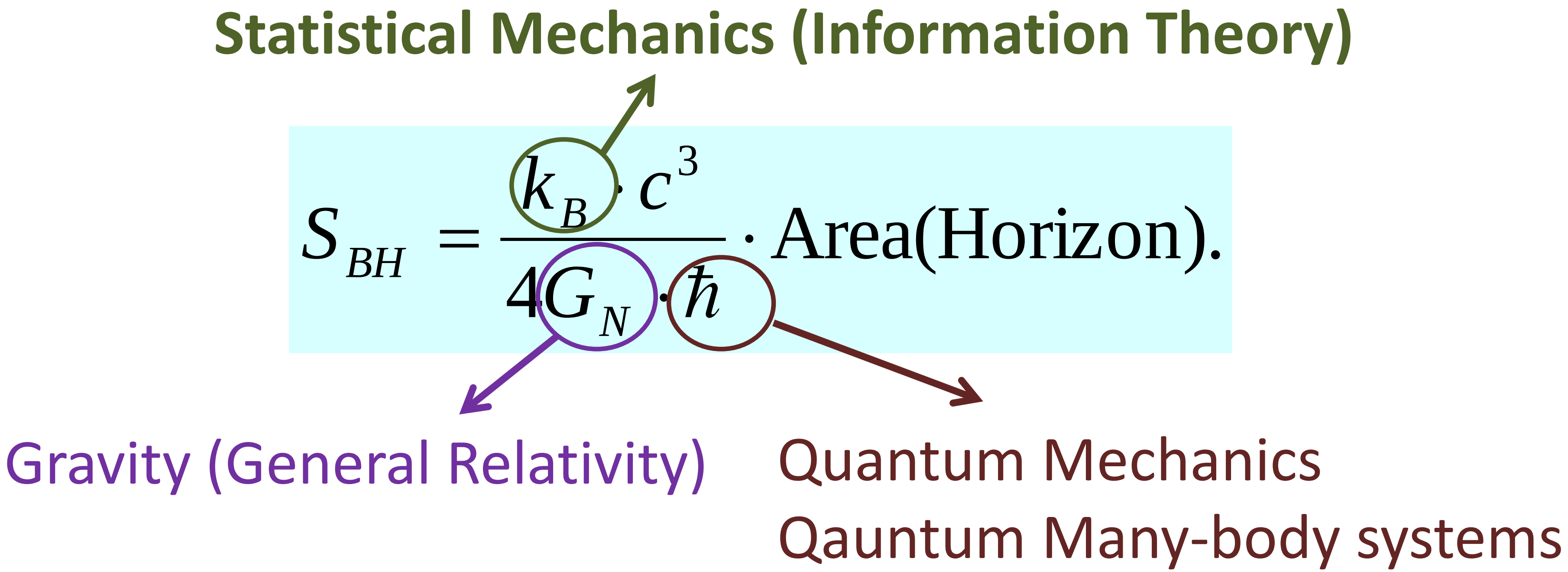
This origin of entropy looks similar to the BH entropy.



The boundary region ∂A ~ the event horizon ?

As we will explain, a complete answer to this historical question is found by considering the AdS/CFT correspondence !

The information hidden inside the horizon is measured by **Bekenstein-Hawking formula** of black hole entropy:



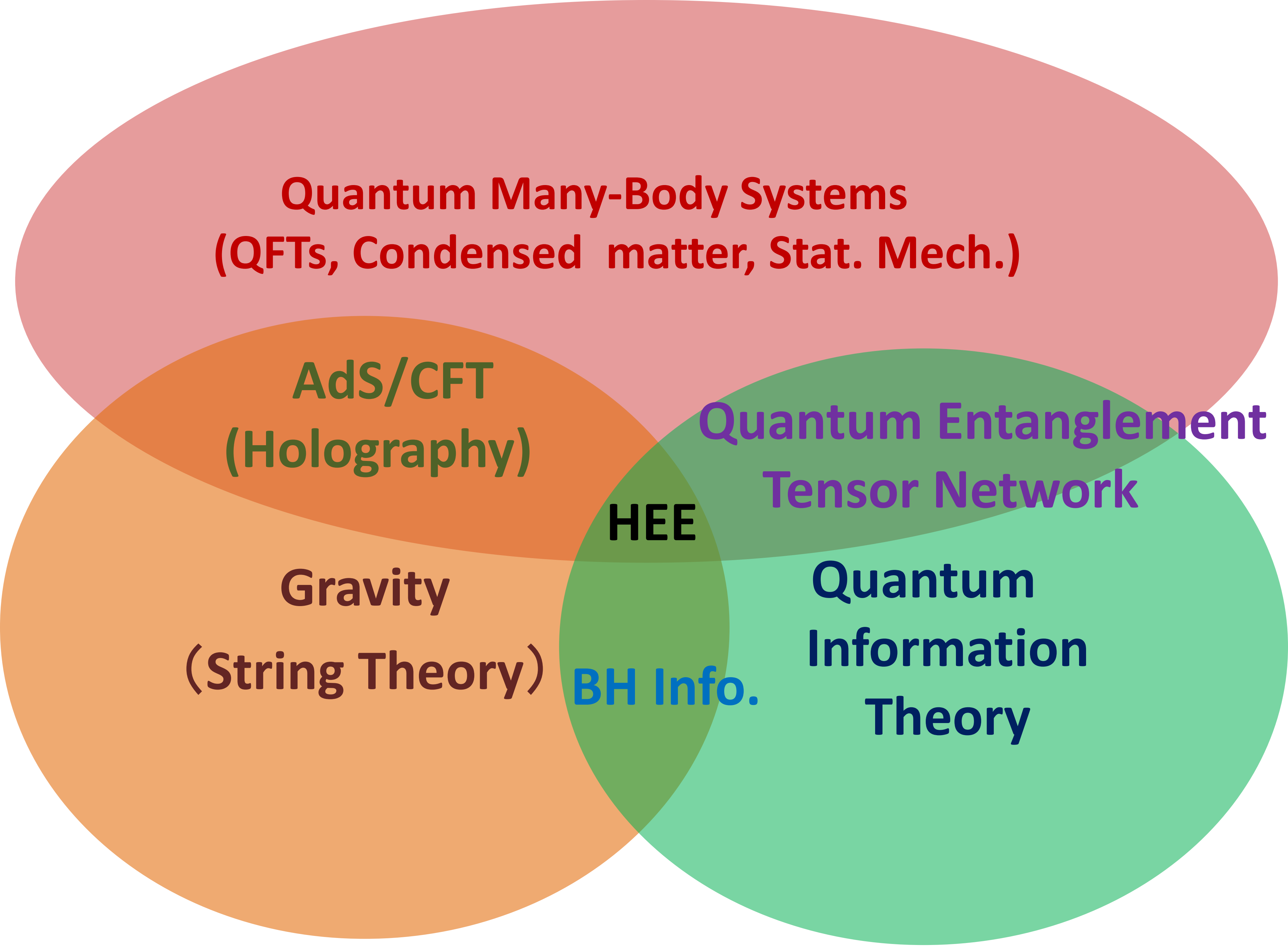
Statistical Mechanics (Information Theory)

$$S_{BH} = \frac{k_B \cdot c^3}{4G_N \cdot \hbar} \cdot \text{Area(Horizon)}.$$

Gravity (General Relativity)

Quantum Mechanics

Qauntum Many-body systems



Quantum Many-Body Systems
(QFTs, Condensed matter, Stat. Mech.)

AdS/CFT
(Holography)

Quantum Entanglement
Tensor Network

HEE

Gravity
（String Theory）

BH Info.

Quantum
Information
Theory

(1-2) Basic Properties of EE

(i) If ρ_{tot} is a **pure state** (i.e. $\rho_{tot} = |\Psi\rangle\langle\Psi|$) and $H_{tot} = H_A \otimes H_B$,

then $S_A = S_B$. ⇒ EE is not extensive !

[Proof]

This follows from the Schmidt decomposition:

$$|\Psi\rangle = \sum_{i=1}^{N} \lambda_i |a_i\rangle_A \otimes |b_i\rangle_B \ , \quad N \leq \min\{|H_A|,|H_B|\}.$$

$$\Rightarrow \mathrm{Tr}[(\rho_A)^n] = \mathrm{Tr}[(\rho_B)^n],$$

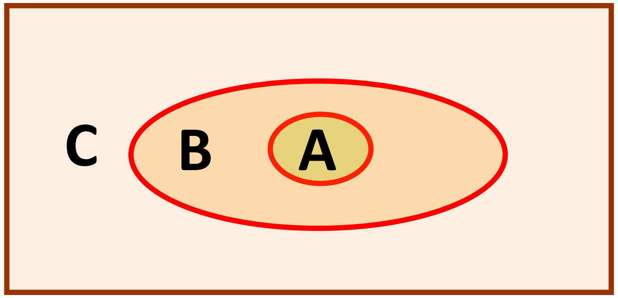
$$\Rightarrow S_A = -\frac{\partial}{\partial n}\mathrm{Tr}[(\rho_A)^n]\Big|_{n\to 1} = S_B \, .$$

(ii) **Strong Subadditivity** (SSA) [Lieb-Ruskai 73]

When $H_{tot} = H_A \otimes H_B \otimes H_C$, for any ρ_{tot} ,

$$S_{A+B} + S_{B+C} \geq S_{A+B+C} + S_B,$$

$$S_{A+B} + S_{B+C} \geq S_A + S_C.$$



Actually, these two inequalities are equivalent .

We can derive the following inequality from SSA:

$$|S_A - S_B| \leq S_{A\cup B} \leq S_A + S_B. \quad (\mathrm{Note: A \cap B \neq \phi \ \ in\ general})$$

Araki-Lieb inequality

Subadditivity

The strong subadditivity can also be regarded as the **concavity** of von-Neumann entropy.

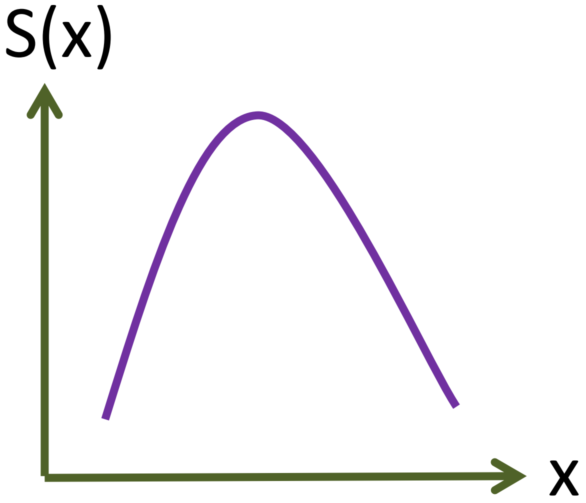
Indeed, if we assume A,B,C are real numbers, then

$$S(A+B)+S(B+C)\geq S(A+B+C)+S(B),$$

$$\Rightarrow 2\cdot S\left(\frac{x+y}{2}\right)\geq S(x)+S(y),$$

$$\Rightarrow \frac{d^2}{dx^2}S(x)\leq 0.$$

(i.e. concave function of x)



Mutual Information

$$I(A,B)=S_A+S_B-S_{A\cup B}\geq 0.$$

This measures an entropic correlation between A and B and is called the mutual information.

(i) The strong subadditivity leads to the relation:

$$I(A,B+C)\geq I(A,B).$$

(ii) The mutual information gives a bound for two point functions:

$$I(A,B)\geq\frac{\left|\langle O_A\cdot O_B\rangle-\langle O_A\rangle\cdot\langle O_B\rangle\right|^2}{2\|O_A\|^2\cdot\|O_B\|^2}$$

(iii) Area law [Bombelli-Koul-Lee-Sorkin 86, Srednicki 93]

EE in QFTs includes UV divergences.

Area Law

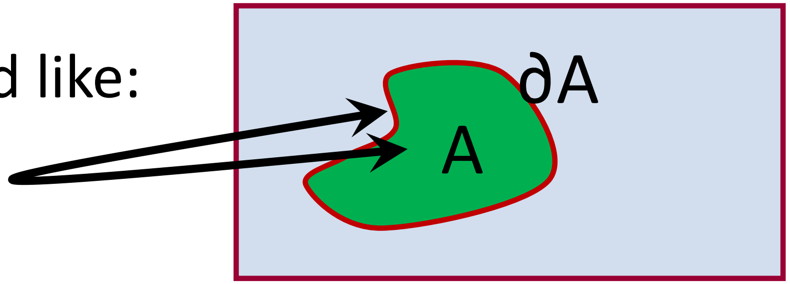
The leading divergence of EE in a d+1 dim. QFT with a UV fixed pt. (i.e. local QFT) is proportional to the area of the (d-1) dim. boundary $\partial\mathrm{A}$:

$$S_A \sim \frac{\mathrm{Area}(\partial \mathrm{A})}{a^{d-1}} + (\text{subleading terms}),$$

$$[a : \text{UV cut off (lattice spacing)}]$$

Intuitively, this property is understood like:

Most strongly entangled

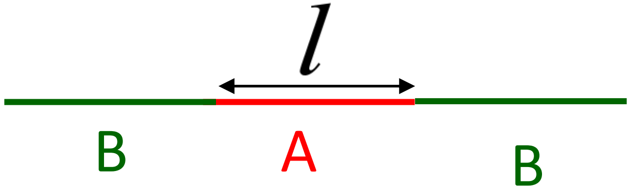


Comments on Area Law

- The area law can be applied for ground states or finite temperature systems. It is violated for highly excited states. (Note $S_A \leq \log(\dim H_A) \approx \mathrm{Vol}(A)$.)

- There are two exceptions:

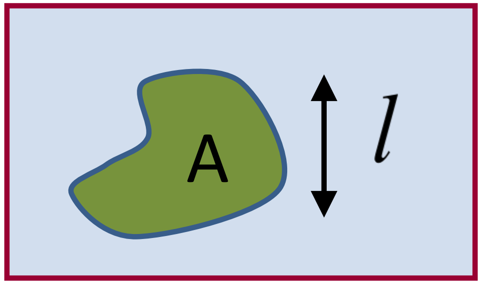
(a) 1+1 dim. CFT $S_A = \frac{c}{3}\log\frac{l}{a}.$



[Holzhey-Larsen-Wilczek 94, Calabrese-Cardy 04]

(b) QFT with Fermi surfaces ($k_F \sim a^{-1}$)

$$S_A \sim \left(\frac{l}{a}\right)^{d-1} \cdot \log\frac{l}{a} + \ldots$$

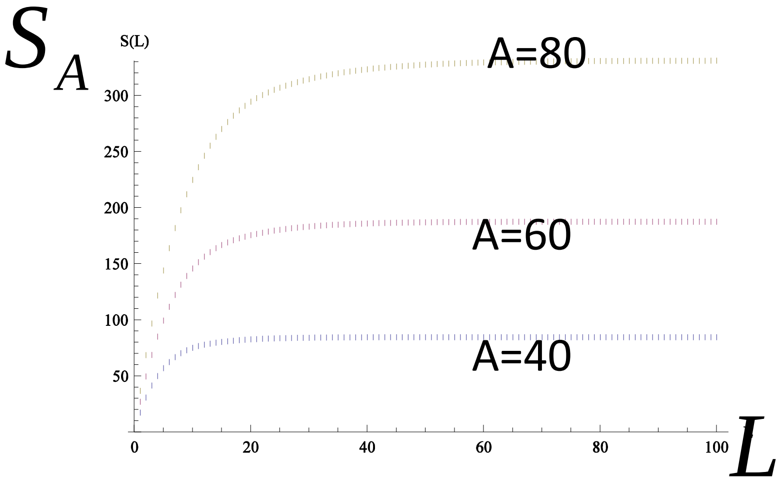


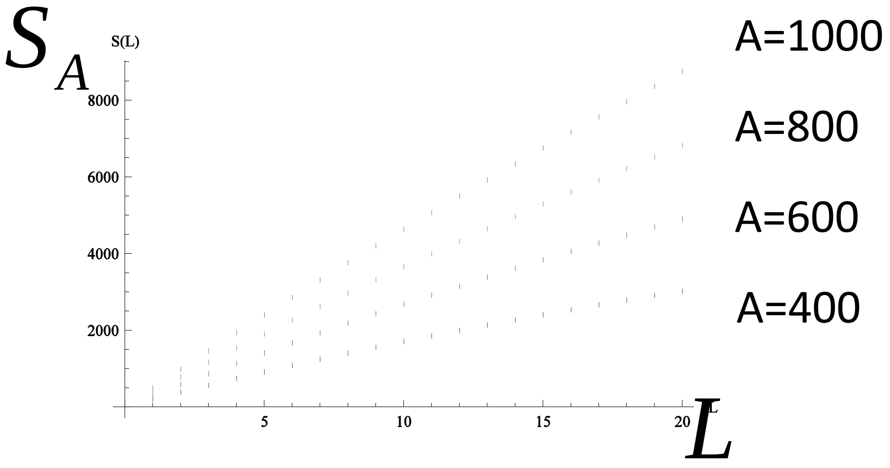
[Wolf 05, Gioev-Klich 05]

Ex. Volume law in Non-local QFTs [Shiba-TT 13]

Consider a 1+1 dim. QFT defined by

$$\boxed{H=\int dx[\dot{\phi}(x)^2+\phi(x)e^{A\sqrt{-\partial^2}}\phi(x)].}$$





$$S_A \cong \begin{cases} AL/2 & (L \ll A) \\ cA^2 & (L \gg A) \end{cases}.$$

Volume law !

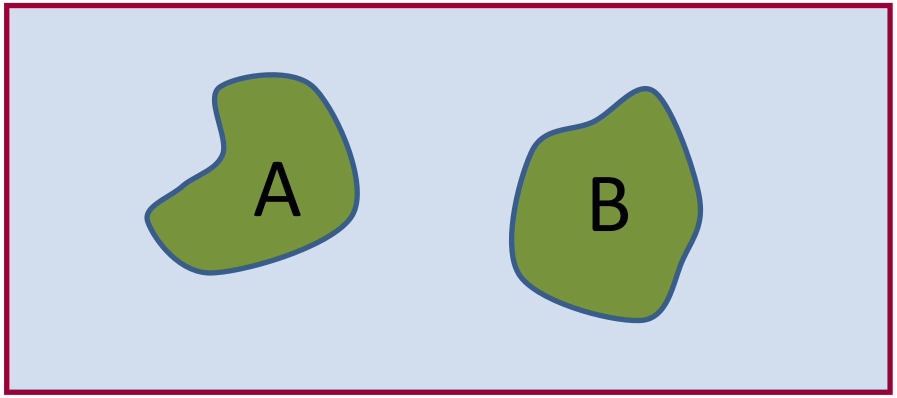
Note: Ground states for generic (non-local) Hamiltonian follow the volume law.

- A rigorous proof of area law is available for free field theories. [e.g. Plenio-Eisert-Dreissig-Cramer 04,05]

- The AdS/CFT predicts the area law for strongly interacting theories as long as the QFT has a UV fixed point.

