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Wavelet techniques in multifractal analysis

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Wavelet Techniques in multifractal analysis

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What is fractal geometry?

Geometry dealing with irregular sets

How can one quantify this irregularity?
Parameters for the analysis of fractals sets

Toy examples: Selfsimilarity exponent

\[ \alpha = \frac{\log 2}{\log 3} \]

Triadic Cantor set

\[ \alpha = \frac{\log 4}{\log 3} \]

Van Koch curve

Box Dimension
Let \( N(\varepsilon) \) be the minimal number of balls of radius \( \varepsilon \) needed to cover the set \( A \)

\[ N(\varepsilon) \sim \varepsilon^{-\dim_B(A)} \]

Advantage:
Computable on experimental data through log-log plot regressions: \( \log(N(\varepsilon)) \) is plotted as a function of \( \log(\varepsilon) \). The slope yields the dimension.
Everywhere irregular signals

Jean Perrin, in his book, “Les atomes” (1913), insists that irregular (nowhere differentiable) functions, far from being exceptional, are the common case in natural phenomena.

Jet turbulence Eulerian velocity signal (Chavara/Laudet/Ciliberto/05)

Fully developed turbulence

Internet Traffic

EUR-USD

Euro vs Dollar (2001-2009)
Red channel

Nature, Sciences, and Arts supply a large variety of everywhere irregular functions.

Challenge: Measure this irregularity and use it for classification and model selection.
Motivation

**Multifractal analysis** : Its purpose is to introduce and study classification parameters for data (functions, measures, distributions, signals, images), which are based on global and local regularity.

A key problem along the 19th century was to determine if a continuous function on $\mathbb{R}$ necessarily has points of differentiability.

A first negative answer was obtained by B. Bolzano in 1830 but was unnoticed.

A second counterexample due to K. Weierstrass settled the issue in 1872.
Weierstrass functions

$$W_H(x) = \sum_{j=0}^{+\infty} 2^{-Hj} \cos(2^j x)$$

$0 < H < 1$

C. Hermite: I turn my back with fright and horror to this appalling wound: Functions that have no derivative

H. Poincaré called such functions “monsters”
Self-similarity for deterministic functions

Weierstrass-Mandelbrot function

\[ W_\alpha(x) = \sum_{-\infty}^{+\infty} 2^{-\alpha j} \sin(2^j x) \quad 0 < \alpha < 1 \]

Exact selfsimilarity :

\[
W_\alpha(2x) = \sum_{-\infty}^{+\infty} 2^{-\alpha j} \sin(2^{j+1} x) \\
= \sum_{-\infty}^{+\infty} 2^{-\alpha (l-1)} \sin(2^l x) \\
= 2^\alpha W_\alpha(x)
\]
Wavelet bases: a tool fitted to selfsimilarity

A wavelet basis on $\mathbb{R}$ is generated by a smooth, well localized, oscillating function $\psi$ such that the $\psi(2^j x - k), \quad j, k \in \mathbb{Z}$ form an orthogonal basis of $L^2(\mathbb{R})$

$\forall f$ such that $\int |f(x)|^2 \, dx < \infty$, $f(x) = \sum_j \sum_k c_{j,k} \psi(2^j x - k)$

where $c_{j,k} = 2^j \int f(x) \psi(2^j x - k) \, dx$

Daubechies Wavelet
Orthogonal basis

Let $f : \mathbb{R} \to \mathbb{R}$. The “energy” of $f$ is $\int |f(x)|^2 \, dx$

Wavelets form an orthogonal basis i.e.

$$\int \psi(2^j x - k) \psi(2^{j'} x - k') \, dx = \begin{cases} 2^{-j} & \text{if } j = j', k = k' \\ 0 & \text{else} \end{cases}$$

$$c_{j,k} = 2^j \int f(x) \psi(2^j x - k) \, dx$$

The energy of $f$ can be recovered from the coefficients

$$\int |f(x)|^2 \, dx = \sum_j \sum_k 2^j |c_{j,k}|^2$$

A. Haar, J. O. Strömberg, Y. Meyer, P.-G. Lemarié, S. Mallat, I. Daubechies, ...

Haar Wavelet (1909)
Why use wavelet bases?

- Characterization of selfsimilarity
- Images are stored by their wavelet coefficients (JPEG 2000)
- Fast decomposition algorithms
- Sparse representations
- Characterization of regularity
- Scaling invariance
Wavelet analysis of the Weierstrass-Mandelbrot function

\[ c_{j,k} = \int W_\alpha(x) \ 2^j \psi(2^j x - k) \, dx \]

\[ = 2^{-\alpha} \int W_\alpha(2x) \ 2^j \psi(2^j x - k) \, dx \]

\[ = 2^{-\alpha} \int W_\alpha(u) \ 2^j \psi(2^{j-1} u - k) \, \frac{du}{2} \]

\[ = 2^{-\alpha} \int W_\alpha(u) \ 2^{j-1} \psi(2^{j-1} u - k) \, du \]

\[ = 2^{-\alpha} \ c_{j-1,k} \]

This scaling invariance of wavelet coefficients justifies the nonstandard normalization used for wavelet coefficients.
Self-similarity: Stochastic processes

Fractional Brownian Motion \((0 < H < 1)\)

Statistical selfsimilarity

\[ \forall a > 0, \quad B_H(a \cdot) \overset{\mathcal{L}}{=} a^H B_H(\cdot) \]

Wavelet counterpart of probabilistic properties:

\[ c_{2j,k} \overset{\mathcal{L}}{=} 2^{-Hj} c_{j,k} \]

Problem: An equality in law among random quantities cannot be checked on a sample path
Fractional Brownian Motions

$H = 0.3$

$H = 0.4$

$H = 0.5$

$H = 0.6$

$H = 0.7$

Challenge: Find a numerically stable way to decide if a real-life signal can be modeled by FBM

