Quantum Amplifiers, Attenuators, and Schroedinger's Cat

R. J. Glauber
Harvard University

Two Microstates:

$|a\rangle$ $|b\rangle$

Evolve Into

$|A\rangle$ $|B\rangle$

Does $|c\rangle = \alpha|a\rangle + \beta|b\rangle$

Evolve Into

$|C\rangle = \alpha|A\rangle + \beta|B\rangle$?

Two Single-Photon States

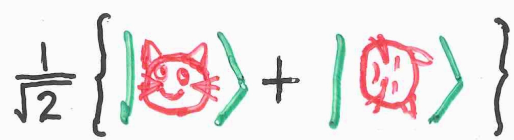
$$|v\rangle = |\updownarrow\rangle \quad , \quad |h\rangle = |\leftrightarrow\rangle$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

Does

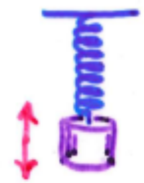
$$|\nearrow\rangle = \frac{1}{\sqrt{2}}\left\{ |\updownarrow\rangle + |\leftrightarrow\rangle \right\}$$

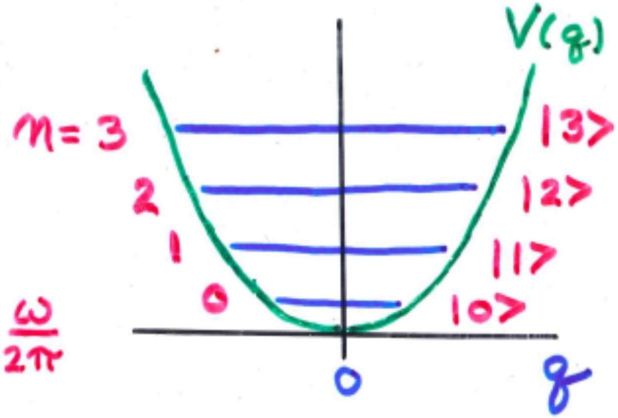
Evolve Into



?

Harmonic Oscillator





Frequency $\nu = \frac{\omega}{2\pi}$

n-Quantum States $|0\rangle, |1\rangle, |2\rangle \cdots$

Coherent States

~ Complex Amplitude α

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Eigenstates of a, $a^\dagger$

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

$$\langle\alpha|\alpha^* = \langle\alpha|a^\dagger$$

Damped Oscillator

Central Oscillator: $a \quad a^{\dagger}$

"Heat Bath" Oscillators: $\{b_k\} \; \{b_k^{\dagger}\}$

Hamiltonian:

$$H = \hbar\omega a^{\dagger} a + \sum_k \hbar \omega_k b_k^{\dagger} b_k + \hbar \sum_k (\lambda_k a^{\dagger} b_k + \lambda_k^{*} b_k^{\dagger} a)$$

Invariance: $\quad a \to a e^{i\theta}$
$\qquad\qquad\quad b_k \to b_k e^{i\theta}$

Conservation Law

$$a^{\dagger} a + \sum_k b_k^{\dagger} b_k = \text{const.}$$

Equations of Motion:

$$\dot{a} = -i\omega a - i\sum_k \lambda_k b_k$$

$$\dot{b}_k = -i\omega_k b_k - i\lambda_k^* a$$

General Form of Solution:

$$a(t) = a(0)\,u(t) + \sum_k b_k(0)\,v_k(t)$$

$$u(0) = 1 \qquad v_k(0) = 0$$

etc.

Coherent state for entire system:

$$\begin{matrix} a \\ b_k \end{matrix} \, |\alpha\{\beta_k\}\rangle = \begin{matrix} \alpha \\ \beta_k \end{matrix} \, |\alpha\{\beta_k\}\rangle$$

Initial state - coherent state

$$\begin{matrix} a(0) \\ b_k(0) \end{matrix} \left| \alpha\{\beta_k\} \right\rangle = \begin{matrix} \alpha \\ \beta_k \end{matrix} \left| \alpha\{\beta_k\} \right\rangle$$

but: $a(t) = a(0)\,u(t) + \sum_k b_k(0)\, v_k(t)$
etc.

Let: $\alpha(t) \equiv \alpha\; u(t) + \sum_k \beta_k\, v_k(t)$
etc.

Then in Heisenberg picture:

$$\begin{matrix} a(t) \\ b_k(t) \end{matrix} \left| \alpha\{\beta_k\} \right\rangle = \begin{matrix} \alpha(t) \\ \beta_k(t) \end{matrix} \left| \alpha\{\beta_k\} \right\rangle$$

But $\quad a(t) = U^{-1}(t)\, a(0)\, U(t) \quad$ etc.