- The UV divergence cancels out in the mutual information.

$$\Rightarrow \; I(A,B) = S_A + S_B - S_{A\cup B} = \text{finite} \geq 0, \quad \text{if} \;\; A \cup B = \phi.$$



- The area law resembles the Bekenstein-Hawking formula of black hole entropy:

$$S_{BH} = \frac{\mathrm{Area(horizon)}}{4G_N}.$$

Actually, the EE can be interpreted not as the total but as a partial (i.e. quantum corrections) contribution to the black hole entropy. [Susskind-Uglm 94]

➡ A more complete understanding awaits the AdS/CFT !

(1-3) Applications of EE to condensed matter physics

$S_A \approx$ Log[``Effective rank'' of density matrix for A]

⇒ This measures how much we can compress the quantum information of ρ_A .

Thus, EE estimates difficulties of computer simulations such as in DMRG etc. [Osborne-Nielsen 01, …]

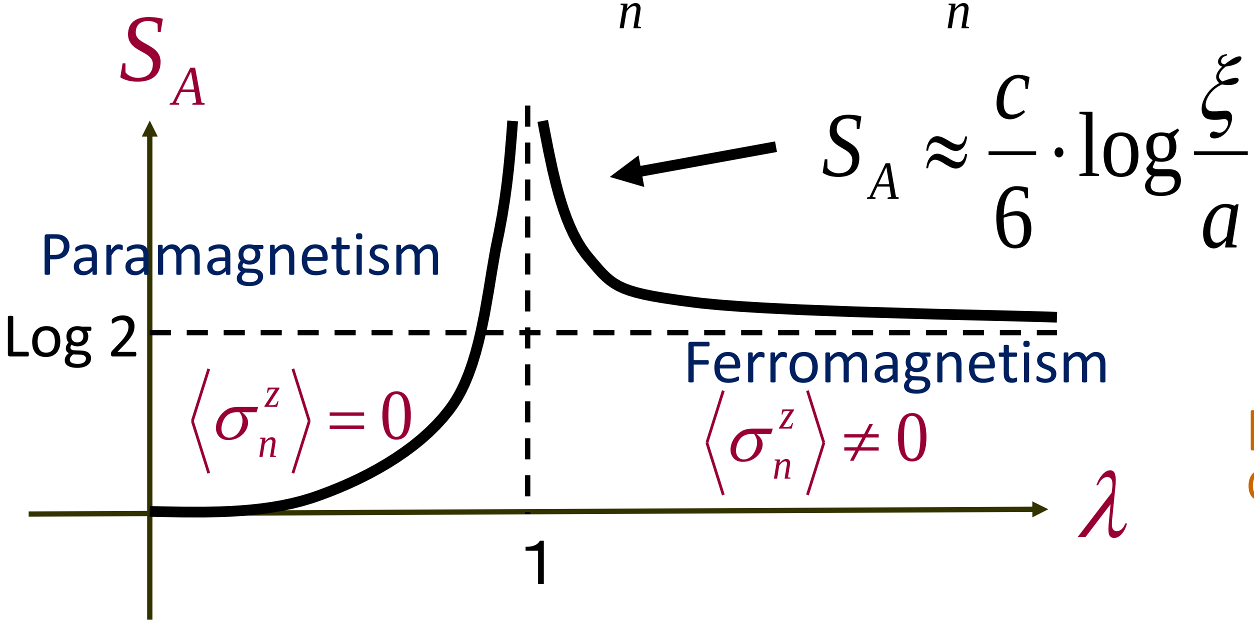
Especially, EE gets divergent at the quantum phase transition point (= quantum critical point).

⇒ **EE = a quantum order parameter !**

Ex. Quantum Ising spin chain

The Ising spin chain with a transverse magnetic field:

$$H = -\sum_n \sigma_n^x - \lambda \sum_n \sigma_n^z \sigma_{n+1}^z$$



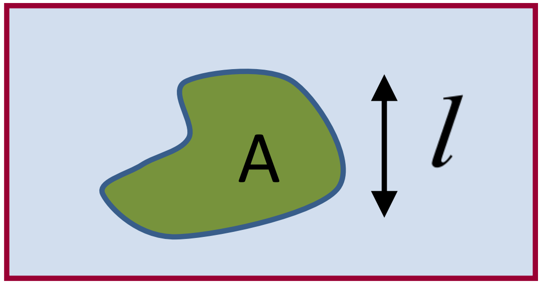
[Vidal-Latorre-Rico-Kitaev 02, Calabrese-Cardy 04]

⇒ This offers a useful numerical method to calculate c.

Topological Entanglement Entropy [Kitaev-Preskill 06, Levin-Wen 06]

In a 2+1 dim. mass gapped theory, EE behaves like

$$S_A = \gamma \cdot \frac{l}{a} + S_{top} \quad .$$



The finite part S_{top} (<0) is invariant under smooth deformations of the subsystem A. ⇒ Topological !

- Top. EE offers us an order parameter of topological systems. (cf. ~~correlation functions~~)
- Recently, Top. EE has been employed to show the existence of spin liquid phases. [Jiang-Wang-Balents 12]

Lecture Plan

① Introduction

② Calculations of EE in QFTs

③ Holographic Entanglement Entropy (HEE)

④ Properties of EE for Excited States

⑤ Holography and Entanglement Renormalization

⑥ Conclusions

② Calculations of EE in QFTs

A basic method of calculating EE in QFTs is so called the **replica method**.

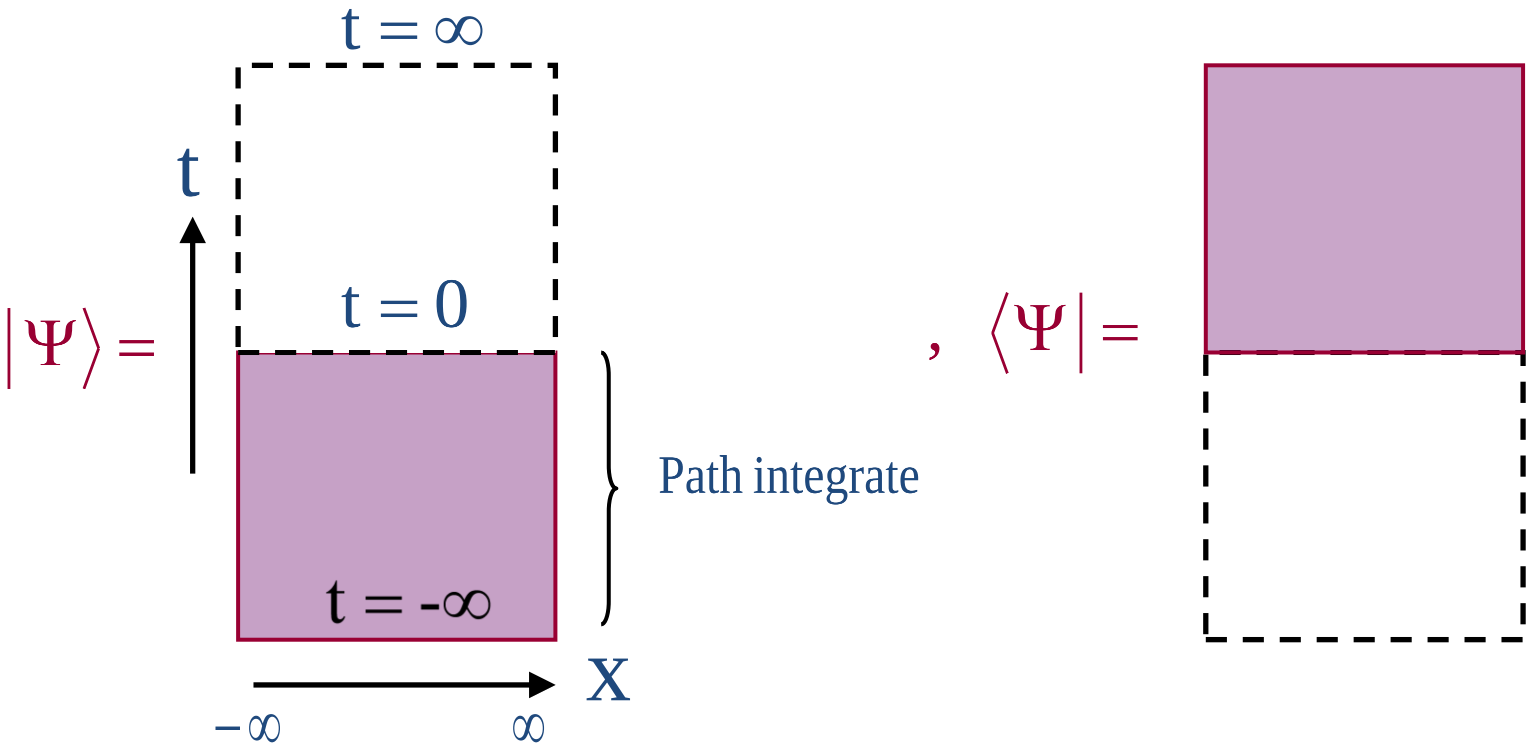
$$S_A = -\frac{\partial}{\partial n} \mathrm{Tr}_A (\rho_A)^n |_{n=1} = -\frac{\partial}{\partial n} \log \mathrm{Tr}_A (\rho_A)^n |_{n=1} \quad .$$

(2-1) 2d CFT

By using this, we can analytically compute the EE in 2d CFTs. [Holzhey-Larsen-Wilczek 94,..., Calabrese-Cardy 04]

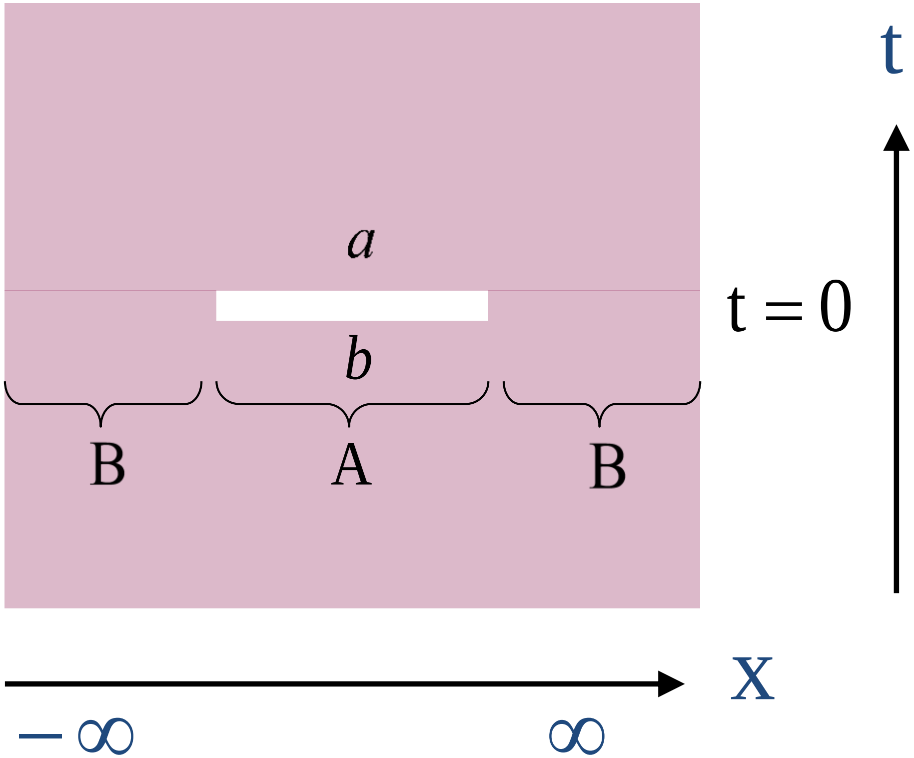
The replica method is also an important method to (often numerically) evaluate EE in more general QFTs.

In the path-integral formalism, the ground state wave function $|\Psi\rangle$ can be expressed in the path-integral formalism as follows:



Next we express $\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$.

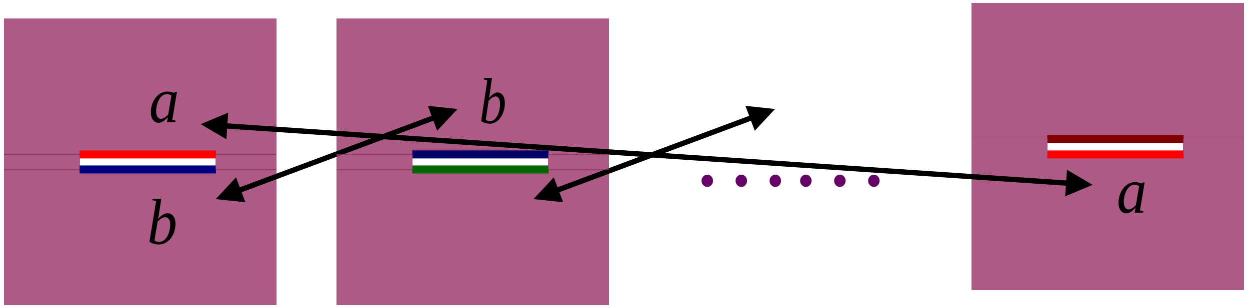
$$[\rho_A]_{ab} =$$



Finally, we obtain a path integral expression of the trace $\mathrm{Tr}\left(\rho_A\right)^n = [\rho_A]_{ab}[\rho_A]_{bc}\cdots[\rho_A]_{ka}$ as follows:

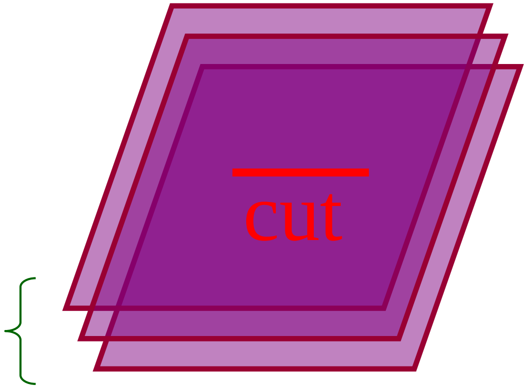
Glue each boundaries successively.

$$\mathrm{Tr}\left(\rho_A\right)^n =$$



$= \text{a path integral over}$

$n\text{-sheeted Riemann surface } \Sigma_{\mathrm{n}}$



n sheets

In this way, we obtain the following representation

$$\mathrm{Tr}\left(\rho_A\right)^n = \frac{Z_n}{(Z_1)^n} \ ,$$

where Z_n is the partition function on the n-sheeted Riemann surface Σ_n.

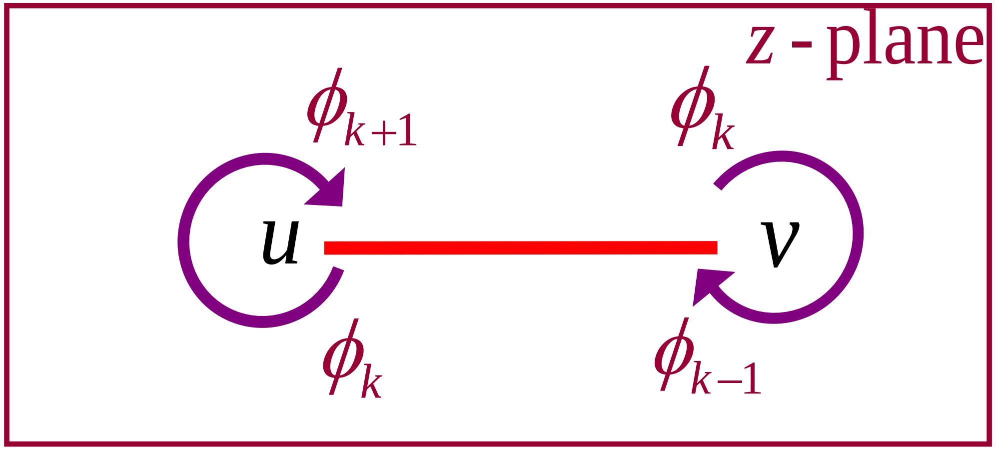
To evaluate Z_n , let us first consider the case where the CFT is defined by a complex free scalar field ϕ .

c=2

It is useful to introduce n replica fields $\phi_1, \phi_2, \cdots \phi_n$ on a complex plane $\Sigma_{n=1} = \mathrm{C}$.

Then we can obtain a CFT equivalent to the one on Σ_n by imposing the boundary condition

$$\phi_k(e^{2\pi i}(z-u)) = \phi_{k+1}(z-u), \qquad \phi_k(e^{2\pi i}(z-v)) = \phi_{k-1}(z-v),$$



By defining $\tilde{\phi}_k = \frac{1}{n}\Sigma_{k=0}^{n-1} e^{2\pi ik/n}\phi_k$, conditions are diagonalized

$$\tilde{\phi}_k(e^{2\pi i}(z-u)) = e^{2\pi ik/n}\tilde{\phi}_k(z-u), \quad \tilde{\phi}_k(e^{2\pi i}(z-v)) = e^{-2\pi ik/n}\tilde{\phi}_k(z-v),$$

Using the orbifold theoretic argument, these twisted boundary conditions are equivalent to the insertion of (ground state) twisted vertex operators at z=u and z=v.

This leads to

$$\mathrm{Tr}\left(\rho_A\right)^n = \prod_{k=0}^{n-1} \left\langle \sigma_{k/n}(u)\sigma_{-k/n}(v) \right\rangle \propto (u-v)^{-\frac{1}{3}(n-1/n)} .$$

$$\sigma_{k/n} : \text{ Twist operator s.t. } \phi \rightarrow e^{2\pi i k/n}\phi$$

$$\text{Conformal dim. : } \Delta(\sigma_{k/n}) = -\frac{1}{2}\left(\frac{k}{n}\right)^2 + \frac{1}{2}\frac{k}{n} .$$

For general 2d CFTs with the central charge c , we can apply a similar analysis. In the end, we obtain

$$\mathrm{Tr}\left(\rho_A\right)^n \propto (u-v)^{-\frac{c}{6}(n-1/n)}.$$

In the end, we obtain

$$S_A = \frac{c}{3}\log\frac{l}{a} \qquad (l \equiv v-u).$$

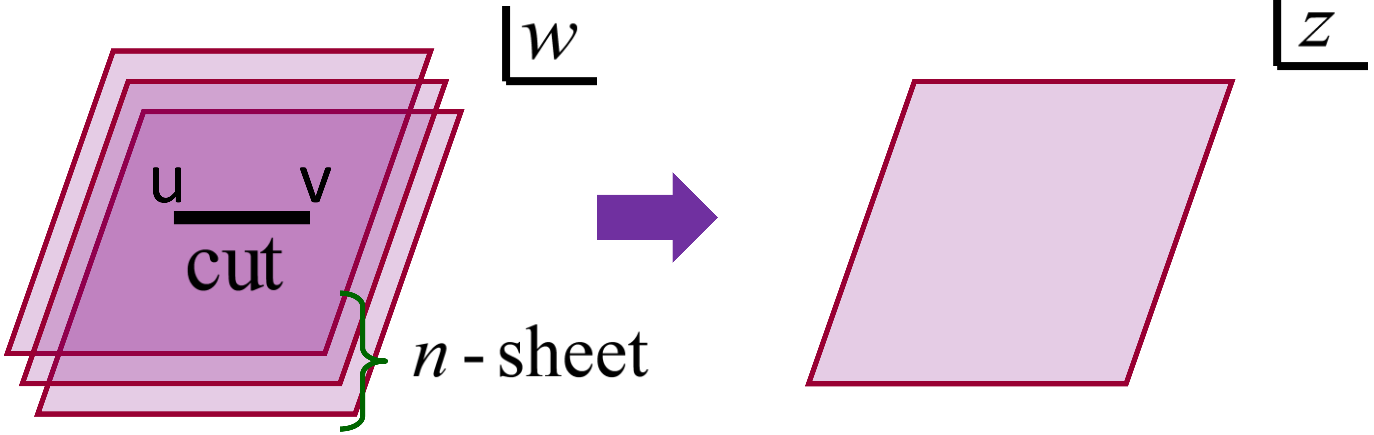
[Holzhey-Larsen-Wilczek 94]

Note: the UV cut off a is introduced such that

$S_A = 0$ at $l = a$.

