Quantum non-equilbrium dynamics in closed systems.

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Non-equilibrium phenomena in CM and ST, ICTP, Trieste, 07/03/2014

United States – Israel
Binational Science Foundation

Brief Outline

- ETH and thermodynamics
- Relaxation in integrable and weakly non-integrable systems
- Emergence of macroscopic Hamiltonian dynamics
  from non-adiabatic response
- Extension of Kibble-Zurek mechanism to dynamical fields.
  Dynamical localization transition near quantum critical
  points.
- Floquet systems

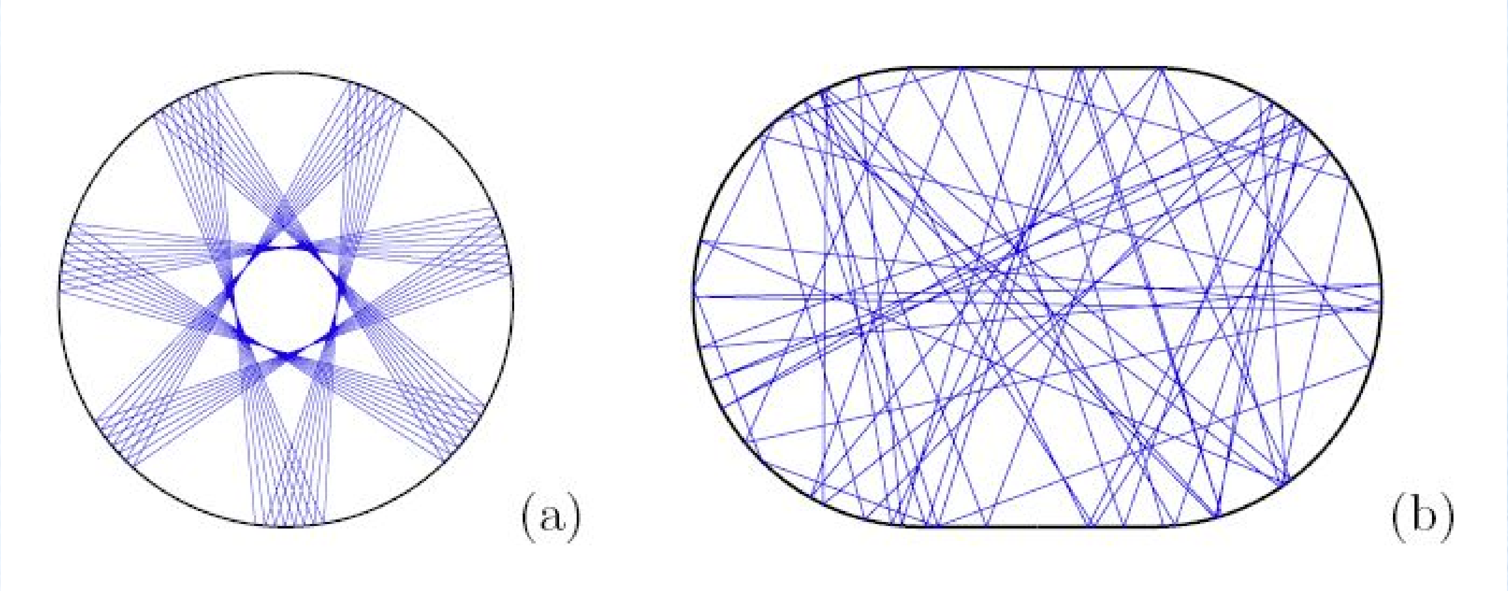
Non-equilibrium quantum systems

J. Von Neumann: Non-equilibrium theory is a “theory of **non-elephants**”.

- Chaos and thermalization in classical and quantum isolated systems, mostly after sudden quenches
- Relaxation in weakly interacting systems. Prethermalization and the generalized Gibbs ensemble.
- Steady states: phases and phase transitions in driven systems (isolated or dissipative)
- Many-body localization: interplay of disorder and ergodicity
- Universal non-equilibrium dynamics.
- Dynamical phase transitions (in time).
- Many more: non-equilibrium Stat. Mech., biological systems, active matter, …

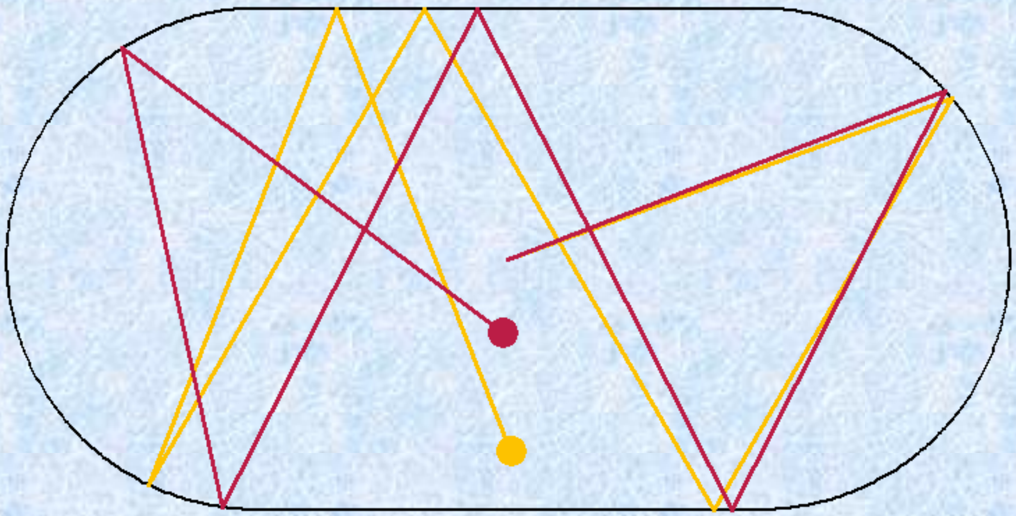
Classical chaotic systems

Regular vs. chaotic motion: like beauty. When we see it we recognize it.



No mathematically rigorous definition of chaos.

Standard practical definition: Divergent trajectories.



No chaos in one-dimension.

$$\frac{p^2}{2m} + V(x) = E$$

Ensures unique relation between *p* and *x*.

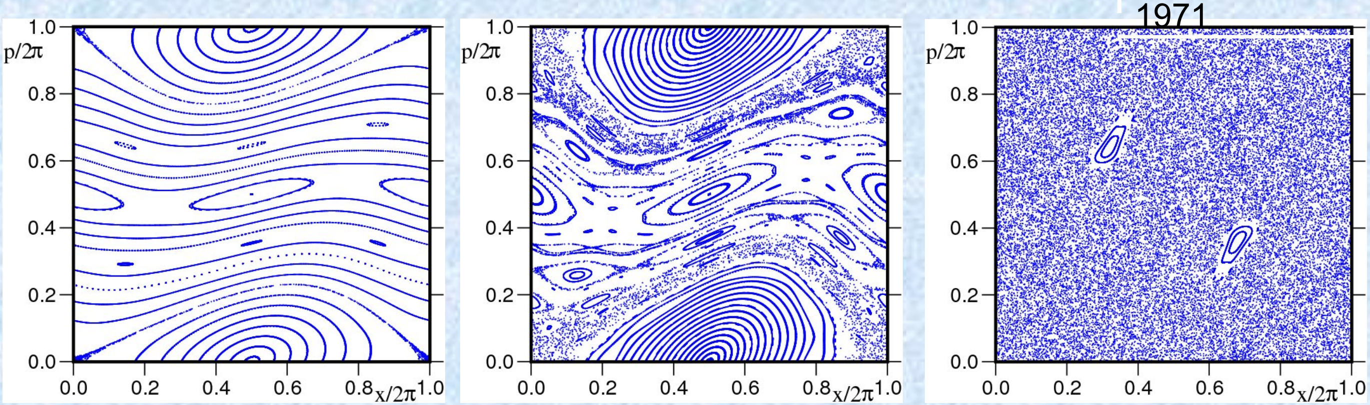
Simplest chaotic system – kicked rotor (kicked Josephson junction). Break energy conservation by kicks.

$$\mathcal{H}(p,\phi,t) = \frac{p^2}{2m} + K\cos(\phi)\delta(t - nT)$$

Equations of motion: Chirikov standard map

$$\phi_{n+1} = \phi_n + Tp_{n+1},\ p_{n+1} = p_n + K\sin(\phi_n)$$

Transition from regular (localized) to chaotic (delocalized) motion as K increases. Chirikov 1971



K=0.5 K=K_g=0.971635 K=5 (images taken from scholarpedia.org)

Quantum systems. Linear equations of motion no chaos?

$$i\hbar\partial_t\psi = \mathcal{H}\psi$$

Linear equations of motion like harmonic chains are non-chaotic.
Any solution is a superposition of normal modes (eigenstates).

von Neumann (1929): chaos (and ergodicity) are encoded in observables.

Wigner (1955), thinking about spectrum of complex nuclei: Hamiltonians of the nuclei are essentially like random matrices.
Any initial structure is rapidly lost once we start diagonalizing it.

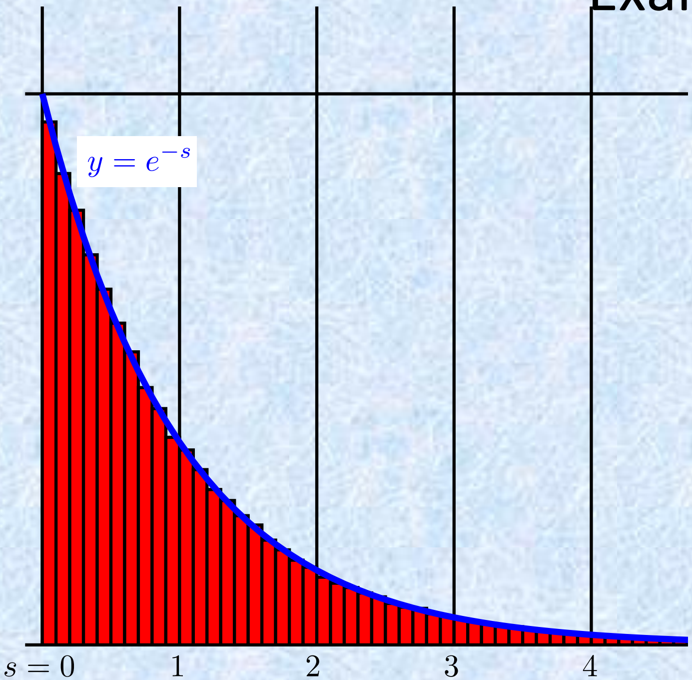
Famous prediction: level repulsion. Wigner-Dyson statistics (Wigner Surmize)

$$P(\omega) = A_\beta \omega^\beta \exp[-B_\beta \omega^2],\ \beta = 1, 2, (4)$$

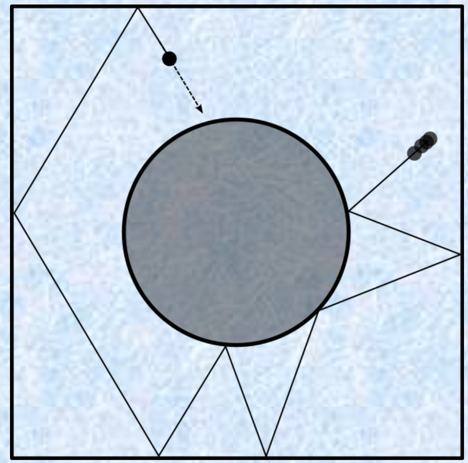
Non-chaotic "generic systems". Expect Poisson statistics (Berry-Tabor conjecture, 1977)

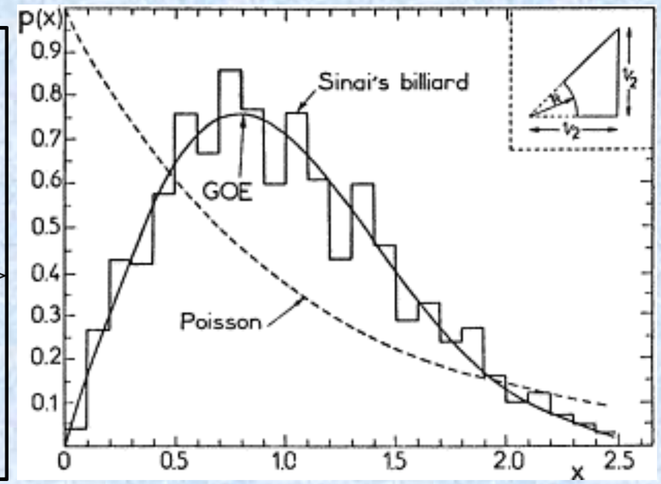
$$P_0(\omega) = \exp[-\omega].$$

Examples of level statistics

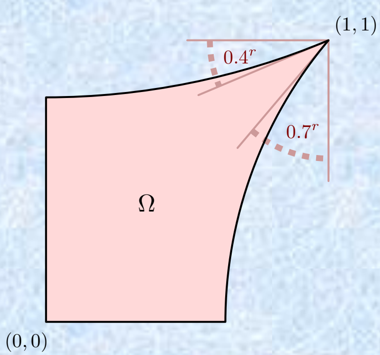


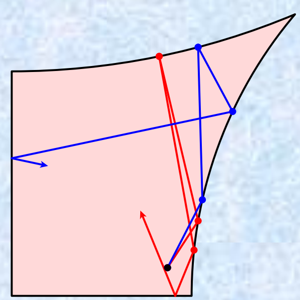
Incommensurate 2D box.
Z. Rudnik, 2008

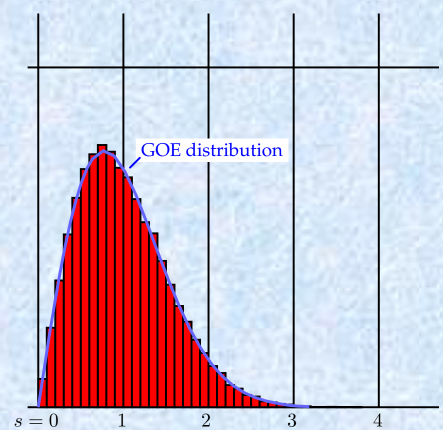




Sinai Billiard. O. Bohigas et. al. 1984







Another chaotic
billiard by Sinai.
Z. Rudnik, 2008

Many-particle systems. Thermalization through eigenstates.