Gaussian processes with stationary increments
From self-similarity to scaling exponents

$B_H(t)$ is the unique centered Gaussian process such that

$$\forall x, y \geq 0, \quad \mathbb{E}(|B_H(x) - B_H(y)|^2) = |x - y|^{2H}$$

It follows that

$$|B_H(x + \delta) - B_H(x)| \sim |\delta|^H$$

$$\int |B_H(x + \delta) - B_H(x)|^p \, dx \sim |\delta|^{Hp}$$

Scale invariance

Kolmogorov scaling function

$$\int |f(x + \delta) - f(x)|^p \, dx \sim |\delta|^{\zeta_f(p)}$$

Advantages:

- Computable on experimental data: log-log plot regressions
- Deterministic quantity that does not depend on the sample path

$\implies$ Turbulence at small scale cannot be modeled by FBM (1950s)
The loose definition of the Kolmogorov scaling function can be given two mathematical interpretations:

- One which makes sense for any (bounded) function

\[ \zeta_f(p) = \lim_{\delta \to 0} \inf \log \left( \frac{\int |f(x+\delta) - f(x)|^p dx}{\log(|\delta|)} \right) \]

- One, which is more precise, under the assumption that \( f \) satisfies an average scale invariance

\[ \int |f(x+\delta) - f(x)|^p dx = |\delta|^{\zeta_f(p) + o(1)} \]

and expresses the fact that the liminf actually is a limit.
Function space interpretation

Lipschitz spaces:
Let $s \in (0, 1)$, and $p \in (1, \infty)$; $f \in Lip(s, L^p(\mathbb{R}^d))$ if $f \in L^p$ and $\exists C > 0$ such that

$$\forall \delta > 0, \quad \| f(x + \delta) - f(x) \|_p \leq C \cdot |\delta|^s$$

$$\iff$$

$$\forall \delta > 0, \quad \int |f(x + \delta) - f(x)|^p dx \leq C \cdot |\delta|^{sp}$$

Therefore, if $\zeta_f(p) < p$,

$$\zeta_f(p) = p \cdot \sup\{s : f \in Lip(s, L^p)\}$$
Wavelets and function spaces: The Sobolev space $H^1(\mathbb{R})$ if and only if $f \in L^2$ and $f' \in L^2$. We define
\[
\| f \|^2 = \int |f(x)|^2 \, dx + \int |f'(x)|^2 \, dx
\]

Fourier:
\[
f(x) = \sum_{n \in \mathbb{Z}} c_n \, e^{2i\pi nx}, \quad f'(x) = \sum_{n \in \mathbb{Z}} c_n \, 2i\pi n \, e^{2i\pi nx}
\]
\[
\| f \|^2 = \sum |c_n|^2 + 4\pi^2 \sum n^2 |c_n|^2
\]

Wavelets:
\[
f(x) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} c_{j,k} \psi(2^j x - k)
\]
\[
f'(x) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} c_{j,k} 2^j \psi'(2^j x - k)
\]
\[
\| f \|^2 \sim \sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} (1 + 2^{2j}) 2^{-j} |c_{j,k}|^2
\]
Wavelets and function spaces:
Sobolev spaces of fractional order

Fractional derivative:
the Fourier transform of $f'$ is $i\xi \hat{f}$

In the Fourier domain, the fractional derivative of order $s$ of $f$ is the operator $(-\Delta)^{s/2}f$ is defined by

$$( -\Delta )^{s/2}f(\xi) = |\xi|^s \hat{f}(\xi)$$

Definition: Sobolev space $L^{p,s}$

$f \in L^{p,s}(\mathbb{R})$ if $f \in L^p(\mathbb{R})$ and $(-\Delta)^{s/2}f \in L^p(\mathbb{R})$
Wavelet estimation of the $L^{p,s}$ norm

$$f = \sum_{j} f_j \quad \text{with} \quad f_j(x) = \sum_{k} c_{j,k} \psi(2^j x - k)$$

In order to estimate $\| f \|_{p,s}$ we consider each “scale” block $f_j$

$$f_j^{(s)}(x) = \sum_{k} c_{j,k} 2^{sj} \psi^{(s)}(2^j x - k)$$

Because of the spatial localization of the wavelets,

$$\int |f_j^{(s)}(x)|^p \, dx \sim \sum_{k} |c_{j,k}|^p 2^{spj} \int |\psi^{(s)}(2^j x - k)|^p \, dx$$

$$\sim \sum_{k} |c_{j,k}|^p 2^{(sp-1)j}$$

Therefore the condition $f \in L^{p,s}$ is close to

$$\exists C, \forall j : \quad 2^{-j} \sum_{k} |c_{j,k}|^p \leq C \cdot 2^{-spj}$$
The wavelet scaling function

\[ \forall p > 0, \quad 2^{-j} \sum_{k} |c_{j,k}|^p \sim 2^{-\zeta_i(p)j} \]

Because of the embeddings between Lipschitz and Sobolev spaces, if \( p \geq 1 \), this definition coincides with Kolmogorov’s scaling function.

