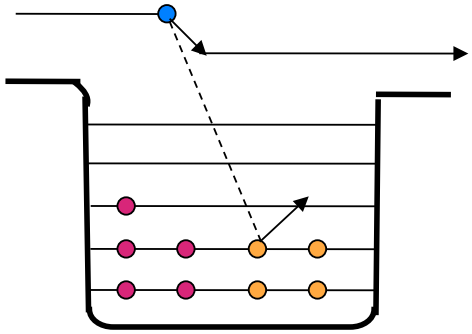
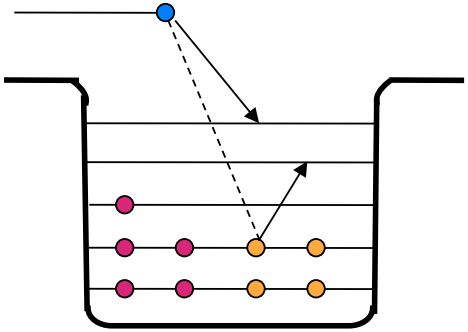
Inelastic scattering and



quantum multistep models



B.V. Carlson -- Depto. de Física
Instituto Tecnológico de Aeronáutica
São José dos Campos SP, Brazil

Inelastic scattering

The single-channel optical model describes the scattering in the elastic channel alone. It is called the spherical optical model because the target may be considered to be spherically symmetric since its structure is never introduced.

Direct reactions that transfer energy as well as momentum are often important. Such inelastic scatterings, in the case of the inert projectiles (n, p, α, d, etc.), leave the target in an excited state and diminish the asymptotic kinetic energy of the projectile. In heavy ion collisions, both the projectile and the target are sometimes excited. To describe low-energy inelastic scattering, we must introduce at least the basic characteristics of the ground and excites states of the target and/or projectile.

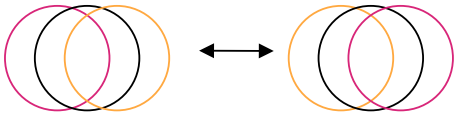
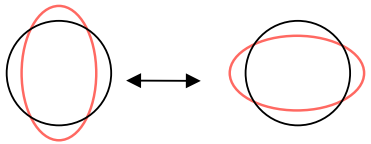
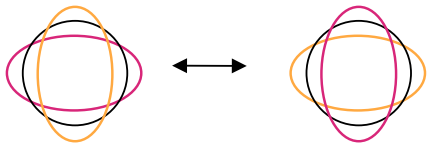
The nature of the ground and excited states of a nucleus are also important factors in detemining the degree to which it is excited in a collision. The states that are most strongly excited in collisions are those that involve collective movement, vibrations and rotations, in particular.

Vibrations

Every nucleus possesses collective vibrational modes of excitation. Their importance in low-energy scattering, however, varies greatly from nucleus to nucleus.

Vibrational modes may be understood qualitatively as shape oscillations of intermixed but incompressible neutron and proton fluids about their equilibrium configuration. The protons and neutrons may oscillate in phase (isoscalar) or out of phase (isovector) with one another.

The simplest modes are:

	Isoscalar	Isovector
$l=1^-$ – dipole	CM displacement	
$l=2^+$ -- quadrupole		

Octupole ($l=3^-$) modes are also common and many others have been observed.

Vibrations – Excitation Energies and states

Isovector vibrations occur ar higher energies than the corresponding isoscalar ones, because of the strong nuclear attraction between protons and neutrons.

Negative parity vibrational modes tend to vary smoothly in energy as a function of the mass number. Positive parity vibrational modes, vary greatly with the mass and depend on the shell structure. The variations in the excitation energies are explained in a microscopic treatment in terms of particle-hole pairs:

Negative parity – particle-hole pairs from two adjacent shells,

Positive parity – particle-hole pairs from same shell, when possible,

otherwise from one shell and from second higher shell.

Ex.: ^{208}Pb – the first excited state is the 3^- octupole state.

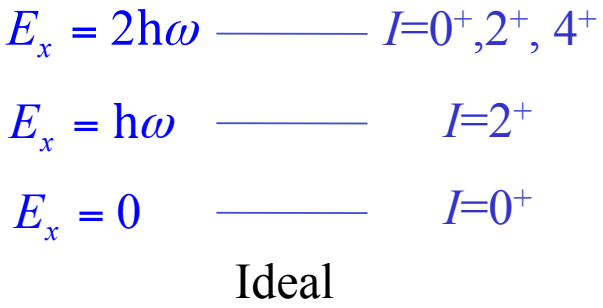
Vibrations are bosonic modes. Multiple excitations are possible but must form symmetric states. Thus an excited state consisting of two $l=2^+$ quadrupole phonons on a $I=0^+$ ground state may have $I=0^+, 2^+, 4^+$.

The states may be written in terms of creation operators $b^\dagger_{I_c N_c}$ as

$$|cI_cN_c\rangle = b^\dagger_{I_cN_c}|0\rangle \qquad \text{and} \qquad |c_1c_2I_cN_c\rangle = \frac{1}{\sqrt{1+\delta_{I_1I_2}}}\left[b^\dagger_{I_1}b^\dagger_{I_2}\right]_{I_cN_c}|0\rangle$$

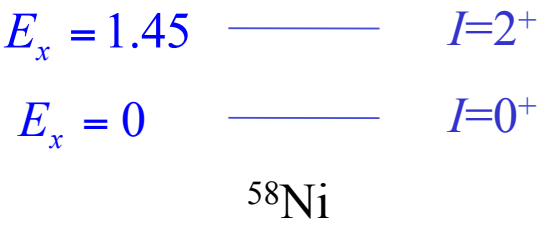
Vibrations – An example

In the simplest case of non-interacting phonons, the spectrum is harmonic. The ideal spectrum of the first few excited quadrupole states on an $I=0^+$ ground state are shown here.



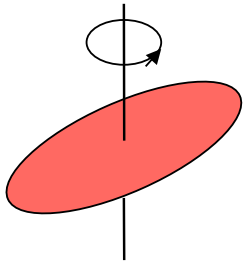
We compare this with the first few excited states of ^{58}Ni (energies in MeV).

The three states that can be interpreted as two-phonon quadrupole states occur close in energy to twice the energy of the one-phonon state and have the correct spin and parity The 1^+ state, however, does not.



Another indicator of the relationship between the states are the branching ratios for their EM decay. The two-phonon 4^+ and 2^+ states decay almost exclusively to the one-phonon 2^+ state, as does the 1^+ state. The 0^+ state decays to several of the others, but principally to the 1^+ state.

Rotations



Many nuclei in the regions between closed shells possess a statically deformed ground state with axial symmetry. The lowest energy excited states of these nuclei are usually rotations about an axis perpendicular to the symmetry axis.

We can approximate the surface of a deformed nucleus as

$$R(\theta') = R_0 \left(1 + \sum_{\lambda} \beta_\lambda \, Y_{\lambda 0}(\theta') \right),$$

where the β_λ, λ=2, 4, 6,... are deformation parameters and the angle θ' is taken with respect to the symmetry axis of the nucleus. The most important of the deformation parameters is β_2.

When $\beta_2 < 0$, the nucleus is oblate. When $\beta_2 > 0$, the nucleus is prolate.

The wave function of a rotational state can be written in terms of an intrinsic wave function χ_K and the rotation matrices D^I_{NK} as

$$\langle \vec{r}_{int} \, | I_c N_c \rangle = \frac{1}{\sqrt{1+\delta_{K0}}} \sqrt{\frac{2I_c+1}{16\pi^2}} \left[\chi_K(\vec{r}_{int}) \, D^{I_c *}_{N_c K}(\hat{r}_{int}) + (-)^{I_c - I_\chi} \, \chi_{-K}(\vec{r}_{int}) \, D^{I_c *}_{N_c,-K}(\hat{r}_{int}) \right]$$

where K is the projection of the angular momentum on the symmetry axis.

Rotations – An example

A rotational band built on a 0^+ ground state consists of states with $I=0^+, 2^+, 4^+, 6^+, \ldots$ A rotational band built on a ground state with spin $I_0 \neq 0$ consists of states with $J=I_0, I_0+1, I_0+2, I_0+3, \ldots$

The excitation energy of a state with angular momentum I is

$$E_x(I) = \frac{\hbar^2}{2\mathscr{I}}\left[I(I+1) - I_0(I_0+1)\right]$$

The nucleus ^{238}U possesses static deformations of $\beta_2=0.198$ and $\beta_4=0.057$. The rotational band based on its 0+ ground state consists of excited states with

$I=2^+$ $E_x=0.045$ MeV = 0.0075 MeV * 2*3,
$I=4^+$ $E_x=0.148$ MeV = 0.0074 MeV * 4*5,
$I=6^+$ $E_x=0.307$ MeV = 0.0073 MeV * 6*7,
$I=8^+$ $E_x=0.518$ MeV = 0.0072 MeV * 8*9,
:
$I=28^+$ $E_x=4.516$ MeV = 0.0056 MeV *28*29, and possibly more.

The electromagnetic decay of each of these states occurs exclusively to the next state of lower energy in the chain.