Where $\quad |t\rangle = U(t)\left|\alpha\{\beta_k\}\right\rangle$ is

the time-dependent Schrödinger

State. Hence

$$\begin{matrix} a(0) \\ b_k(0) \end{matrix} |t\rangle = \begin{matrix} \alpha(t) \\ \beta_k(t) \end{matrix} |t\rangle \quad .$$

and solution is

$|t\rangle = (\text{c-number phase factor}) \cdot \left|\alpha(t)\{\beta_k(t)\}\right\rangle$

Find $\rho(t)$ -Density Operator

If $\rho(0) = |\alpha\{\beta_k\}\rangle\langle\alpha\{\beta_k\}|$,

then $\rho(t) = |\alpha(t)\{\beta_k(t)\}\rangle\langle\alpha(t)\{\beta_k(t)\}|$

where $\alpha(t)$, $\{\beta_k(t)\}$ satisfy same equations of motion as $a(t)$, $\{b_k(t)\}$,

i.e. $\alpha(t) = \alpha\, u(t) + \sum_k \beta_k v_k(t)$ etc.

Partial Density Op. for Central Oscillator

$$\rho_A(t) = \mathrm{Trace}_B\, \rho(t)$$
$$= |\alpha(t)\rangle\langle\alpha(t)|$$

(Pure coh. state)

Including Mixtures:

$$\rho_A(t) = \int |\alpha(t)\rangle\langle\alpha(t)|\; P(\{\beta_k\}) \prod_k d^2\beta_k$$

e.g.

"Heat Bath" Oscillators in Chaotic States

$$\rho_A(t) = \prod_k \left(\frac{1}{\pi\langle n_k\rangle}\right) \int e^{-\sum_k \frac{|\beta_k|^2}{\langle n_k\rangle}} |\alpha(t)\rangle\langle\alpha(t)| \prod_k d^2\beta_k$$

Result:

$$\rho_A(t) = \int P(\alpha\, 0|\gamma\, t)\, |\gamma\rangle\langle\gamma|\, d^2\gamma$$

where

$$P(\alpha\, 0|\gamma\, t) = \frac{1}{\pi D(t)}\, e^{-\frac{|\gamma - \alpha u(t)|^2}{D(t)}}$$

Dispersion: $D(t) = \sum_k \langle n_k\rangle |v_k(t)|^2$

For $\langle n_k\rangle = 0$ $\qquad D(t) = 0$

$$P(\alpha\, 0|\gamma\, t) = \delta^{(2)}(\gamma - \alpha u(t))$$

Pure state - Has Poincaré recurrences, etc.

Find $u(t)$, $\{v_k(t)\}$

W. W. Approximation :

$$u(t) = e^{-i\omega' t - \kappa t}$$

where $\omega' = \omega + \delta\omega$

$$\delta\omega - i\kappa = \lim_{\epsilon \to 0} \sum_k \frac{|\lambda_k|^2}{\omega - \omega_k + i\epsilon}$$

Note $\quad |u(t)|^2 + \sum_k |v_k(t)|^2 = 1 \quad$ Cons. Law

v_k have resonant peak at $\omega_k = \omega + \delta\omega = \omega'$

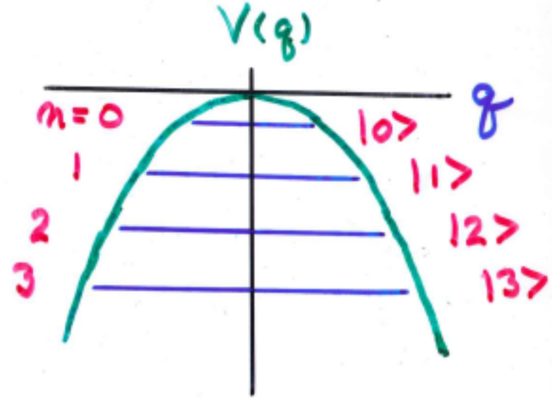
Hence

$$
\begin{aligned}
D(t) &= \sum \langle n_k \rangle |v_k(t)|^2 \\
&\approx \langle n_{\omega'} \rangle \sum |v_k(t)|^2 \\
&\approx \langle n_{\omega'} \rangle (1 - |u(t)|^2) \\
&\approx \langle n_{\omega'} \rangle (1 - e^{-2\kappa t})
\end{aligned}
$$

Then

$$P(\alpha\, 0 | \gamma\, t) = \frac{1}{\pi D(t)}\, e^{-\frac{|\gamma - \alpha e^{-(i\omega' + \kappa)t}|^2}{D(t)}}$$

Inverted Oscillator



$$H_a = -\frac{1}{2}(p^2 + \omega^2 q^2)$$

$$= -\hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

i.e. $\omega \to -\omega$

$$a(t) = a(0)\, e^{+i\omega t}$$

$$a^\dagger(t) = a^\dagger(0)\, e^{-i\omega t}$$

a and $a^\dagger$ interchange roles

Amplifying Oscillator: Couple Inverted Oscillator to a set of Ordinary Oscillators

Amplifier Hamiltonian:

$$H = -\hbar\omega a^\dagger a + \sum_k \hbar\omega_k b_k^\dagger b_k + \hbar\sum_k (\lambda_k a b_k + \lambda_k^* b_k^\dagger a^\dagger)$$

Invariance: $a \to a e^{i\theta}$, $b_k \to b_k e^{-i\theta}$

Conservation Law:

$$a^\dagger a - \sum_k b_k^\dagger b_k = \text{const.}$$

Equations of Motion:

$$\dot{a} = i\omega a - i\sum \lambda_k^* b_k^\dagger$$

$$\dot{b}_k = -i\omega_k b_k - i\lambda_k^* a^\dagger$$

Solutions take form:

$$a(t) = a(0)\,U(t) + \sum_k b_k^\dagger(0)\,V_k(t)$$

where $U(0) = 1$ $V_k(0) = 0$

etc.