J. Von Neumann (1929), J. Deutch (1991), M. Srednicki (1994), M. Rigol et. al. (2008)

Extension of Wigner ideas : Ergodic Hamiltonians within a Thouless energy shell
$E_T = \hbar/\tau_D,\ \tau_D = L^2/D$ looks like a random matrix.

This shell contains Exponentially many levels. Hence recover Wigner-Dyson statistics, all eigenstates are statistically the same, so each one is a good microcanonical ensemble, …

$$\mathcal{H} = E\mathcal{I} + \mathrm{e}^{-S(E)/2}\mathcal{R},\ P(\mathcal{R}) \propto \exp\left[-\frac{1}{4}Tr[\mathcal{R}^2]\right].$$

Eigen-functions are random vectors in a large-dimensional space.

Observables (M. Srednicki 1996, M. Rigol et. al. 2008)

$$\mathcal{O}_{mn} = \mathcal{O}(E)\delta_{mn} + \mathrm{e}^{-S(E)/2}\sigma_{mn}$$

Natural extension beyond the Thouless energy shell

$$\mathcal{O}_{mn} = \mathcal{O}(E)\delta_{mn} + \mathrm{e}^{-S(\bar{E})/2}f(E_n - E_m)\sigma_{mn}$$

Numerical checks

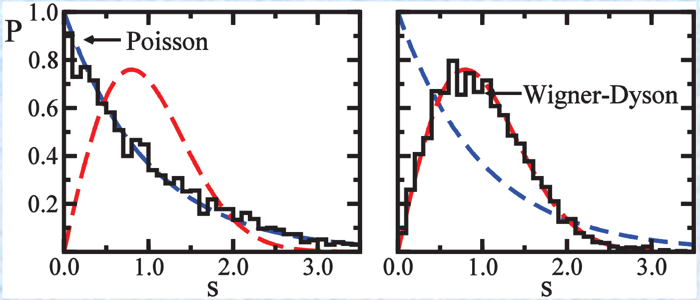


FIG. 1: (Color online) Level spacing distribution for the Hamiltonian in Eqs. (1) with $L = 15$, 5 spins up, $\omega = 0$, $\epsilon_d = 0.5$, $J_{xy} = 1$, and $J_z = 0.5$ (arbitrary units); bin size = 0.1. (a) Defect on site $d = 1$;(b) defect on site $d = 7$. The dashed lines are the theoretical curves.

Interacting spin-chain with a single impurity

(A. Gubin and L. Santos, 2012)

$$H_z = \sum_{i=1}^{L} \omega_i S_i^z = \left(\sum_{i=1}^{L} \omega S_i^z \right) + \epsilon_d S_d^z$$

$$H_{\mathrm{NN}} = \sum_{i=1}^{L-1} \left[J_{xy} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + J_z S_i^z S_{i+1}^z \right]$$

Onset of quantum chaos is the same as onset of thermalization.

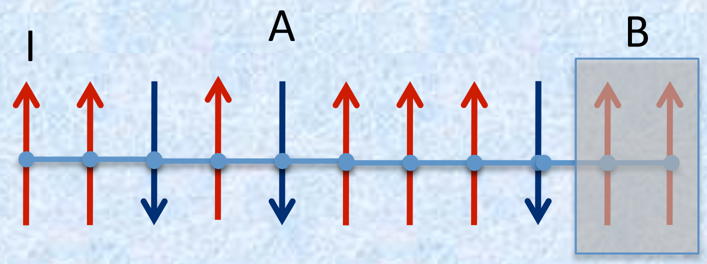
Thermalization: dephasing (= energy measurement) + ETH

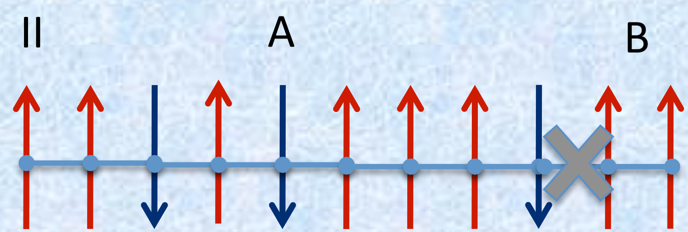
Thermalization: due to ETH delocalization initial states in the space of eigenstates (projection to exponentially many random vectors).

Parallels with many-body localization (Basko, Aleiner, Altshuler, 2006), D. Huse et. al. 2008+

How robust are the eigenstates?

Prepare the system in the energy eigenstate





Setup I: trace out few spins (do not touch them but no access to them)

Setup II: suddenly cutoff the link

Same reduced density matrix at *t=0*

Define two relevant entropies (both are conserved in time after quench):

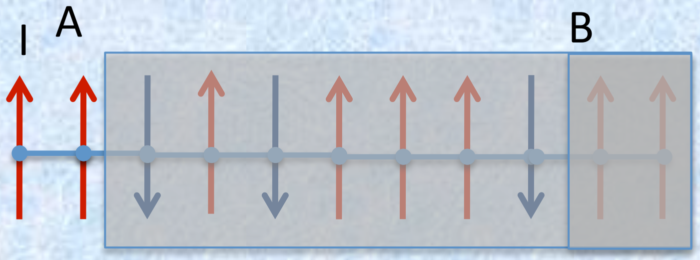
$$S_{A,vn} = -\mathrm{Tr}[\rho_A \log(\rho_A)]$$

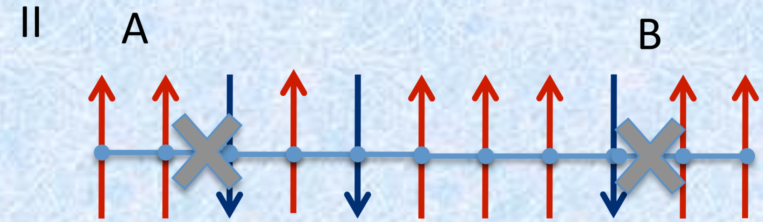
Entanglement (von Neumann's entropy)

$$S_{A,d} = -\sum_n \rho_{A,nn} \log(\rho_{A,nn})$$

Diagonal (measure of delocalization), entropy of time averaged density matrix after quench

Both entropies are conserved in time in both setups





$$S_{A,vn} = -\mathrm{Tr}[\rho_A \log(\rho_A)]$$

$$S_{A,d} = -\sum_n \rho_{A,nn} \log(\rho_{A,nn})$$

B>>A, trace out most of the system. Eigenstate is a typical state so expect that

$$\rho_A \sim \exp[-\beta H_A] \;\Rightarrow S_{A,vn} \approx S_{A,d} \approx S_{eq}$$

B<<A, trace out few spins, perhaps only one (say out of 10^{22}). What happens?

$$S_{A,vn} = S_{B,vn} \sim N_B$$

Remove one spin. Barely excite the system. Density matrix is almost diagonal (stationary)?

$S_{A,d} \cancel{\approx S_{A,vn}}?$ Wrong (in nonintegrable case)

Use ETH: Perform quench, deposit non extensive energy δE. Occupy all states in this energy shell.

$$S_{A,d} \approx \log(\Omega(E_A)\delta E) \approx S_{A,eq}$$

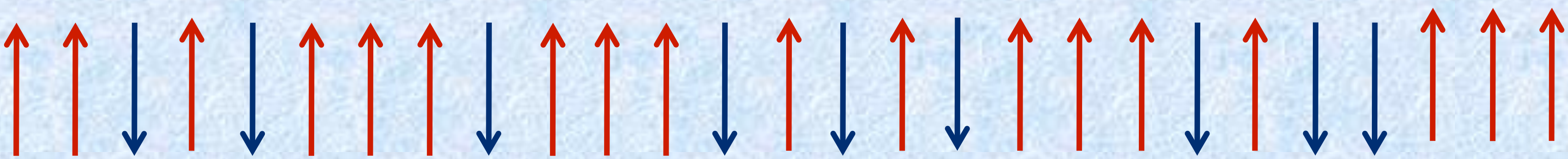
Cutting one spin is enough to recreate equilibrium (microcanonical) density matrix. (Numerical check L. Santos, M. Rigol, A.P. 2012

Statistical mechanics can be recovered from one postulate:

All states within a narrow shell are equally occupied

Thermodynamics (including this postulate) can be recovered from ETH plus dephasing (relaxation to the diagonal ensemble).

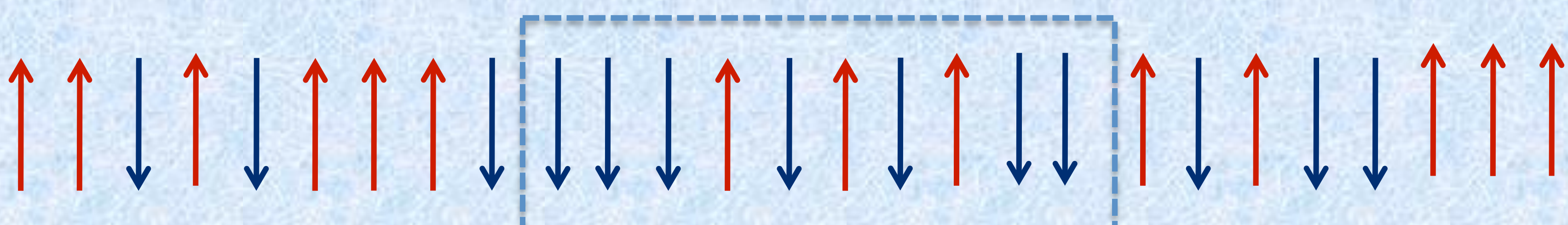
Imagine the most integrable many-body system: collection of non-interacting Ising spins. The dynamics is very simple: each spin is conserved.



A typical state with a fixed magnetization is thermal. Stat. Mech. works

Quantum typicality: Typical many-body state of a big system (Universe) locally look like thermal (von Neumann 1929, Popescu et. al. 2006, Goldstein et. al. 2009)

Flip 10 spins in the middle doing a Rabi pulse



By performing a local quench we create a very atypical state, which is not thermal, whether the system is integrable or not. In an integrable system this state will never thermalize.