Advantages:

- Effectively computable on experimental data through log-log plot regressions with respect to the scale parameter.
- Extends the Kolmogorov scaling function to \( p \in (0, 1) \).
- Independent of the (smooth enough) wavelet basis.
- Deformation invariant.
- Deterministic for large classes of stochastic processes.
Wavelets in several variables

In 2D, wavelets used are of tensor product type:

\[ \psi^1(x, y) = \psi(x)\varphi(y) \]

\[ \psi^2(x, y) = \varphi(x)\psi(y) \]

\[ \psi^3(x, y) = \psi(x)\psi(y) \]

**Notations**

Dyadic squares:

\[ \lambda = \left[ \frac{k}{2^j}, \frac{k + 1}{2^j} \right] \times \left[ \frac{l}{2^j}, \frac{l + 1}{2^j} \right] \]

Wavelet coefficients

\[ c_\lambda = 2^{2j} \int \int f(x, y) \psi^i \left( 2^i(x, y) - (k, l) \right) \, dx \, dy \]

Natural normalization for scaling invariance
Computation of 2D wavelet coefficients
Computation of 2D wavelet coefficients
Computation of 2D wavelet coefficients
Computation of 2D wavelet coefficients
Computation of 2D wavelet coefficients
Computation of 2D wavelet coefficients
Computation of 2D wavelet coefficients
Computation of 2D wavelet coefficients
Notations for wavelets on $\mathbb{R}^d$

Dyadic cubes

If $k = (k_1, \ldots, k_d)$, \[ \lambda = \left[ \frac{k_1}{2^j}, \frac{k_1 + 1}{2^j} \right] \times \cdots \times \left[ \frac{k_d}{2^j}, \frac{k_d + 1}{2^j} \right] \]

Wavelets

$\psi_\lambda(x) = \psi^i(2^j x - k) \quad i = 1, \ldots, 2^d - 1$

Wavelet coefficients

$c_\lambda = 2^{dj} \int f(x) \psi^j(2^j x - k) \, dx$

Dyadic cubes at scale $j$

$\Lambda_j = \{ \lambda : \, |\lambda| = 2^{-j} \}$

Wavelet expansion of $f$

$f(x) = \sum_j \sum_{\lambda \in \Lambda_j} c_\lambda \psi_\lambda(x)$
The wavelet scaling function on $\mathbb{R}^a$

$$\forall p > 0, \quad 2^{-dj} \sum_{\lambda \in \Lambda_j} |c_{\lambda}|^p \sim 2^{-\zeta_f(p)j}$$

With two variants: The first one is

$$\zeta_f(p) = \liminf_{j \to +\infty} \frac{\log \left( 2^{-dj} \sum_{\lambda \in \Lambda_j} |c_{\lambda}|^p \right)}{\log(2^{-j})}$$

This liminf definition coincides with Kolmogorov's scaling function

$$\forall p > 0, \quad \zeta_f(p) = \sup \left\{ s : f \in B_p^{s/p, \infty} \right\}$$
Scale invariance

The second variant is:

$$\zeta_f(p) = \lim_{j \to +\infty} \frac{\log \left( 2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \right)}{\log(2^{-j})}$$

It means that quantities defined at each scale $j$ satisfy some scale invariance property in the limit of small scales.

This condition is usually met in signal and image processing.

**Additional range of parameters available for classification**: $p \in (0, 1)$

Indeed, $L^p$ or Sobolev spaces do not make sense for $p < 1$. One has to use wavelet based substitutes.
The role of the wavelet scaling function

\[ \zeta_f(p) = \sup \{ s : f \in L^{p,s/p} \} \quad (p \geq 1) \]

\[ = \sup \{ s : f \in B^{s,p,\infty}_p \} \quad (p > 0) \]

- If \( \zeta_f(1) > 1 \), then \( f \in BV \)
- If \( f \) is a measure, then \( \zeta_f(1) \geq 0 \)
- If \( \zeta_f(1) > 0 \), \( f \) then belongs to \( L^1 \)
- If \( \zeta_f(2) > 0 \), then \( f \in L^2 \)

Besov spaces have an alternative definition which is not based on a wavelet expansion

\[ \implies \]

The wavelet scaling function is independent of the (smooth enough) wavelet basis chosen
Log-log plot regression

Power-law behavior: Key requirement for the numerical derivation of the scaling function (here at $p = 1$)

Data courtesy Vivienne Investissement
Wavelet scaling functions of synthetic images

Scaling functions
\[
\begin{align*}
\zeta_f(p) &= 1 \\
\zeta_f(p) &= 2 - \frac{\log 4}{\log 3} \approx 0.74
\end{align*}
\]

Disk
Van Koch snowflake
$j_1 - j_2 = 6$

$\zeta(q) = 0.21717$

DWT fonction d'échelle $\zeta(q)$

$\zeta(q=1) = -0.21717$
\[ j_1 = \xi \]
\[ j_2 = 5 \]

DWT fonction d'échelle \( \varsigma(q) \)

\[ \varsigma(q=1) = -0.06422 \]
DWT fonction de structure $\log_2 S(j, q=1)$

\[ j_1 = 4 \quad j_2 = 7 \quad \zeta(q=1) = 0.72218 \]

DWT fonction d'échelle $\zeta(q)$

\[ \zeta(q=1) = 0.72218 \]
DWT fonction de structure $\log_2 S(j,q=1)$

$\begin{array}{c|c}
 j \backslash q & \zeta(q=1) = 0.53726 \\
 3 & 5.5 \\
 6 & 7.5 \\
\end{array}$

DWT fonction d'échelle $\zeta(q)$

$\zeta(q=1) = 0.53726$
Uniform Hölder regularity

Hölder spaces: Let $\alpha \in (0, 1)$; $f \in C^\alpha(\mathbb{R}^d)$ if
\[ \exists C, \forall x, y : |f(x) - f(y)| \leq C |x - y|^{\alpha} \]

If $\alpha > 1$, $f \in C^\alpha(\mathbb{R})$ if and only if $f([\alpha]) \in C^{\alpha-[\alpha]}$

If $\alpha < 0$, $f \in C^\alpha(\mathbb{R})$ if and only if $f = g(-[\alpha])$ with $g \in C^{\alpha-[\alpha]}$

The uniform Hölder exponent of $f$ is
\[ H^{\text{min}}_f = \sup \{ \alpha : f \in C^\alpha(\mathbb{R}^d) \} \]

Numerical computation

Let $\omega_j = \sup_{\Lambda_j} |c_{\lambda}|$ then
\[ H^{\text{min}}_f = \liminf_{j \to +\infty} \frac{\log(\omega_j)}{\log(2^{-j})} \]