The generalized optical potential -- vibrations

The simplest manner of extending the optical potential to take into account either static deformation or the dynamical deformation of a vibrational mode is to modify the radii of the terms in the potential accordingly.

In its simplest form, a vibrational mode of a nucleus may be taken as a shape oscillation about a spherical equilibrium mode. The radii of the terms in the potential may be expressed as

$$R_i = R_{0i}\left(1 + \sum_{\lambda\mu} a_{\lambda\mu} Y_{\lambda\mu}(\hat{r})\right) \quad \text{with} \quad a_{\lambda\mu} = \frac{\beta_\lambda}{\sqrt{2\lambda+1}}(b^\dagger_{\lambda\mu} + (-)^\mu b_{\lambda-\mu})$$

where $b^\dagger_{\lambda\mu}$ and $b_{\lambda\mu}$ are the phonon creation/annihilation operators and the β_λ are the amplitudes of the shape oscillations.

We may then expand the optical potential in the creation/annihilation operators as

$$U_{opt}(r,\hat{r}) = U_{opt}(r) + \sum_i \frac{\partial U_{opt}}{\partial R_i} R_{0i} \sum_{\lambda\mu} a_{\lambda\mu} Y_{\lambda\mu}(\hat{r})$$

The potential is sometimes expanded to second order in the operators. The second order potential permits single-step transitions to two-phonon states.

The generalized optical potential -- rotations

The optical potential for a deformed nucleus may also be obtained by expanding the deformed potential radii

$$R_i(\theta') = R_{0i}\left(1 + \sum_\lambda \beta_\lambda Y_{\lambda 0}(\theta')\right),$$

in a Taylor series in the deformation parameters, β_λ. However, when the deformations are large, it is better to expand it directly in multipoles as

$$U_{opt}(r,\hat{r}') = \sum_\lambda U_\lambda(r) Y_{\lambda 0}(\hat{r}') \quad \text{with} \quad U_\lambda(r) = \int d\Omega' U_{opt}(r,\theta') Y_{\lambda 0}(\theta').$$

The moments $U_{\lambda\mu}(r)$, with $\mu \neq 0$, vanish in the body-fixed frame. The body-fixed angles $\hat{r}'$ are related to the space fixed ones $\hat{r}$ through the collective angular coordinates of the nucleus, $\hat{r}_{\text{int}}$. This implies that

$$Y_{\lambda 0}(\hat{r}') = \sum_\mu Y_{\lambda\mu}(\hat{r}) D^\lambda_{\mu 0}(\hat{r}_{int}) = \sum_\mu Y_{\lambda\mu}(\hat{r}) Y^*_{\lambda\mu}(\hat{r}_{int}).$$

The optical potential in the rotational model may thus be expanded as

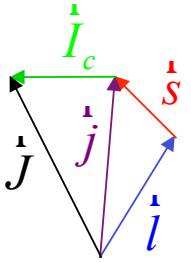
$$U_{opt}(\vec{r},\hat{r}_{int}) = \sum_{\lambda\mu} U_\lambda(r) Y_{\lambda\mu}(\hat{r}) Y^*_{\lambda\mu}(\hat{r}_{int}).$$

The generalized optical potential, in both models, couples the relative motion to the internal degrees of freedom of the target.

Coupled-channels partial wave expansion

To properly take into account the angular momentum of the target, the spin-angular functions must be coupled to the target states to form target-spin-angular functions of total angular momentum J and projection M,

$$\mathscr{Y}^{JM}_{lsjc}(\hat{r}) = \sum_{nN_c} \langle jnI_cN_c|JM\rangle \, \mathscr{Y}^{jn}_{ls}(\hat{r}) \, |I_cN_c\rangle \, .$$



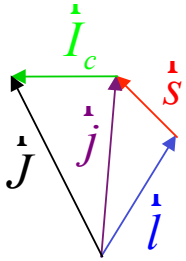
The functions also depend on the internal target coordinates. In terms of these, the scattering wave function may be expanded in a sum over both the excited states and angular momenta,

$$\Psi = 4\pi \sum_{\substack{l\,jcJM \\ l'j'c'}} \mathscr{Y}^{JM}_{l'sj'c'}(\hat{r}) i^{l'} \psi^{J}_{l'j'c',l\,jc}(r) \frac{e^{i\sigma_{lc}}}{k_c r} \mathscr{Y}^{JM\dagger}_{lsjc}(\hat{k}) \, .$$

The most significant difference here is that the partial wave functions depend on two sets of indices, l, j, c and l', j', c'. For a particle with spin in the spherical optical model, we have two indices l and l', in principal, for each value of the total angular momentum j. For particles of spin 0 or spin ½, parity conservation reduces the two, l and l', to have the same value. The partial wave functions and S-matrix elements are then uncoupled scalar quantities. Here, we should look more carefully to see how the channels could be coupled.

Coupled partial waves

To analyze the partial waves that can couple, we must consider all possible combinations of the orbital angular momentum l, the spin s, the channel angular momentum j and the target spins I_c that can sum to a given value J of the total angular momentum and possess a given value of the parity, π.



Consider a simple example: a spin-½ nucleon incident on a 0^+ ground state that can be excited to a 2^+ target state. We have

For the $J^\pi=1/2^+$ channel:

0+ ground state: $l=0, j=1/2$

2+ excited state: $l=2, j=3/2$

$l=2, j=5/2$

3 coupled channels

For the $J^\pi=1/2^-$ channel:

0+ ground state: $l=1, j=1/2$

2+ excited state: $l=1, j=3/2$

$l=3, j=5/2$

For the $J^\pi=5/2^+$ channel:

0+ ground state: $l=2, j=5/2$

2+ excited state: $l=0, j=1/2$

$l=2, j=3/2$

$l=2, j=5/2$

$l=4, j=7/2$

$l=4, j=9/2$

6 coupled channels

For the $J^\pi=5/2^-$ channel:

0+ ground state: $l=3, j=5/2$

2+ excited state: $l=1, j=1/2$

$l=1, j=3/2$

$l=3, j=5/2$

$l=3, j=7/2$

$l=5, j=9/2$

The coupled equations

When the partial-wave expansion is substituted in the Schrödinger equation, it reduces to a set of coupled equations for each value of J^π,

$$\frac{\hbar^2}{2\mu}\left\{\frac{d^2}{dr^2}-\frac{l'(l'+1)}{r^2}+k_{c'}^2\right\}\psi^J_{l'j'c',ljc}(r)-\sum_{l''j''c''}\mathscr{U}^J_{l'j'c',l''j''c''}(r)\psi^J_{l''j''c'',ljc}(r)=0,$$

where the potential matrix elements are those of the target-spin-angular functions, $\mathscr{U}^J_{l'j'c',ljc}(r)=\int d^3r_{int}\,d\Omega\,\mathscr{Y}^{JM\dagger}_{l'sj'c'}(\hat{r})U_{opt}(\vec{r},\vec{r}_{int})\mathscr{Y}^{JM}_{lsjc}(\hat{r})$.

The matrix elements are independent of M due to rotational invariance and symmetric under interchange of indices, if the system is time-reversal invariant.

If we group the matrix elements of the coupled equations into matrices,

$$l'\,\delta_{l'l}\delta_{j'j}\delta_{c'c}\to L_J\,,\qquad k_{c'}\,\delta_{l'l}\delta_{j'j}\delta_{c'c}\to K_J\,,$$

$$\psi^J_{l'j'c',ljc}(r)\to\Psi_J(r)\,,\qquad \mathscr{U}^J_{l'j'c',ljc}(r)\to U_J(r),$$

we may write the coupled equations for each value of J^π as a matrix equation

$$\left\{\frac{d^2}{dr^2}-\frac{L_J(L_J+1)}{r^2}+K_J^2-\frac{2\mu}{\hbar^2}U_J(r)\right\}\Psi_J(r)=0.$$

The scattering amplitude and S-matrix

We may also introduce the target-spin angular functions into the matrix representation of the partial wave decomposition, but as a vector rather than as a matrix,

$$\mathscr{Y}^{JM}_{lsjc}(\hat{r}) = \langle \hat{r} | lsjcJM \rangle \rightarrow \langle \hat{r} | JM \rangle .$$

The wave function may then be written as

$$\Psi = \frac{4\pi}{r} \sum_{JM} \langle \hat{r} | JM \rangle \, i^{L_J} \Psi_J(r) \, e^{i\sigma_J} \, K_J^{-1} \, \langle JM | \hat{k} \rangle .$$

Conceptually, obtaining the scattering amplitude is now straightforward. As before, the wave function must be integrated numerically from the origin to beyond the range of the nuclear potential. There, it is matched to either Coulomb or free waves (in matrix form),

$$\Psi_J(r) \rightarrow \frac{i}{2} \left(H_J^-(r) - H_J^+(r) e^{i\sigma_J} \overline{S}_J e^{i\sigma_J} \right) e^{-i\sigma_J} \qquad \text{where} \qquad \overline{S}^J_{l'j'c',ljc} \rightarrow \overline{S}_J .$$

Substituting this expression in the partial wave expansion and analyzing its asymptotic form, we obtain the scattering amplitude,

$$\overline{f}(\theta) = \frac{4\pi}{2i} \sum_{JM} \langle \hat{r} | JM \rangle \left(e^{i\sigma_J} \overline{S}_J \, e^{i\sigma_J} - 1_J \right) K_J^{-1} \, \langle JM | \hat{k} \rangle .$$

The matrix elements of the scattering amplitude, $\overline{f}_{\nu' N_c' c', \nu N_c c}$, are labeled by the target state and the projections of the projectile and target spins.