Fundamental thermodynamic relation for open and closed systems

Start from a stationary state. Consider some dynamical process

$$E = \sum_n E_n \rho_{nn} \Rightarrow \Delta E = \sum_n E_n \Delta\rho_{nn} + \Delta E_n \rho_{nn}$$

$$S_d = -\sum_n \rho_{nn} \log(\rho_{nn}) \Rightarrow \Delta S_d = -\sum_n \Delta\rho_{nn} \log(\rho_{nn}) + \cancel{\Delta\rho_{nn}}$$

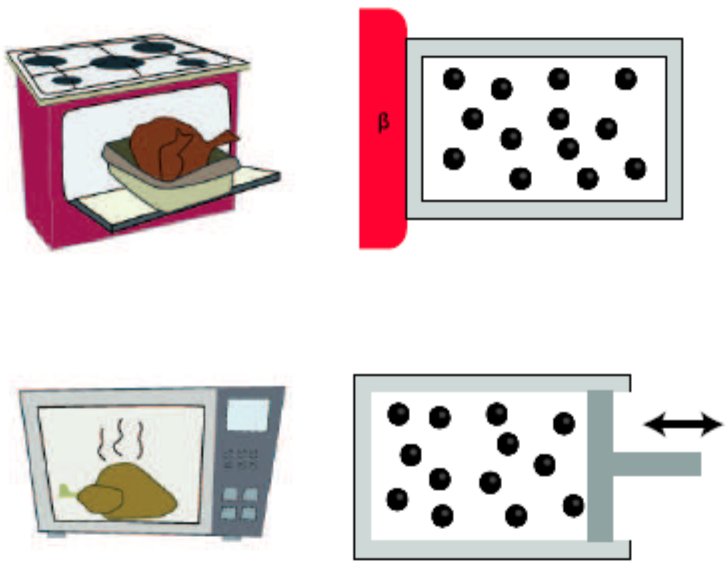
Assume initial Gibbs Distribution

$$\rho_{nn} = \frac{1}{Z}\exp[-\beta E_n] \Rightarrow \log(\rho_{nn}) = -\beta E_n - \log(Z)$$

Combine together

$$\Delta E = \left.\frac{\partial E}{\partial \lambda}\right|_{S_d} \Delta\lambda + \frac{1}{T}\Delta S_d \Leftrightarrow \Delta E = T\Delta S - \mathcal{F}_\lambda \Delta\lambda,\ \mathcal{F}_\lambda = -\left.\frac{\partial E}{\partial \lambda}\right|_S$$

Recover fundamental relation with the only assumption of Gibbs distribution



Imagine we are umping some energy into an isolated system. Does fundamental relation still apply?

$$S_d = -\sum_n \rho_{nn} \log(\rho_{nn}) = -\int dE\, \Omega(E)\rho(E) \log(\rho(E))$$

$$\rho(E)\Omega(E) = W(E) \quad - \quad \text{Energy distribution}$$

$$\log(\Omega(E)) = S_m \quad - \quad \text{microcanonical entropy}$$

$$S_d(\epsilon) \quad = \quad \int d\epsilon W(\epsilon) S_{micro}(\epsilon) - \int d\epsilon W(\epsilon) \ln(W(\epsilon))$$

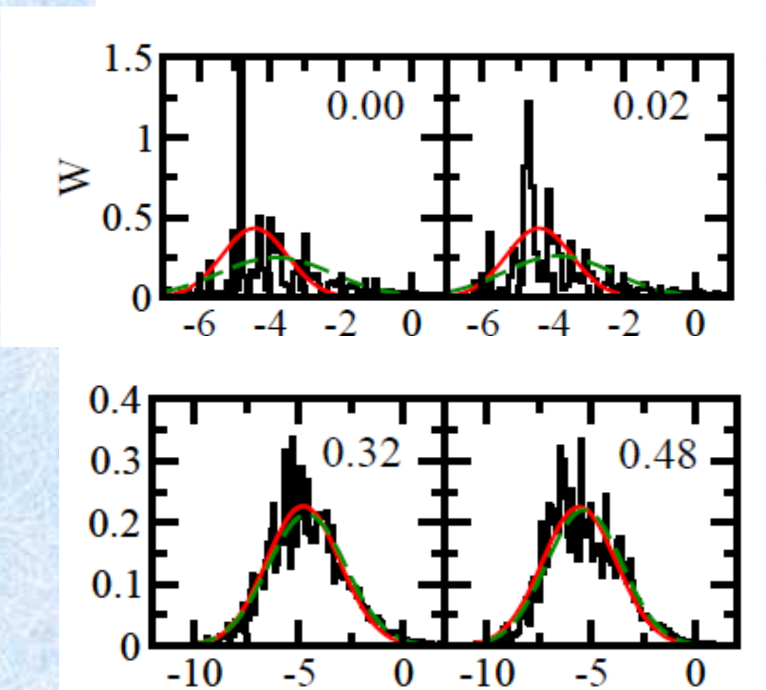
Recover fundamental relation if the density matrix is not exponentially sparse.

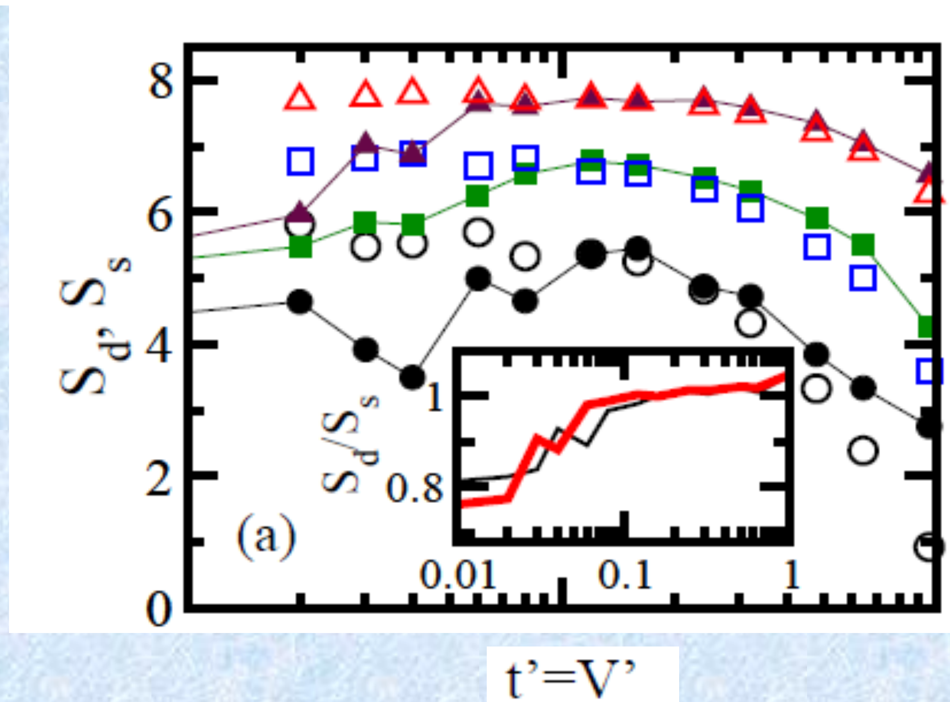
Density matrix is not sparse after any quench by local operators due to ETH.

Example: hardcore bosons in 1D (with L. Santos and M. Rigol)

$$H_B = \sum_{j=1}^{L} \Big[-t \left(\hat{b}_j^\dagger \hat{b}_{j+1} + \text{H.c.} \right) - t' \left(\hat{b}_j^\dagger \hat{b}_{j+2} + \text{H.c.} \right) + \quad (4)$$

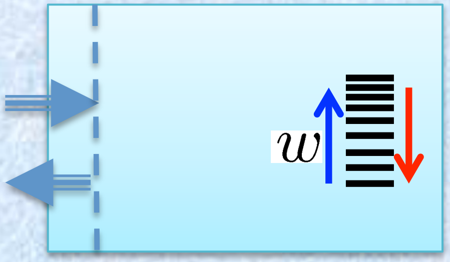
$$V \left(\hat{n}_j^b - \frac{1}{2} \right) \left(\hat{n}_{j+1}^b - \frac{1}{2} \right) + V' \left(\hat{n}_j^b - \frac{1}{2} \right) \left(\hat{n}_{j+2}^b - \frac{1}{2} \right) \Big]$$





ETH and delocalization in the Hilbert space come together!

Fluctuation theorems (Bochkov, Kuzovlev, Jarzynski, Crooks)



Initial stationary state + time reversability:

$$p_{mn} = p_{nm} = |U_{mn}|^2$$

Eigenstate thermalization hypothesis: microscopic probabilities are smooth (independent on m,n)

$$P_E(w) = \sum_m p_{mn} \delta(E_m - E_n - w) = p_{mn} \Omega(E + w)$$

$$\tilde{P}_{E+w}(-w) = p_{nm} \Omega(E)$$

$$\frac{P_E(w)}{\tilde{P}_{E+w}(-w)} = \frac{\Omega_{\lambda+\delta\lambda}(E+w)}{\Omega_\lambda(E)} \approx \exp[\beta w - \beta \Delta F]$$

Bochkov, Kuzivlev, 1979, Crooks 1998

Integrate Crooks equality get

$$\langle \exp[-\beta w] \rangle = \exp[-\beta \Delta F]$$

Jarzynski equality, 1997

$$w = w_{\rm ad} + Q, \;\; w_{\rm ad} = \Delta F, \;\; \langle \exp[-\beta \Delta Q] \rangle = 1$$

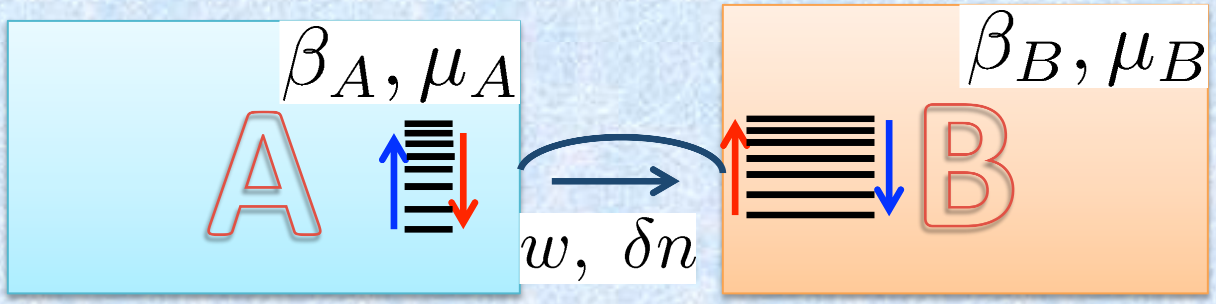
Jarzynski equality allows one to separate heating from adiabatic work for an arbitrary protocol. Hard to measure if work is large (e.g. extensive)

Incremental heating, do cumulant expansion

$$\langle \exp[-\beta \delta Q] \rangle = 1 \;\Rightarrow\; \langle \delta Q \rangle \approx \frac{\beta}{2} \langle \delta Q^2 \rangle_c$$

Infinitesimal version of the second law + Einstein like relations from ETH

Open systems



Two (or more conserved quantities): from ETH recover (non-equilibrium) Onsager relations

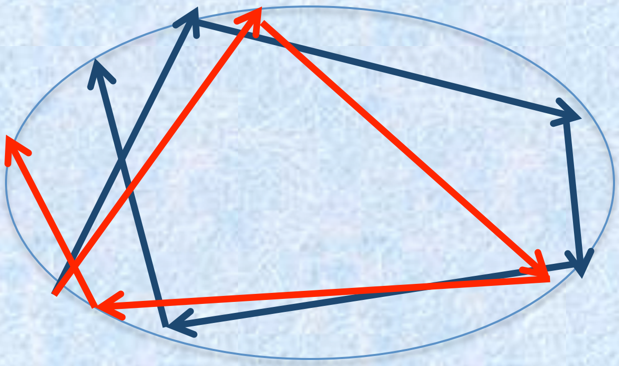
$$\begin{pmatrix} w \\ \delta n \end{pmatrix} = \begin{pmatrix} \delta w^2 & \langle w \delta n \rangle_c \\ \langle w \delta n \rangle_c & \delta n^2 \end{pmatrix} \begin{pmatrix} \Delta\beta \\ \Delta\lambda \end{pmatrix}$$

Microcanonical ensemble + locality of interactions: recover statistical mechanics.

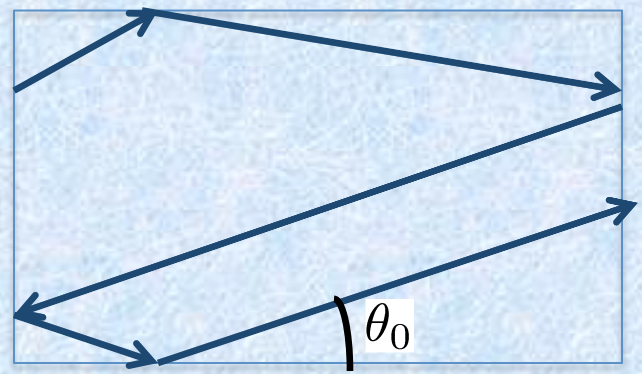
ETH + dephasing: recover thermodynamics

- Second law of thermodynamics (ETH is not needed)
- Fundamental thermodynamic relation
- Fluctuation-dissipation relations
- Einstein relations (and universal microwave heating laws)
- Fluctuation theorems (Jarzynski, Crooks)
- Onsager relations
- Detailed balance
- Finite size corrections to temperature
- Delocalization in periodically driven (Floquet) systems. Divergence of the Magnus (Baker-Campbell-Hausdorff) expansion in the thermodynamic limit in ergodic systems (P. Ponte et. al. 2014)
- Nontrivial long-range correlations in systems with flowing currents

Integrable vs. Non-integrable systems



Chaotic system: rapid (exponential) relaxation to microcanonical ensemble



Integrable system: relax to constraint equilibrium:

$$P(x,p) \to \delta(p-p_0)\delta(\theta-\theta_0)$$

Quantum language: in both cases relax to the diagonal ensemble

$$\rho_{mn} \to \rho_{nn}\delta_{mn}$$

$$\rho_{nn} = \frac{1}{Z}\exp[-\sum_n \beta_n I_n]$$

Integrable systems: generalized Gibbs ensemble (Jaynes 1957, Rigol 2007, J. Cardy, F. Essler, P. Calabrese, J.-S. Caux, E. Yuzbashyan …) . What if integrability is slightly broken?