General CFTs [Calabrese-Cardy 04]

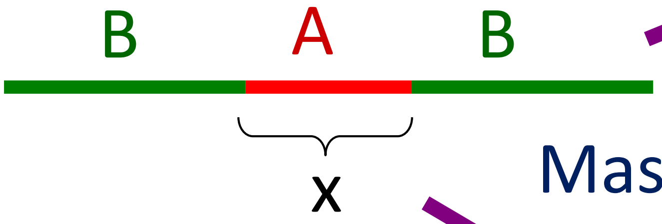
Consider the conformal map: $z^n = \frac{w-u}{w-v}$.



$$T(w) = \left(\frac{dz}{dw}\right)^2 \underbrace{T(z)}_{=0} + \frac{c}{12}\underbrace{\{z,w\}}_{\text{Schwarzian derivative}} = \frac{c(1-n^{-2})}{24}\cdot\frac{(v-u)^2}{(w-u)^2(w-v)^2}.$$

$$\Rightarrow \Delta_{\text{each sheet}} = \frac{c(1-n^{-2})}{24}, \qquad \Delta_{\text{tot}} = n\Delta_{\text{each sheet}} = \frac{c(n-1/n)}{24}.$$

More general results in 2d CFT [Calabrese-Cardy 04]



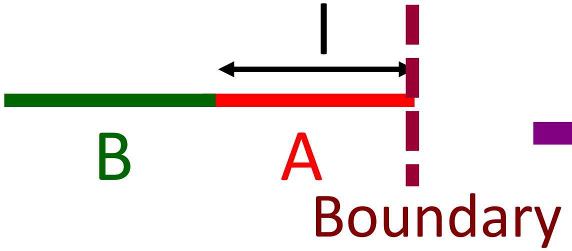
$$S_A = \frac{c}{3}\log\frac{x}{a}$$

Mass gap

$$S_A = \frac{c}{3}\log\frac{\xi}{a}$$

Finite Temp.

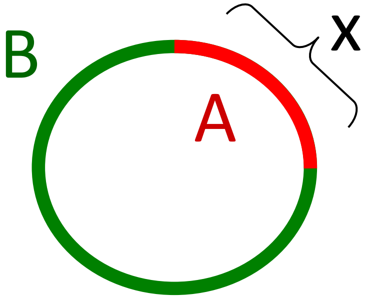
$$S_A = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh\left(\frac{\pi x}{\beta}\right)\right)$$



$$S_A = \frac{c}{6}\log\left(\frac{l}{a}\right) + \underline{\log g},$$

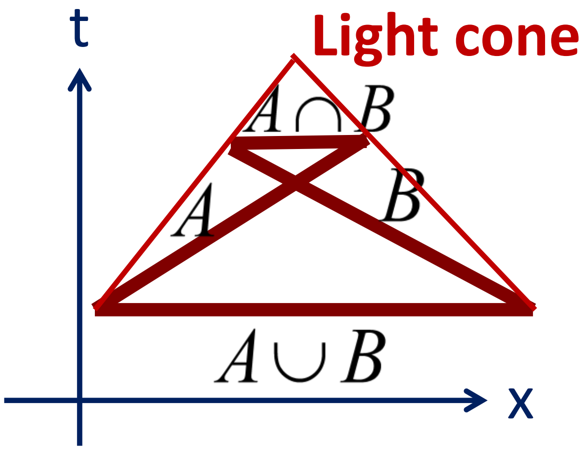
Boundary Entropy

[Affleck-Ludwig 91]



$$S_A = \frac{c}{3}\log\left(\frac{L}{\pi a}\sin\left(\frac{\pi x}{L}\right)\right)$$

Entropic C-theorem [Casini-Huerta 04]



Consider a relativistic QFT.

We have $S_A + S_B \geq S_{A\cup B} + S_{A\cap B}$,

$$l_A \cdot l_B = l_{A\cup B} \cdot l_{A\cap B} \quad .$$

We set $l_{A\cup B} = e^a, \quad l_{A\cap B} = e^b, \quad l_A = l_B = e^{(a+b)/2}$.

$$\Rightarrow 2\cdot S\left(\frac{a+b}{2}\right) \geq S(a) + S(b),$$

$$\Leftrightarrow \frac{\partial^2 S(x)}{\partial x^2} = \frac{1}{3}\cdot\frac{\partial C(x)}{\partial x} \leq 0 \quad \text{(entropic c - theorem).}$$

(2-2) Higher dimensional CFT

We can still apply the replica method:

$$S_A = -\frac{\partial}{\partial n}\log[\mathrm{Tr}\left(\rho_A\right)^n]\Big|_{n=1} = -\frac{\partial}{\partial n}\log\left[\frac{Z_n}{(Z_1)^n}\right]\Bigg|_{n=1} \quad .$$

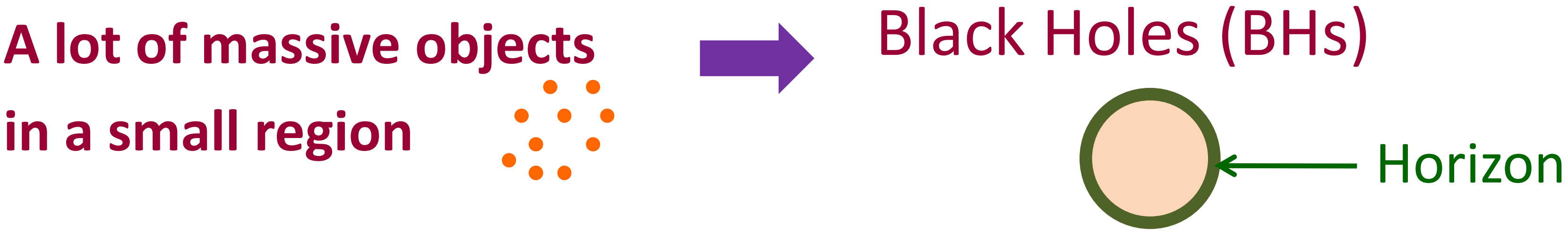
However, in general, there is no analytical way to calculate Z_n . (`Twist operators' get non-local !)
Thus in many cases, numerical calculations are needed.

➡ One motivation to explore the holographic analysis !

③ Holographic Entanglement Entropy

(3-1) What is ``Holography’’ ?

In the presence of gravity,

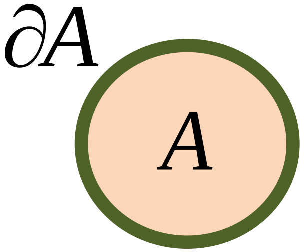


The information hidden inside BHs is measured by the Bekenstein-Hawking black hole entropy:

$$S_{BH} = \frac{\text{Area(Horizon)}}{4G_N}$$

This consideration leads to the idea of entropy bound:

$$S(A) \leq \frac{\mathrm{Area}(\partial \mathrm{A})}{4G_N}$$



(S(A) = the entropy in a region A)

➡ The degrees of freedom in gravity are proportional to the area instead of the volume !

cf. In non-gravitational theories, the entropy is proportional to volume.

Motivated by this, holographic principle has been proposed ['t Hooft 93, Susskind 94, ….]:

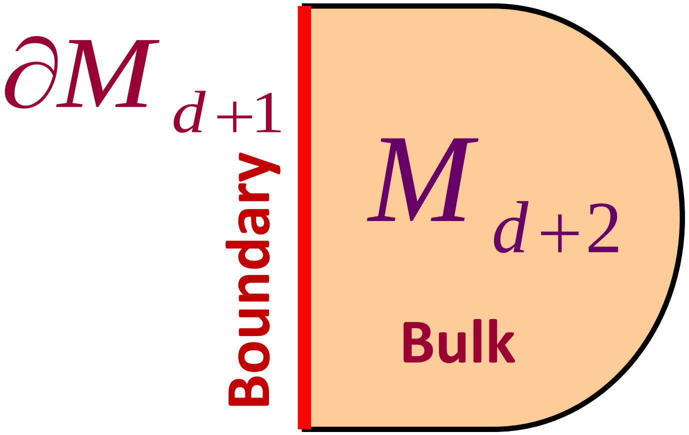
Often, lives on the boundary of (d+2) dim. spacetime

Holographic Principle

(d+2) dim. Quantum gravity

Equivalent ⟺

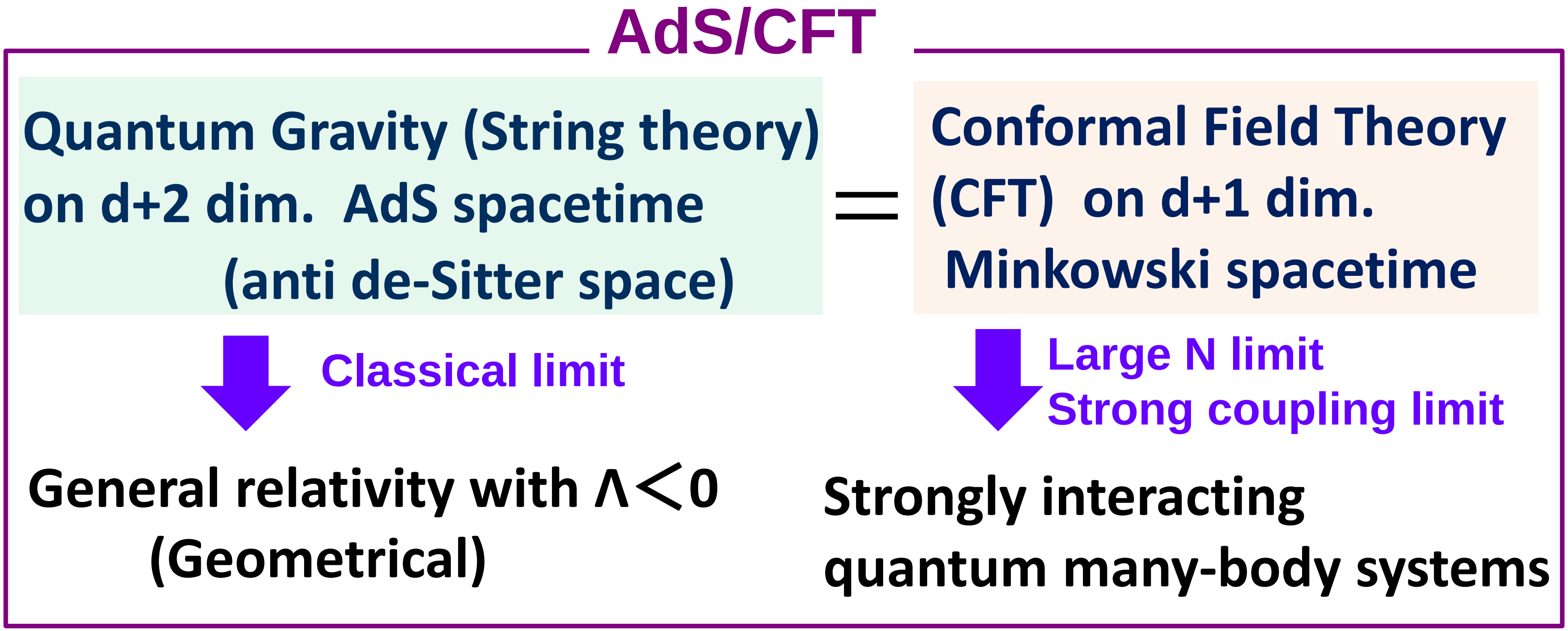
(d+1) dim. Non-gravitational theory (e.g. QM, QFT, CFT, etc.)



Note: Holography offers us a non-perturbative definition of quantum gravity !

(3-2) AdS/CFT (the best example of holography)

[Maldacena 97]



AdS/CFT

Quantum Gravity (String theory) on d+2 dim. AdS spacetime (anti de-Sitter space)

$=$

Conformal Field Theory (CFT) on d+1 dim. Minkowski spacetime

Classical limit

General relativity with $\Lambda < 0$ (Geometrical)

Large N limit
Strong coupling limit

Strongly interacting quantum many-body systems

$\mathrm{IIB\,string\ \ on\ \ AdS_5 \times S^5 \ \Leftrightarrow\ 4D\ N=4\ \ SU(N)\ \ SYM}$

$$\mathrm{SO(2,4)} \;=\; \text{4D conformal symmetry}$$

$$\mathrm{SO(6)} \;=\; \text{R - symmetry of } \mathrm{N=4\,SYM}$$

$$\frac{R_{AdS}}{l_{Planck}} \quad \propto \quad N^{1/4}$$

$$\frac{R}{l_{String}} \;=\; (Ng_{YM}^2)^{1/4} \equiv \lambda^{1/4}.$$

(i) small quantum gravity corrections = large N CFT

(ii) small stringy corrections = strong coupled CFT

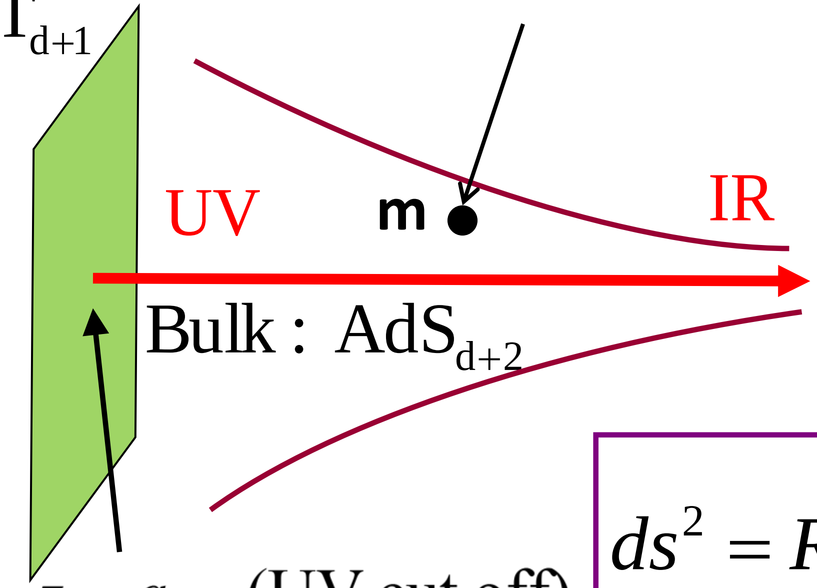
In this lecture, we mainly ignore both of these corrections.

Therefore we concentrate on strongly coupled large N CFT.

The meaning of the extra dimension

$$E_{\text{particle}} = m\sqrt{-g_{tt}} \propto \frac{m}{z}.$$

$\text{Boundary}: \ \text{CFT}_{\text{d+1}}$



UV

m

IR $\ z >> 1$

$z \sim$ length scale

$\text{Bulk}: \ \text{AdS}_{\text{d+2}}$

$z = a$ (UV cut off)

$$\boxed{ds^2 = R^2 \cdot \frac{dz^2 - dt^2 + d\vec{x}^2}{z^2}}$$

The radial direction z corresponds to the length scale in CFT under the RG flow. (1/z ~ Energy Scale)

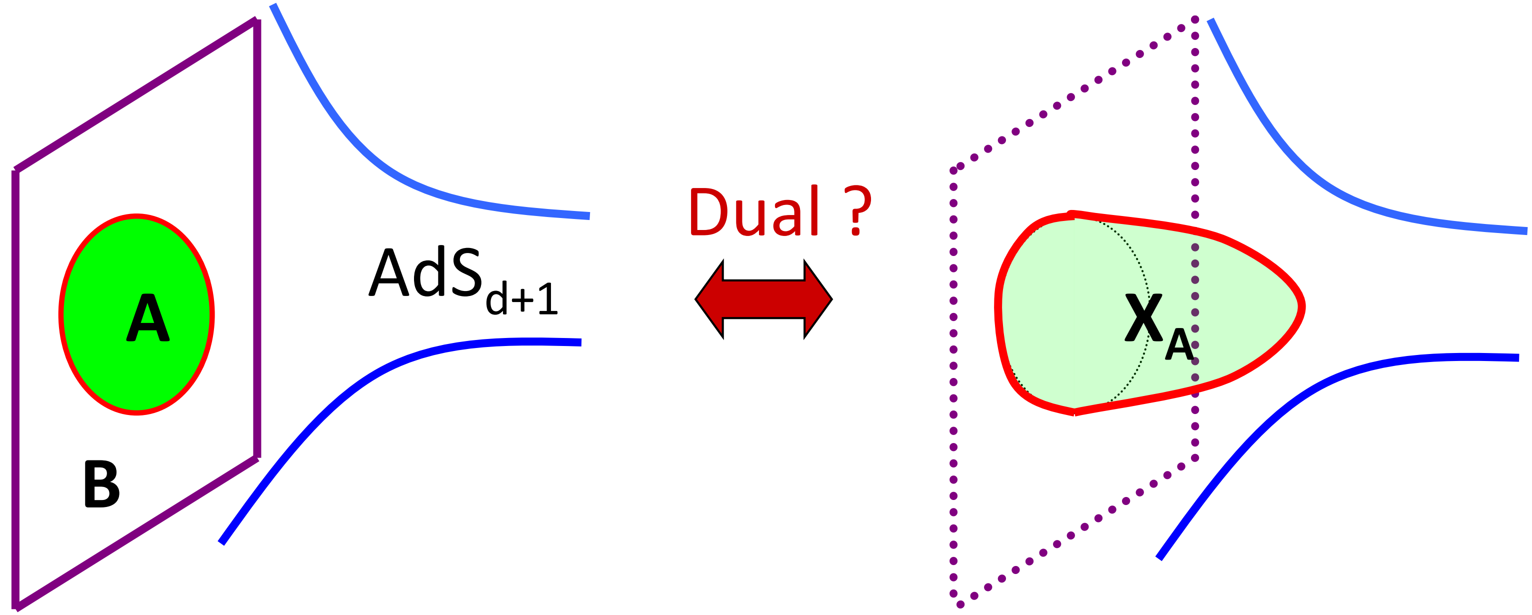
Bulk to boundary relation

The basic principle in AdS/CFT to calculate physical quantities is the bulk to boundary relation [GKP-W 98]:

$$\boxed{Z_{Gravity}(M) = Z_{CFT}(\partial M).}$$

Quantum Information in AdS ?