Flux normalization

The cross section can be defined in terms of a ratio of current densities or fluxes. When energy is removed from the relative motion, as in inelastic scattering, the relative velocity and the corresponding flux are reduced. To correct for this, we must multiply the scattering amplitude by a factor of

$$\sqrt{v_f/v_i} = \sqrt{k_f/k_i}$$

We may do this by defining first the normalized S-matrix,

$$S_J = K_J^{1/2}\,\bar{S}_J\,K_J^{-1/2},$$

and then defining the normalized scattering amplitude in its terms as

$$\begin{aligned} f(\theta) &= \frac{4\pi}{2i}\sum_{JM}\langle\hat{r}|JM\rangle\left(e^{i\sigma_J}\,S_J\,e^{i\sigma_J}-1_J\right)K_J^{-1}\langle JM|\hat{k}\rangle \\ &= f_C(\theta)+\frac{4\pi}{2i}\sum_{JM}\langle\hat{r}|JM\rangle\,e^{i\sigma_J}\,(S_J-1_J)\,e^{i\sigma_J}\,K_J^{-1}\langle JM|\hat{k}\rangle \end{aligned}$$

where the Coulomb amplitude $f_C(\theta)$ is now a matrix, diagonal in the spin projections and state indices, but different for each of the target states due to the difference in the relative motion.

Angular distributions and cross sections

The angular distributions for an unpolarized beam and target are obtained by averaging the squared amplitude over the initial spin projections and summing over the final ones. Denoting the initial state by c_0 and its spin by I_0, the differential elastic cross section is

$$\frac{d\sigma_{el}}{d\Omega} = \frac{1}{(2s+1)(2I_0+1)} \sum_{\substack{\nu' N_0' \\ \nu N_0}} \left| f_{\nu' N_0' \nu N_0}(\theta) \right|^2 .$$

The differential inelastic cross section to an excited state c with spin I_c is

$$\frac{d\sigma_c}{d\Omega} = \frac{1}{(2s+1)(2I_0+1)} \sum_{\substack{\nu' N_c' \\ \nu N_0}} \left| f_{\nu' N_c' \nu N_0}(\theta) \right|^2 .$$

For neutrons, the integrated elastic cross section is

$$\sigma_{el} = \frac{1}{2(2I_0+1)} \frac{\pi}{k_{c_0}^2} \sum_{\substack{l' j' l j \\ J}} (2J+1) \left| S^J_{l' j' c_0, l j c_0} - \delta_{l' l} \delta_{j' j} \right|^2 .$$

For charged or neutral particles, the inelastic cross section to an excited state c with spin I_c is

$$\sigma_c = \frac{1}{(2s+1)(2I_0+1)} \frac{\pi}{k_{c_0}^2} \sum_{\substack{l' j' l j \\ J}} (2J+1) \left| S^J_{l' j' c, l j c_0} \right|^2 \qquad c \neq c_0 .$$

Absorption cross sections

Just as in the spherical optical model, we may associate an elastic absorption cross section σ_r with the flux lost from the elastic channel,

$$\sigma_r = -\frac{1}{v}\oint \vec{j}_0 \cdot d\vec{a} \qquad \text{where} \qquad \vec{j}_0 = \frac{\hbar}{2i\mu}\left(\Psi_0^\dagger \nabla\Psi_0 - (\nabla\Psi_0)^\dagger \Psi_0\right),$$

with Ψ_{c_0} being the ground-state component of the wave function. This cross section includes the flux lost to inelastic scattering as well as absorption.

We may also define a total absorption cross section σ_{abs} (which is smaller than the elastic one) as the flux lost from all of the channels together,

$$\sigma_{abs} = -\frac{1}{v}\oint \sum_c \vec{j}_c \cdot d\vec{a}, \quad \text{where} \quad \vec{j}_c = \frac{\hbar}{2i\mu}\left(\Psi_c^\dagger \nabla\Psi_c - (\nabla\Psi_c)^\dagger \Psi_c\right),$$

with Ψ_c the component of the wave function of state *c*.

We have for the inelastic channels

$$\frac{1}{v}\oint \vec{j}_c \cdot d\vec{a} = \sigma_c \qquad c \neq c_0 \qquad \text{so that} \qquad \sigma_r = \sigma_{abs} + \sum_{c\neq c_0} \sigma_c .$$

That is, the elastic absorption cross section is the sum of the total absorption cross section and the inelastic excitation cross sections.

Cross sections and transmission coefficients

Uing the asymptotic form of the wave function, the elastic absorption cross section may be calculated,

$$\sigma_r = \frac{1}{(2s+1)(2I_0+1)}\frac{\pi}{k^2}\sum_{\substack{l'j'lj\\J}}(2J+1)\left(\delta_{l'l}\delta_{j'j}-\left|S^J_{l'j'c_0,ljc_0}\right|^2\right).$$

It is a sum of the contributions of the elastic S-matrix elements.

The total absorption cross section may be reduced to a similar form,

$$\sigma_{abs} = \frac{1}{(2s+1)(2I_0+1)}\frac{\pi}{k^2}\sum_{ljJ}(2J+1)\,T^J_{ljc_0,ljc_0},$$

where we have introduced the coupled-channel transmission coefficients, which in matrix form are

$$T_J = 1_J - S_J^\dagger S_J.$$

For neutrons, we may define the total cross section as the sum of the elastic and the elastic absorption ones,

$$\sigma_{tot} = \sigma_{el}+\sigma_r = \frac{1}{2I_0+1}\frac{\pi}{k^2}\sum_{ljJ}(2J+1)(1-\mathrm{Re}\,S^J_{ljc_0,ljc_0}).$$

The total cross section measures the flux lost from the incident plane wave. It takes into account scattering of any type.

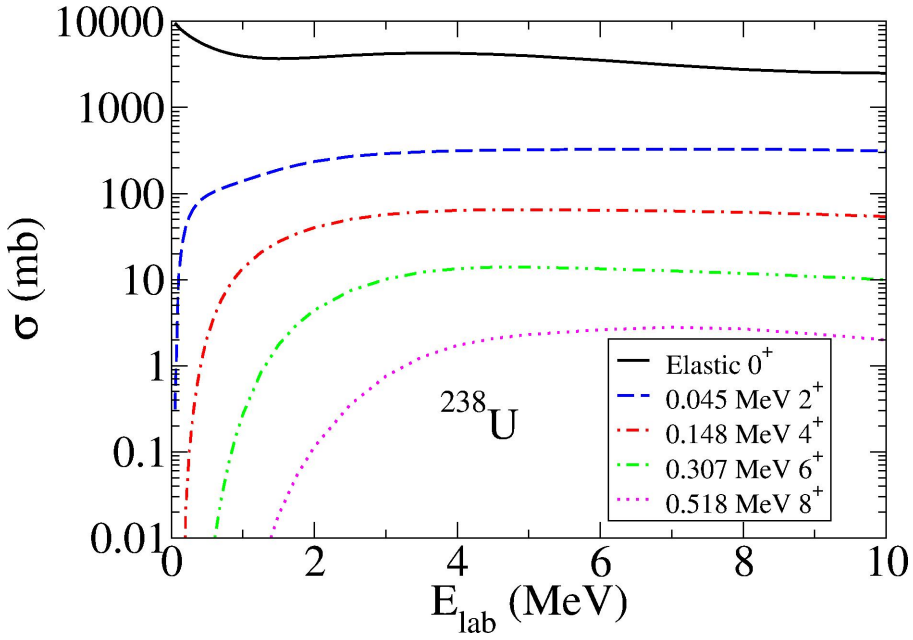
An example – Rotational excitation of ^{238}U

Let us consider excitation of the ground state rotational band of ^{238}U through the 8^+ state. For J=1/2, 1+2+2+2+2=9 coupled channels are involved. For large values of the total angular momentum, we have 1+5+9+13+17=45 coupled channels in each partial wave.

The cross sections of the first excited states increase rapidly above their thresholds. The cross sections of the more highly excited states increase more smoothly.

All of the cross sections decrease very slowly at high energy.

The high energy values of the cross sections decrease by a factor of about 5 for each state as one ascends the rotational band in excitation energy.