$H^{\text{min}}_f > 0 \implies f$ is continuous

$H^{\text{min}}_f < 0 \implies f$ is not locally bounded
Validity of jump models in finance

Euro vs. USD
2001-2009

$H_{min}^f$
local estimation
(3 months window)

Data supplied by Vivienne Investissement
Classification based on the uniform Hölder exponent

Heartbeat intervals

Healthy

Heart failure

\[ H_f^{\text{min}} = -0.06 \] \hfill \[ H_f^{\text{min}} = -0.55 \]
Analysis of paintings: Van Gogh challenge

(in collaboration with D. Rockmore)

Arles and Saint-Rémy period  Van Gogh  Paris period
Challenge : Date

Paris period - Arles, Saint-Rémy period - Unknown
Canals : Red vs. Saturation

$h_{min} - h_{min}$; $j=[1,4]/[1,4]$ – RGB 0/1 – CH 2/1
Drawbacks

Classification only based on the wavelet scaling function proved insufficient in several occurrences (turbulence,...)

New ideas initiated by U. Frisch and G. Parisi are required

Beyond scale invariance: Multifractal Analysis

Giorgio Parisi

Uriel Frisch
Pointwise regularity

Let $f$ be a locally bounded function $\mathbb{R}^d \rightarrow \mathbb{R}$ and $x_0 \in \mathbb{R}^d$; $f \in C^\alpha(x_0)$ if there exist $C > 0$ and a polynomial $P$ of degree less than $\alpha$ such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha$$

The Hölder exponent of $f$ at $x_0$ is

$$h_f(x_0) = \sup \{\alpha : f \in C^\alpha(x_0)\}$$

Heuristics: $$|f(x) - f(x_0)| \sim |x - x_0|^{h_f(x_0)}$$
Constant Hölder exponents

FBMs with Hölder exponents 0.3, 0.4, 0.5, 0.6 and 0.7
Varying Hölder exponent

\[ h_f(x) = x \]

(Constructions of K. Daoudy, J. Lévy-Véhel and Y. Meyer)
Difficulty to use directly the pointwise regularity exponent for classification

For classical models, such exponents are extremely erratic

Lévy processes

multiplicative cascades

The function $h$ is random and everywhere discontinuous

Goal: Recover some information on $h(x)$ from (time or space) averaged quantities which is:

- numerically computable on a sample path of the process, or on real-life data,
- deterministic (independent of the sample path)
Hausdorff dimension

\( \varepsilon \)-coverings

Definition: Let \( A \subset \mathbb{R}^d \); an \( \varepsilon \)-covering of \( A \) is a family \( \mathcal{C}_\varepsilon = \{B_i\}_{i \in \mathbb{N}} \) of sets \( B_i \) such that

\[ \forall i, \text{diam} (B_i) \leq \varepsilon \]

\[ A \subset \bigcup B_i \]

Example 1: If \( A \) is a smooth curve, “economical” \( \varepsilon \)-coverings satisfy

\[ \sum_{i} (\text{diam} (B_i)) \sim L(A) \]
Example 2: If $A$ is a smooth surface, “economical” $\varepsilon$-coverings satisfy

$$\sum_{i} (\text{diam}(B_i))^2 \sim S(A)$$
Hausdorff dimension

The Hausdorff dimension of $A$ is the exponent $\dim(A)$ satisfying

for $\varepsilon$ small enough, $\inf_{C_{\varepsilon}} \sum_{i} (\operatorname{diam}(B_i))^{\dim(A)} \sim 1$.

Convention: $\dim(\emptyset) = -\infty$.

If the $(B_i)$ have the same width $\varepsilon$, their number $N(\varepsilon)$ satisfies

$$N(\varepsilon) \sim \varepsilon^{-\dim(A)}$$

$\implies$ The Hausdorff dimension is smaller than the box dimension.

Advantage: Good mathematical properties

Drawback: Not computable for real-life data
Spectrum of singularities

The isohölder sets of $f$ are the sets

$$E_H = \{ x_0 : \ h_f(x_0) = H \}$$

Let $f$ be a locally bounded function

The spectrum of singularities of $f$ is

$$D_f(H) = \dim (E_H)$$

(by convention, $\dim (\emptyset) = -\infty$)

Classification based on a (potentially) infinite number of fractal sets

Can one estimate the spectrum of singularities of signals and images?

Can it yield new classification parameters?
Wavelet leaders

Let $\lambda$ be a dyadic cube; $3\lambda$ is the cube of same center and three times wider.

Let $f$ be a locally bounded function; the wavelet leaders of $f$ are the quantities

$$d_\lambda = \sup_{\lambda' \subset 3\lambda} |c_{\lambda'}|$$
Computation of 2D wavelet leaders
Wavelet characterization of pointwise smoothness

Let $\lambda_j(x_0)$ denote the dyadic cube of width $2^{-j}$ that contains $x_0$

**Theorem** If $H_f^{\min} > 0$, then

$$\forall x_0 \in \mathbb{R}^d \quad h_f(x_0) = \lim \inf_{j \to +\infty} \frac{\log(d_{\lambda_j(x_0)})}{\log(2^{-j})}.$$ 

**Sketch of proof:**

A bounded function $f : \mathbb{R}^d \to \mathbb{R}$ belongs to $f \in C^\alpha(x_0)$ ($\alpha > 0$) if there exist $C > 0$ and a polynomial $P$ of degree less than $\alpha$ such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha$$

$$\iff$$

$$\sup_{x \in B(x_0, r)} |f(x) - P(x - x_0)| \leq C \cdot r^\alpha$$
Wavelet characterization of pointwise smoothness

\[ \sup_{x \in B(x_0, r)} |f(x) - P(x - x_0)| \leq C \cdot r^\alpha \iff \| f - P \|_{L^\infty(B(x_0, r))} \leq C \cdot r^\alpha \]

One uses the “wavelet characterization” of \( L^\infty \):

If \( f \in L^\infty \) then \( \sup_{\lambda} |c_\lambda| \leq C \)

- |x - x_0|^h
  - \( h = 0.6 \)

- |x - x_0|^h \sin \left( \frac{1}{|x - x_0|^{\beta}} \right)
  - \( h = 0.6, \beta = 1 \)
Associated scaling function and function spaces

Wavelet scaling function
\[ 2^{-dj} \sum_{\lambda \in \Lambda_j} |c_{\lambda}|^p \sim 2^{-\zeta_f(p)j} \]

Leader scaling function
\[ 2^{-dj} \sum_{\lambda \in \Lambda_j} |d_{\lambda}|^p \sim 2^{-\eta_f(p)j} \]

Two possible definitions depending whether the quotient of logarithms only has a liminf or a real limit.