Normal-Ordered Characteristic Function

for A-Oscillator

$$\chi_N(\mu) = \mathrm{Trace}\left\{ \rho(t)\, e^{\mu a^\dagger} e^{-\mu a} \right\}$$
$$= \mathrm{Trace}\left\{ \rho(0)\, e^{\mu a^\dagger(t)} e^{-\mu a(t)} \right\}$$

Fourier Transform is P-distrib. in

$$\rho(t) = \int P(\gamma t)\, |\gamma\rangle\langle\gamma|\, d^2\gamma$$

For an initial coherent state

$$\rho(0) = |\alpha\rangle\langle\alpha|$$

We find

$$P(\alpha 0|\gamma t) = \frac{1}{\pi n(t)}\, e^{-\frac{|\gamma - \alpha U(t)|^2}{n(t)}}$$

with

$$n(t) = |U(t)|^2 - 1 + \sum_k \langle n_k\rangle |V_k(t)|^2$$

or

$$= \sum_k (1 + \langle n_k\rangle)\, |V_k(t)|^2$$

W.-W. Approximation:

$$U(t) = e^{i\omega'' t + \kappa t}$$

where $\omega'' = \omega - \delta\omega$ with $\delta\omega$, κ same as for damped case

Now

$$|U(t)|^2 - \sum_k |V_k(t)|^2 = 1 \qquad \text{cons. law}$$

and

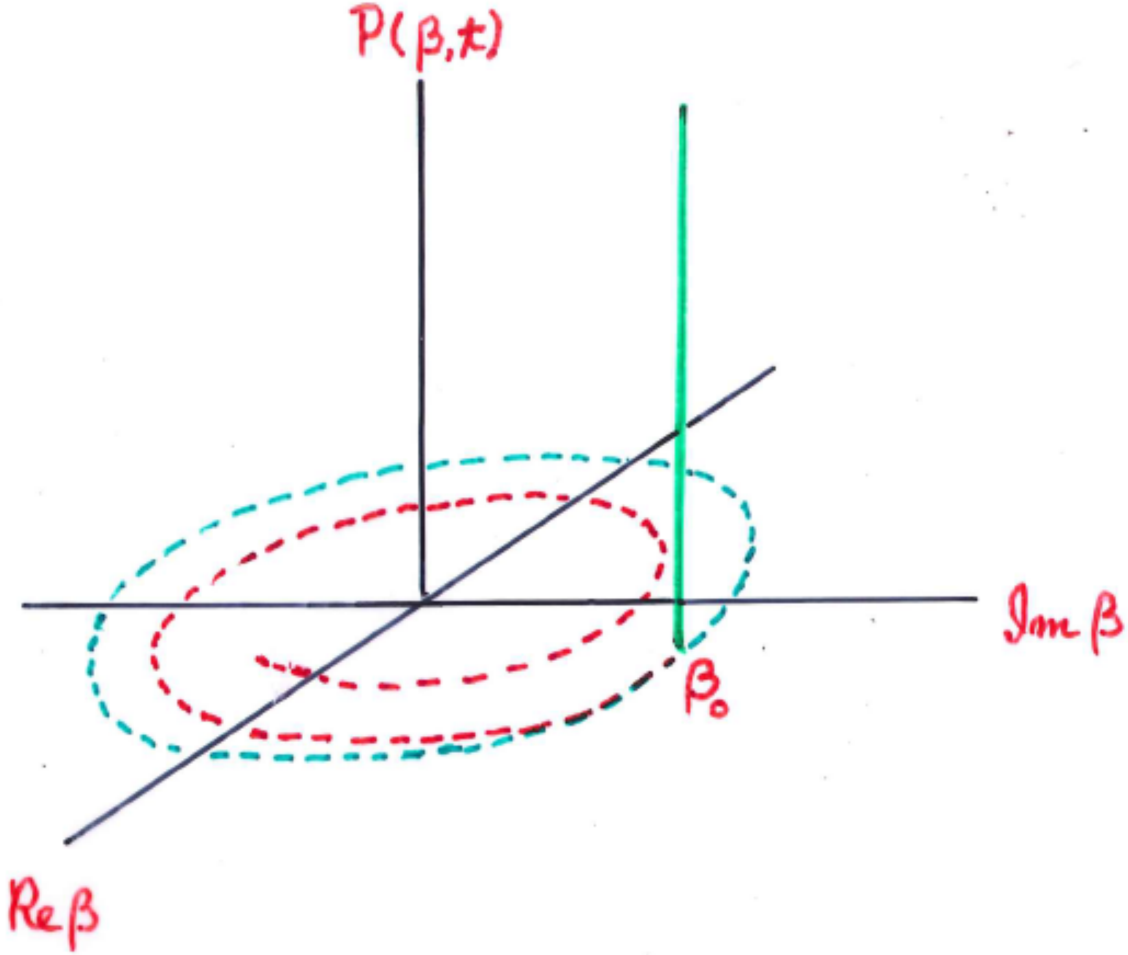
$$\sum_k \langle n_k\rangle |V_k|^2 \approx \langle n_{\omega''}\rangle \sum |V_k|^2$$
$$\approx \langle n_{\omega''}\rangle \left(|U(t)|^2 - 1\right)$$