A Basic Question: Which region in the AdS does encode the `information in a certain region' of the CFT ?



Region A in CFT_d

Region X_A in AdS_{d+1}

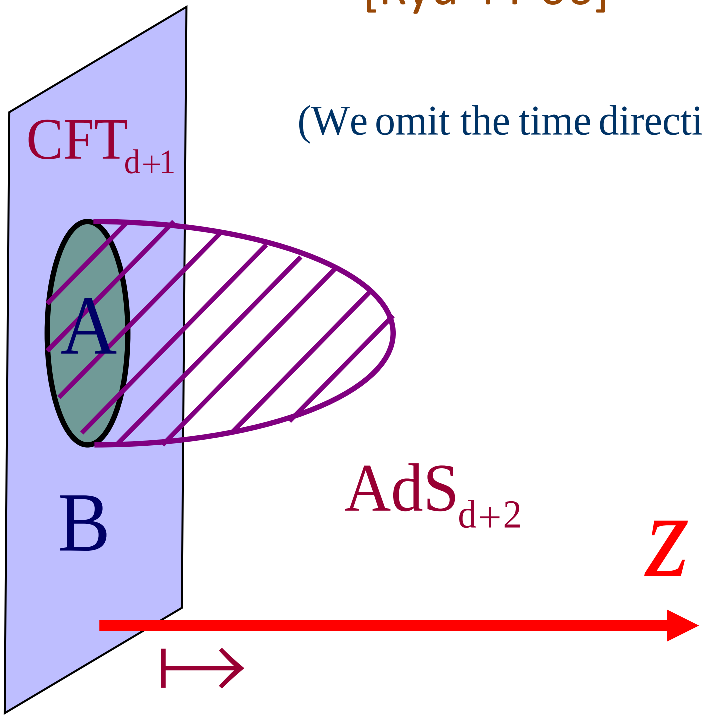
➡ Consider the entanglement entropy S_A which measures the amount of information !

(3-3) Holographic Entanglement Entropy (HEE)

Holographic Entanglement Entropy Formula

[Ryu-TT 06]

$$\boxed{S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N}}$$



(We omit the time direction.)

γ_A is the minimal area surface (codim.=2) such that

$\partial A = \partial \gamma_A$ and $A \sim \gamma_A$.

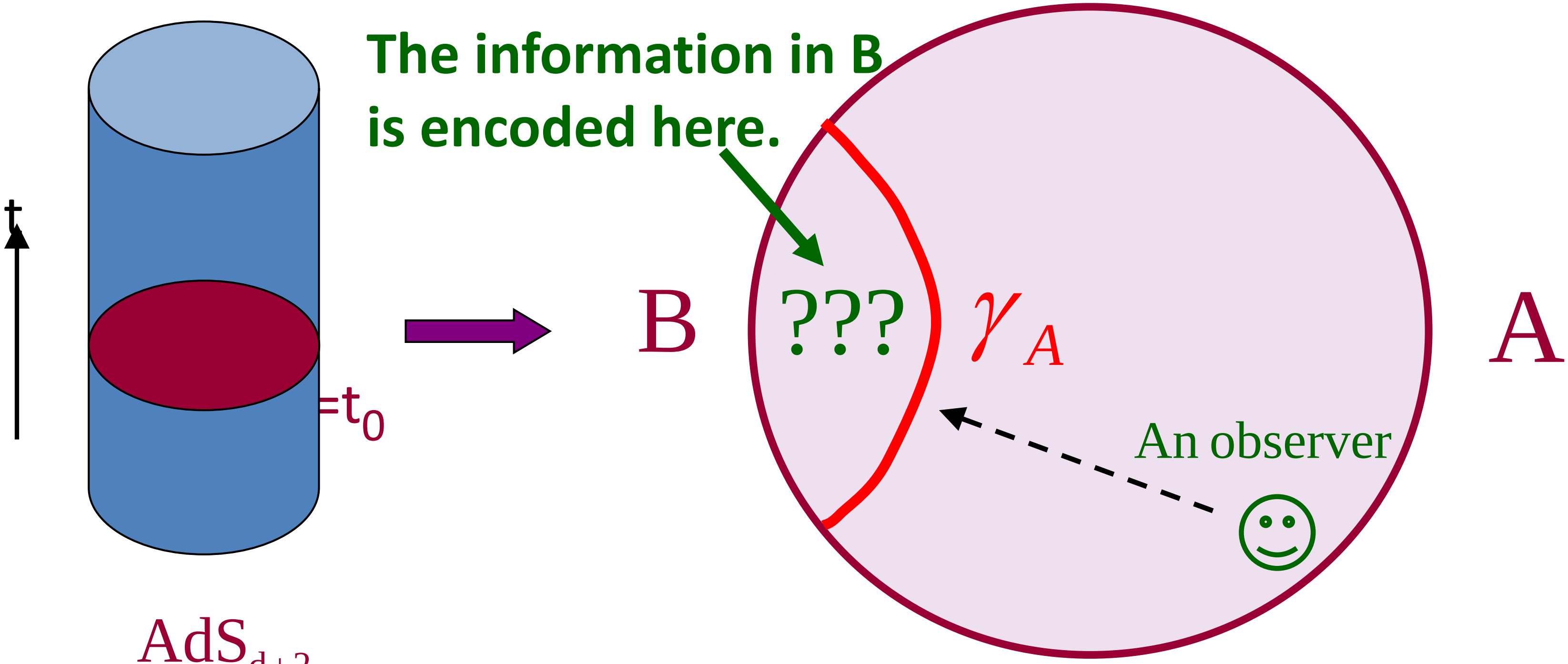
homologous

$z > a$ (UV cut off)

$$ds^2_{AdS} = R^2_{AdS} \frac{-dt^2 + \sum_{i=1}^{d-1} dx_i^2 + dz^2}{z^2} .$$

Motivation of HEE

Here we employ the global coordinate of AdS space and take its time slice at t=t_0.



The information in B is encoded here.

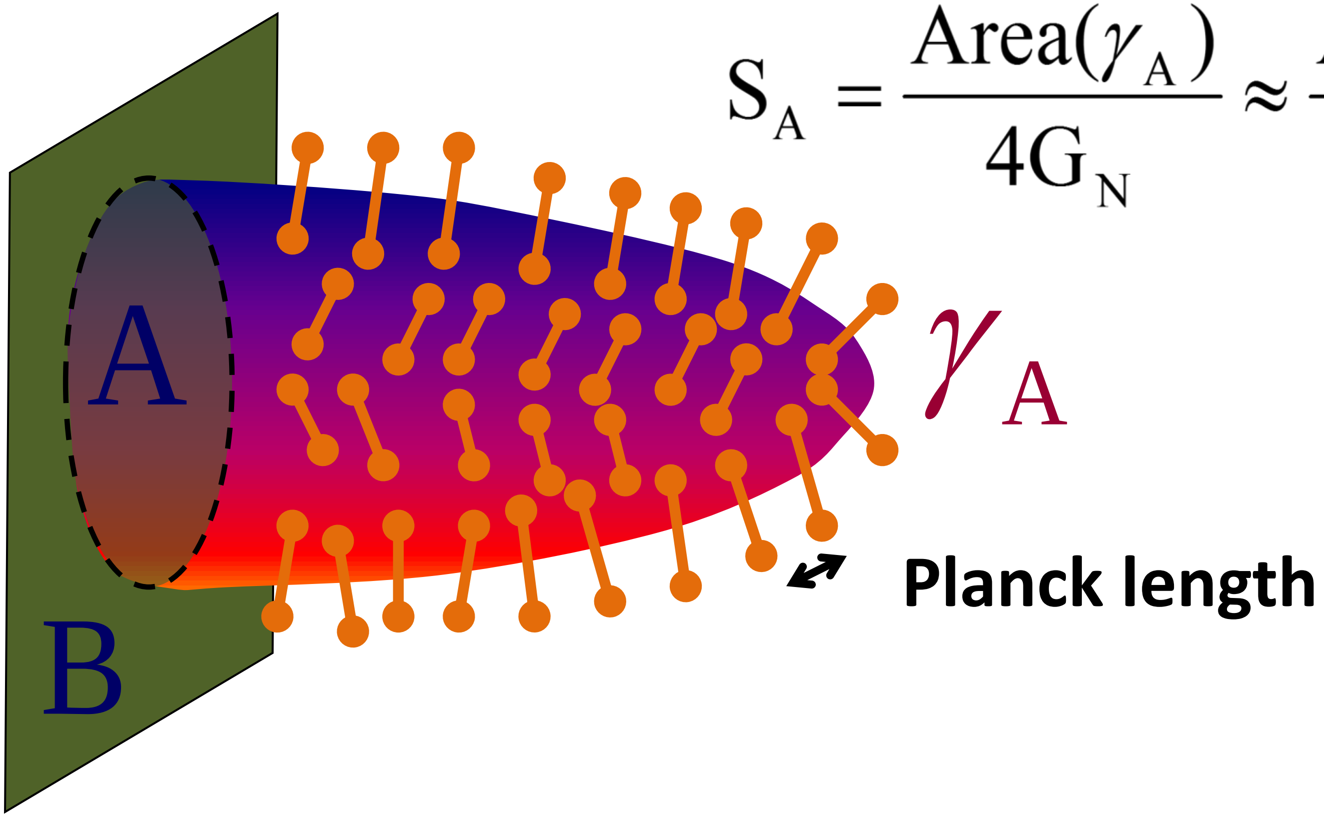
AdS_{d+2}

in global Coordinate

The HEE suggests that

A spacetime in gravity

= Collections of bits of quantum entanglement



$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N} \approx \frac{\mathrm{Area}(\gamma_A)}{l_{pl}^2}.$$

One Possiblity: **Entanglement Renormalization (MERA)** will be discussed later.

Comments

- If backgrounds are time-dependent, we need to employ extremal surfaces in the Lorentzian spacetime instead of minimal surfaces. If there are several extremal surfaces we should choose the smallest one.

[Hubeny-Rangamani-TT 07]

- In the presence of black hole horizons, the minimal surfaces wrap the horizon as the subsystem A grows enough large.

⇒ Reduced to the Bekenstein-Hawking entropy, consistently.

(3-4) Verifications of HEE

- Confirmations of basic properties:

Area law, Strong subadditivity (SSA), Conformal anomaly,….

- Direct Derivation of HEE from AdS/CFT:

(i) Pure AdS, A = a round sphere [Casini-Huerta-Myers 11]

(ii) Euclidean AdS/CFT [Lewkowycz-Maldacena 13, Faulkner 13, cf. Fursaev 06]

(iii) Disjoint Subsystems [Headrick 10, Faulkner 13, Hartman 13]

(iv) General time-dependent AdS/CFT → Not yet.

[But, evidences of SSA: Allais-Tonni 11, Callan-He-Headrick 12, Wall 13]

- Corrections to HEE beyond the supergravity limit:

[Higher derivatives: Hung-Myers-Smolkin 11, de Boer-Kulaxizi-Parnachev 11,…..]

[1/N effect: Barrella-Dong-Hartnoll-Martin 13,…]

[Higher spin gravity: de Boer-Jottar 13, Ammon-Castro-Iqbal 13]

Leading divergence and Area law

For a generic choice of γ_{A}, a basic property of AdS gives

$$\mathrm{Area}(\gamma_{\mathrm{A}}) \sim R^{d} \cdot \frac{\mathrm{Area}(\partial\gamma_{\mathrm{A}})}{a^{d-1}} + (\text{subleading terms}),$$

where R is the AdS radius.

Because $\partial\gamma_{\mathrm{A}} = \partial \mathrm{A}$, we find

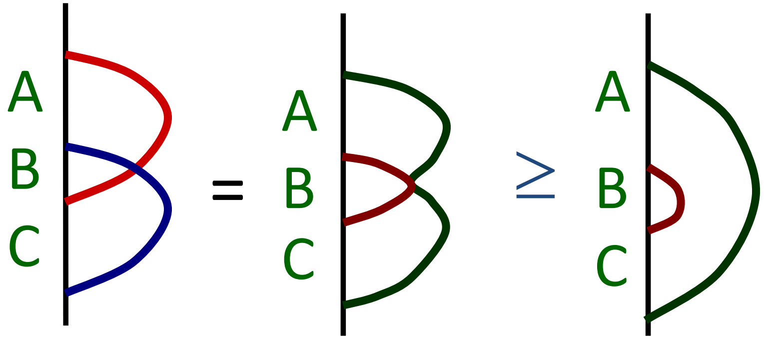
$$S_A \sim \frac{\mathrm{Area}(\partial \mathrm{A})}{a^{d-1}} + (\text{subleading terms}).$$

This agrees with the known area law relation in QFTs.

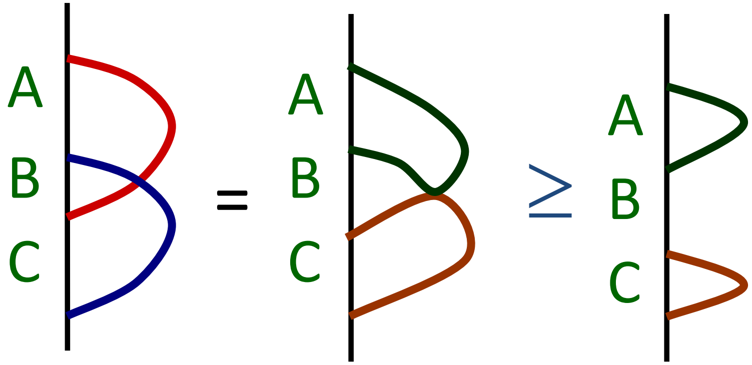
Holographic Strong Subadditivity

The holographic proof of SSA inequality is very quick !

[Headrick-TT 07]



$$\Rightarrow S_{A+B}+S_{B+C}\geq S_{A+B+C}+S_{B}$$



$$\Rightarrow S_{A+B}+S_{B+C}\geq S_{A}+S_{C}$$

Note: This proof can be applied if $S_A=\underset{\gamma_A}{\mathrm{Min}}\left[F(\gamma_A)\right]$, for any functional F.

⇒ higher derivative corrections

HEE from AdS3/CFT2

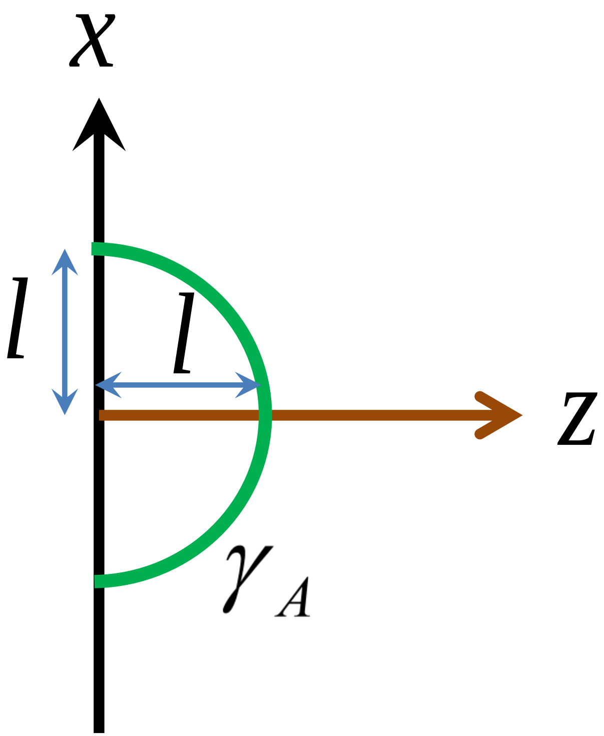
In AdS3/CFT2, the HEE is given by the geodesic length in the AdS3:

$$ds^2 = R^2 \cdot \frac{dz^2 - dt^2 + dx^2}{z^2}.$$

This is explicitly evaluated as follows:

$$x = \sqrt{l^2 - z^2} \quad \Rightarrow \quad ds^2_{circle} = \frac{l^2 dz^2}{z^2\sqrt{l^2 - z^2}}.$$

$$L(\gamma_A) = 2R\int_a^l dz \frac{l}{z\sqrt{l^2 - z^2}} = 2R\log\frac{2l}{a}.$$



Finally, the HEE is found to be

$$S_A = \frac{L(\gamma_A)}{4G_N^{(3)}} = \frac{2R}{4G_N^{(3)}} \log\left(\frac{2l}{a}\right) = \frac{c}{3}\log\left(\frac{2l}{a}\right),$$

where we employed the famous relation

$$c = \frac{3R}{2G_N^{(3)}}.$$

[Brown-Henneaux 86]

In this way, HEE reproduces the 2 dim. CFT result.

Finite temperature CFT

Consider a 2d CFT in the high temp. phase $\frac{l}{\beta} >> 1$.

⇒ The dual gravity background is the BTZ black hole:

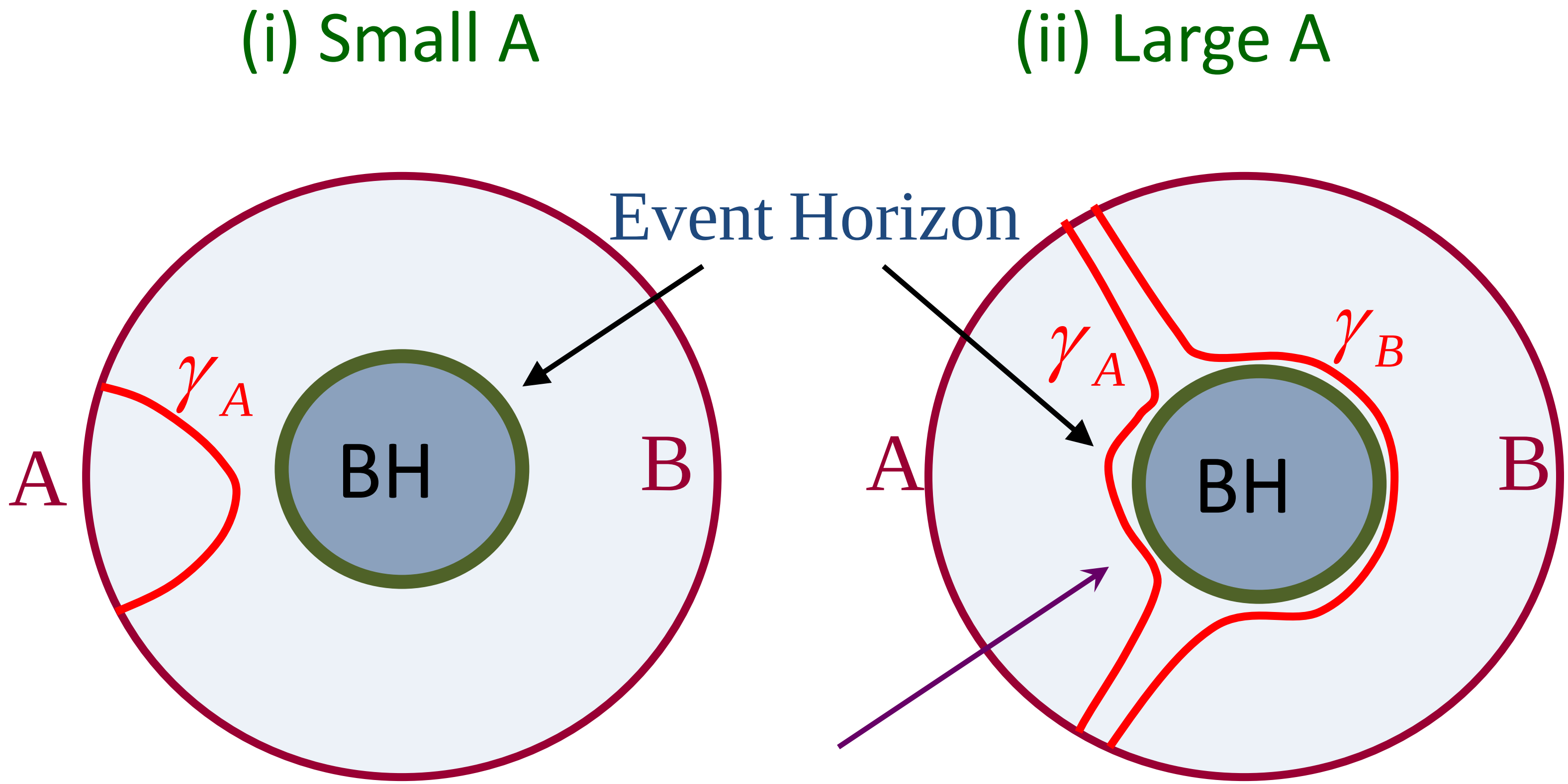
$$ds^2 = -(r^2 - r_H^2)dt^2 + \frac{R^2}{r^2 - r_H^2}dr^2 + r^2 d\phi^2,$$

$$\text{where} \quad \phi \sim \phi + 2\pi, \qquad \frac{L}{\beta} = \frac{r_H}{R} >> 1.$$

$$\Rightarrow \quad S_A = \frac{c}{3}\log\left(\frac{\beta}{a}\sinh\left(\frac{\pi l}{\beta}\right)\right) .$$

agrees with the 2d CFT result.