Oscillation spaces: Let \( p > 0 \); \( f \in \mathcal{O}_p^s (\mathbb{R}^d) \) if
\[ \exists C, \forall j : 2^{-dj} \sum_{\lambda \in \Lambda_j} |d_{\lambda}|^p \leq 2^{-sp_j} \]
\[ \forall p > 0, \quad \eta_f(p) = p \cdot \sup \{ s : p \in \mathcal{O}_p^s \} \]
\( \mathcal{O}_p^s \) and \( B_p^{s;\infty} \) coincide if \( s > d/p \)

If \( s < d/p \), one can derive information on the inter-scales correlations of wavelet coefficients
The nature of pointwise singularities

Comparing Besov and Oscillation spaces which contain $f$ yields information on the type of pointwise singularities present in the function

**Cusp singularities** : $f(x) = C|x - x_0|^\alpha + o(|x - x_0|^\alpha)$
When the Besov and oscillation spaces that contain $f$ coincide

**Chirp singularities** : $f(x) = C|x - x_0|^\alpha \sin\left(\frac{1}{|x - x_0|^{\beta}}\right) + o(|x - x_0|^\alpha)$
When the Besov and oscillation spaces that contain $f$ differ

**Heuristic justification** : For cusp-like singularities, the supremum in the leaders in reached at the root of the dyadic tree

$$2^{-dj} \sum_{\lambda \in \Lambda_j} |d_\lambda|^p$$

For cusps : $d_\lambda := \sup_{\lambda' \subset 3\lambda} |c_{\lambda'}| \sim c_\lambda$
Estimation of the $p$-variation

Let $f : [0, 1]^d \rightarrow \mathbb{R}$ be a continuous function. If $A \subset [0, 1]^d$, the first order oscillation of $f$ on $A$ is

$$\text{Os}_f(A) = \sup_A f - \inf_A f$$

**Definition**: Let $p \geq 1$; $f$ belongs to $V^s_p$ if

$$\exists C \quad \forall j \quad 2^{-dj} \sum_{\lambda \in \Lambda_j} (\text{Os}_f(3\lambda))^p \leq C 2^{-spj}$$

**Definition**: $f$ has bounded $p$-variation if

$$\exists C \quad \forall j \geq 0 \quad \sum_{\lambda \in \Lambda_j} (\text{Os}_f(3\lambda))^p \leq C$$

Therefore if $f \in V^d_p$, then $f$ has bounded $p$-variation
Estimation of the $p$-variation

**Theorem**: Let $p \geq 1$; then

$$\forall \varepsilon > 0 \quad C^\varepsilon \cap V^s_p \leftrightarrow O^s_p \leftrightarrow V^{s+\varepsilon}_p$$

**Corollary**: If $H_{f min} > 0$, one can determine whether $f$ has bounded $p$-variation or not by inspecting the value of its leader scaling function at $p$, indeed:

- If $\zeta_f(p) > d$, then $f$ has bounded $p$-variation
- If $\zeta_f(p) < d$, then the $p$-variation of $f$ is unbounded
Quadratic variation

Wavelet criterium

If $H_f^{\min} > 0$ and $\zeta_f(2) > 1$ then $f$ has bounded quadratic variation

Euro vs. USD
2001-2009

$\zeta(2) = 1.0053$
Multifractal formalism:

The Frisch-Parisi derivation revisited

Leader structure function: \( T_{p,j} = 2^{-dj} \sum_{\lambda \in \Lambda_j} |d\lambda|^p \sim 2^{-\eta_f(p)j} \)

Contribution to \( T_{p,j} \) of the points \( x_0 \) such that \( h(x_0) = H \):

\[
\sim 2^{-dj} \cdot \left(2^{-j}\right)^{D_f(H)} \cdot (2^{-Hj})^p = (2^{-j})^{d+Hp-D_f(H)}
\]

\( \eta_f(p) = \inf_H (d + Hp - D_f(H)) \)

\( \eta_f \) is expected to be the Legendre transform of \( D_f \)

Conjecture: \( D_f(H) = \inf_{p \in \mathbb{R}} (d + Hp - \eta_f(p)) \)

We define the leader spectrum as:

\( L_f(H) = \inf_{p \in \mathbb{R}} (d + Hp - \eta_f(p)) \)

Theorem: If \( f \in C^\varepsilon(\mathbb{R}^d) \) for an \( \varepsilon > 0 \) then \( \forall H \in \mathbb{R}, \ D_f(H) \leq L_f(H) \)
Classification parameters deduced from the multifractal spectrum

- $H^{\text{min}}$ : Minimal regularity
- $H^{\text{max}}$ : Maximal regularity
- $c_1$ : Position of the max
- $c_2$ : Width
- $c_3$ : Asymmetry
Monohölder vs. Multifractality

Modeling using Fractional Brownian Motion is reasonable.