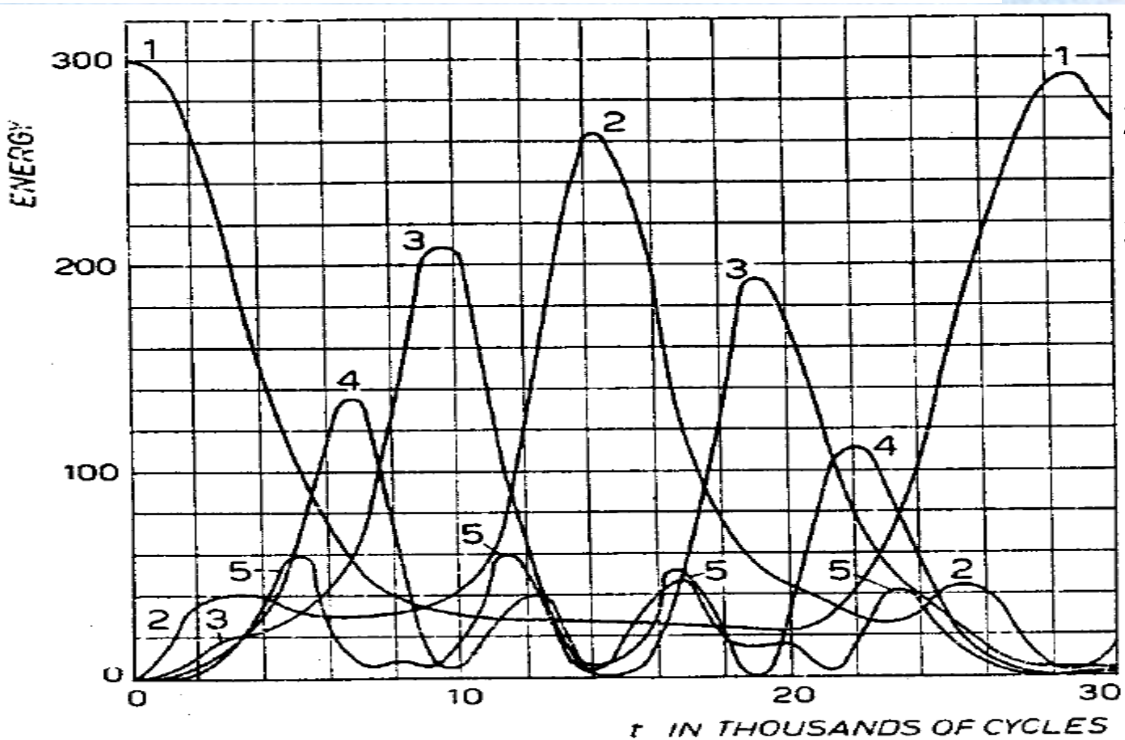
Relaxation to equilibrium in integrable, and nearly integrable systems. Prethermalization

Fermi-Pasta-Ulam problem

$$\ddot{x}_n = (x_{n+1} - 2x_n + x_{n-1}) + \beta[(x_{n+1} - x_n)^3 - (x_n - x_{n-1})^3]$$

STUDIES OF NON LINEAR PROBLEMS

E. FERMI, J. PASTA, and S. ULAM
Document LA-1940 (May 1955).



Slow variables

$$E_q = \frac{1}{2}(|p_q|^2 + |\omega_q x_q|^2)$$

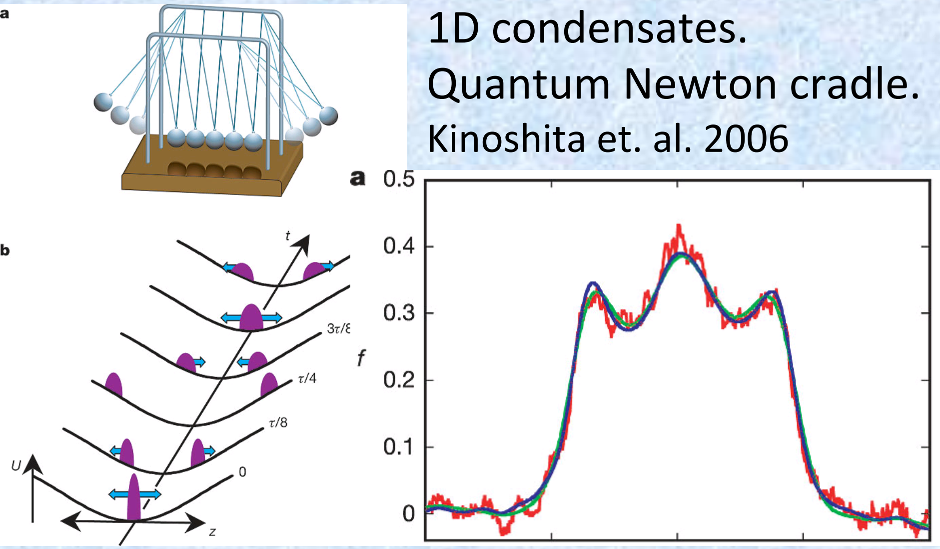
Expectation:

1. Excite single normal mode
2. Follow dynamics of energies
3. Eventual energy equipartition

Found:

1. Quasiperiodic motion
2. Energy localization in q-space
3. Revivals of initial state
4. No thermalization!

Recent experiments on prethermalization



1D condensates.
Quantum Newton cradle.
Kinoshita et. al. 2006

Prethermalization of condensates on atom chips. Gring et. al. 2012

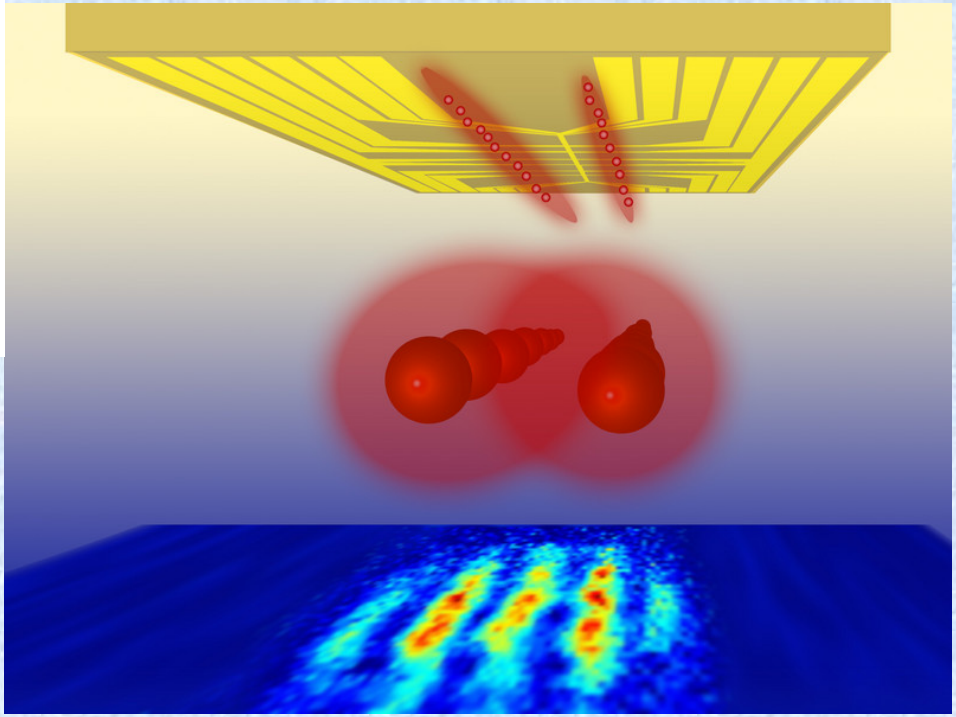
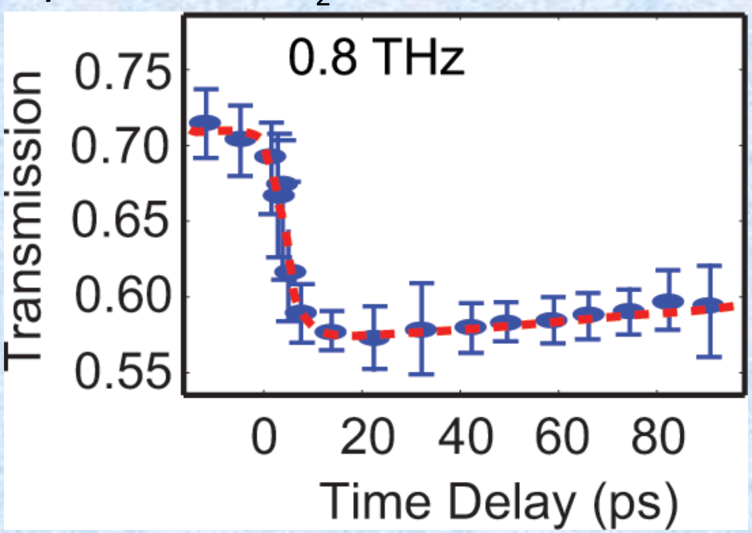


Image from J. Schmiedmayer page

No equilibration between symmetric and anti-symmetric modes.

Transmission coefficient for pump-probe dynamics in VO_2 oxide

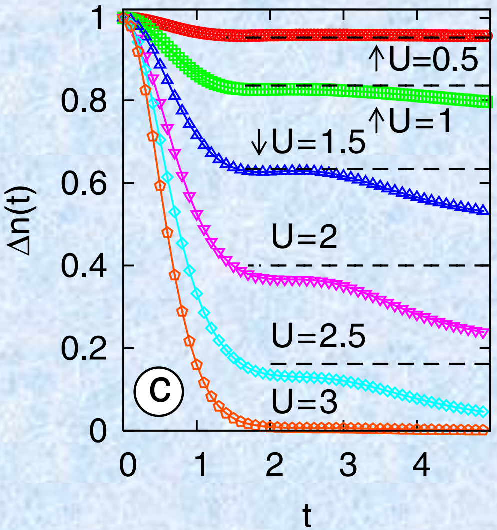


M. K. Liu, et al, Nature 487, 345 (2012)

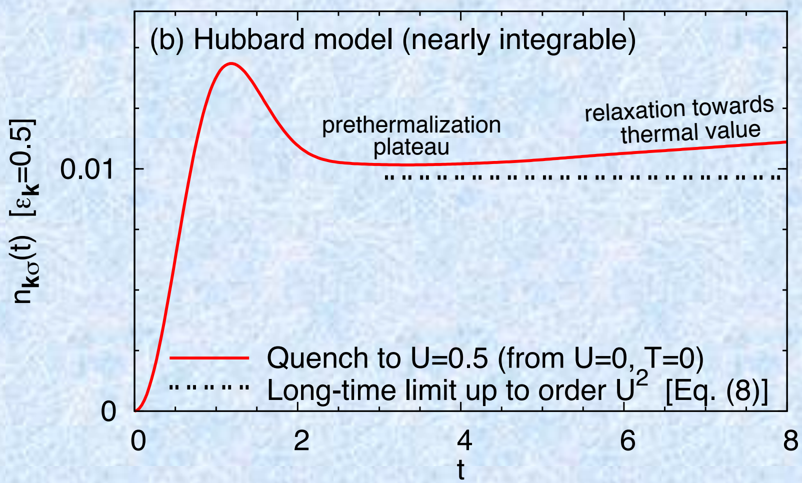
Theory.

- Original proposal by Berges et. al. (2004). Some of it is well known from long time ($T_e \neq T_{\mathrm{lattice}}$ in semiconductor lasers) Pinned the term prethermalization.
- Apparent connections to Kolmogorov turbulence (V. Gurarie, 1994).
- Possible understanding through renormalization group (T. Gasenser, J. Berges, L. Mathey, A.P., A. Mitra, R. Vosk, E. Altman, … recent)
- Connection to quantum kinetic equations and GGE (S. Kehrein, M. Kollar, M. Eckstein,… recent).