Geometric Interpretation



(i) Small A

(ii) Large A

Event Horizon

When A is large (i.e. high temperature), γ_A wraps a part of horizon. This leads to the thermal contribution $S_A \approx (\pi/3)c\,lT$ to the entanglement entropy.

Note : $S_A \neq S_B$ due to the BH .

Disconnected Subsystem and Phase Transition

$$A = A_1 \cup A_2$$

[Headrick 10]

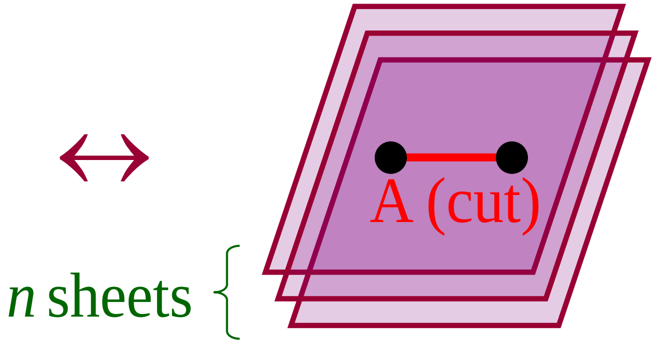
This is consistent with the CFT calculations done in [Furukawa-Pasquier-Shiraishi 08, Calabrese-Cardy-Tonni 09] .

Derivation of HEE Formula

Let us try to derive the HEE from the bulk-boundary relation of AdS/CFT. ⇒ We employ the replica method.

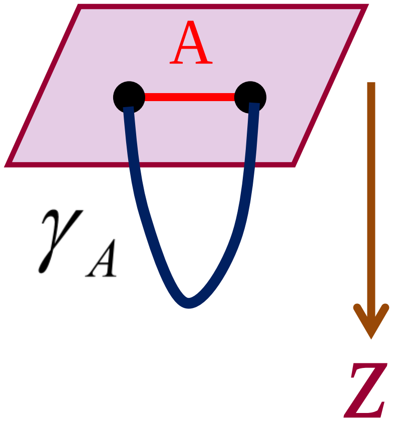
In the CFT side, the (negative) deficit angle $2\pi(1-n)$ is localized on ∂A:

$$\mathrm{Tr}_A[\rho_A^n] \quad \leftrightarrow$$



Naïve Assumption : The AdS dual is given by extending the deficit angle into the bulk AdS. [Fursaev 06]

⇒ The curvature is delta functionally localized on the deficit angle surface:



$$R = 4\pi(n-1)\cdot\delta(\gamma_A) + \dots.$$

$$\Downarrow$$

$$S_{gravity} = \frac{1}{16\pi G_N}\int dx^{d+2}\sqrt{g}R + \dots \rightarrow \frac{\text{Area}(\gamma_{\text{A}})}{4G_N}\cdot(n-1).$$

$$\Downarrow$$

$$S_A = -\frac{\partial}{\partial n}\log \mathrm{tr}_A \rho_A^n = -\frac{\partial}{\partial n}\log\left(\frac{\mathrm{Z_n}}{(\mathrm{Z_1})^n}\right) = \frac{\text{Area}(\gamma_{\text{A}})}{4G_N}.$$

$$\delta S_{gravity} = 0 \rightarrow \gamma_A = \text{minimal surface}\,!$$

However, this argument is not exactly correct ! [Headrick 10]

⇒ Indeed, $\mathrm{tr}_A \rho_A^n$ (Renyi entropy) does not agree with the known 2d CFT results for n=2,3,...

Recently, it was explained that this naïve argument gives correct results only in the n→1 limit [Lewkowycz-Maldacena 13]:

$\mathrm{tr}_A \rho_A^n$ should be dual to a smooth geometry without any deficit angles. However, the difference becomes

$$[\mathrm{tr}_A \rho_A^n]_{\mathrm{Smooth}} - [\mathrm{tr}_A \rho_A^n]_{\mathrm{Singular}} = O\left((n-1)^2\right) \qquad (n \to 1).$$

owing to the Einstein eq. ⇒ Correct results for EE !

But not for Renyi entropy !

Higher derivative corrections to HEE

Consider stringy corrections but ignore loop corrections in AdS. (⇔deviations from strongly coupled limit, but still large N in CFT)

[Hung-Myers-Smolkin 11, de Boer-Kulaxizi-Parnachev 11,.. Dong, Camps 13]

Ex. Gauss-Bonnet Gravity

$$S_{GBG}=-\frac{1}{16G_N}\int dx^{d+2}\sqrt{g}\left[R-2\Lambda+\lambda R_{AdS}^2 L_{GB}\right]$$

$$L_{GB}\equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2.$$

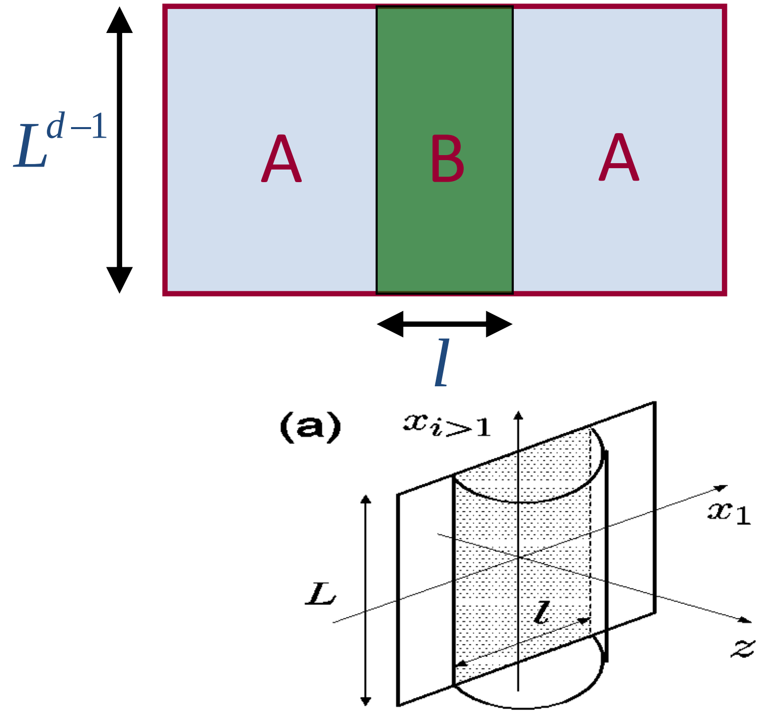
➡

$$S_A=\underset{\gamma_A}{\mathrm{Min}}\left[\frac{1}{4G_N}\int_{\gamma_A}dx^d\sqrt{h}(1+2\lambda R_{AdS}^2 R)\right].$$

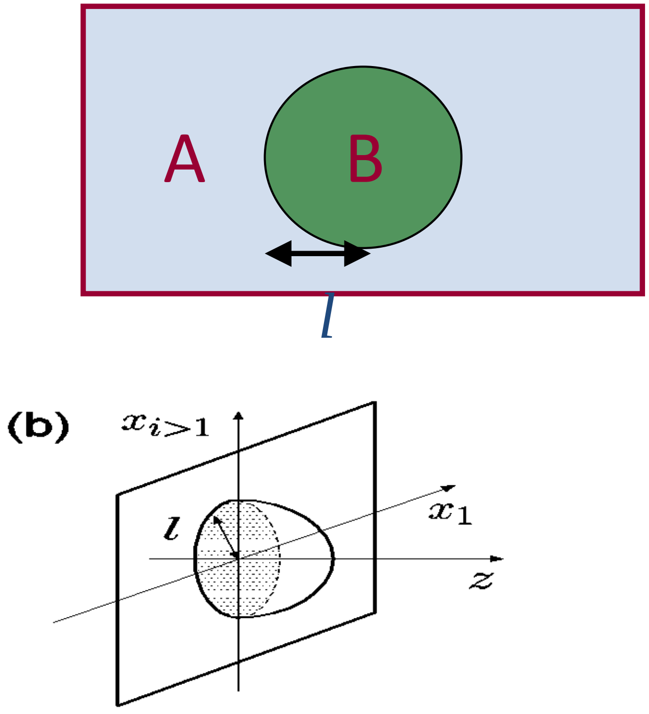
(3-5) HEE in Higher dim.

Consider the HEE in the Poincare metric dual to a CFT on $R^{1,d}$. We concentrate on the following two examples:

(a) Strip



(b) Circular disk



Entanglement Entropy for (a) Infinite Strip from AdS

$$S_A = \frac{R^d}{2(d-1)G_N^{(d+2)}}\left[\underline{\left(\frac{L}{a}\right)^{d-1}} - C\cdot\underline{\left(\frac{L}{l}\right)^{d-1}}\right],$$

where $C = 2^{d-1}\pi^{d/2}\left(\Gamma\left(\frac{d+1}{2d}\right)\Big/\Gamma\left(\frac{1}{2d}\right)\right)^{d}.$

Area law divergence

This term is finite and does not depend on the UV cutoff.

d=1 (i.e. AdS3) case:

$$S_A = \frac{R}{2G_N^{(3)}}\log\frac{l}{a} = \frac{c}{3}\log\frac{l}{a}.$$

Agrees with 2d CFT results

[Holzhey-Larsen-Wilczek 94 ; Calabrese-Cardy 04]

Entanglement Entropy for (b) Circular Disk from AdS

[Ryu-TT 06]

$$S_A = \frac{\pi^{d/2} R^d}{2G_N^{(d+2)}\Gamma(d/2)}\left[\underline{p_1\left(\frac{l}{a}\right)^{d-1}} + p_3\left(\frac{l}{a}\right)^{d-3} + \cdots \right.$$

$$\left. \cdots + \begin{cases} p_{d-1}\left(\frac{l}{a}\right) + \underline{p_d} & (\text{if } d = \text{even}) \\ p_{d-2}\left(\frac{l}{a}\right)^2 + \underline{q\log\left(\frac{l}{a}\right)} & (\text{if } d = \text{odd}) \end{cases}\right],$$

Area law divergence

$$\text{where} \quad p_1 = (d-1)^{-1},\ p_3 = -(d-2)/[2(d-3)],....$$

$$.... \quad q = (-1)^{(d-1)/2}(d-2)!!/(d-1)!! \quad .$$

A universal quantity which characterizes odd dim. CFT

⇒ Satisfy 'C-theorem'

[Myers-Sinha 10; closely related to F-theorem Jafferis-Klebanov-Pufu-Safdi 11]

Conformal Anomaly (central charge)

2d CFT $c/3 \cdot \log(l/a)$

4d CFT $-4a \cdot \log(l/a)$

[Ryu-TT 06, Solodukhin 08,10, Lohmayer-Neuberger-Schwimmer-Theisen 09, Dowker 10, Casini-Huerta, 10, Myers-Sinha 10, Casini-Hueta-Myers 11]

④ Properties of EE for Excited States

(4-1) One Simple Motivation

1st law of thermodynamics: **$T \cdot dS = dE$**

Temp. Information Energy

⇒ Can we find an analogous relation in any quantum systems which are far from the equilibrium ?

Something like: **$T_{ent} \cdot dS_A = dE_A$** ??

Information in A Energy in A

= EE

Can we observe EE ??

(4-2) Holographic Calculation of EE for excited states

Consider an asymptotically AdSd+2 background

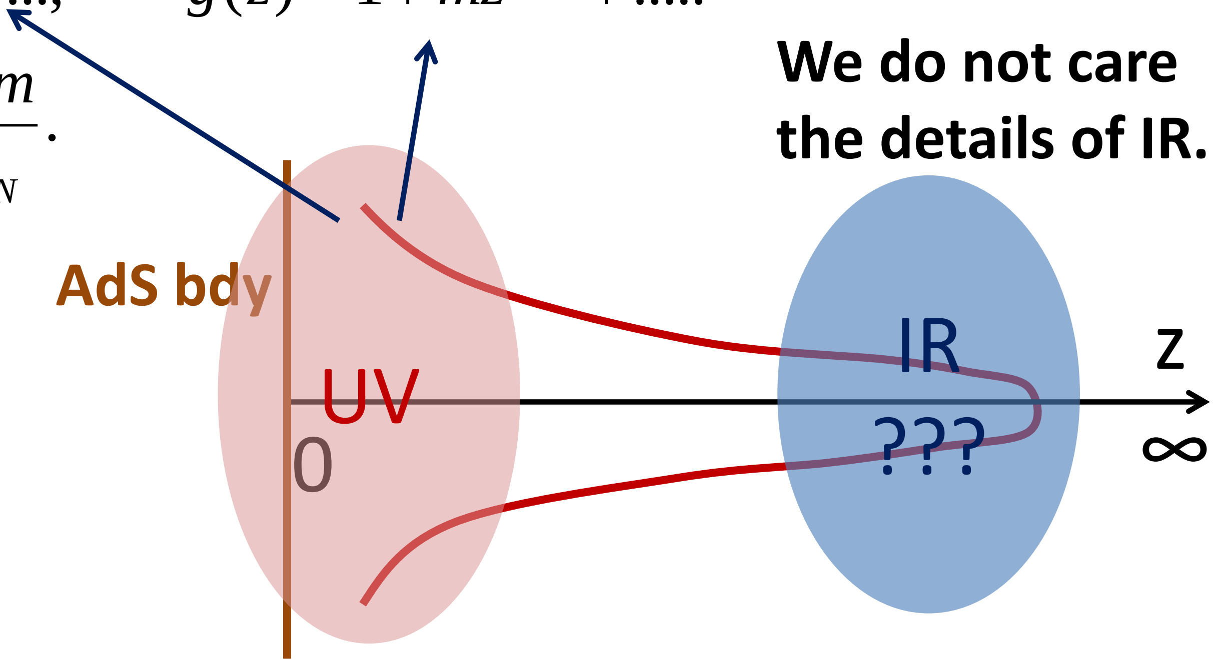
(= an excited state in CFTd+1):

$$ds^2 = \frac{R^2}{z^2}\left(-f(z)dt^2 + g(z)dz^2 + \sum_{i=1}^{d} dx_i^2\right) ,$$

$$f(z) = 1 - mz^{d+1} + ..., \qquad g(z) = 1 + mz^{d+1} + ...$$

$$\Rightarrow \varepsilon = T_{tt} = \frac{dR^{d+1}m}{16\pi G_N} .$$

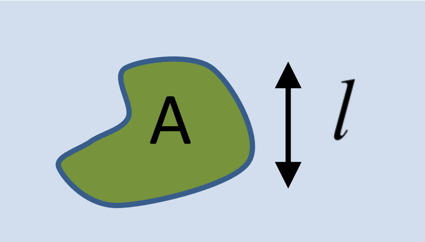
Energy density



We do not care the details of IR.

Assume the size l of subsystem A is small such that

$$ml^{d+1} << 1,$$



then we can show $T_{ent} \cdot \Delta S_A = \Delta E_A,$

where $\Delta E_A = \int_A dx^d T_{tt}.$

The `entanglement temperature' is given by $T_{ent} = \frac{c}{l}.$

The constant c is universal in that it only depends on the shape of the subsystem A: $e.g. \quad c = \frac{d+2}{2\pi} \quad \text{when A} = \text{a round sphere.}$

Holographic Prediction

Consider exited states in a CFT which has approximately translational and rotational invariance.

If the subsystem A is small enough such that

$$T_{tt} \cdot l^{d+1} << R^d / G_N \approx O(N^2),$$

then the following `1st law' like relation is satisfied:

$$\underbrace{T_{ent} \cdot \Delta S_A}_{\textbf{Info.}} = \underbrace{\Delta E_A}_{\textbf{Energy}}, \qquad T_{ent} \equiv \frac{c}{l} \ ,$$

Note: The constant c depends only on the geometry of A.

More Recent Progresses

(1) The first law can be simply expressed as follows: [Blanco-Casini-Hung-Myers 13, Wong-Klich-Pando Zayas-Vaman 13]

$$\Delta S_A = \Delta H_A , \quad (\rho_A \equiv e^{-H_A}).$$

(2) The perturbative Einstein eq. is equivalent to a constraint of HEE:
$$\left(\partial_l^2 - \partial_l - \partial_{\vec{x}}^2 - \frac{3}{l^2}\right)\Delta S_A(t,\vec{x},l) = \langle O\rangle\langle O\rangle$$

[Nozaki-Numasawa-Prudenziati-TT 13, Bhattacharaya-TT 13]

(3) Moreover, the first law was shown to be equivalent to the perturbative Einstein eq.

[Lashkari-McDermott-Raamsdonk 13, Faulkner-Guica-Hartman-Myers-Raamsdonk 13]

(4-3) Large Subsystem Limit of EE [Nozaki-Numasawa-TT 14]

If the size of subsystem A is large (or equally for largely excited states), then the details of the IR geometry are very important.

⇒ We cannot expect any universal properties like the first law.

⇒ We will study solvable explicit examples (i.e. massless free scalar field theory) by direct field theory calculations.