Data from http://mawi.wide.ad.jp/mawi/
Cascade models : Binomial cascade

We obtain a measure $\mu$ on the interval $[0, 1]$ :

$$\mu(I) = \text{amount of sand that fell on } I$$

Courtesy of Jean-François Colonna, LACTAMME
Cascade models : Binomial cascade

Intervals of length $\left(\frac{1}{2}\right)^n$:

far left : $\mu(I) = \left(\frac{1}{4}\right)^n = |I|^2$

far right : $\mu(I) = \left(\frac{3}{4}\right)^n = |I|^{{\log(4/3)}/{\log 2}}$

average : $\mu(I) = \left(\frac{1}{4}\right)^{n/2} \left(\frac{3}{4}\right)^{n/2} = |I|^{{\log(4/\sqrt{3})}/{\log 2}}$
Cascade models: Binomial cascade

Repartition function of the measure $\mu$:

$$f(x) = \mu([0, x]) = \text{amount of sand in } [0, x]$$

$$f(x + \delta) - f(x) = \mu([x, x + \delta]) \sim \delta^{h(x)}$$

$$h(x) \in \left[ \frac{\log(4/3)}{\log 2}, 2 \right]$$

Spectrum of singularities of $f$:

$$d_f(H) = \dim(\{x : h(x) = H\})$$
Multiplicative model: Mandelbrot Cascades

$W_{11}$  $W_{12}$

$W > 0$

$i = 1$
Multiplicative model: Mandelbrot Cascades
Multiplicative model: Mandelbrot Cascades
Multiplicative model : Mandelbrot Cascades

\[ P(W) \]

\[ W > 0 \]

\[ i = 6 \]

\[ W_{11}, W_{12}, W_{21}, W_{22}, W_{23}, W_{24}, W_{31}, W_{32}, W_{33}, W_{34}, W_{35}, W_{36}, W_{37}, W_{38} \]

...
Multiplicative model: Mandelbrot Cascades

\[ P(W) \]

\[ W_{i>0} \]

\[ W_{11} \rightarrow W_{12} \]
\[ W_{21} \rightarrow W_{22} \rightarrow W_{23} \rightarrow W_{24} \]
\[ W_{31} \rightarrow W_{32} \rightarrow W_{33} \rightarrow W_{34} \rightarrow W_{35} \rightarrow W_{36} \rightarrow W_{37} \rightarrow W_{38} \]

\[ \ldots \]

\[ \ldots \]
Multifractal models for signals

The repartition function of a probability measure is not appropriate in signal modeling. One can make it oscillate randomly by taking $f(t)$ for time change

$$X(t) = B_H(f(t))$$

Fractional Brownian motion in multifractal time proposed by B. Mandelbrot for financial modeling

Alternatives :
- Orthonormal wavelet series whose coefficients are derived from the measure : A. Arneodo, J. Barral and S. Seuret
- ...
Model refutation

(joint work with Bruno Lashermes)

Jet turbulence Eulerian velocity signal (ChavarriaBaudetCiliberto95)

Log-normal vs. She-Leveque model
Princeton Experiment : Stylometry issues

- Experiment design:
  - same Painter (Charlotte Casper) does Original and Copies
  - a series of 7 small paintings,
  - different set of materials (various brushes, grounds, paints)
  - Original and Copies with same set of materials
  - high resolution digitalisation, under uniform conditions.

- Description:

<table>
<thead>
<tr>
<th>Pair</th>
<th>Ground</th>
<th>Paint</th>
<th>Brushes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>CP Canvas</td>
<td>Oils</td>
<td>S &amp; H</td>
</tr>
<tr>
<td>2</td>
<td>CP Canvas</td>
<td>Acrylics</td>
<td>S &amp; H</td>
</tr>
<tr>
<td>3</td>
<td>Smooth CP Board</td>
<td>Oils</td>
<td>S &amp; H</td>
</tr>
<tr>
<td>4</td>
<td>Bare linen canvas</td>
<td>Oils</td>
<td>S</td>
</tr>
<tr>
<td>5</td>
<td>Chalk and Glue</td>
<td>Oils</td>
<td>S</td>
</tr>
<tr>
<td>6</td>
<td>CP Canvas</td>
<td>Acrylics</td>
<td>S</td>
</tr>
<tr>
<td>7</td>
<td>Smooth CP Board</td>
<td>Oils</td>
<td>S</td>
</tr>
</tbody>
</table>
Charlotte2’s Original & Copy
Charlotte2 MF
Charlotte2 MF
Charlotte2 MF

![Charlotte2 MF Diagrams](image-url)
Charlotte2 MF
Charlotte2 MF

Charlotte2

$q$-dependence of the multifractal spectrum $\zeta(q)$ (left) and the singularity spectrum $D(h)$ (right) for the original and copy versions of Charlotte2.
Multifractal analysis of the USD-Euro change rate

Legendre spectra
Uniform Hölder regularity

Hölder spaces: Let $\alpha \in (0, 1)$; $f \in C^\alpha$ if

$$\exists C, \forall x, y : \quad |f(x) - f(y)| \leq C \cdot |x - y|^{\alpha}$$

The uniform Hölder exponent of $f$ is

$$H_{f}^{\text{min}} = \sup \{ \alpha : f \in C^\alpha \}$$

Numerical computation

Let $\omega_j = \sup_{\lambda \in \Lambda_j} |c_\lambda|$ then

$$H_{f}^{\text{min}} = \lim \inf_{j \to +\infty} \frac{\log(\omega_j)}{\log(2^{-j})}$$

$H_{f}^{\text{min}} > 0 \implies f$ is continuous

$H_{f}^{\text{min}} < 0 \implies f$ is not locally bounded
Is $H_{\text{min}} > 0$ fulfilled in applications?

Internet Traffic

\[ H_{\text{min}} = -0.46 \]
Heartbeat Intervals

\[ H_{\text{min}} = -0.55 \]
$h_{\text{min}} = -0.18079$
$h_{\text{min}} = -0.23042$
Pointwise regularity with negative exponents?