Fermi surface discontinuity for the interaction quench. M. Eckstein et al 2009

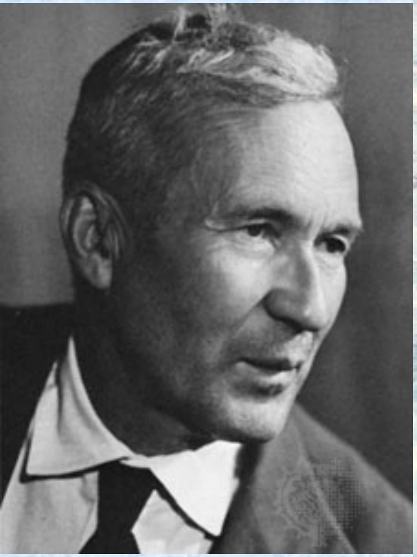


Explanation through the GGE ensemble.
M. Kollar et al 2011



No general theoretical framework yet, but a few ideas are very promising.

More familiar examples of GGE: Kolmogorov turbulence



A. N. Kolmogorov

Images from Wikipedia

Pump energy at long wavelength. Dissipate at short wavelength. Non-equilibrium steady state

Scaling solution of the Navier Stokes equations

$\partial v + (v\nabla)v = -\nabla p + \nu \triangle v$

$\nabla v = 0$

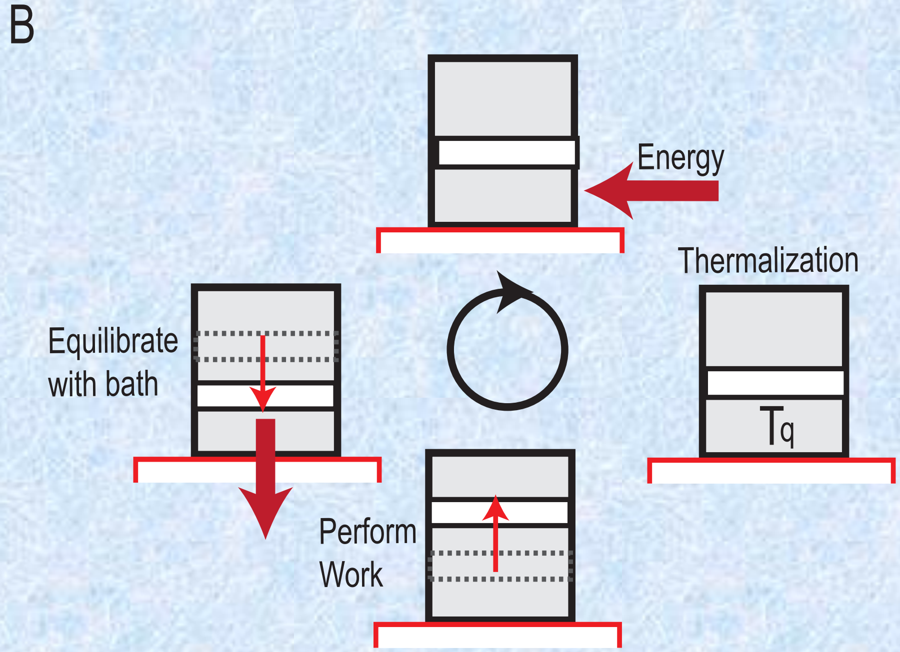
$$E_k \propto C\nu^{2/3}k^{-5/3}$$

$$E(k) \simeq P^{2/3}k^{-5/3}\rho^{1/3}$$

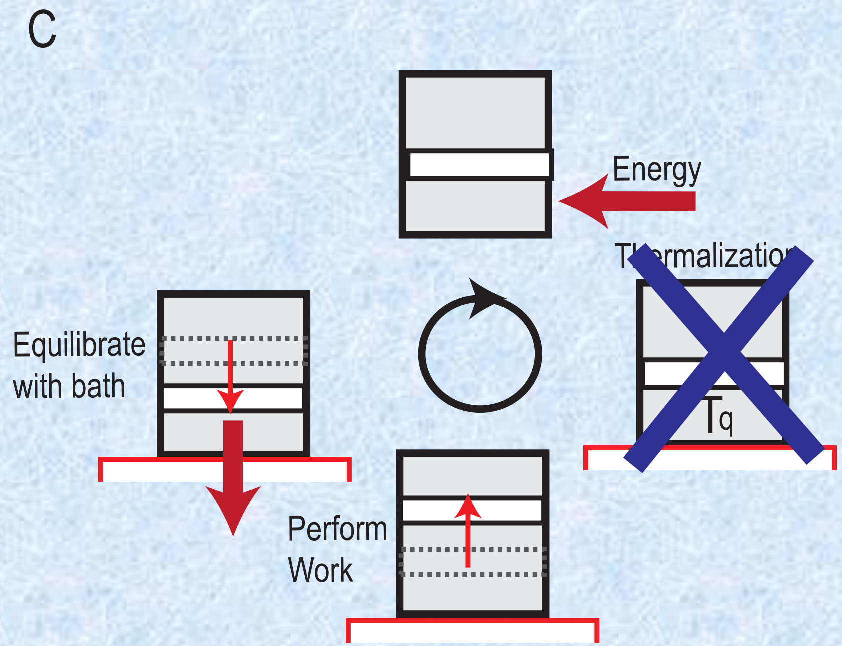
This energy can be thought of as the mode dependent temperature. A particular type of GGE.

Zakharov, L'vov, Fal'kovich: derived this solution from the kinetic equations

Potential applications to non-ergodic engines (work cycle faster than thermalization time)



Ergodic engine

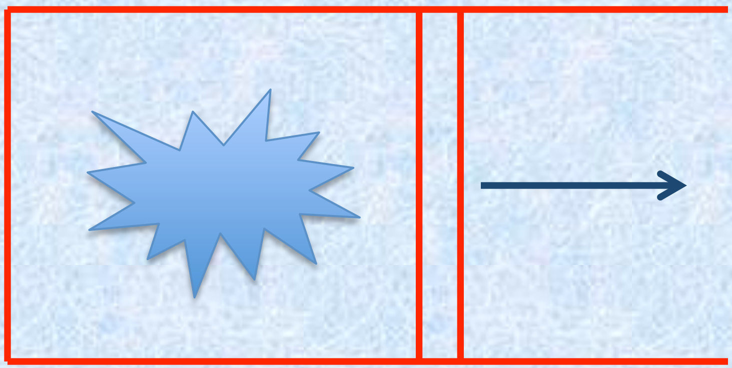


Non-ergodic engine

Can beat the second law and have much higher efficiency.
Example: electric engines vs. combustion engines.

Reason for higher efficiency of non-ergodic engines. Need to release less entropy to the environment (P. Mehta and A.P., 2012).

Ideal gas engine (Lenoir cycle)



I) Deposit energy at constant volume
II) Push piston until pressure drops to equilibrium
III) Relax back to equilibrium at constant pressure

If the energy is induced along the x-axis and the work cycle is shorter than the tehrmalization time get a higher efficiency.

Closely related research: information theory (Szilard engine), Informational thermodynamics (realizing Maxwell's daemons), information and reversibility.

Gauge transformations in quantum systems

$$|\psi(\vec{\lambda})\rangle = U^\dagger(\vec{\lambda})|\Psi\rangle$$

Canonical transformations = equations of motion.
Gauge potential = momentum operator

$$i\partial_\lambda|\psi(\vec{\lambda})\rangle = i\partial_\lambda U^\dagger(\vec{\lambda})|\Psi\rangle = -\mathcal{A}_\lambda|\psi\rangle,\ \mathcal{A}_\lambda = iU^\dagger\partial_\lambda U,\ \mathcal{A}_\lambda^\dagger = \mathcal{A}_\lambda$$

If $|\psi(\vec{\lambda})\rangle$ is a family of ground states $A_\lambda = \langle\mathcal{A}_\lambda\rangle_0$ is the Berry connection

Hamiltonian equations of motion in a moving frame

$$i\partial_t|\psi\rangle = i\dot{\lambda}_a\partial_{\lambda_a}U^\dagger|\Psi\rangle + U^\dagger H|\Psi\rangle = (U^\dagger H U - \dot{\lambda}_a\mathcal{A}_a)|\psi\rangle$$

Special instantaneous frame, where U diagonalizes instantaneous Hamiltonian.

This frame is convenient to study the non-adiabatic response perturbatively.

General theory of non-adiabatic response
(with L. D'Alessio, 2013)

$$i\partial_t|\psi\rangle = (U^\dagger H U - \dot\lambda_a \mathcal{A}_a)|\psi\rangle = (\tilde H - \dot\lambda_a \mathcal{A}_a)|\psi\rangle$$

$$i\partial_t \rho = [\rho, \tilde H - \dot\lambda_a \mathcal{A}_a]$$

Go to the interaction (adiabatic Heisenberg) picture with respect to $\tilde H$. Use standard perturbation theory

$$\rho(t) = \rho_0 + i\int_0^t dt' \dot\lambda_a(t')[\mathcal{A}_a(t'), \rho_0]$$

Calculate expectation values of observables

$$M_b(t) \equiv \langle -\partial_b H\rangle = M_b^0 - \int_0^t dt' \int_0^\beta d\tau \dot\lambda_a(t')\langle \mathcal{M}_b(t)\mathcal{M}_a(t'+i\tau)\rangle_0$$

Expand the rate near *t'=t.*

$$\partial\mathcal{M}_b = -\partial_{\lambda_b} H$$

Simple expression which contains a lot

$$M_b(t) = M_b^{(0)} + F_{ba}\dot{\lambda}_a - \eta_{ba}\dot{\lambda}_a - \kappa_{ba}\ddot{\lambda}_b - F'_{ba}\ddot{\lambda}_a$$

Off-shell

$$F_{ba} = i\langle[\mathcal{A}_b, \mathcal{A}_a]\rangle_0$$

Berry curvature

$$\kappa_{ba} = \frac{1}{2}\int_0^\beta d\tau \langle \mathcal{A}_b(-i\tau)\mathcal{A}_a(0) + \mathcal{A}_a(-i\tau)\mathcal{A}_b(0)\rangle_0$$

Mass tensor

$$\kappa_{ba} \approx \frac{\beta}{2}\langle \mathcal{A}_b(0)\mathcal{A}_a(0) + \mathcal{A}_a(0)\mathcal{A}_b(0)\rangle_0 = \beta g_{ba}$$

High-temperature (classical limit). Reduces to the equipartition theorem

On-shell: F' – asymmetric friction,

$$\eta_{ba} = \pi\beta \sum_{n\neq m} \rho_m^0 \langle m|\mathcal{M}_b|n\rangle\langle n|\mathcal{M}_a|m\rangle \delta(\mathcal{E}_n - \mathcal{E}_m)$$

Friction tensor

Positive temperature guarantees simultaneous positivity of *η, κ* (second law of thermodynamics)

Usual setup: $\lambda(t)$ is an external parameter:
Quenches, non-adiabatic dynamics, Floquet systems, ...

What if $\lambda(t)$ itself is a macroscopic dynamical field?
E.g. a magnet coupled to a spring, large N order parameter, superconducting gap, center of mass (angle) of a large object, ...

(Barankov, Levitov, Spivak; Yuzbashyan et. al.; Andreev, Gurarie, Radzihovsky; Chandran et. al., …)

Need to solve coupled self-consistent equations of motion.

$$H_{tot}(\lambda) = H_0(\lambda, p_\lambda) + H(\lambda, \{\hat{s}_j\})$$

$$\frac{d\lambda}{dt} = \frac{\partial H_0}{\partial p_\lambda}, \quad \frac{dp_\lambda}{dt} = -\frac{\partial H_0}{\partial \lambda} - \langle \psi(t)|\partial_\lambda H|\psi(t)\rangle$$

$$i\partial_t|\psi(t)\rangle = H(\lambda(t))|\psi(t)\rangle$$

Adiabatic limit: Born-Oppenheimer approximation.
Berry (1989) – quantum correction to the Born-Oppenheimer force, given by the Fubini-Study metric.