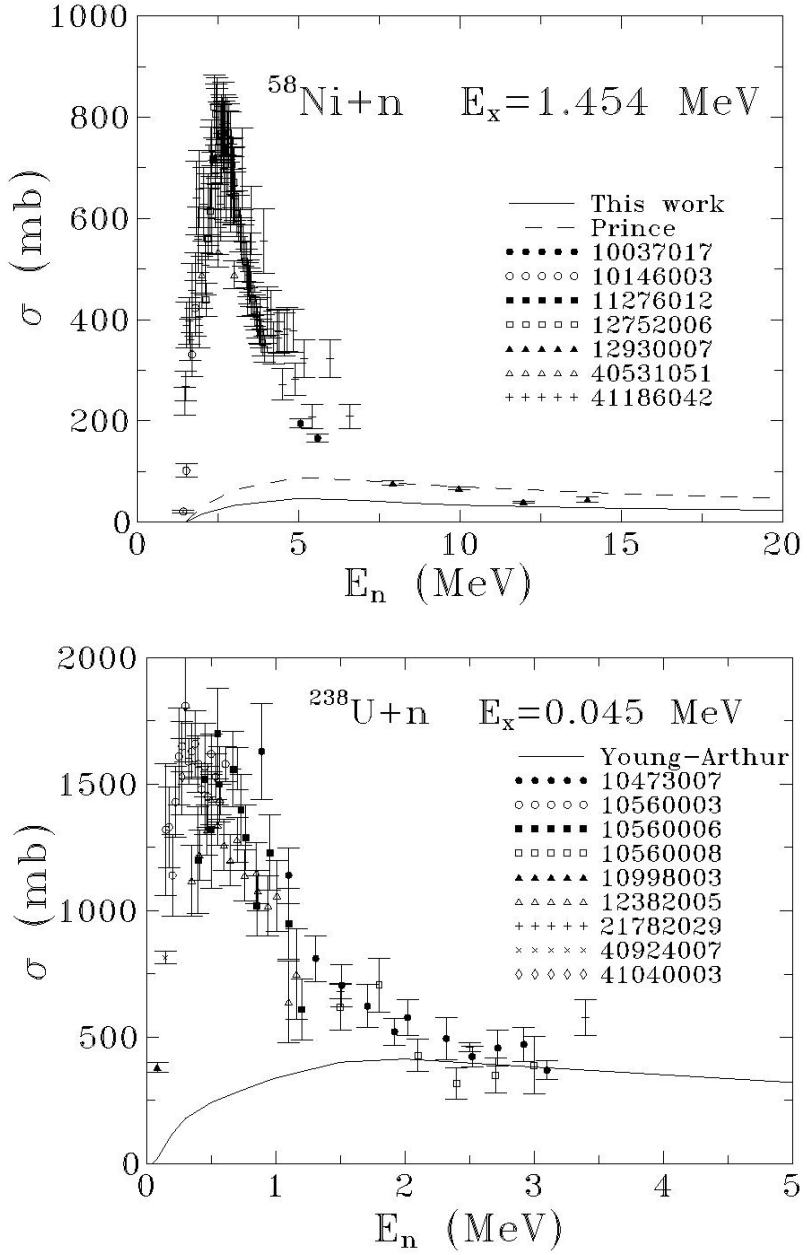


Comparison with experiment

Inelastic cross sections are dominated by the contribution from the compound nucleus at low energies, as seen here for the first excited states of ^{58}Ni and ^{238}U.

The two calculations of the ^{58}Ni inelastic cross section use the same value of β_2=0.2, yet yield cross sections that differ by almost a factor of two due to differences in the optical potentials.

The cross section for excitation of the rotational state in ^{238}U is 5 to 10 times greater than that of the vibrational state in ^{58}Ni, mainly due to the factor of 30 difference in their excitation energies.



The Lippmann-Schwinger equation -- I

The integral representation of the wave equation, the Lippmann-Schwinger equation,

$$\Psi = \Psi_0 + \left(E^+ - H_0\right)^{-1} U' = \Psi_0 + G_0^+ U' \Psi, \quad \text{where} \quad (E - H_0)\Psi_0 = 0,$$

is often very useful for the analysis and solution of scattering problems. Here, G_0^+ is the outgoing-wave Green's function and Ψ_0 a wave function with an incoming wave boundary condition. It is usually most convenient to place the single-channel optical potential in the H_0 of the equation and only the couplings between states in U'.

For the single-channel optical model, we can define incoming/outgoing-wave solutions, $h_{lc}^{j\pm}(r)$, of the wave equation,

$$\left\{ \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + k_c^2 - \frac{2\mu}{\hbar^2}\left(U_{cen,c}(r) + d_l^j\, U_{so,c}(r)\right) \right\} h_{lc}^{j\pm}(r) = 0,$$

where the spin-orbit factor is $d_l^j = d_{so}\left(j(j+1) - l(l+1) - s(s+1)\right)/2.$

Asymptotically, these solutions behave as incoming/outgoing Coulomb (free) waves,

$$h_{lc}^{j\pm}(r) \to H_{lc}^{\pm}(r) = e^{\mp i\sigma_{lc}}\left(G_{lc}(r) \pm iF_{lc}(r)\right).$$

The Lippmann-Schwinger equation -- II

The solution to the single-channel Schrödinger equation that is regular at the origin is given in terms of the incoming/outgoing solutions and the S-matrix as

$$\psi_{lc}^{j+}(r) = \frac{i}{2}\left(h_{lc}^{j-}(r) - h_{lc}^{j+}(r)e^{2i\sigma_{lc}}S_{0lc}^{j}\right) = \psi_{lc}^{j}(r)e^{i\sigma_{lc}},$$

which is just the single-channel wave function of the partial wave expanson. We have merely relabeled the S-matrix as S_0.

The single-channel Green's function may be decomposed in partial waves as

$$G_{0c}^{+}(\vec{r},\vec{r}') = \frac{1}{rr'}\sum_{ljn}\mathscr{Y}_{ls}^{jn}(\hat{r})\,g_{lc}^{j+}(r,r')\,\mathscr{Y}_{ls}^{jn\dagger}(\hat{r}'),$$

where

$$g_{lc}^{j+}(r,r') = -\frac{2\mu}{\hbar^2 k_c}\psi_{lc}^{j+}(r_<)\,h_{lc}^{j+}(r_>).$$

The complete single-channel Green's function for the coupled-channels problem may then be composed as

$$\begin{aligned} G_0^{+}(\vec{r},\vec{r}') &= \sum_{cN_c} G_{0c}^{+}(\vec{r},\vec{r}')\,|I_cN_c\rangle\langle I_cN_c| \\ &= \frac{1}{rr'}\sum_{\substack{ljc\\JM}}\mathscr{Y}_{lsj}^{JM}(\hat{r})\,g_{lc}^{j+}(r,r')\,\mathscr{Y}_{lsj}^{JM\dagger}(\hat{r}'). \end{aligned}$$

The Lippmann-Schwinger equation -- III

In terms of the channel matrices, the Green's function G_0^+ takes the form

$$G_0^+(\vec{r},\vec{r}') = \frac{1}{rr'}\sum_{JM} \langle \hat{r}|JM\rangle \, G_{0J}^+(r,r') \, \langle JM|\hat{r}'\rangle \,,$$

where we have grouped the appropriate Green's functions in diagonal matrices,

$$g_{lc}^{j+}(r,r')\delta_{l'l}\delta_{j'j}\delta_{c'c} \to G_{0J}^+(r,r') \,.$$

In terms of these, we can write the contribution to the Lippmann-Schwinger equation of each partial wave as

$$\Psi_J(r) = \Psi_{0J}(r) + \int_0^\infty dr' \, G_{0J}^+(r,r') \, U_J'(r') \, \Psi_J(r') \,,$$

Substituting the large-*r* expressions for the wave functions,

$$\Psi_{\substack{J \\ 0J}} \to \frac{i}{2}\left(H_J^-(r) \quad - \quad H_J^+(r) e^{i\sigma_J} \overline{S}_{\substack{J \\ 0J}} e^{i\sigma_J}\right) e^{-i\sigma_J}$$

and using flux conservation to normalize the S-matrix, $S_J = K_J^{1/2}\,\overline{S}_J\,K_J^{-1/2}$,, we obtain

$$S_J = S_{0J} + 2i\frac{2\mu}{\hbar^2} K_J^{-1/2} \int_0^\infty dr' \, \Psi_{0J}(r') \, U_J'(r') \, \Psi_J(r') \, K_J^{-1/2} \,.$$

The distorted-wave Born approximation (DWBA)

The Lippmann-Schwinger equation,

$$\Psi_J(r) = \Psi_{0J}(r) + \int_0^\infty dr'\, G^+_{0J}(r,r')\, U'_J(r')\, \Psi_J(r')\,,$$

here in partial wave form, contains the wave function $\Psi_J(r)$ on both the right and left sides of the equation. This can be used to advantage when the coupling potential U' is small. We then expect the wave function Ψ_J to be little different from the uncoupled one Ψ_{0J}, so that we have, to first order,

$$\Psi^{(1)}_J(r) = \Psi_{0J}(r) + \int_0^\infty dr'\, G^+_{0J}(r,r')\, U'_J(r')\, \Psi_{0J}(r')\,.$$

The corresponding DWBA S-matrix is

$$S^{(1)}_J = S_{0J} + 2i\frac{2\mu}{\hbar^2} K_J^{-1/2} \int_0^\infty dr' \Psi_{0J}(r')\, U'_J(r')\, \Psi_{0J}(r')\, K_J^{-1/2}\,.$$

The DWBA approximation may be extended to higher orders by substituting the solution of the previous order in the Lippmann-Schwinger equation. The second-order solution, for example, is obtained by substituting the first order solution in the integral equation. However, the DWBA is usually not used above the second-order.