Pointwise Hölder regularity: \( f \in C^\alpha(x_0) \) if
\[
|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha
\]

If \( \alpha < 0 \), this definition implies that, outside of \( x_0 \), \( f \) is locally bounded
\[ \implies \text{it could be used only to define isolated singularities of negative exponent} \]

Which definition of pointwise regularity would allow for negative exponents?

Clue: The definition of pointwise Hölder regularity can be rewritten
\[
f \in C^\alpha(x_0) \iff \sup_{B(x_0,r)} |f(x) - P(x - x_0)| \leq Cr^\alpha
\]

Definition (Calderón and Zygmund): Let \( f \in L^p(\mathbb{R}^d) \); \( f \in T_{\alpha}^p(x_0) \) if there exists a polynomial \( P \) such that for \( r \) small enough,
\[
\left( \frac{1}{r^d} \int_{B(x_0,r)} |f(x) - P(x - x_0)|^p \, dx \right)^{1/p} \leq Cr^\alpha
\]
The $p$-exponent

**Definition:** Let $f \in L^p(\mathbb{R}^d); f \in T^p_\alpha(x_0)$ if there exists a polynomial $P$ such that for $r$ small enough,

$$ \left( \frac{1}{r^d} \int_{B(x_0, r)} |f(x) - P(x - x_0)|^p dx \right)^{1/p} \leq Cr^\alpha $$

The $p$-exponent of $f$ at $x_0$ is $h_p(x_0) = \sup \{ \alpha : f \in T^p_\alpha(x_0) \}$

The $p$-spectrum of $f$ is $d^p(H) = \dim \{ x_0 : h_p(x_0) = H \}$

Remarks:

- The case $p = +\infty$ corresponds to pointwise Hölder regularity
- The normalization is chosen so that a cusp $|x - x_0|^\alpha$ has the same $p$-exponent for all $p : h_p(x_0) = \alpha$ (as long as $\alpha \geq -d/p$)
How can one check that the data belong to $L^p$?

The wavelet scaling function is informally defined by

$$
\forall p > 0 \quad 2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \sim 2^{-\zeta_f(p)j}
$$

- If $\zeta_f(p) > 0$, then $f \in L^p$
- If $\zeta_f(p) < 0$, then $f \not\in L^p$
Properties of $p$-exponents

Gives a mathematical framework to the notion of negative regularity exponents

The $p$-exponent satisfies: $h_p(x_0) \geq -\frac{d}{p}$

$p$-exponents may differ:

Theorem: Let $f$ be an $L^1$ function, and $x_0 \in \mathbb{R}^d$. Let

$$p_0 = \sup\{p : f \in L^p_{\text{loc}}(\mathbb{R}^d) \text{ in a neighborhood of } x_0\}$$

The function $p \to h_p(x_0)$ is defined on $[1, p_0)$ and possesses the following properties:

1. It takes values in $[-d/p, \infty]$.
2. It is a decreasing function of $p$.
3. The function $r \to h_{1/r}(x_0)$ is concave.

Furthermore, Conditions 1 to 3 are optimal.
When do $p$-exponents coincide?

**Notation :** $h_{p,\gamma}(x_0)$ denotes the $p$-exponent of the fractional integral of $f$ of order $s$ at $x_0$

**Definition :** $f$ is a cusp of exponent $h$ at $x_0$ if $\exists p, \gamma > 0$ such that

- $h_p(x_0) = h$
- $h_{p,\gamma}(x_0) = h + \gamma$

**Theorem :** This notion is independent of $p$ and $\gamma$
Types of pointwise singularities (Yves Meyer)

Typical pointwise singularities:

**Cusps**: \( f(x) - f(x_0) = |x - x_0|^H \)

After one integration:

\[ f^{(-1)}(x) - f^{(-1)}(x_0) \sim \frac{1}{H} |x - x_0|^{H+1} \]

**Oscillating singularity**: \( f(x) - f(x_0) = |x - x_0|^H \sin \left( \frac{1}{|x - x_0|^{\beta}} \right) \)

After one integration:

\[ f^{(-1)}(x) - f^{(-1)}(x_0) = \frac{|x - x_0|^{H+(1+\beta)}}{\beta} \cos \left( \frac{1}{|x - x_0|^{\beta}} \right) + \cdots \]

More generally, after a fractional integration of order \( s \),

- If \( f \) has a cusp at \( x_0 \), then \( h_{I^s f}(x_0) = h_f(x_0) + s \)
- If \( f \) has an oscillating singularity at \( x_0 \), then
  \[ h_{I^s f}(x_0) = h_f(x_0) + (1 + \beta)s \]
Further classification

Two types of oscillating singularities:

“Full singularities” : \(|x - x_0|^H \sin\left(\frac{1}{|x - x_0|^\beta}\right)\) (p-exponents coincide)

“Skinny singularities” : \(|x - x_0|^H 1_{E_\gamma}\) where

\[E_\gamma = \bigcup \left[\frac{1}{n}, \frac{1}{n} + \frac{1}{n^\gamma}\right]\] for \(\gamma > 1\)

(p-exponents differ)

Characterization

Oscillating singularities : \(F(x) = |x - x_0|^H g\left(\frac{1}{|x - x_0|^\beta}\right) + r(x)\)

Where \(g\) is indefinitely oscillating

For “full singularities” \(g\) is “large at infinity”

For “skinny singularities” \(g\) is “small at infinity”
Wavelet derivation of the $p$-exponent

**Definition:** Let $f : \mathbb{R}^d \to \mathbb{R}$ be locally in $L^p$; the $p$-leaders of $f$ are

$$d^p_\lambda = \left( \sum_{\lambda' \subset 3\lambda} |c_{\lambda'}|^p 2^{d(j-j')} \right)^{1/p}$$

where $j'$ is the scale associated with the subcube $\lambda'$ included in $3\lambda$ (i.e. $\lambda'$ has width $2^{-j'}$).