Our goal: go beyond adiabatic approximation.

$$H_{tot}(\vec{\lambda}) = H_0(\vec{\lambda}) + H(\vec{\lambda})$$

The Hamiltonian H_0 is just to build intuition, e.g.

$$H_0(\vec{\lambda}) = \frac{\vec{p}_{\vec{\lambda}}^{\,2}}{2m} + V(\vec{\lambda})$$ $m \to \infty$ means $\vec{\lambda}$ is an external field

$$m_{ba}\dot{\lambda}_a = p_b$$

$$\dot{p}_b = -\partial_{\lambda_b} V + M_b(t) = -\partial_{\lambda_b} V - \langle \psi(t) | \partial_{\lambda_b} H | \psi(t) \rangle$$

$$(m_{ba} + \kappa_{ba} + F'_{ba})\ddot{\lambda}_a + (\eta_{ba} - F_{ba})\dot{\lambda}_a = -\partial_{\lambda_b} V + M_b^0$$

Zero temperature + gap: recover macroscopic Hamiltonian (Newtonian) dynamics. Can always find a canonical momentum

These equations without dissipation can be rewritten in the Lagrangian and Hamiltonian form

$$(m_{ba} + \kappa_{ba})\ddot{\lambda}_a - F_{ba}\dot{\lambda}_a = -\partial_{\lambda_b} V + M_b^0$$

$$\mathcal{L} = \frac{1}{2}\,\dot{\lambda}_\nu\,(m+\kappa)_{\nu\mu}\,\dot{\lambda}_\mu + \dot{\lambda}_\mu\,A_\mu(\vec{\lambda}) - V(\vec{\lambda}) - E_0(\vec{\lambda})$$

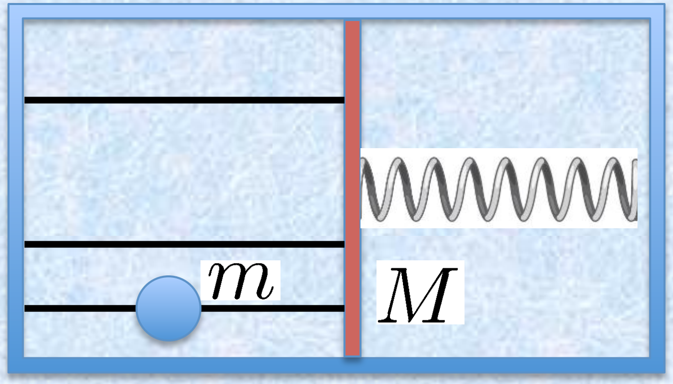
$E_0(\vec{\lambda}) \;=\; \langle 0_\lambda | H(\vec{\lambda}) | 0_\lambda \rangle$ - responsible for the Casimir force

$$p_\nu \equiv \frac{\partial \mathcal{L}}{\partial \dot{\lambda}_\nu} = (m_{\nu\mu} + \kappa_{\nu\mu})\dot{\lambda}_\mu + A_\nu(\vec{\lambda})$$

$$\mathcal{H} \equiv \dot{\lambda}_\nu\,p_\nu - \mathcal{L} = \frac{1}{2}(p_\nu - A_\nu)(m+\kappa)^{-1}_{\nu\mu}(p_\mu - A_\mu) + V(\vec{\lambda}) + E_0(\vec{\lambda})$$

Emergent Hamiltonian dynamics with minimal coupling to the gauge fields.
Without mass renormalization term: M. Berry 1989

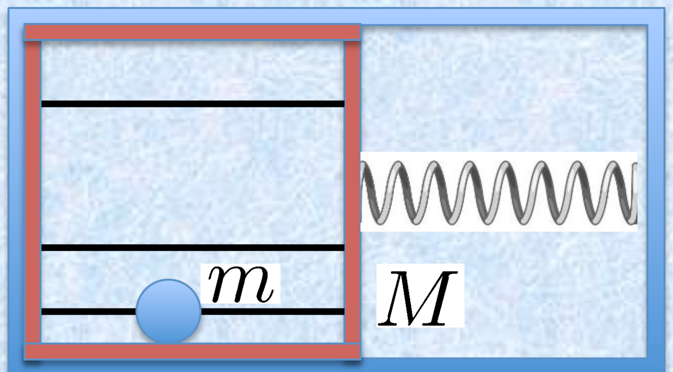
Example: particle in a box



Massless membrane.

What mass will we measure?

$$M = 2\hbar^2 \sum_{n\neq 0} \frac{\langle n|\partial_x H|0\rangle^2}{(E_n - E_0)^3} = 2\sum_{n\neq 0} \frac{V^2\psi_0(0)^2\psi_n(0)^2}{(E_n - E_0)^3} = m\frac{16}{\pi^2}\sum_{n=2}^{\infty}\frac{n^2}{(n^2-1)^3} = m\left(\frac{1}{3} - \frac{1}{2\pi^2}\right) \approx 0.28m$$



$$M = 2\sum_{n\neq 0} \frac{V^2(\psi_0(0)^2\psi_n(0)^2 + \psi_0(L)^2\psi_n(L)^2 - 2\psi_0(0)\psi_0(L)\psi_n(0)\psi_n(L))}{(E_n - E_0)^3} = m\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{16n^2}{(4n^2-1)^3} = m$$

Mass of two walls is not equal to the sum of masses of each wall measured separately!. Mass of photons coming.

Mass renormalization near a quantum critical point

$$\kappa_\lambda = \int_0^\infty d\tau \langle \mathcal{A}_\lambda(0) \mathcal{A}_\lambda(\tau) \rangle_0 = 2 \sum_{n \neq 0} \frac{\langle n | \partial_\lambda H | 0 \rangle^2}{(E_n - E_0)^3}$$

Mass has a singularity near QCP.

Scaling dimension of the mass density

$$[\mu] = d - z - \frac{2}{\nu}$$

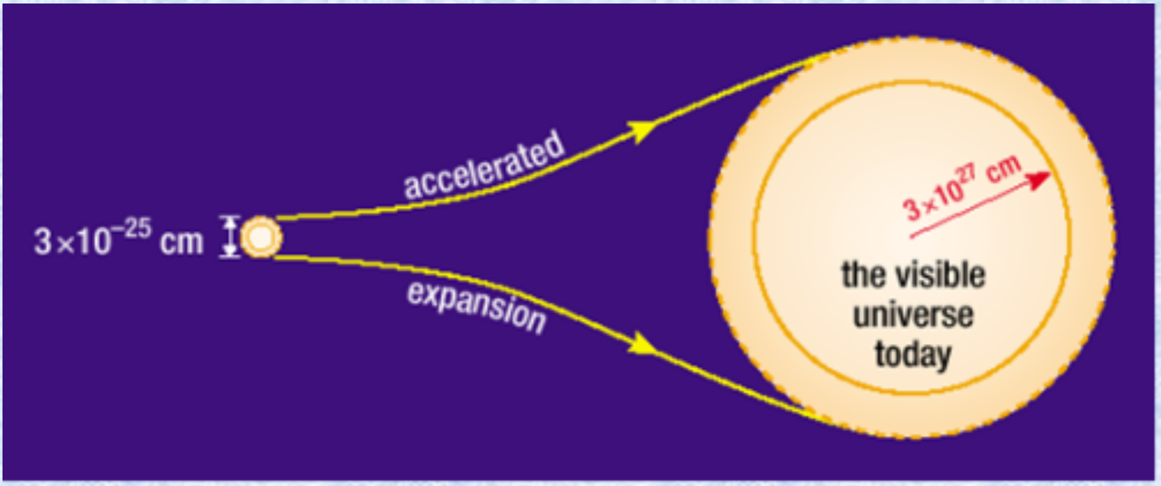
Mass diverges for

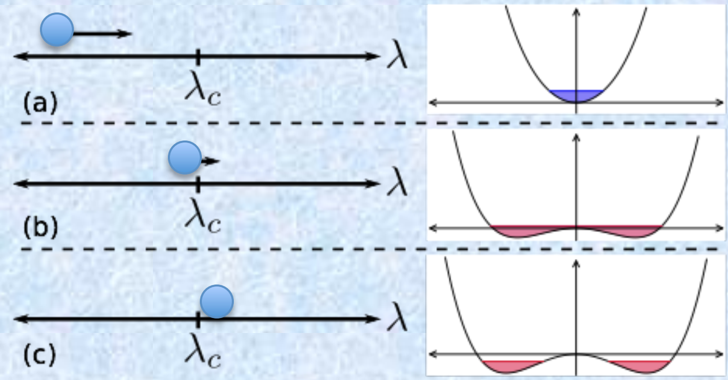
$$d < d_c = z + \frac{2}{\nu}$$

$$z = 1,\, \nu = 1/2 \;\Rightarrow d_c = 5, \quad z = 1,\, \nu = 1, \Rightarrow d_c = 3$$

Below critical dimension it can be hard to pass through QCP

From a snowball to inflation in cosmology (hypothetical scenario)



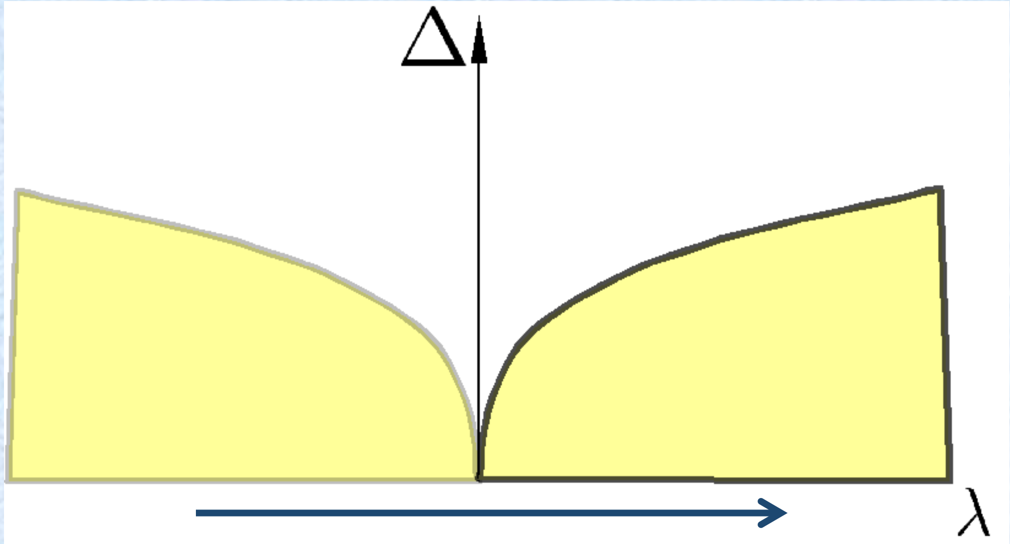


$$H_\phi = \int d^d x \left[|\Pi(x)|^2 + |\nabla\phi(x)|^2 + \lambda|\phi(x)|^2 + u|\phi(x)|^4 \right] + H_0(\lambda)$$

Scalar field λ rolls affecting the Higgs mass. Near QCP (zero Higgs mass) it gets very heavy and dynamically localizes.

The Kibble-Zurek type scaling argument

$$H_\phi = \int d^dx \left[|\Pi(x)|^2 + |\nabla\phi(x)|^2 + \lambda|\phi(x)|^2 + u|\phi(x)|^4\right] + H_0(\lambda)$$



Scaling dimension of velocity

$$[d\lambda/dt] = [\lambda] - [t] = z + 1/\nu$$

Divergent KZ correlation length

$$\xi_{KZ} = \frac{1}{|\dot\lambda|^{\frac{\nu}{z\nu+1}}}$$

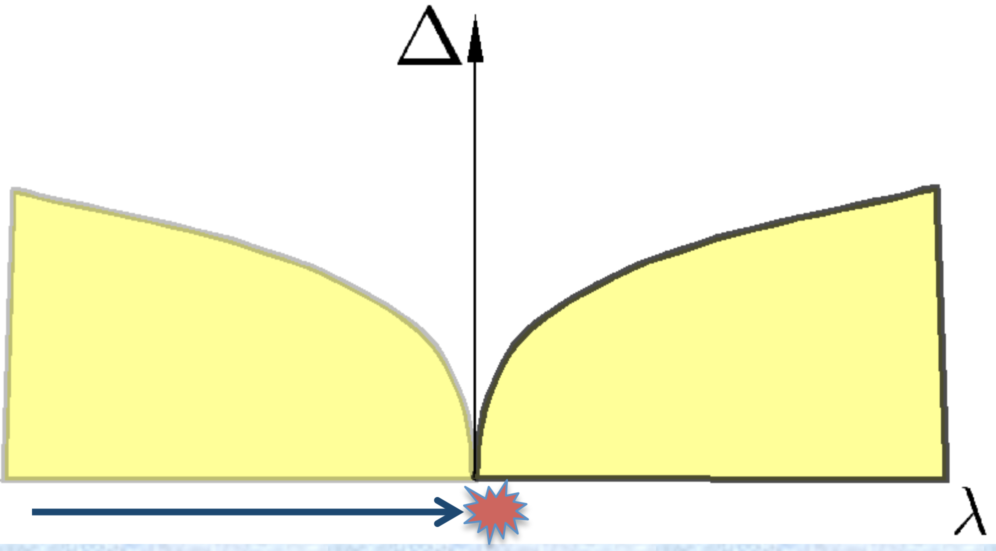
Characteristic gap $\quad \Delta \sim 1/\xi_{KZ}^z$

Energy (heat) density $\quad Q/L^d \sim \Delta\xi_{KZ}^d \sim |\dot\lambda|^{\frac{(d+z)\nu}{z\nu+1}}$

Initial kinetic energy $\quad K/L^d \sim \mu\dot\lambda^2$

Can expect localization if $\quad Q > K$

Localization from the Kibble-Zurek



$$K < Q \;\Leftrightarrow\; \mu\dot{\lambda}^2 < |\dot{\lambda}|^{\frac{(d+z)\nu}{z\nu+1}}$$

Localization is expected if we absorb more energy than it has

$$\frac{(d+z)\nu}{z\nu+1} < 2 \;\Leftrightarrow\; d < z + \frac{2}{\nu}$$

The slower the system goes the more likely it is localized

Same criterion for the localization as from the mass divergence.