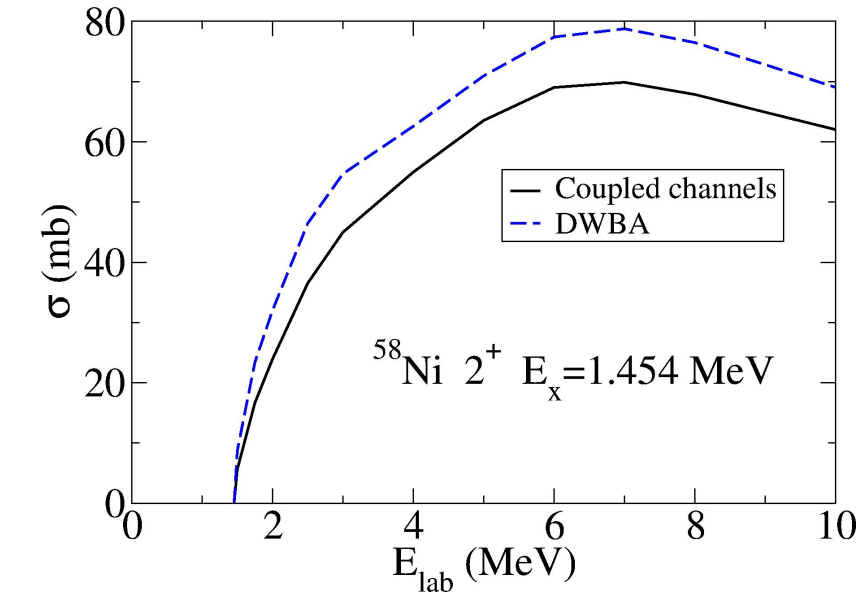
Limit of the DWBA

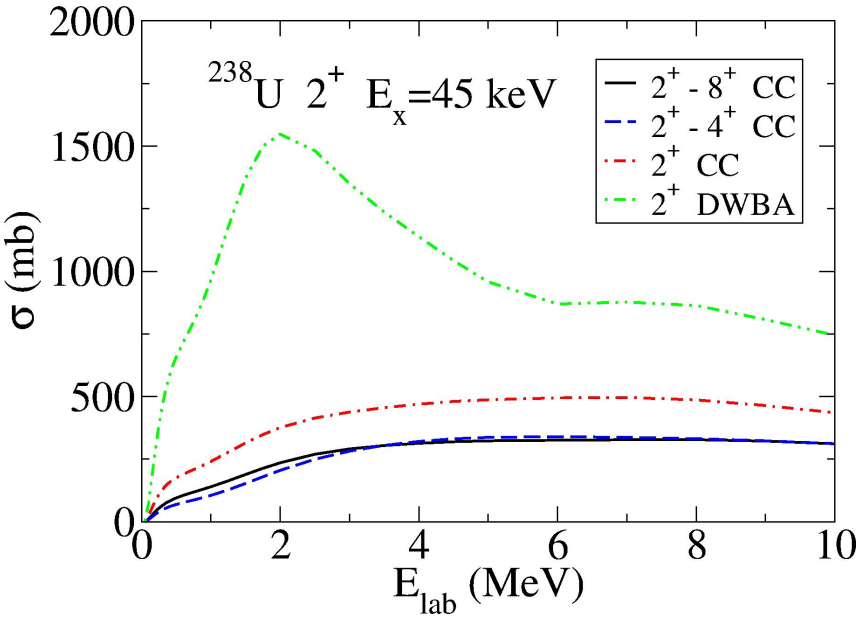
Two examples give us an idea of when the DWBA might be applied to inelastic scattering.

The DWBA provides a reasonable approximation to excitation of a vibrational state such as the 2^+ one in ^{58}Ni. However, it greatly overestimates the excitation of a strongly-excited rotational state, such as the 2^+ one in ^{238}U.

In general, the DWBA overestimates the inelastic cross section, since it does not take into account transitions back to the ground state.

In the case of ^{238}U, we note that transitions to other states of the rotational band can also be important.





Quantum models of preequilibrium emission

1) Quantum mechanical models separate the energy spectrum of a particle into a continuum component P and a bound state component Q. The angular distribution expected of the two is quite different. The continuum component can be forward peaked but the bound state component must necessarily be symmetric about 90^{o}.

2) The semi-classical models are limited to a simplistic particle-hole description of nuclear structure, due to the fact that they are based on a particle picture of the evolution of the collision. This is compensated in part by the use of single-particle and hole energies rather than coordinates and momenta, as in an intra-nuclear cascade model. However, these models cannot take into account the coherent effects that lead to nuclear collectivity. These effects are quantum in nature. They can be taken into account in part by considering a quantum partition of the particle-hole structure of the semi-classical models,

$$\begin{aligned} P &= P_1 + P_3 + P_5 + P_7 + \ldots, \\ Q &= Q_3 + Q_5 + Q_7 + \ldots \end{aligned}$$

and taking into account the interactions within each exciton class.

Multistep Compound (MSC) models - I

The simplest example of a MSC model is in fact a semiclassical exciton one. There are two principal differences from the standard exciton model:
1) The densities of states contain only bound single-particle states;
2) Emission requires a transition that raises one of the nucleons to an unbound, continuum state.

The densities of bound states can be calculated in much the same manner as the densities of states that take into account the maximum hole energy. With B being the maximum energy of a bound particle, we have

$$\omega_B(p,h,E) = \frac{g^n}{p!h!(n-1)!}\sum_{j=0}^{p-1}(-1)^j\binom{p}{j}(E-jB)^{n-1}\,\theta\,(E-jB)$$

when $E < pB$ and

$$\omega_B(p,h,E) = \frac{g^n}{p!h!}\sum_{j=0}^{p-1}\sum_{m=0}^{h-1}(-1)^j\binom{p}{j}\frac{[(p-j)B]^{p+m}}{(p+m)!(h-m-1)!}(E-pB)^{h-m-1}$$

when $E > pB$. The terms correcting for the maximum hole energy should also be included when the energy is sufficiently high.

H. Feshbach, A. Kerman , S. Koonin, Ann Phys. (N.Y.) **125** (1988) 166.

Multistep Compound (MSC) models - II

The differential width for emission of a particle of energy ε_c is the sum of three contributions, from transitions that increase the exciton number by two, leave it the same or decrease it by two. The subsequent emission also decreases the exciton number by one. We define the corresponding densities of available states as

$$\omega_{B+}(n,\varepsilon_c) = \int_{\varepsilon_c+B}^{E} d\varepsilon\, \omega_B(0,2,\varepsilon-\varepsilon_c-B)\,\omega_B(0,1,\varepsilon)\,\frac{\omega_B(p,h-1,E-\varepsilon)}{\omega_B(n,E)}$$

$$\omega_{B0}(n,\varepsilon_c) = \int_{\varepsilon_c+B}^{E} d\varepsilon \left(\omega_B(1,0,\varepsilon-\varepsilon_c-B)\,\omega_B(2,0,\varepsilon)\,\frac{\omega_B(p-2,h,E-\varepsilon)}{\omega_B(n,E)} \right.$$

$$\left. +\omega_B(0,1,\varepsilon-\varepsilon_c-B)\,\omega_B(1,1,\varepsilon)\,\frac{\omega_B(p-1,h-1,E-\varepsilon)}{\omega_B(n,E)} \right)$$

and

$$\omega_{B-}(n,\varepsilon_c) = \omega_B(2,1,\varepsilon_c+B)\,\frac{\omega_B(p-2,h-1,E-\varepsilon_c-B)}{\omega_B(n,E)}$$

Note that the exciton number of the final configuration is n+1, n-1 and n-3, respectively.

Multistep Compound (MSC) models - II

We now write the differential widths for emission in terms of the densities of available states as

$$\frac{d\lambda_{e+}}{d\varepsilon_c}(n,E) = \frac{2\pi}{\hbar}|M_c|^2 g_c(\varepsilon_c)\omega_{B+}(n,\varepsilon_c)$$

$$\frac{d\lambda_{e0}}{d\varepsilon_c}(n,E) = \frac{2\pi}{\hbar}|M_c|^2 g_c(\varepsilon_c)\omega_{B0}(n,\varepsilon_c)$$

and

$$\frac{d\lambda_{e-}}{d\varepsilon_c}(n,E) = \frac{2\pi}{\hbar}|M_c|^2 g_c(\varepsilon_c)\omega_{B-}(n,\varepsilon_c)$$

where

$$g_c(\varepsilon_c) = (2s+1)V\frac{(2m)^{3/2}\sqrt{\varepsilon_c}}{(2\pi)^2\hbar^3}$$

is the density of states of the particle in the continuum.

Note that the bound state to continuum matrix element $|M_c|^2$ contains a 1/V ¨normalization" of the continuum state that will cancel the factor of V in the density of states, as V (the volume) becomes large.