**Theorem:** (C. Melot)
If $\eta_f(p) > 0$, then

$$\forall x_0 \in \mathbb{R}^d \quad h_f p(x_0) = \lim_{j \to +\infty} \inf \frac{\log(d^p_{\lambda_j(x_0)})}{\log(2^{-j})}.$$
p-leaders and negative regularity

\[ h_p(x_0) = -0.1 \]

\[ \hat{h}(x_0) = 0.00 \]

\[ \hat{h}(x_0) = -0.08 \]

\[ \hat{h}(x_0) = -0.10 \]
$p$-Multifractal Formalism

The $p$-scaling function is defined informally by

$$\forall q \in \mathbb{R}, \quad 2^{-dj} \sum_{\lambda \in \Lambda_j} |d^p_\lambda|^q \sim 2^{-\eta_p(q)j}$$

$$\eta_p(q) = \liminf_{j \to +\infty} \frac{\log \left( 2^{-dj} \sum_{\lambda \in \Lambda_j} |d^p_\lambda|^q \right)}{\log(2^{-j})}$$

Stability properties:
- Invariant with respect to deformations
- independent of the wavelet basis

The $p$-Legendre Spectrum is

$$L_p(H) = \inf_{q \in \mathbb{R}} (d + Hq - \eta_p(q))$$
\[ p_0 = +\infty \quad h_X^{(\text{min})} = 0 \]

\[ p_0 = 10 \quad h_X^{(\text{min})} = -0.2 \]

\[ p_0 = 2.5 \quad h_X^{(\text{min})} = -0.3 \]
\[ p_0 = +\infty \quad h_X^{(\text{min})} = 0 \]

\[ p_0 = 10 \quad h_X^{(\text{min})} = -0.2 \]

\[ p_0 = 2.5 \quad h_X^{(\text{min})} = -0.3 \]
\[ p_0 = +\infty \quad h_X^{(\min)} = 0 \]

\[ p_0 = 10 \quad h_X^{(\min)} = -0.2 \]

\[ p_0 = 2.5 \quad h_X^{(\min)} = -0.3 \]
\begin{align*}
p_0 &= +\infty \quad h_X^{(\min)} = 0 \\
p_0 &= 10 \quad h_X^{(\min)} = -0.2 \\
p_0 &= 2.5 \quad h_X^{(\min)} = -0.3
\end{align*}
\[ p_0 = +\infty \Rightarrow h_X^{(\text{min})} = 0 \]

\[ p_0 = 10 \Rightarrow h_X^{(\text{min})} = -0.2 \]

\[ p_0 = 2.5 \Rightarrow h_X^{(\text{min})} = -0.3 \]
\[ p_0 = +\infty \quad h_X^{(\text{min})} = 0 \]

\[ p_0 = 10 \quad h_X^{(\text{min})} = -0.2 \]

\[ p_0 = 2.5 \quad h_X^{(\text{min})} = -0.3 \]
\[ p_0 = +\infty \quad h_X^{(min)} = 0 \]

\[ p_0 = 10 \quad h_X^{(min)} = -0.2 \]

\[ p_0 = 2.5 \quad h_X^{(min)} = -0.3 \]
\[ p_0 = +\infty \quad h_X^{(\text{min})} = 0 \]

\[ p_0 = 10 \quad h_X^{(\text{min})} = -0.2 \]

\[ p_0 = 2.5 \quad h_X^{(\text{min})} = -0.3 \]
$p_0 = +\infty \quad h_X^{(\text{min})} = 0$

$p_0 = 10 \quad h_X^{(\text{min})} = -0.2$

$p_0 = 2.5 \quad h_X^{(\text{min})} = -0.3$
\[ p_0 = +\infty \quad h_X^{(\min)} = 0 \]

\[ p_0 = 10 \quad h_X^{(\min)} = -0.2 \]

\[ p_0 = 2.5 \quad h_X^{(\min)} = -0.3 \]
Advantages and drawbacks of multifractal analysis based on the $p$-exponent

- Allows to deal with larger collections of data
- The estimation is not based on a unique extremal value, but on an $l^p$ average $\Rightarrow$ better statistical properties
- Systematic bias in the estimation of $p$-leaders
Beyond selfsimilarity? Historic photographic paper classification
Directions of research and open problems

1. Give an interpretation of the leader scaling function for $p < 0$
2. Obtain genericity results that would take into account the whole scaling function (and not only $p > 0$)
3. Extend multifractality results for solutions of PDEs
4. Develop a multifractal formalism that would take into account directional regularity (M. Clausel, S. Roux, B. Vedel)
5. Develop a local multifractal analysis: Pertinent in theory (Markov processes) and applications (finance)
7. Go beyond selfsimilarity
$H_{f}^{\text{min}} < 0$ : Spaces of distributions

Let $s > 0$; the fractional integral of order $s$ is the operator defined by

$$\mathcal{I}^s(f)(\xi) = (1 + |\xi|^2)^{-s/2} \hat{f}(\xi)$$

The application of $\mathcal{I}^s$ to $f$ shifts $H_{f}^{\text{min}}$ by $s$: $H_{\mathcal{I}^s f}^{\text{min}} = H_{f}^{\text{min}} + s$

A first solution if $H_{f}^{\text{min}} < 0$ consists in performing a fractional integration of $f$ of order $s > -H_{f}^{\text{min}}$

In practice one performs a pseudo-fractional integration of order $s > -H_{f}^{\text{min}}$ which amounts to replace the wavelet coefficients $c_\lambda$ by $2^{-sj}c_\lambda$

The new function $f_s$ thus obtained is uniform Hölder, with uniform regularity exponent $H_{f}^{\text{min}} + s > 0$

$\implies$ One can compute the wavelet leaders of $f_s$
$\implies$ Mathematical results concerning the multifractal formalism are valid