Hence dispersion is

$$n(t) = (1 + \langle n_{\omega''}\rangle)(e^{2\kappa t} - 1)$$

For initial coherent state

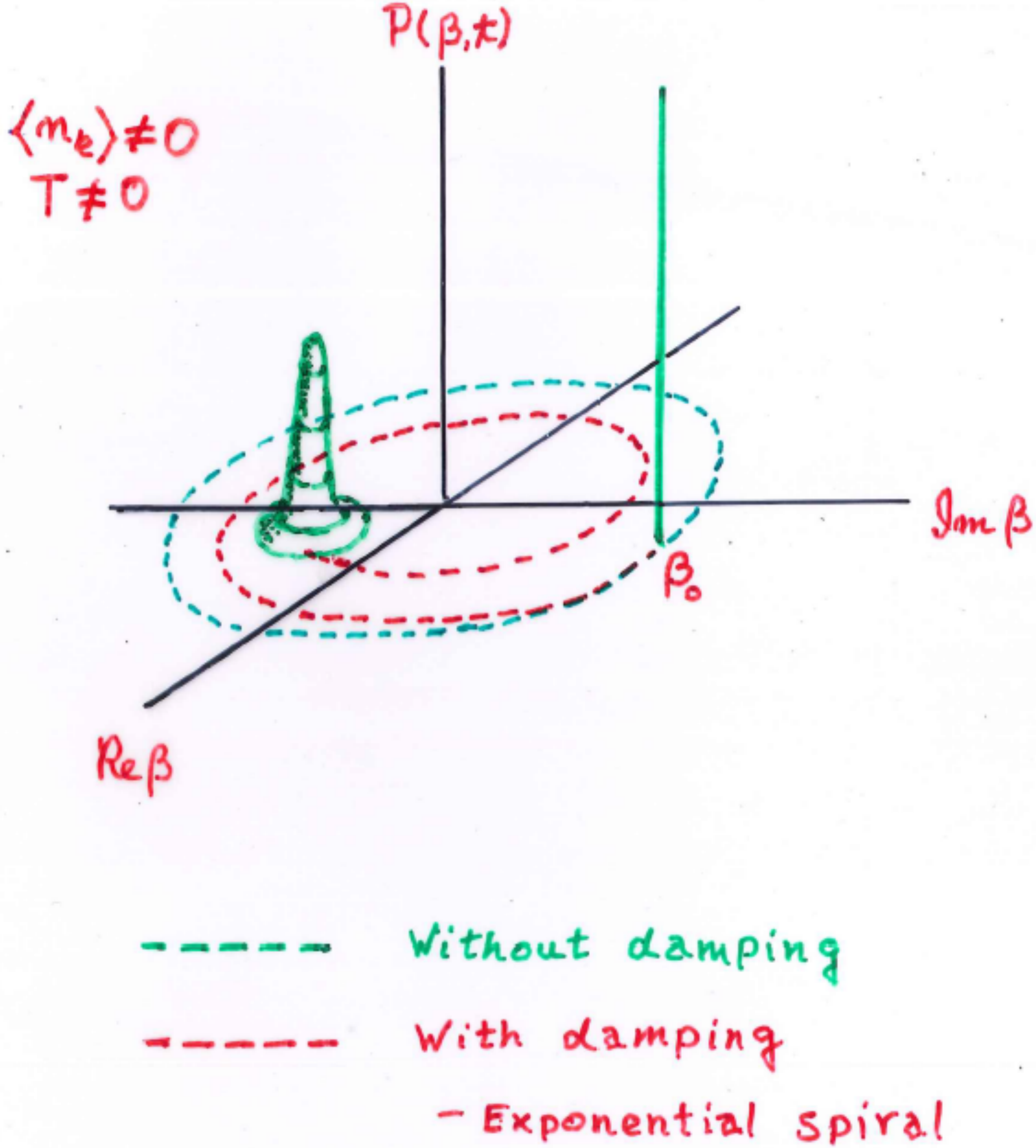
$$P(\alpha 0|\gamma t) = \frac{1}{\pi n(t)} e^{-\frac{|\gamma - \alpha e^{[i\omega'' + \kappa]t}|^2}{n(t)}}$$