Not really surprising from understanding the scaling theory.

Check numerically. Transverse field Ising model

$$H = \frac{p_\lambda^2}{2\mu L} + H_{\mathrm{TFI}}(\lambda) - E_{\mathrm{gs}}(\lambda)$$

$$H_{\mathrm{TFI}}(\lambda) = -\sum_j (1-\lambda) s_j^z + s_j^x s_{j+1}^x, \quad E_{\mathrm{gs}}(\lambda) = \langle 0 | H_{\mathrm{TFI}} | 0 \rangle$$

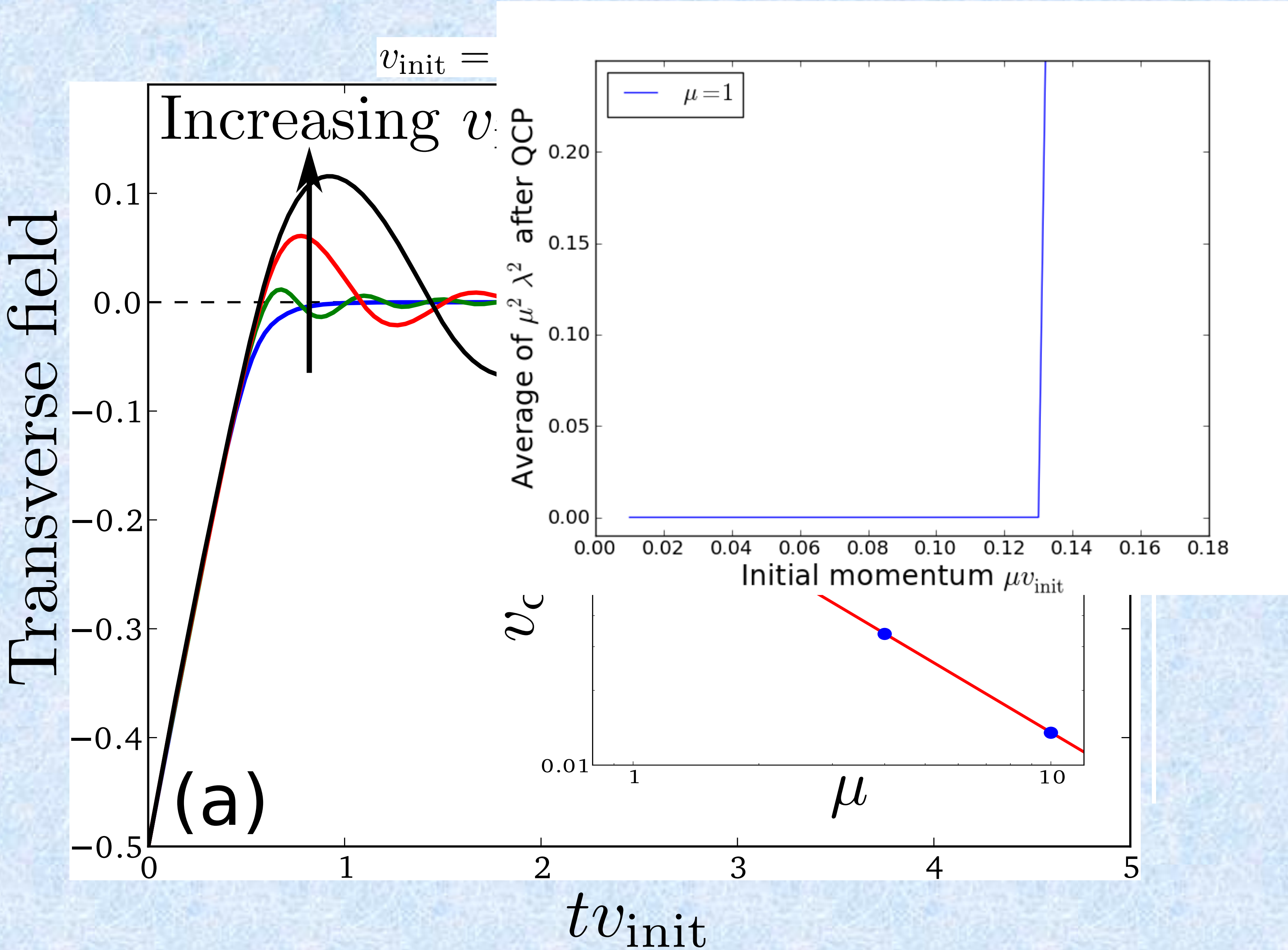
First subtract the GS energy so that the field moves in a flat potential (later revise this assumption).

Trapping condition

$$K < Q \;\Leftrightarrow\; \mu\dot{\lambda}^2 < |\dot{\lambda}|^{\frac{(d+z)\nu}{z\nu+1}} = |\dot{\lambda}|, \quad d = z = \nu = 1$$

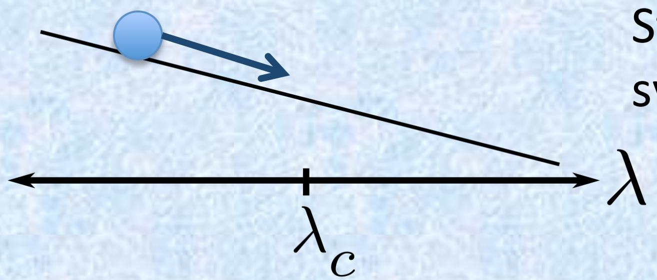
Expect trapping when $\mu|\dot{\lambda}| < \mathrm{const} \sim 1$

Observe sharp transition to the trapping regime at $\mu v_{\text{init}} \approx 0.13$



Finite slope

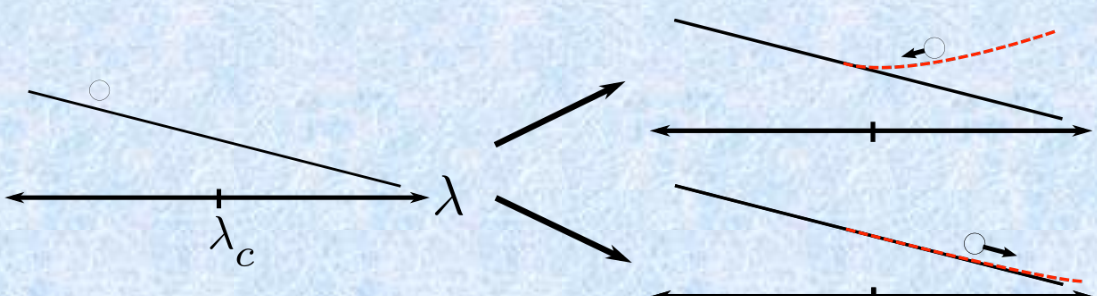
$$H = \frac{p_\lambda^2}{2\mu L} + H_{\mathrm{TFI}}(\lambda) - E_{\mathrm{gs}}(\lambda) - \alpha\lambda$$



Start from the rest at some λ_{ini} and release the system. What will happen?

Naïve answer: will roll down, perhaps stumble a bit near QCP and move on. Wrong!

The system can be truly self-trapped due to heating



Expect two scenarios:
Untrapped (adiabatic)
Trapped (enough heating)

Start far from QCP: not too fast

$$K \sim \mu \dot{\lambda}^2 \sim \alpha \lambda_{\text{init}} \lesssim Q \sim |\dot{\lambda}|, \Rightarrow \ \alpha \lambda_{\text{init}} \lesssim \sqrt{\frac{\alpha \lambda_{\text{init}}}{\mu}}$$

$$\lambda_{\text{init}} \lesssim \frac{1}{\mu \alpha}$$

Start near QCP : not too slow

$$\mu \to \kappa \sim \frac{1}{\lambda_{\text{init}}^2} \ \Rightarrow \ \lambda_{\text{init}} \lesssim \frac{1}{\mu \alpha} \to \lambda_{\text{init}} \lesssim \frac{\lambda_{\text{init}}^2}{\alpha} \quad \Rightarrow \quad \lambda_{\text{init}} > \alpha$$

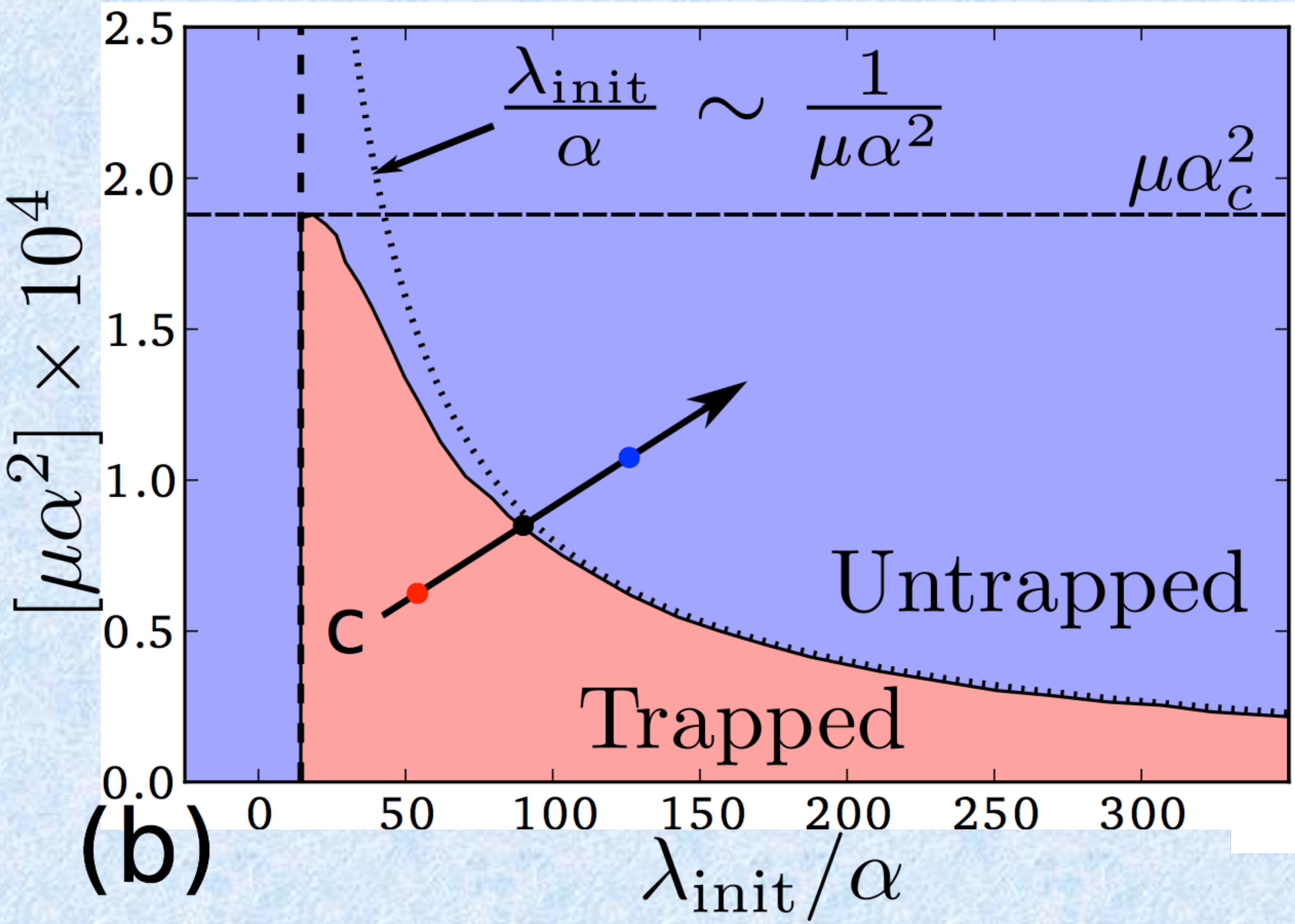
Expect trapping when

$$1 \lesssim \frac{\lambda_{\text{init}}}{\alpha} \lesssim \frac{1}{\mu \alpha^2}$$