Multistep Compound (MSC) models - III

To normalize the emission widths to standard optical quantities, we require that the emission rates leading to the same final exciton class and emission channel sum to the optical quantity,

$$\frac{d\lambda_{e+}}{d\varepsilon_c}(n-1)+\frac{d\lambda_{e0}}{d\varepsilon_c}(n+1)+\frac{d\lambda_{e-}}{d\varepsilon_c}(n+3)=(2s+1)\frac{\mu\varepsilon_c\sigma(\varepsilon_c)}{\pi^2\hbar^3}$$

which furnishes

$$\frac{2\pi}{\hbar}|M_c|^2 g_c(\varepsilon_c)=\frac{(2s+1)\mu\varepsilon_c\sigma(\varepsilon_c)/(\pi^2\hbar^3)}{\omega_{B+}(n-1,\varepsilon_c)+\omega_{B0}(n+1,\varepsilon_c)+\omega_{B-}(n+3,\varepsilon_c)}$$

The internal transition widths are limited to bound-state densities but have the same form as before,

$$\lambda_+(n,E)\omega_B(n,E)=\lambda_-(n+2,E)\omega_B(n+2,E)=$$

$$=\frac{2\pi}{\hbar}|M|^2\int_0^E d\varepsilon\,(\omega_B(2,1,\varepsilon)\omega_B(1,0,\varepsilon)\omega_B(p-1,h,E-\varepsilon)$$

$$+\omega_B(1,2,\varepsilon)\omega_B(0,1,\varepsilon)\omega_B(p,h-1,E-\varepsilon))$$

H. Nishioka, J.J.M. Verbaarschot, H.A. Weidenmüller, S. Yoshida, Ann. Phys. (N.Y.) **183** (1988) 166.
M. Herman, G. Reffo, H.A. Weidenmüller, Nucl. Phys. A **536** (1992)124.

Multistep Compound (MSC) models - IV

The initial configuration will be the 2p-1h exciton class, if the energy is sufficiently low for bound 2p-1h states to exist. Assuming that to be the case, we must still take into account the competition with the multistep direct reactions.

$$\sigma_B = \sigma_{abs} \frac{\left\langle V_{cb}^2 \right\rangle^2 \omega_B(2,1,E)}{\left\langle V_{cb}^2 \right\rangle \omega_B(2,1,E) + \left\langle V_{cc}^2 \right\rangle \omega_C(2,1,E)}$$

As the energy increases, the contribution to the MSC process decreases. Flux can also enter the MSC from the 3p-2h, 4p-3h, etc. MSD stages. However, the higher stages tend to emit fewer pre-equilibrium particles.

This model can be extended to a two-component model and can also be extended to include angular momentum conservation. The MSC model in the EMPIRE-3.1 code conserves angular momentum.

The complete MSC model contains terms in which a nucleon is emitted and then reabsorbed. Such terms could in principal change the exciton number by four. They are usually not included in model calculations.

Multistep Direct (MSD) models

Multistep direct reaction models treat the case in which a nucleon remains in the continuum. As one particle is singled out, they are models of the "leading particle" type. However, the MSD is one of the few pre-equilibrium models that permits a more realistic description of nuclear structure than the simple single-particle particle-hole one.

Although they do not treat secondary collisions and emissions, as the exciton models can and the HMS model does automatically, they are able to describe the cross section and angular distribution of the fast particle with more accuracy than any other model.

The first MSD model was proposed by Feshbach, Kerman and Koonin and is known as the FKK model. A similar model, but based on a more rigorous development, was proposed by Nishioka,Weidenmüller and Yoshida. Calculations of single and two-step processes were first performed by Tamura, Udagawa and Lenske. The model in the EMPIRE-3.1 code follows most closely their version of the MSD.

H. Feshbach, A. Kerman, S. Koonin, Ann. Phys. (N.Y.) **125** (1980) 429.
H. Nishioka, H.A. Weidenmüller, S. Yoshida, Ann. Phys. (N.Y.) **183** (1988) 166.
T. Tamura, T. Udagawa, H. Lenske, Phys. Rev. C **26** (1982) 379.

Multistep Direct (MSD) models - I

We begin by writing the Hamiltonian as a sum of three terms,

$$H = H^{opt} + H^{int} + V^{res}$$

where

- H^{opt} is the energy-averaged optical model Hamiltonian that describes the relative motion of the projectile and the target;
- H^{int} is the intrinsic Hamiltonian of the target; and
- V^{res} is the residual interaction between the projectile and the target.

If the residual interaction were strong, we would consider solving the Schrödinger equation

$$(E - H^{opt} - H^{int} - V^{res}) \left| \psi^{(+)} \right\rangle = 0$$

by projecting onto the states of the intrinsic Hamiltonian

$$\left| \psi^{(+)} \right\rangle = \sum_{\mu} |\mu) \left| \psi_{\mu}^{(+)} \right\rangle \qquad \text{where} \qquad H^{int} |\mu) = \varepsilon_{\mu} |\mu)$$

to obtained the coupled equations

$$(E - H^{opt} - \varepsilon_{\mu}) \left| \psi_{\mu}^{(+)} \right\rangle - \sum_{\nu} V_{\mu\nu}^{res} \left| \psi_{\nu}^{(+)} \right\rangle = 0$$

The Coupled Channels model

Once the coupled channels equations are solved, the scattering amplitude can be calculated as

$$T_{\mu 0} = T_0 \delta_{\mu,0} + \sum_{\nu} \left\langle \phi_{\mu}^{(-)} \middle| V_{\mu\nu}^{res} \middle| \psi_{\nu}^{(+)} \right\rangle$$

where the optical wave function is the solution to

$$\left\langle \phi_{\mu}^{(-)} \middle| (E - H^{opt} - \varepsilon_{\mu}) = 0 \right.$$

and the differential cross section is

$$\frac{d^2\sigma}{dE d\Omega}(E_{\mu}, \Omega_{\mu} \leftarrow E, \Omega_0) = |T_{\mu 0}|^2$$

The reaction cross section is the difference between the flux that enters in the elastic channel and that which exits in it,

$$\sigma_r = -\frac{1}{v} \oint \vec{j} \cdot \vec{da} \qquad \text{while} \qquad \sigma_{\mu} = \int |T_{\mu 0}|^2 d\Omega$$

so that

$$\sigma_r = \sigma_{abs} + \sum_{\mu \neq 0} \sigma_{\mu}$$

Multistep Direct (MSD) models - II

For the most part, the residual interaction couples the states weakly. We then consider the equation in its Lippmann-Schwinger form

$$\left|\psi^{(+)}\right\rangle = \left|\phi^{(+)}\right\rangle + G^{+}(E)V^{res}\left|\psi^{(+)}\right\rangle$$

which we write in terms of coupled channels as

$$\left|\psi_{\mu}^{(+)}\right\rangle = \left|\phi_{0}^{(+)}\right\rangle \delta_{\mu,0} + \sum_{\nu} G_{\mu}^{+}(E)V_{\mu\nu}^{res}\left|\psi_{\nu}^{(+)}\right\rangle$$

where the propagator is

$$G_{\mu}^{+}(E) = \frac{1}{E^{+} - H^{opt} - \varepsilon_{\mu}}$$

When the coupling is weak, we can obtain a good approximation to the solution by back substitution,

$$\begin{aligned}\left|\psi_{\mu}^{(+)}\right\rangle = & \left|\phi_{0}^{(+)}\right\rangle \delta_{\mu,0} + G_{\mu}^{+}(E)V_{\mu 0}^{res}\left|\phi_{0}^{(+)}\right\rangle \\ & + \sum_{\nu} G_{\mu}^{+}(E)V_{\mu\nu}^{res}G_{\nu}^{+}(E)V_{\nu 0}^{res}\left|\phi_{0}^{(+)}\right\rangle + \ldots\end{aligned}$$

Multistep Direct (MSD) models - III

The scattering amplitude to the state μ is given by

$$T_{\mu 0} = T_0 \delta_{\mu,0} + \sum_{\nu} \left\langle \phi_\mu^{(-)} \middle| V_{\mu\nu}^{res} \middle| \psi_\nu^{(+)} \right\rangle$$

where T_0 is the elastic optical scattering amplitude. The amplitude for scattering inelastically to the state μ is then

$$\begin{aligned} T_{\mu 0} &= \left\langle \phi_\mu^{(-)} \middle| V_{\mu 0}^{res} \middle| \phi_0^{(+)} \right\rangle + \left\langle \phi_\mu^{(-)} \middle| \sum_{\nu} V_{\mu\nu}^{res} G_\nu^+(E) V_{\nu 0}^{res} \middle| \phi_0^{(+)} \right\rangle \\ &\quad + \left\langle \phi_\mu^{(-)} \middle| \sum_{\nu,\kappa} V_{\mu\nu}^{res} G_\nu^+(E) V_{\nu\kappa}^{res} G_\kappa^+(E) V_{\kappa 0}^{res} \middle| \phi_0^{(+)} \right\rangle + \ldots \end{aligned}$$

The differential cross section is

$$\frac{d^2\sigma}{dE d\Omega}(E_\mu, \Omega_\mu \leftarrow E, \Omega_0) = |T_{\mu 0}|^2$$

In order to calculate this expression, we must first simplify it.