We will focus on the following quantities:

$$\Delta S_A^{(n)}\left[\left|O\right\rangle\right] = S_A^{(n)}\left[\left|O\right\rangle\right] - S_A^{(n)}\left[\left|\mathrm{vac}\right\rangle\right] ,$$

$$\text{Excited state}: \quad \left|O\right\rangle \equiv O(x)\left|\mathrm{vac}\right\rangle.$$

$$S_A^{(n)}\left[\left|\Psi\right\rangle\right] = \text{Entanglement } n\text{-th Renyi Entropy}$$

$$\text{for the state} \left|\Psi\right\rangle$$

Replica Method for Excited States

We want to calculate $\mathrm{Tr}\left(\rho_A\right)^n$ for

$$\rho_{tot}(t,x) = e^{-iHt} e^{-\varepsilon H} O(x) \left| \mathrm{vac} \right\rangle \left\langle \mathrm{vac} \right| O(x) e^{-\varepsilon H} e^{iHt}$$

$$= O(\tau_e, x) \left| \mathrm{vac} \right\rangle \left\langle \mathrm{vac} \right| O(\tau_l, x),$$

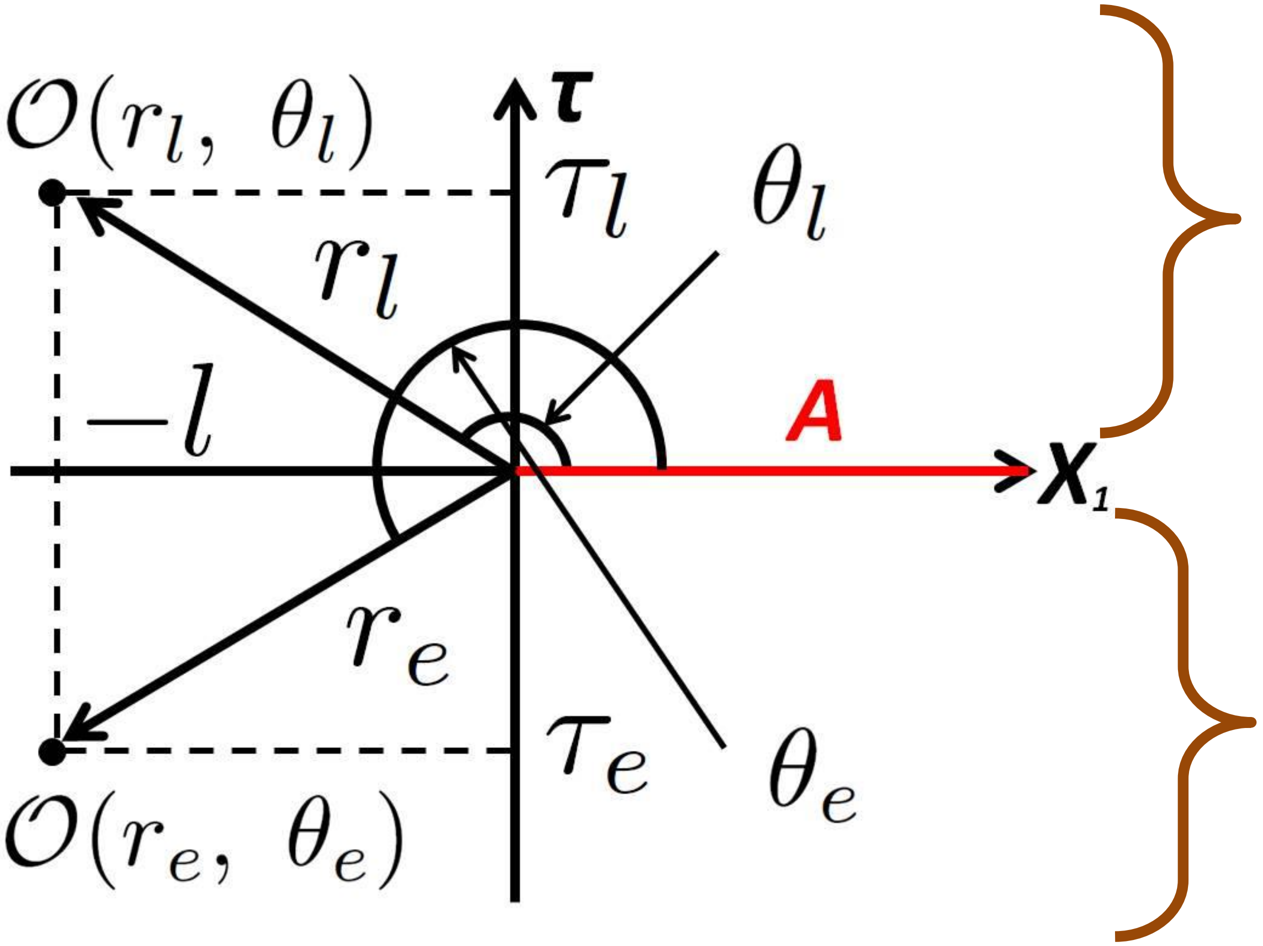
$$(\tau_e \equiv -\varepsilon - it, \qquad \tau_l \equiv -\varepsilon + it),$$

where ε is the UV regulator for the operator.

Note: τ denotes the Euclidean time. We compute the time evolution via the Euclidean analytical continuation.

Consider a $d+1$ dim. CFT.

$(\tau, x_1, x_2, \cdots, x_d) \in \mathrm{R}^{d+1} \quad \Rightarrow \quad$ We set $x_1 + i\tau = re^{i\theta}$.

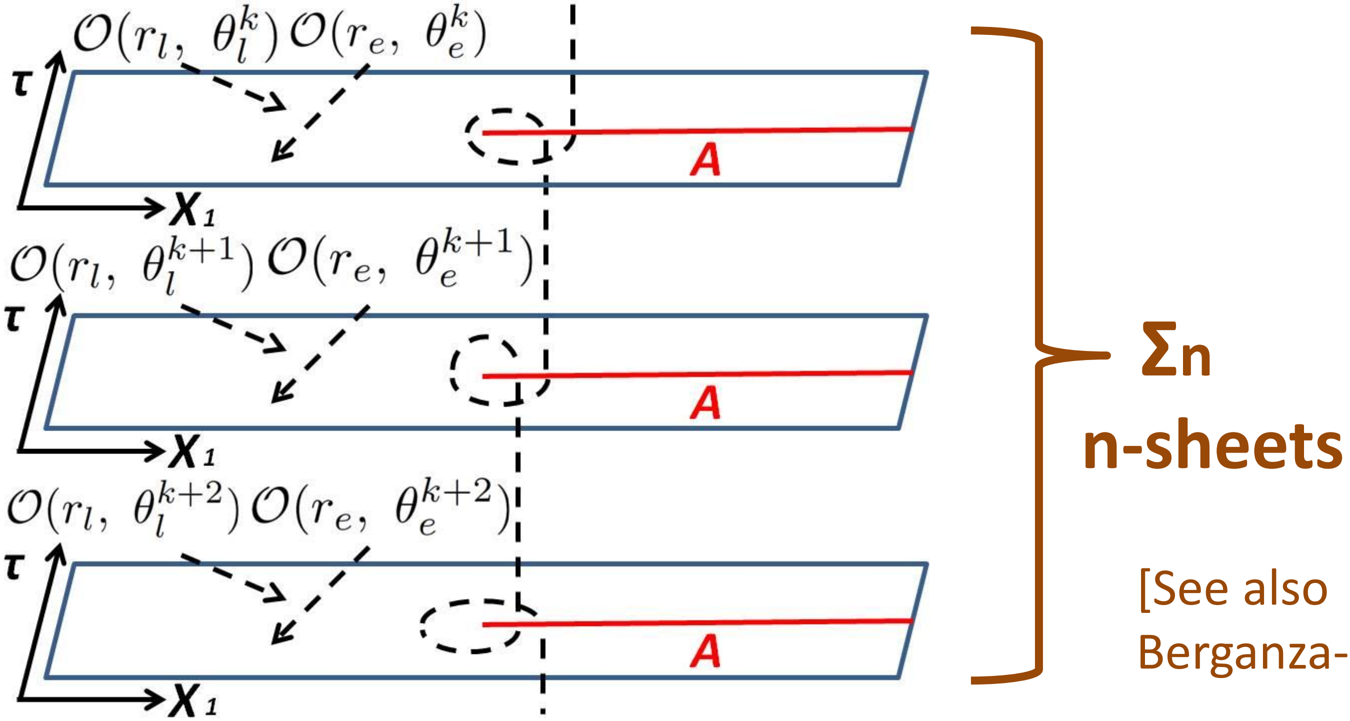


$\langle \mathrm{vac} | O(\tau_l, x)$

$O(\tau_e, x) | \mathrm{vac} \rangle$

In this way, the (Renyi) EE can be expressed in terms of correlation functions (2n-point function etc.) on Σn :

$$\Delta S_A^{(n)} = \frac{1}{1-n}\cdot\Big[\log\big\langle O(r_l,\theta_l^n)O(r_e,\theta_e^n)\cdots O(r_l,\theta_l^1)O(r_e,\theta_e^1)\big\rangle_{\Sigma_n} - n\cdot\log\big\langle O(r_l,\theta_l\ )O(r_e,\theta_e\ )\big\rangle_{\Sigma_1}\Big].$$



[See also Berganza-Alcaraz-Sierra 11]

Explicit Calculations of (Renyi) Entanglement Entropy

We focus on the free massless scalar field theory

$$S=\int d^{d+1}x\left[\partial_\mu\phi\partial^\mu\phi\right]$$

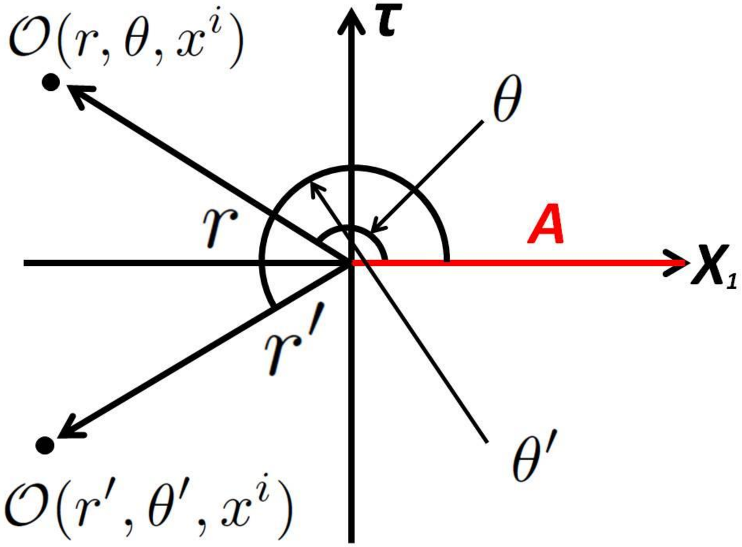
and calculate 2n-pt functions using the Green function:

$$G_n[(r,\theta,\vec{x});(s,\varphi,\vec{y})]=\frac{1}{4n\pi^2 rs(a-1/a)}\cdot\frac{a^{1/n}-a^{-1/n}}{a^{1/n}+a^{-1/n}-2\cos\left((\theta-\varphi)/n\right)},$$

where $\frac{a}{1+a^2}\equiv\frac{rs}{|\vec{x}-\vec{y}|^2+r^2+s^2}.$

The operator O is chosen as

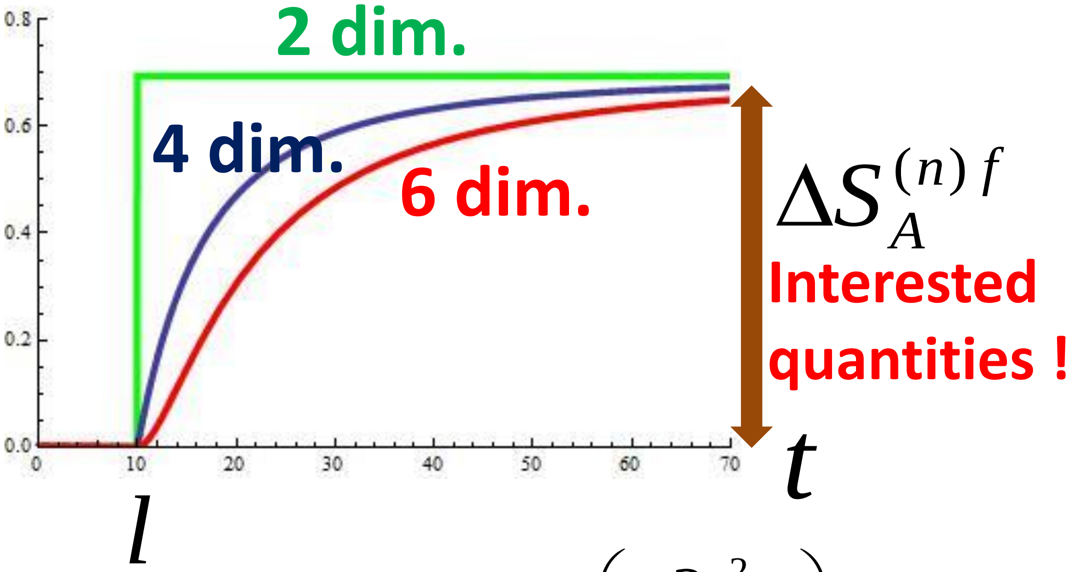
$$O_k=:\phi^k: \qquad .$$

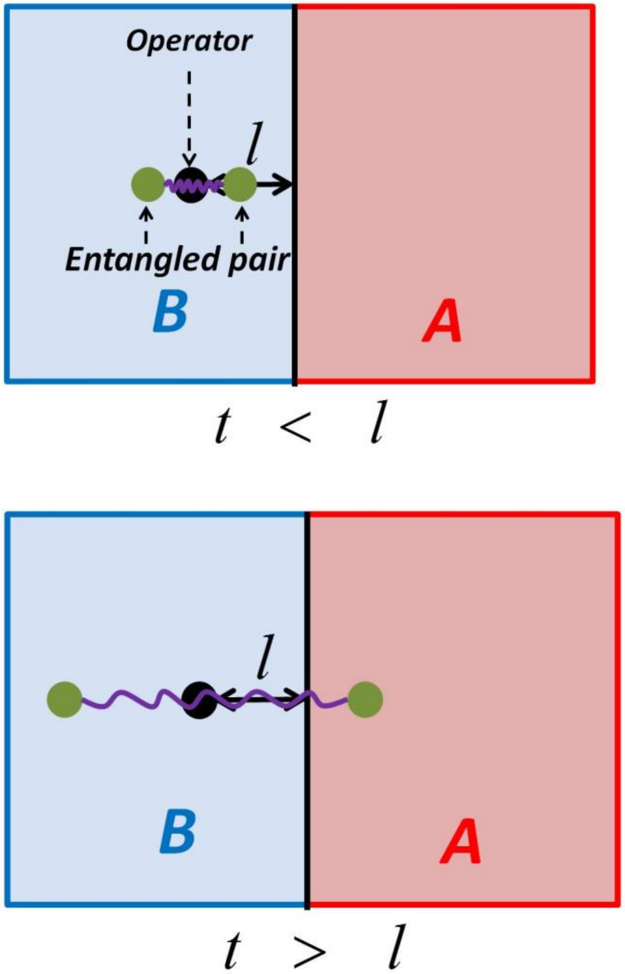


Time evolution in free massless scalar theory

$\Delta S_A^{(2)}$ for $O = :\phi:$ (i.e. $k=1$)

$$\begin{pmatrix} \text{We chose } x_1 = -l \text{ with } l = 10 \\ \text{and } x_2 = \cdots = x_d = 0. \end{pmatrix}$$





$$E.g. \quad \Delta S_{A(4\text{dim})}^{(2)} = \log\left(\frac{2t^2}{t^2+l^2}\right).$$

Our conjecture:
(Monotonicity)

$$\boxed{\frac{d}{dt}\Delta S_A^{(n)} \geq 0.}$$

$\Delta S_A^{(n)f}$ for $O=\phi^k$ in $d+1>2$ dim.

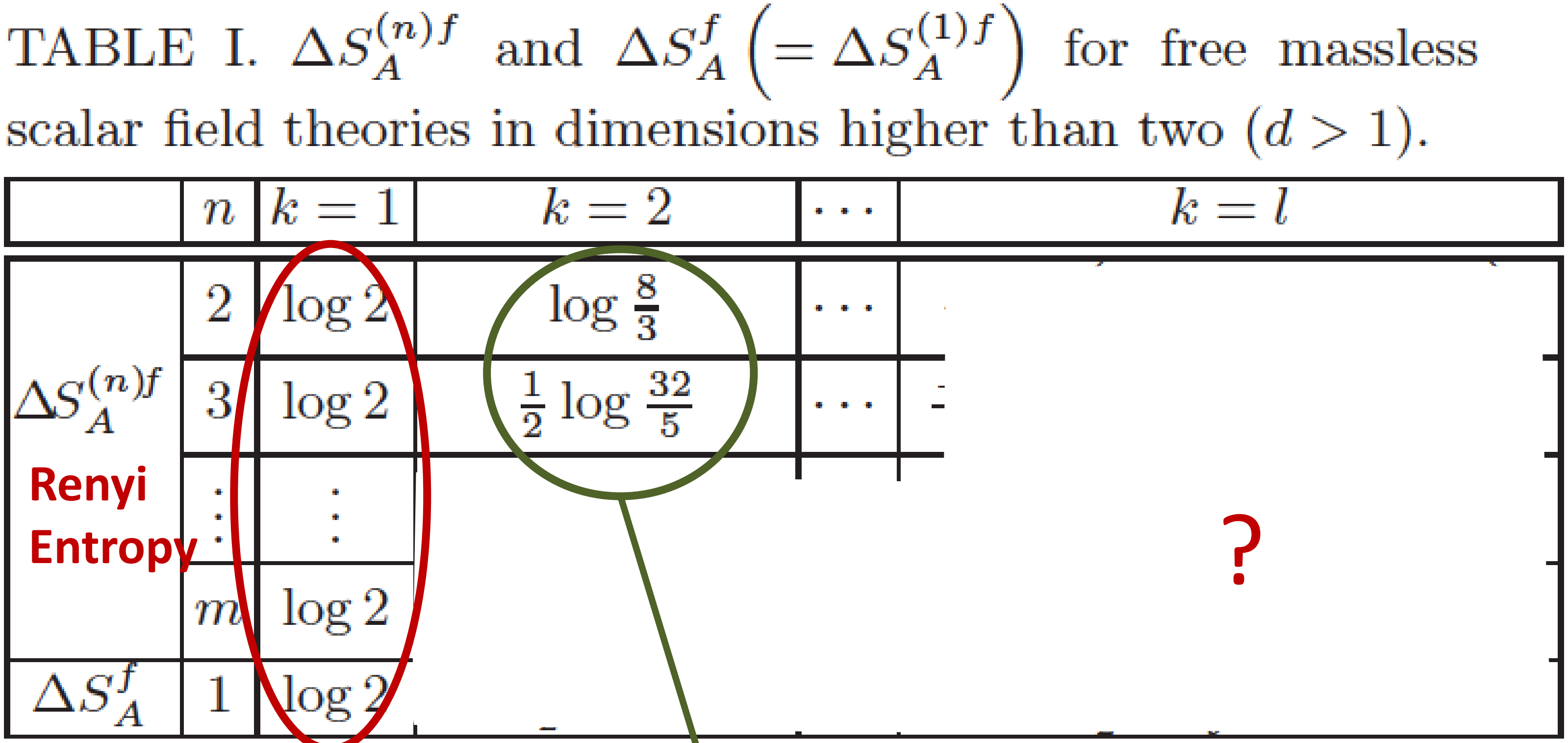


TABLE I. $\Delta S_A^{(n)f}$ and $\Delta S_A^f \left(= \Delta S_A^{(1)f}\right)$ for free massless scalar field theories in dimensions higher than two ($d > 1$).

	n	$k=1$	$k=2$	$\cdots$	$k=l$
$\Delta S_A^{(n)f}$	2	$\log 2$	$\log \frac{8}{3}$	$\cdots$	
	3	$\log 2$	$\frac{1}{2}\log\frac{32}{5}$	$\cdots$	
	$\vdots$	$\vdots$			
	m	$\log 2$			
ΔS_A^f	1	$\log 2$			

Renyi Entropy

?