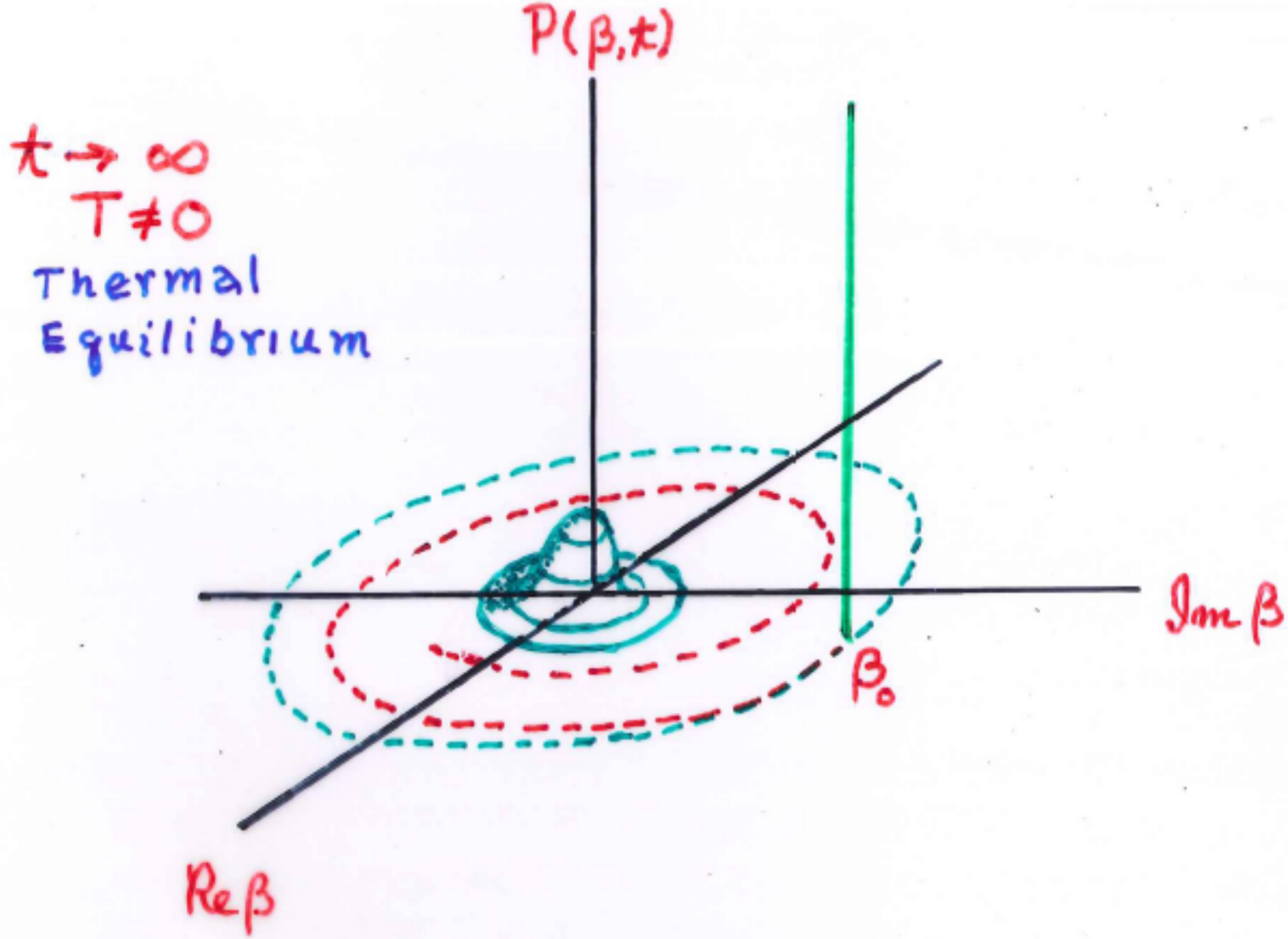
- - - - - Without damping

- - - - - With damping

– Exponential spiral