Simplifying the MSD model - I

We first note that the optical wave functions vary slowly with energy compared to the variation of the intrinsic wave functions. We denote this with a Roman index that refers to the energy in the region of the intrinsic state, $\mu \rightarrow m$, etc.
To analyze the statistical properties of the cross section, we focus on the matrix elements of the intrinisic Hamiltonian.

If the residual interaction is a two-body one and the ground state is a pure 0p-0h state, then the one-step matrix element

$$\left\langle \phi_{\mu}^{(-)} \middle| V_{\mu 0}^{res} \middle| \phi_{0}^{(+)} \right\rangle$$

will only couple to the 1p-1h component of the state μ. Likewise, assuming that the 1p-1h component has not evolved into others during propagation, the two-step matrix element

$$\left\langle \phi_{m}^{(-)} \middle| \sum_{\nu} V_{\mu\nu}^{res} G_{n}^{+}(E) V_{\nu 0}^{res} \middle| \phi_{0}^{(+)} \right\rangle$$

will only couple to the 2p-2h (and 0p-0h) of the state μ. In the same manner, we expect the three-step amplitude to couple principally to the 3p-3h states.

Simplifying the MSD model - II

If we perform an average of the cross section over the states in a small range of energy, we expect that cross terms, such as

$$\frac{1}{N}\sum_{\mu\in\Delta E}\left\langle \phi_0^{(+)}\right| V_{0\mu}^{res}\left|\phi_m^{(-)}\right\rangle\left\langle \phi_m^{(-)}\right|\sum_{\nu} V_{\mu\nu}^{res}G_n^{+}(E)V_{\nu 0}^{res}\left|\phi_0^{(+)}\right\rangle$$

will tend to vanish due to the incoherence of the amplitudes of the states involved. We thus expect the average cross section to be an incoherent sum of the n-step cross sections.

$$\begin{aligned}\frac{d^2\sigma}{dE_m d\Omega} = & \overline{\left|\left\langle \phi_m^{(-)}\right| V_{\mu 0}^{res}\left|\phi_0^{(+)}\right\rangle\right|^2} \\ & + \overline{\left|\left\langle \phi_m^{(-)}\right|\sum_{\nu} V_{\mu\nu}^{res}G_n^{+}(E)V_{\nu 0}^{res}\left|\phi_0^{(+)}\right\rangle\right|^2} \\ & + \overline{\left|\left\langle \phi_m^{(-)}\right|\sum_{\nu,\kappa} V_{\mu\nu}^{res}G_n^{+}(E)V_{\nu\kappa}^{res}G_k^{+}(E)V_{\kappa 0}^{res}\left|\phi_0^{(+)}\right\rangle\right|^2} + \ldots\end{aligned}$$

Average density operators

It will be useful to be more precise about the average in energy. We can write the microscopic instrinsic-state density operator as

$$\hat{\rho}_{micro}(\varepsilon) = -\frac{1}{\pi}\mathrm{Im}\left[\frac{1}{\varepsilon - H^{int} + i\eta}\right] = \sum_{\mu}|\mu)\delta(\varepsilon - \varepsilon_{\mu})(\mu|$$

We assume that when we average this over a distribution of width larger than the average level spacing,

$$\hat{\rho}(\varepsilon) = \int d\varepsilon' g\left(\varepsilon - \varepsilon'\right)\hat{\rho}_{micro}(\varepsilon')$$

it reduces to a sum of terms for each of the np-nh configurations spaces,

$$\hat{\rho}(\varepsilon) = \sum_{n}\hat{\rho}_{n}(\varepsilon) \qquad \text{with} \qquad \hat{\rho}_{n}(\varepsilon) = \sum_{c=npnh}|c)P_{c}(\varepsilon)(c|$$

where the spectroscopic densities are given by

$$P_{c}(\varepsilon) = -\frac{1}{\pi}\mathrm{Im}\left[\int d\varepsilon' g\left(\varepsilon - \varepsilon'\right)\left(c\left|\frac{1}{\varepsilon - H^{int} + i\eta}\right|c\right)\right]$$

Simplifying the MSD model - III

We next expand the residual interaction in multipoles V_λ. Since only 1p-1h configurations are excited in the one-step process, the average one-step cross section involves an average over transitions to 1p-1h states m around the excitation energy ε_m with form factors

$$F_\lambda^{m0} = (m|V_\lambda|0)$$

However, we will assume that the interaction can be represented in terms of average state-independent form factors F_λ and nuclear transition operators O_λ,

$$V^{res}(\vec{r},\xi) = \sum_\lambda F_\lambda(\vec{r})\, O_\lambda(\xi)$$

We define the transition strength function S_λ in terms of the average density operator as

$$\left(c\left|O_\lambda^\dagger, \hat{\rho}(\varepsilon) O_\lambda\right|c\right) = \delta_{\lambda\lambda'} S_\lambda(\varepsilon, c)$$

This describes the transition rate per unit energy from the state c to the states c' centered about ε with angular momentum transfer Δl=λ.

Simplifying the MSD model - IV

To determine the average form factor, we might consider an average like the following

$$\begin{aligned} F_\lambda(\vec{r}) &= \left(c\left|O_\lambda^\dagger \hat{\rho}(\varepsilon) V^{res}\right|c\right) / S_\lambda(\varepsilon, c) \\ &= \sum_{c'} F_\lambda^{c'c}(\vec{r}) P_{c'}(\varepsilon) \left|(c'|O_\lambda|c)\right|^2 / \sum_{c'} P_{c'}(\varepsilon) \left|(c'|O_\lambda|c)\right|^2 \end{aligned}$$

However, this is still very specific – it depends on the configuration c. Instead, a global average over all np-nh configurations is used,

$$F_\lambda(\vec{r}) = \mathrm{Tr}\left[\hat{\rho} O_\lambda^\dagger \hat{\rho} V^{res}\right] / \mathrm{Tr}\left[\hat{\rho} O_\lambda^\dagger \hat{\rho} O_\lambda\right]$$

We can now write the one-step cross section as

$$\frac{d^2\sigma^{(1)}}{dE_m d\Omega} = \sum_\lambda S_\lambda(\varepsilon_m) \left|\left\langle \phi_m^{(-)} \right| F_\lambda \left| \phi_0^{(+)} \right\rangle\right|^2$$

where $E_m = E - \varepsilon_m$

The MSD two-step cross section - I

To evaluate the two-step cross section

$$\overline{\left| \left\langle \phi_m^{(-)} \right| \sum_{\nu} V_{\mu\nu}^{res} G_n^{+}(E) V_{\nu 0}^{res} \left| \phi_0^{(+)} \right\rangle \right|^2}$$

we assume that a typical term can be factorized into an optical part

$$\left\langle \phi_0^{(+)} \right| F_{\lambda_1'}^{\dagger} G^{opt\,\dagger}(E_n') F_{\lambda_2'}^{\dagger} \left| \phi_m^{(-)} \right\rangle \left\langle \phi_m^{(-)} \right| F_{\lambda_2} G^{opt}(E_n) F_{\lambda_1} \left| \phi_0^{(+)} \right\rangle$$

and an intrinsic part

$$\left(0 \left| O_{\lambda_1'}^{\dagger} G^{int\,\dagger}(\varepsilon_n') O_{\lambda_2'}^{\dagger} \hat{\rho}(\varepsilon_m) O_{\lambda_2} G^{int}(\varepsilon_n) O_{\lambda_1} \right| 0 \right)$$

This approximation neglects the evolution of the intrinsic states and restricts the intrinsic propagator to its 1p-1h part. We also assume that the cross terms of these components average to zero.

The MSD two-step cross section - II

With the averaging assumption, we have

$$\left(0\left|O^{\dagger}_{\lambda_1'}G^{int\,\dagger}(\varepsilon_n')O^{\dagger}_{\lambda_2'}\hat{\rho}(\varepsilon_m)O_{\lambda_2}G^{int}(\varepsilon_n)O_{\lambda_1}\right|0\right)$$

$$\rightarrow \sum_{c_1}|a_{c1}|^2(\varepsilon_n)\left(0\left|O^{\dagger}_{\lambda_1}\right|c_1\right)\left(c_1\left|O^{\dagger}_{\lambda_2}\hat{\rho}(\varepsilon_m)O_{\lambda_2}\right|c_1\right)\left(c_1\left|O_{\lambda_1}\right|0\right)$$

where the $|a_{c1}|^2$ are the coefficients of the 1p-1h term in the states at ε_n.