EE

EPR state !

What do these mean ?
(Log[rational number] ?)

$\Delta S_A^{(n)f}$ for $O=\phi^k$ in $d+1>2$ dim.

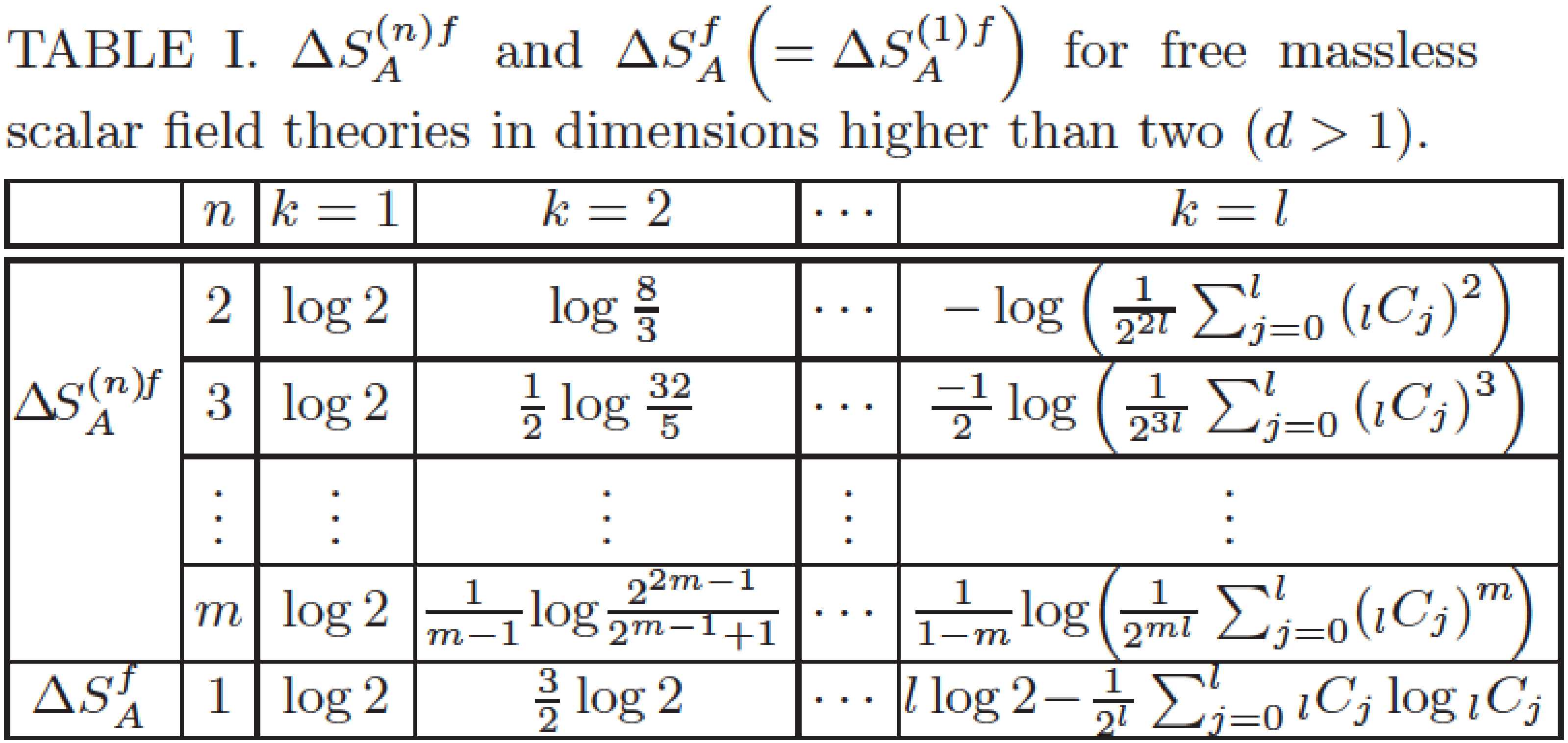


TABLE I. $\Delta S_A^{(n)f}$ and $\Delta S_A^f \left(=\Delta S_A^{(1)f}\right)$ for free massless scalar field theories in dimensions higher than two ($d>1$).

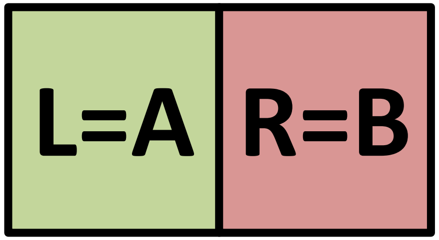
	n	$k=1$	$k=2$	$\cdots$	$k=l$
$\Delta S_A^{(n)f}$	2	$\log 2$	$\log \frac{8}{3}$	$\cdots$	$-\log\left(\frac{1}{2^{2l}}\sum_{j=0}^{l}\left({}_lC_j\right)^2\right)$
	3	$\log 2$	$\frac{1}{2}\log\frac{32}{5}$	$\cdots$	$\frac{-1}{2}\log\left(\frac{1}{2^{3l}}\sum_{j=0}^{l}\left({}_lC_j\right)^3\right)$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	m	$\log 2$	$\frac{1}{m-1}\log\frac{2^{2m-1}}{2^{m-1}+1}$	$\cdots$	$\frac{1}{1-m}\log\left(\frac{1}{2^{ml}}\sum_{j=0}^{l}\left({}_lC_j\right)^m\right)$
ΔS_A^f	1	$\log 2$	$\frac{3}{2}\log 2$	$\cdots$	$l\log 2-\frac{1}{2^l}\sum_{j=0}^{l}{}_lC_j\log{}_lC_j$

$${}_lC_j \equiv \frac{l!}{(l-j)!\cdot j!}$$

General interpretation

First , notice that in free CFTs, there are definite (quasi) particles moving at the speed of light.

$$\Rightarrow \quad \phi \approx \underset{\text{left-moving}}{\phi_L} + \underset{\text{right-moving}}{\phi_R} \quad .$$



$$\phi^n|\mathrm{vac}\rangle \approx \sum_{j=0}^{k} {}_kC_j \cdot (\phi_L)^j \cdot (\phi_R)^{k-j}|\mathrm{vac}\rangle$$

$$= 2^{-k/2}\sum_{j=0}^{k}\sqrt{{}_kC_j}\,|j\rangle_L|k-j\rangle_R .$$

Normalized as $\langle j|j'\rangle = \delta_{j,j'}$

By tracing out the subsystem B(=R), we find that the matrix ρA is the k+1 times k+1 diagonal matrix:

$$\rho_A = 2^{-k} \cdot \begin{pmatrix} {}_kC_0 & & & \\ & {}_kC_1 & & \\ & & \ddots & \\ & & & {}_kC_k \end{pmatrix}.$$

$$\Rightarrow \quad \Delta S_A^{(n)f} = \frac{1}{1-n} \log\left[2^{-nk} \sum_{j=0}^{k} ({}_kC_j)^n\right],$$

$$\Delta S_A^{f} = k \log 2 - 2^{-k} \sum_{j=0}^{k} {}_kC_j \cdot \log[{}_kC_j].$$

This agrees with explicit results from the replica method.

It suggests that $\Delta S_A^{(n)f}$ is `topological invariant’ w.r.t. A.

(4-4) Free Scalar CFT in 2d

In two dimension (d=1), the left and right decomposition is exact !

In the massless scalar field theory, there is one subtlety: the conformal dimension of ϕ is vanishing.

Thus $O =: \phi^k :$ is not a good local operator !

⇒ Instead, we can consider

$$O_1 =: e^{i\alpha\phi} : \quad \text{or} \quad O_2 =: e^{i\alpha\phi} : + : e^{-i\alpha\phi} : \qquad .$$

In the case $O_1 =: e^{i\alpha\phi}:$ we find the result is trivial:

$$\Delta S_A^{(n)f} = 0.$$

This is simply because $O_1 = e^{i\alpha\phi_L}\left|\mathrm{vac_L}\right\rangle \otimes e^{i\alpha\phi_R}\left|\mathrm{vac_R}\right\rangle$ is a direct product state.

On the other hand,

$$O_2 = e^{i\alpha\phi_L}\left|\mathrm{vac_L}\right\rangle \otimes e^{i\alpha\phi_R}\left|\mathrm{vac_R}\right\rangle + e^{-i\alpha\phi_L}\left|\mathrm{vac_L}\right\rangle \otimes e^{-i\alpha\phi_R}\left|\mathrm{vac_R}\right\rangle$$

$$\approx \left|\uparrow\right\rangle_L \left|\uparrow\right\rangle_R + \left|\downarrow\right\rangle_L \left|\downarrow\right\rangle_R$$

is the EPR state and indeed we find $\Delta S_A^{(n)f} = \log 2.$

(4-5) Interacting Rational CFTs

We can show that the late time EE in 2d rational CFTs is given by $\Delta S_A^{(n)f} = \log d_O$.

[He-Numasawa-Watanabe-TT 14]

Here do is so called the quantum dimension:

$$d_O \equiv \frac{S_{I,O}}{S_{I,I}} \quad ,$$

which measures degrees of freedom of an operator O.

Example: Ising model

3 conformal blocks: $[I],\ [\sigma],\ [\varepsilon].$

$$[I]\otimes[I]=[I], \quad [\varepsilon]\otimes\ [\varepsilon]=[I]. \quad \Rightarrow d_I = d_\varepsilon = 1.$$

$$[\sigma]\otimes[\sigma]=[I]\oplus[\varepsilon]. \quad \Rightarrow d_\sigma = \sqrt{2}.$$

$$\text{Thus we find}: \ \Delta S_A^{(n)}[I] = \Delta S_A^{(n)}[\varepsilon] = 0,$$

$$\Delta S_A^{(n)}[\sigma] = \log\sqrt{2}.$$

⑤ Holography and Entanglement Renormalization

(5-1) Outline

We may obtain a metric from a CFT as follows:

a CFT state ⇒ EE = Minimal Areas ⇒ metric

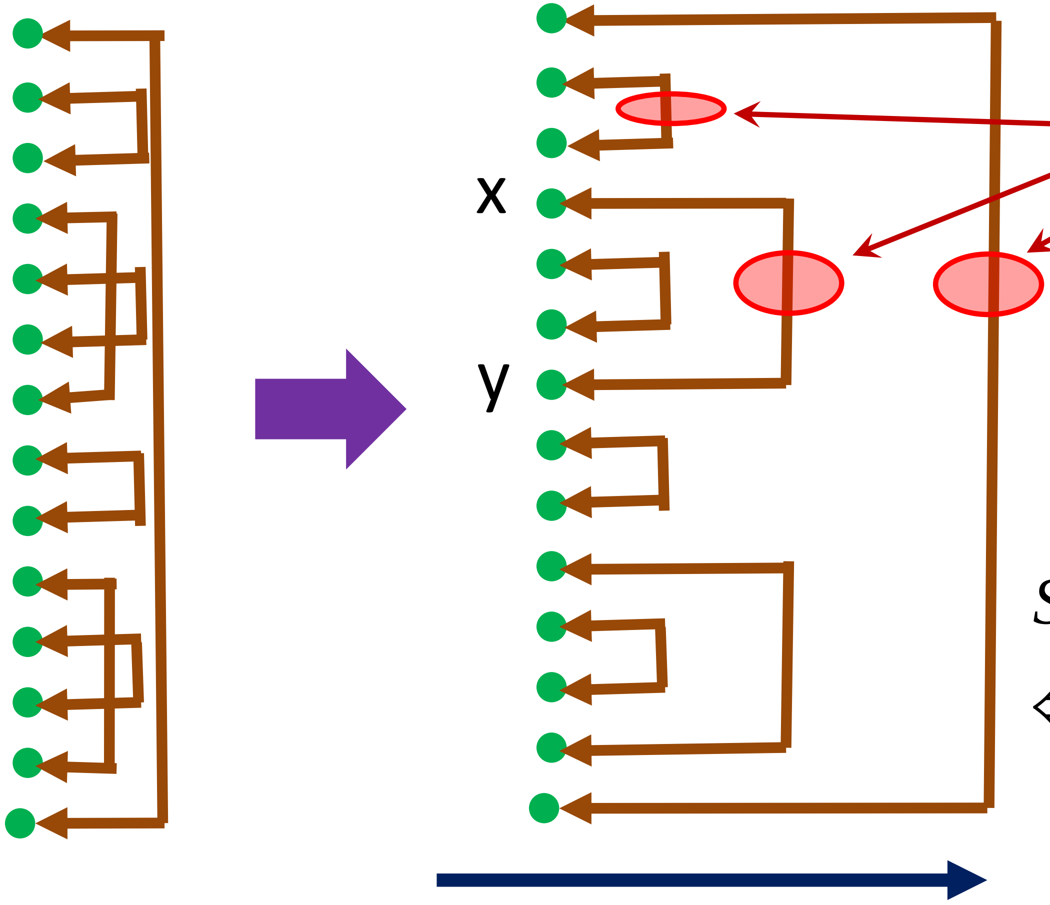
$|\Psi\rangle$ S_A $\mathrm{Area}(\gamma_A)$ $g_{\mu\nu}$

One candidate of such frameworks is so called the entanglement renormalization (MERA) [Vidal 05] as pointed out by [Swingle 09].

[cf. Other approach to emergent gravity: Raamsdonk 09, Lee 09]

Basic Idea

Classify the entanglement between spins by their ranges.



1 qubit

= an unit area in QG

Dynamics of $\left|\Psi\right\rangle$

$=$Waves of Entanglemet

$=$Gravitational waves

$$S_A = S(x, y, t)$$

$$\Leftrightarrow g_{\mu\nu}\left((x+y)/2, z = y - x, t\right)$$

Extra dimension (z=y-x)

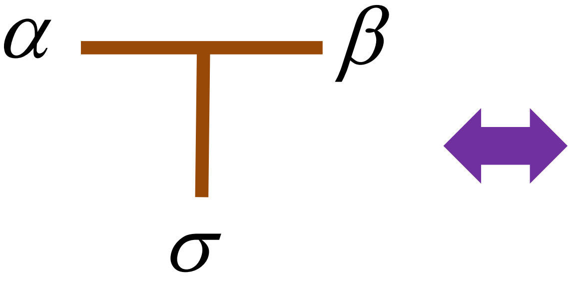
(5-2) Tensor Network (TN)

[See e.g. the review Cirac-Verstraete 09]

Recently, there have been remarkable progresses in numerical algorithms for quantum lattice models, based on so called **tensor product states**.

This leads to various **nice variational ansatzs** for the ground state wave functions in various quantum many-body systems.

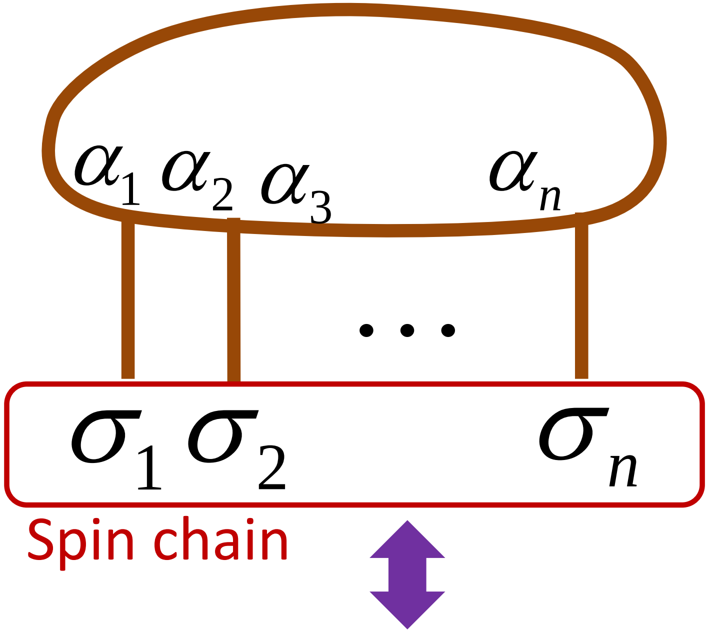
⇒ An ansatz is good if it respects quantum entanglement of the true ground state.



$$M_{\alpha\beta}(\sigma)$$

Ex. Matrix Product State (MPS)

[DMRG: White 92,…,
Rommer-Ostlund 95,..]