Trapping is possible only if

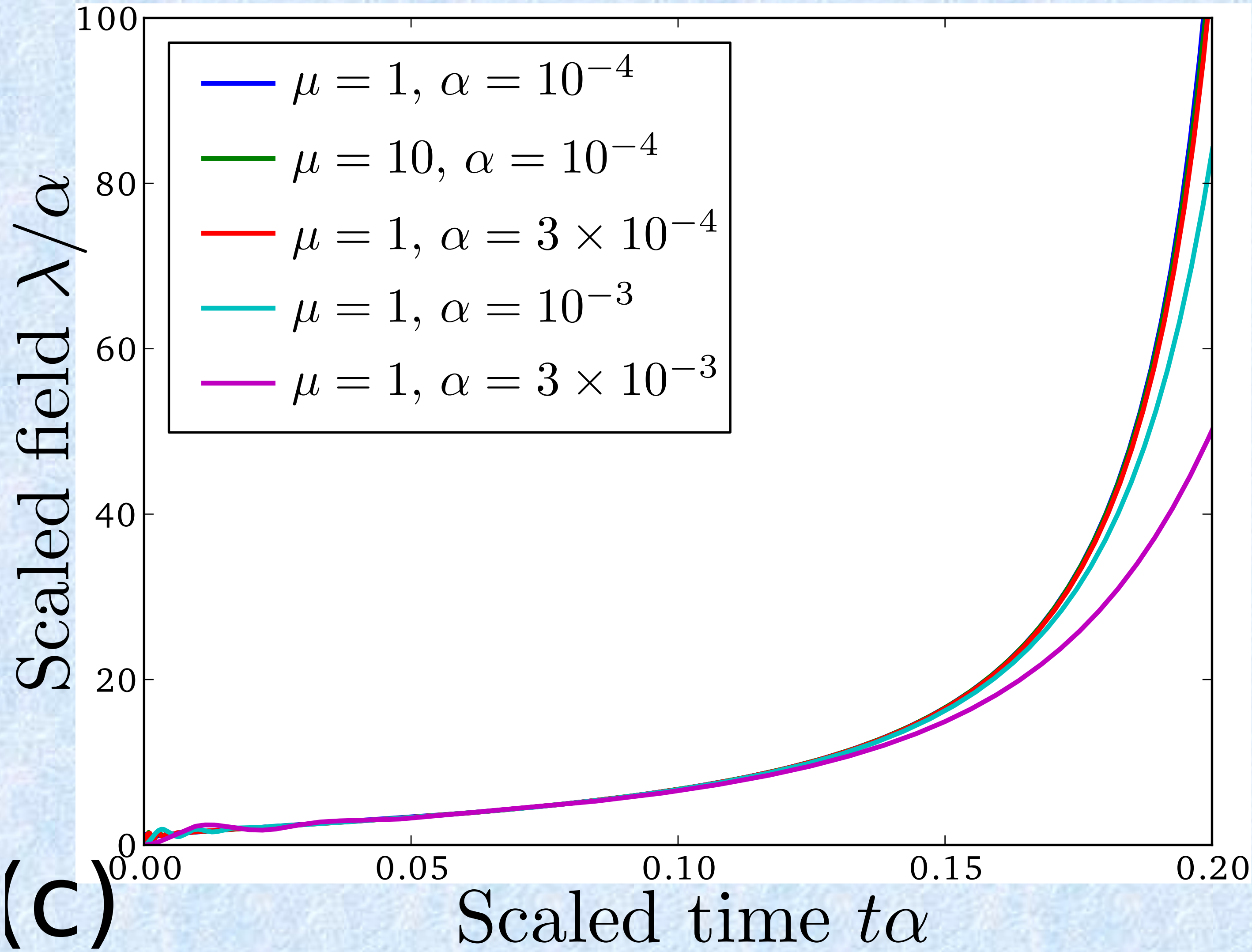
$$\mu \alpha^2 \ll 1$$

Numerical phase diagram



Numerical constants are not very small, but this is quite typical.

Interesting non-equilibrium dynamics if start near QCP.
Bare mass is irrelevant and can be set to zero.

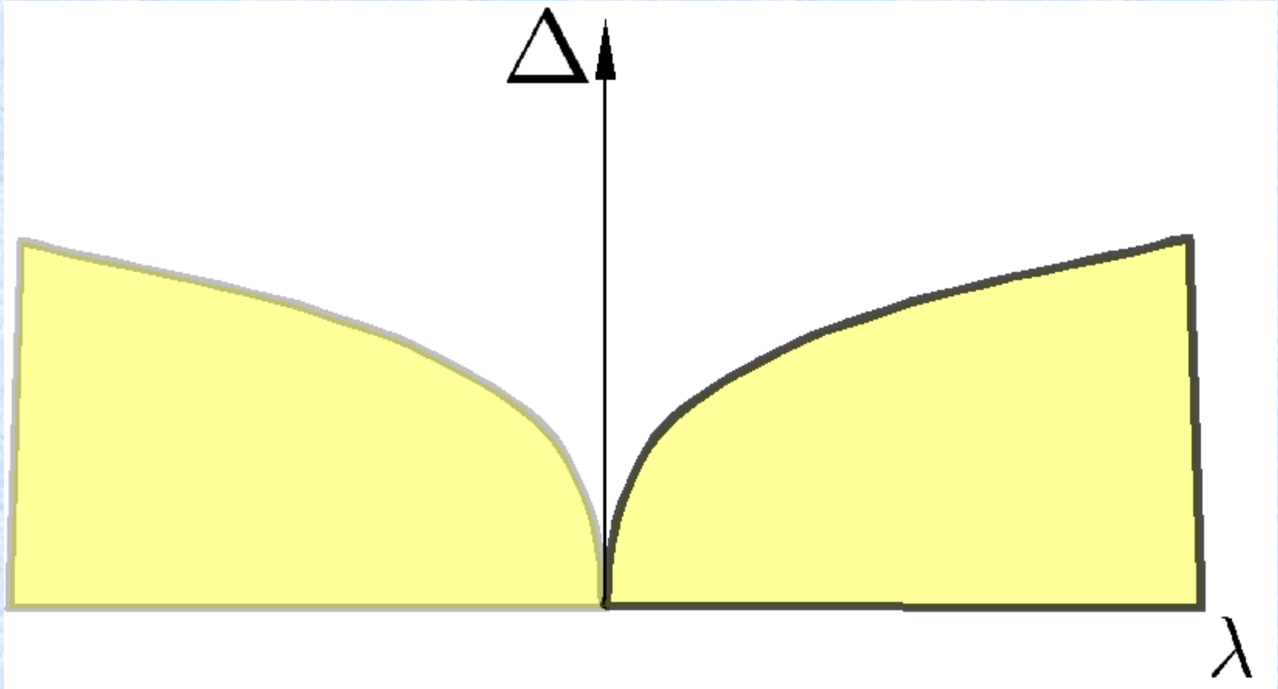


Except for transients and long times have a full scaling collapse

Outlook: dynamic trapping is consistent
with thermodynamic trapping

$$H_{tot}(\lambda) = H_0(\lambda, p_\lambda) + H(\lambda, \{\hat{s}_j\})$$

Consider a fixed energy state. Equilibrium: maximize entropy



The entropy is maximized near QCP where excitations are cheapest.

From scaling expect entropy maximum near QCP for finite range of slopes.

Typical phase diagram of cuprates

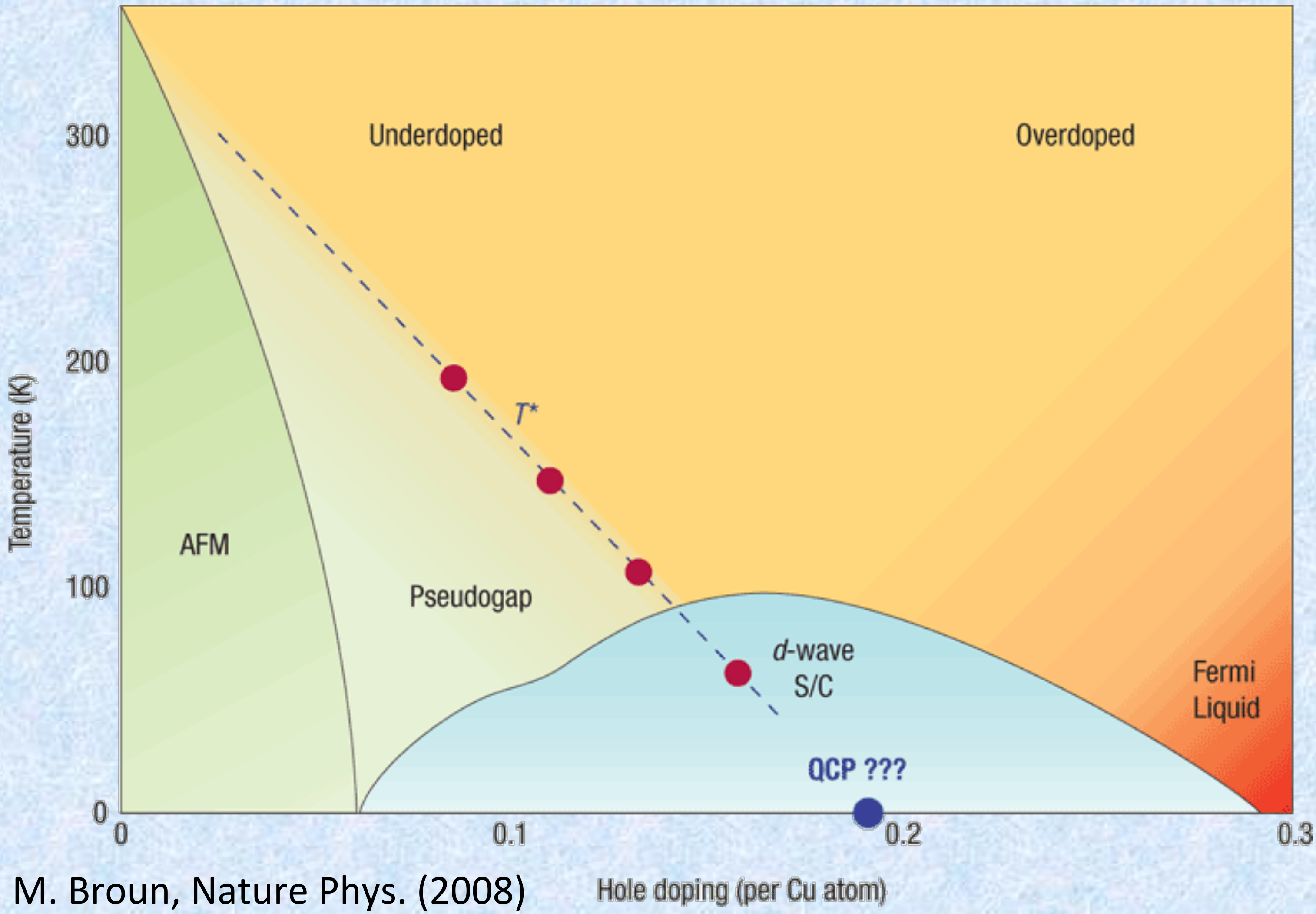


Image: D. M. Broun, Nature Phys. (2008)

QCP is a natural place for localizing a macroscopic DOF like an order parameter.
Similar to the order by disorder scenario.

Summary

- ETH -> thermodynamics
- Relaxation in weakly nonintegrable systems through prethermalization
- Recover macroscopic Hamiltonian dynamics from time scale separation
- Divergence of mass near critical points. Dynamical self-trapping near quantum critical points.

F. Essler, talk at KITP, 2012

Let I_m be **local** (in space) integrals of motion $[I_m, I_n]=[I_m, H(h)]=0$

Define GGE density matrix by: $\rho_{gG}=\exp(-\Sigma\, \lambda_m\, I_m)/Z_{gG}$

λ_m **fixed** by $\mathrm{tr}[\rho_{gG}\, I_m]= \langle\psi(0)|\, I_m\, |\psi(0)\rangle$

Reduced density matrix of B: $\rho_{gG,B}=\mathrm{tr}_A\, \rho_{gG}$

Prove for a particular (transverse field Ising) model $\rho_B(\infty)= \rho_{gG,B}$

Works both for equal and non-equal time correlation functions.
Need only integrals of motion, which "fit" to the subsystem

$$I_n = \sum_j I_n(j, j+1, \ldots, j+\ell_n)$$

If we can not measure I_n – have too many fitting parameters.

What if integrability is slightly broken?