We approximate this as

$$\sum_{c_1}P_{c1}(\varepsilon_n)\left(0\left|O^{\dagger}_{\lambda_1}\right|c_1\right)\left(c_1\left|O_{\lambda_1}\right|0\right)\frac{\sum_{c_1}P_{c1}(\varepsilon_n)\left(c_1\left|O^{\dagger}_{\lambda_2}\hat{\rho}(\varepsilon_m)O_{\lambda_2}\right|c_1\right)}{\sum_{c_1}P_{c1}(\varepsilon_n)}$$

$$\rightarrow S_{\lambda_1}(\varepsilon_n)S_{\lambda_2}(\varepsilon_m,\varepsilon_n)$$

where the second term is

$$S_{\lambda}(\varepsilon_m,\varepsilon_n)=\mathrm{Tr}\left[\hat{\rho}_1(\varepsilon_n)O^{\dagger}_{\lambda}\hat{\rho}(\varepsilon_m)O_{\lambda}\right]/\mathrm{Tr}[\hat{\rho}_1(\varepsilon_n)]$$

The MSD two-step cross section - III

With these approximations, the two-step cross section can be written as

$$\frac{d^2\sigma^{(2)}}{dE_m d\Omega} = \sum_{\lambda_1 \lambda_2} \int d\varepsilon_n \, S_{\lambda_2}(\varepsilon_m, \varepsilon_n) \, S_{\lambda_1}(\varepsilon_n) \left| \left\langle \phi_m^{(-)} \right| F_{\lambda_2} G^{opt}(E_n) F_{\lambda_1} \left| \phi_0^{(+)} \right\rangle \right|^2$$

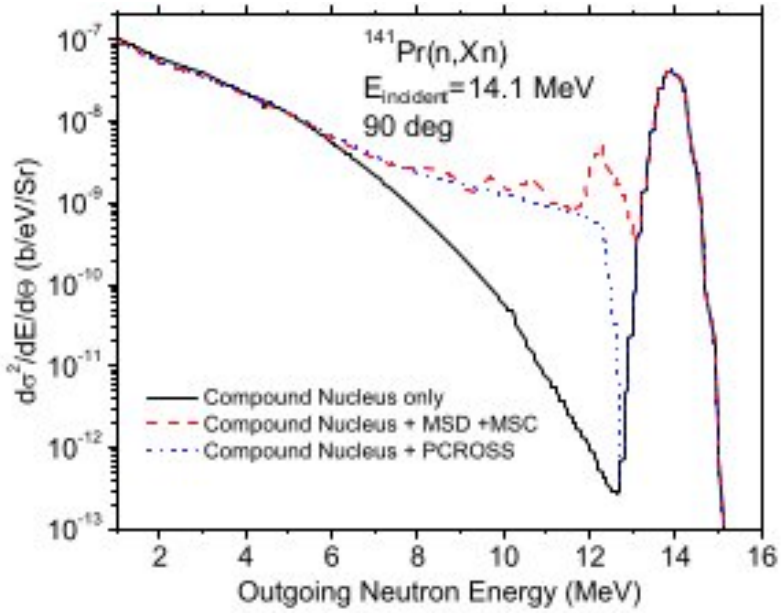
Although the transition strength function $S_\lambda(\varepsilon_m,\varepsilon_n)$ is defined so as to describe the 1p-1h to 2p-2h transitions, it is usually approximated in terms of the 1p-1h transition strength function as

$$S_\lambda(\varepsilon_m, \varepsilon_n) \approx S_\lambda(\varepsilon_m - \varepsilon_n)$$

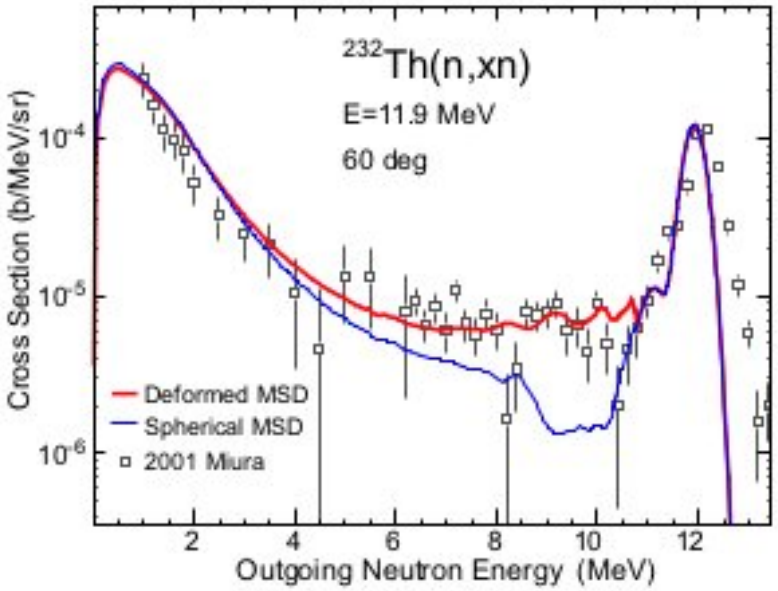
Further terms in the series could be calculated, this has only been done in the FKK approximation to date.

Although the transition strength factor can (and has been) calculated using a simple set of particle-hole basis states, it is the point at which more information on the nuclear structure can be included in the calculation. It is usually calculated using the RPA or qRPA approximation, which diagonalizes the Hamiltonian in an extended 1p-1h basis

Examples of MSD calculations



This calculation compares an exciton model calculation (using PCROSS) with an MSD one. Note the structure furnished by the MSD calculation.

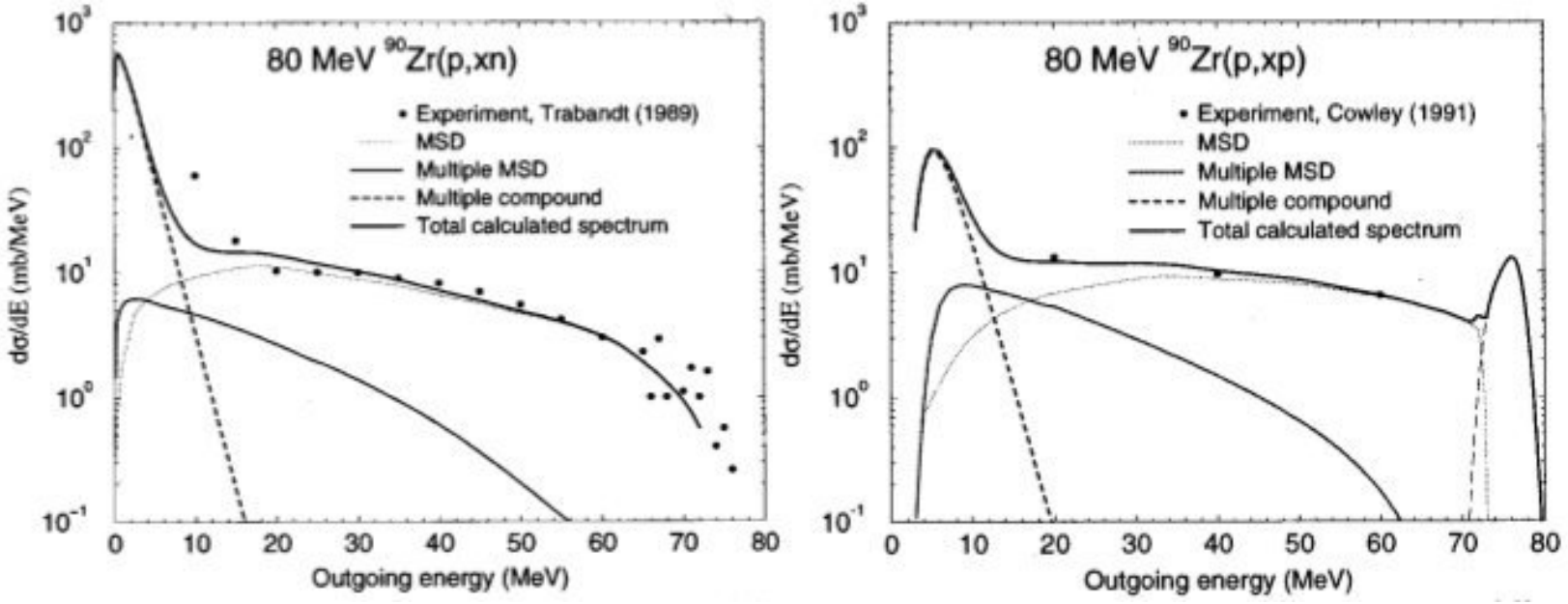


This calculation shows the excellent results obtainable from an MSD calculation when the nuclear structure is well described.

M. Herman et al., NDS **108** (2007) 2655.

An example of an MSD-FKK calculations

This example shows one of the few calculations of both inelastic scattering and charge exchange using an MSD model. The calculation was extended to the four-step process using a particle-hole basis of states and the FKK approximation, which obtains the MSD series through a convolution of double differential distributions.



A.J. Koning, M.B. Chadwick, Phys. Rev. C **56** (1997) 970.

Summary

Direct reactions that transfer energy as well as momentum are often quite important. Such inelastic scatterings, in the case of the inert projectiles, leave the target in an excited state and diminish the asymptotic kinetic energy of the projectile.

The states that are most strongly excited in inelastic collisions are those that involve collective movement, vibrations and rotations, in particular. When the coupling between states is strong, a coherent coupled channels claculation is necessary describe the reaction. When the coupling is weak, the distorted-wave Born approximation is often sufficient.

Quantum mechanical models separate the energy spectrum of a particle into a continuum component and a bound state component. The continuum component is usually forward peaked but the bound state component must necessarily be symmetric about 90^{o}. These models succeed in taking into account the coherent effects that lead to nuclear collectivity as well as the incoherent single-particle excitations at slightly higher excitation energies.