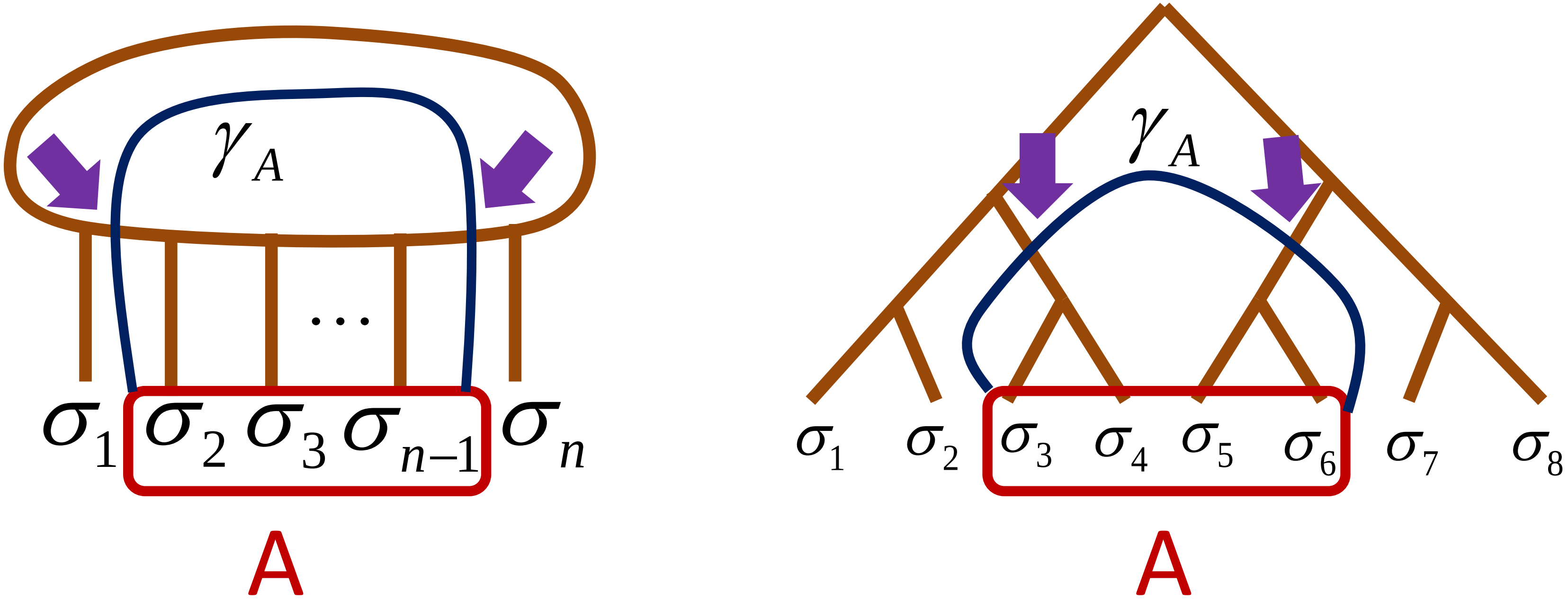
$$\alpha_i = 1,2,\ldots,\chi,$$

$$\sigma_i = \uparrow \text{ or } \downarrow .$$

$$|\Psi\rangle = \sum_{\sigma_1,\sigma_2,\cdots,\sigma_n} \mathrm{Tr}[M(\sigma_1)M(\sigma_2)\cdots M(\sigma_n)] \, \underset{n\ \mathit{Spins}}{|\sigma_1,\sigma_2,\cdots,\sigma_n\rangle}$$

MPS and TTN are not good near quantum critical points (CFTs) because EE in CFTs are too large to describe:

$$S_A \leq 2\log\chi \quad (<< \log L \sim S_A^{CFT}).$$



In general,

$$\boxed{\begin{aligned} &S_A \sim N_{\text{int}} \cdot \log\chi, \\ &N_{\text{int}} \equiv \min[\#\text{Intersections of } \gamma_A]. \end{aligned}}$$

(5-3) AdS/CFT and (c)MERA

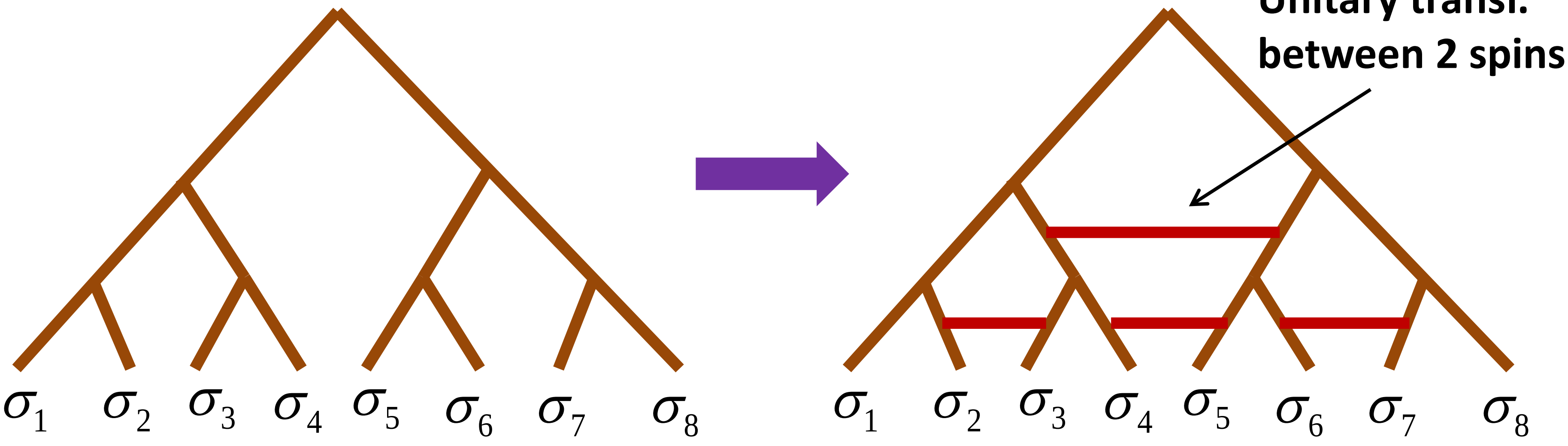
<u>MERA</u> (**M**ultiscale **E**ntanglement **R**enormalization **A**nsatz):

An efficient variational ansatz for CFT ground states

[Vidal 05 (for a review see 0912.1651)].

To respect its large entanglement in a CFT,

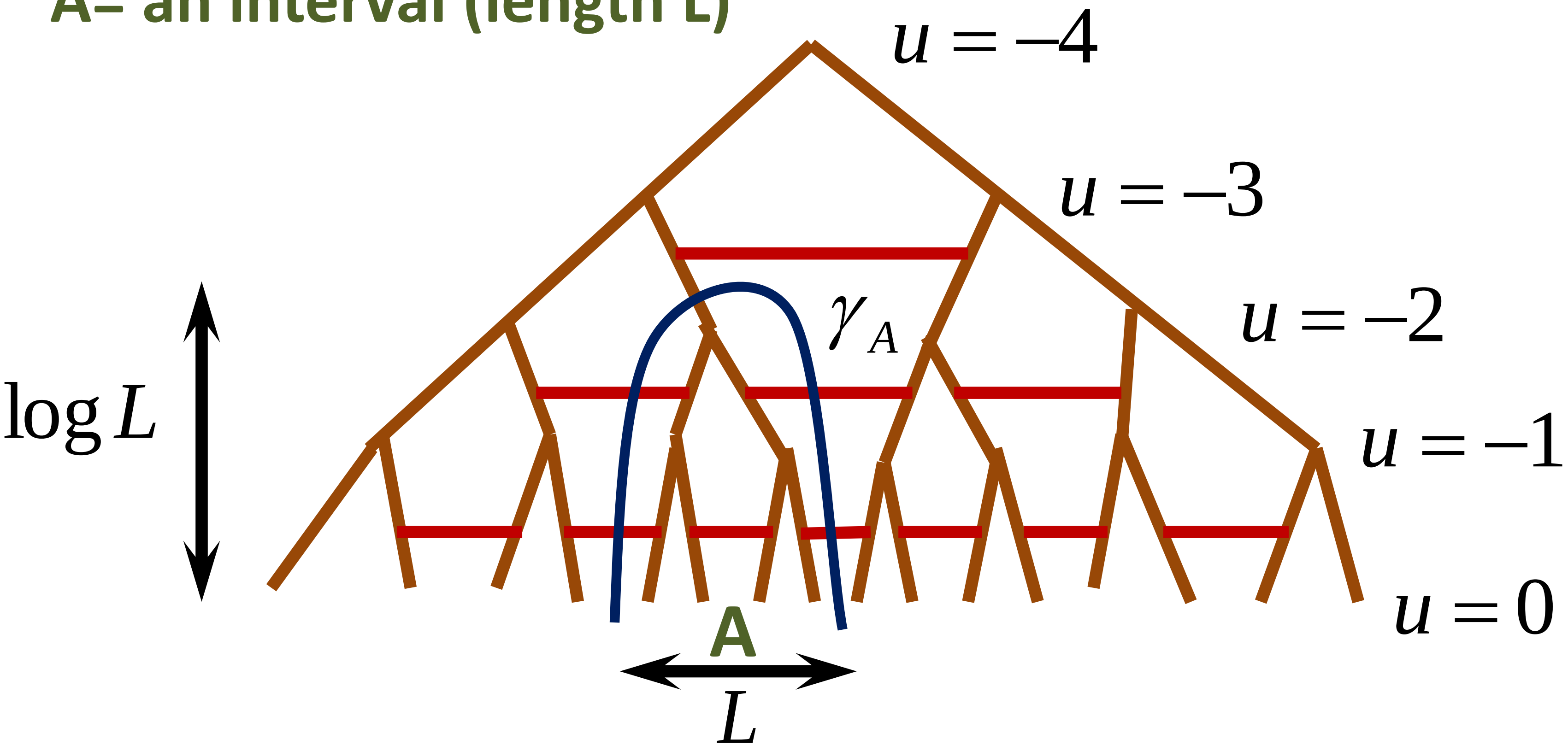
we add **(dis)entanglers**.



Unitary transf.
between 2 spins

Calculations of EE in 1+1 dim. MERA

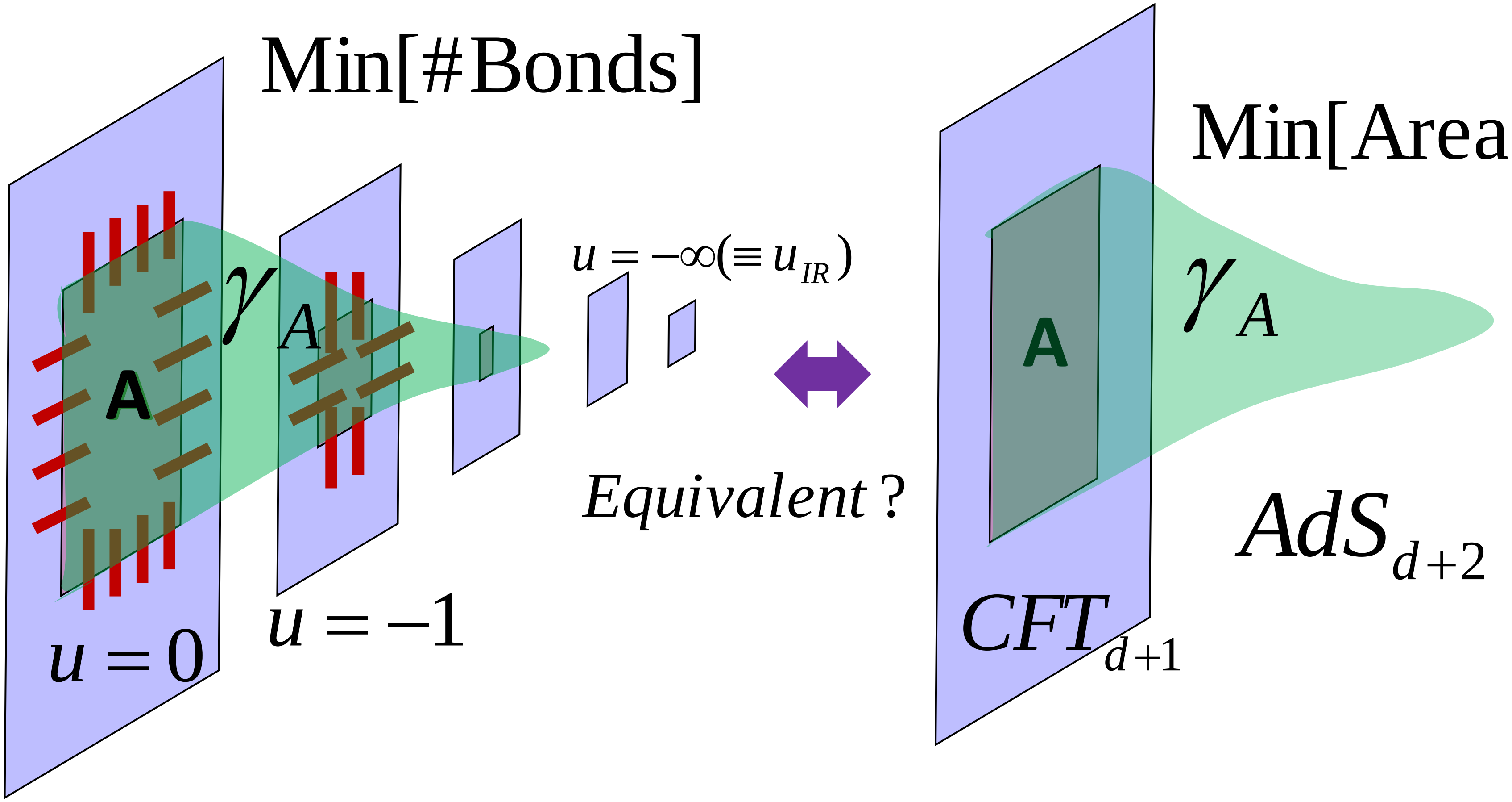
A= an interval (length L)



$$S_A \propto \mathrm{Min}[\#\mathrm{Bonds}] \propto \log L$$

$\Rightarrow$ agrees with 2d CFTs.

A conjectued relation to AdS/CFT [Swingle 09]



$$\text{Metric} = ds^2 + \frac{e^{2u}}{\varepsilon^2}(-dt^2 + d\vec{x}^2) = \frac{dz^2 - dt^2 + d\vec{x}^2}{z^2},$$

where $z = a \cdot e^{-u}$.

Now, to make the connection to AdS/CFT clearer, we want to consider the MERA for QFTs.

Continuous MERA (cMERA)

[Haegeman-Osborne-Verschelde-Verstraete 11]

$$\underbrace{|\Psi(u)\rangle}_{\text{True ground state (highly entangled)}} = P \cdot \exp\left(-i \int_{-\infty}^{u} ds [K(s) + L] \right) \cdot \underbrace{|\Omega\rangle}_{\text{IR state (no entanglement)}} ,$$

$$\Rightarrow \text{Real space renormalization flow: length scale} \sim a \cdot e^{-u}.$$

K(u) : disentangler, L: scale transformation

Conjecture

$$d+1 \ \text{dim. cMERA} = \text{gravity on } \mathrm{AdS}_{d+2} \qquad z = a \cdot e^{-u}.$$

(5-4) Emergent Metric from cMERA [Nozaki-Ryu-TT 12]

We focus on gravity duals of translational invariant static states, which are not conformal in general.

We conjecture that the metric in the extra direction is given by using the Bures metric (or Fisher information metric):

$$\boxed{g_{uu}du^2 = N\cdot\left(1-\left|\left\langle\Psi(u)\,|\,e^{iLdu}\,|\,\Psi(u+du)\right\rangle\right|^2\right).}$$

$N^{-1}\equiv\int dx^d\cdot\int_0^{\Lambda e^u} dk^d =$ The total volume of phase space at energy scale u.

Bures Metric

The **Bures distance** between two states is defined by

$$D(\psi_1,\psi_2)=1-\left|\langle\psi_1|\psi_2\rangle\right|^2.$$

More generally, for two mixed states ρ1 and ρ2,

$$D(\rho_1,\rho_2)=1-\mathrm{Tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}.$$

When the state depends on the parameters {ξi},
the **Bures metric (Fisher information metric)** is defined as

$$D[\psi(\xi),\psi(\xi+d\xi)]=g_{ij}d\xi^i d\xi^j.$$

⇒ Reparameterization invariant
(in our case: coordinate u)

Explicit metric in toy example: free scalar

$$\boxed{ds^2_{Gravity} = g_{uu} du^2 + \frac{e^{2u}}{\varepsilon^2} \cdot d\vec{x}^2 - g_{tt} dt^2}$$

(i) Massless scalar (E=k)

$$g_{uu} = \frac{1}{4} \quad \Rightarrow \quad \text{the pure } AdS$$

(ii) Massive scalar

$$g_{uu} = \frac{e^{4u}}{4(e^{2u} + m^2/\Lambda^2)^2}.$$

Capped off in the IR z<1/m

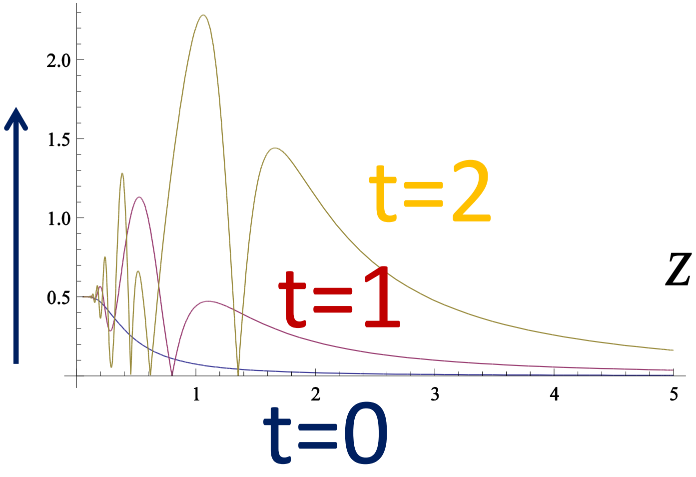
$$\Rightarrow ds^2 = \frac{dz^2}{z^2} + \left(\frac{1}{\Lambda^2 z^2} - \frac{m^2}{\Lambda^2}\right)(d\vec{x}^2 - dt^2).$$

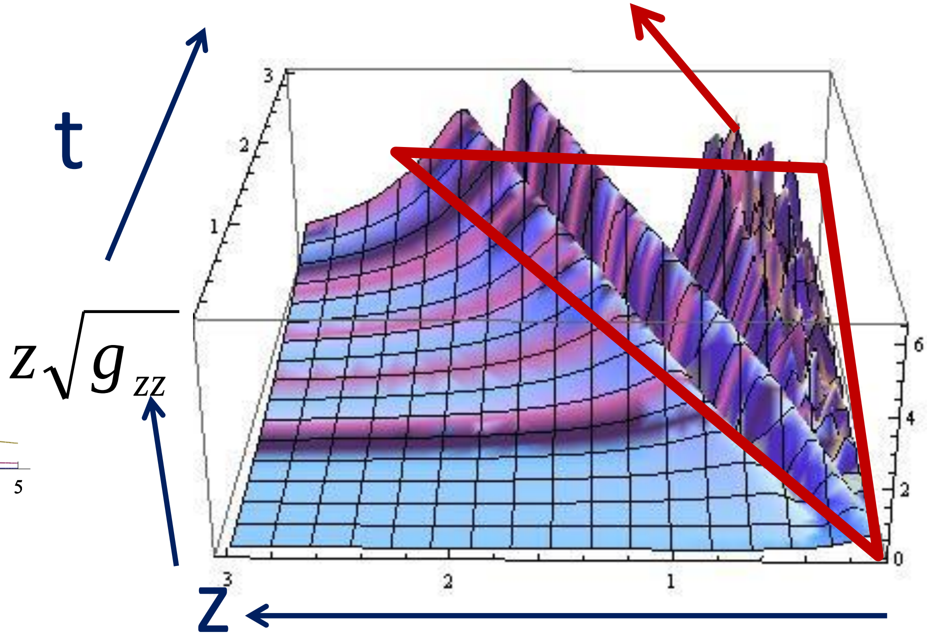
Time dependent metric from the 2d Quantum Quench

Looks like gravitational waves.

[Mollabashi-Nozaki-Ryu-TT 13]

$$z\sqrt{g_{zz}} = g(u)$$





We can also analytically confirm the linear growth: SA∝t because g(u)∝t at late time in any dim.

This is consistent with the 2d CFT result [Calabrese-Cardy 05].

and with the holographic result (any d). [Hartman-Maldacena 13]