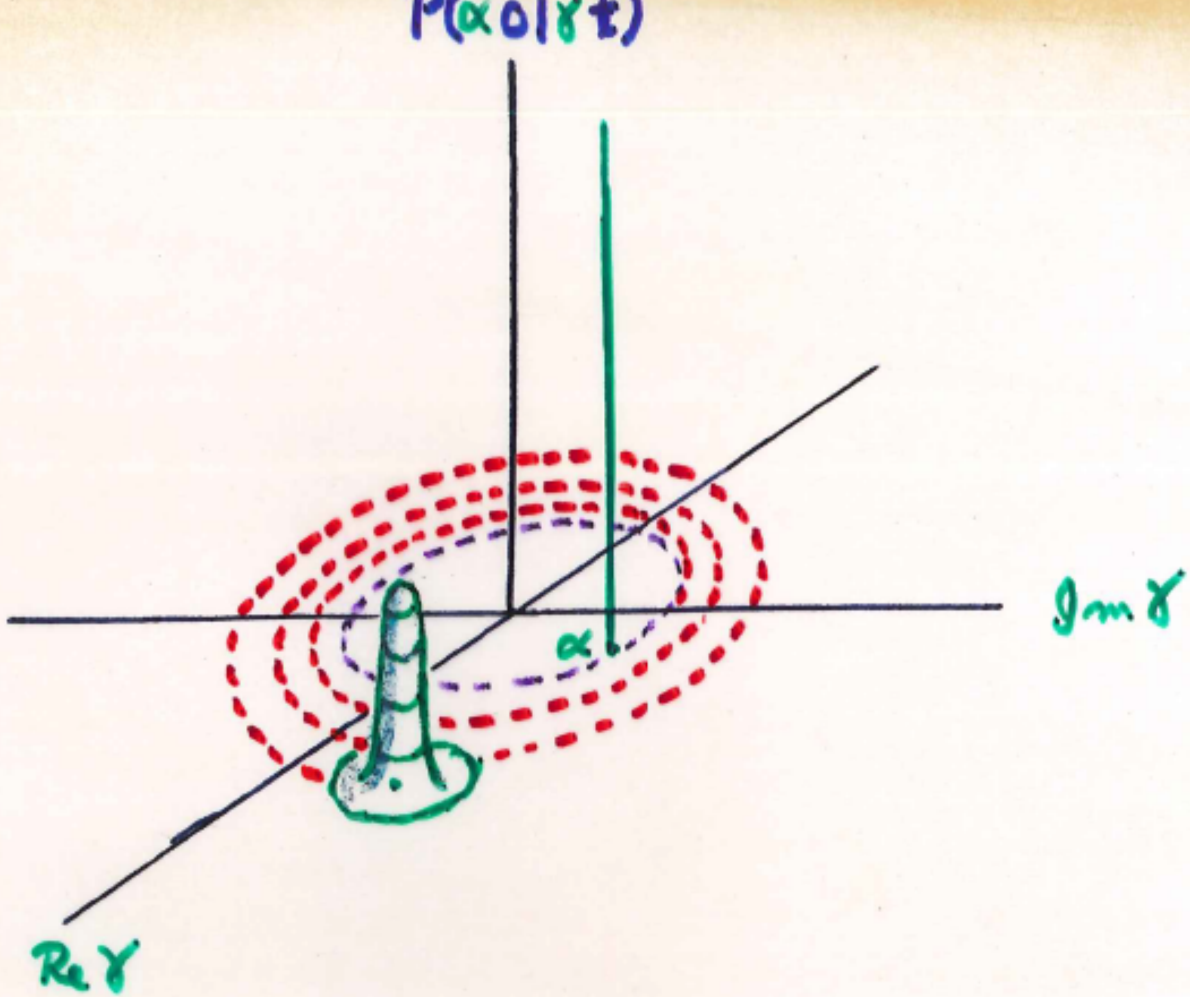
P(β,t)
⟨n_k⟩≠0
T≠0
Im β
Re β
β₀
Without damping
With damping
- Exponential spiral



- - - - - Without damping

- - - - - With damping

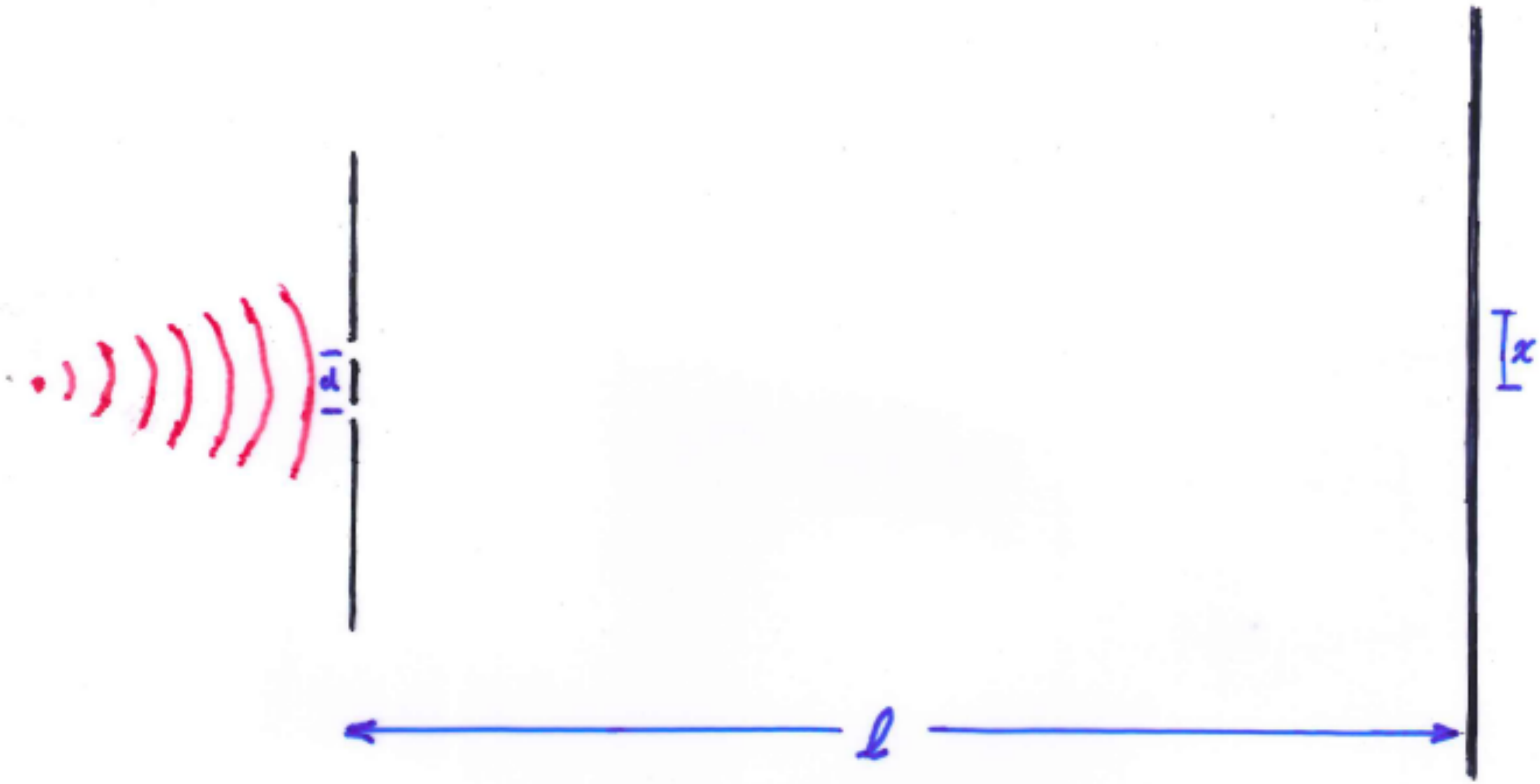
\- Exponential spiral



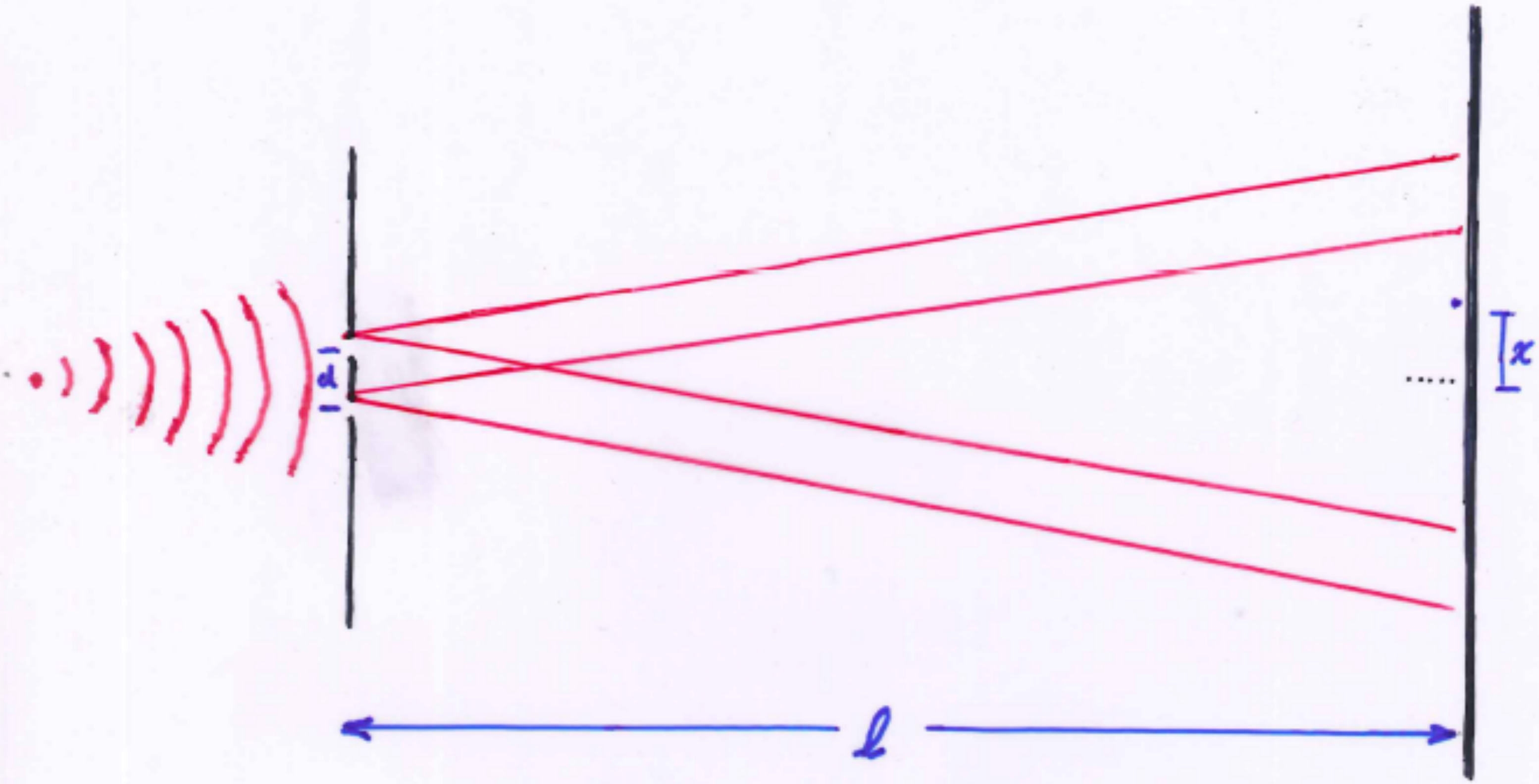
- - - - - without amplification

- - - - - with amplification

Exponential Spiral


d
x
ℓ

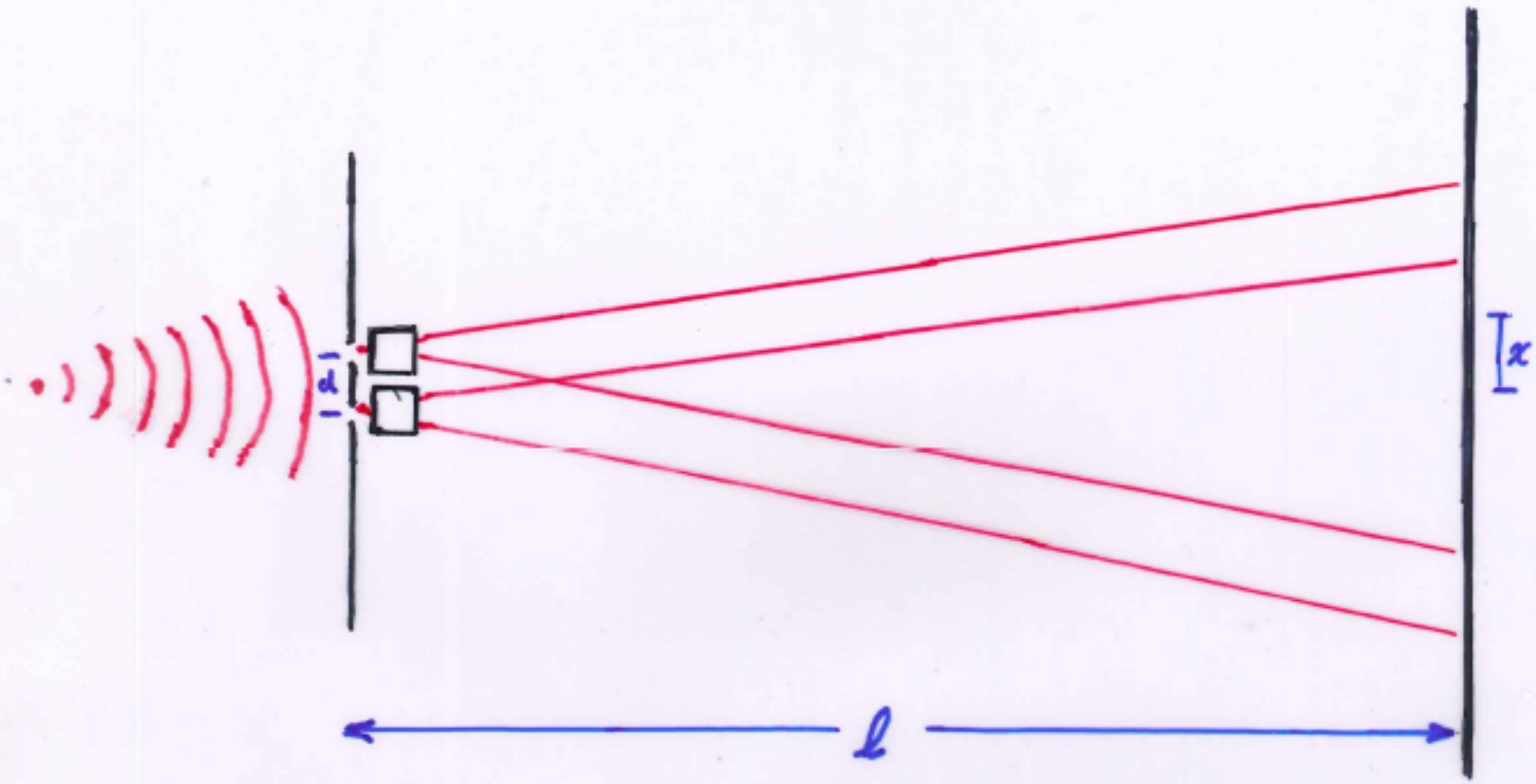
Young's Exp't



Field $\sim \cos\frac{\pi x d}{\lambda \ell}$

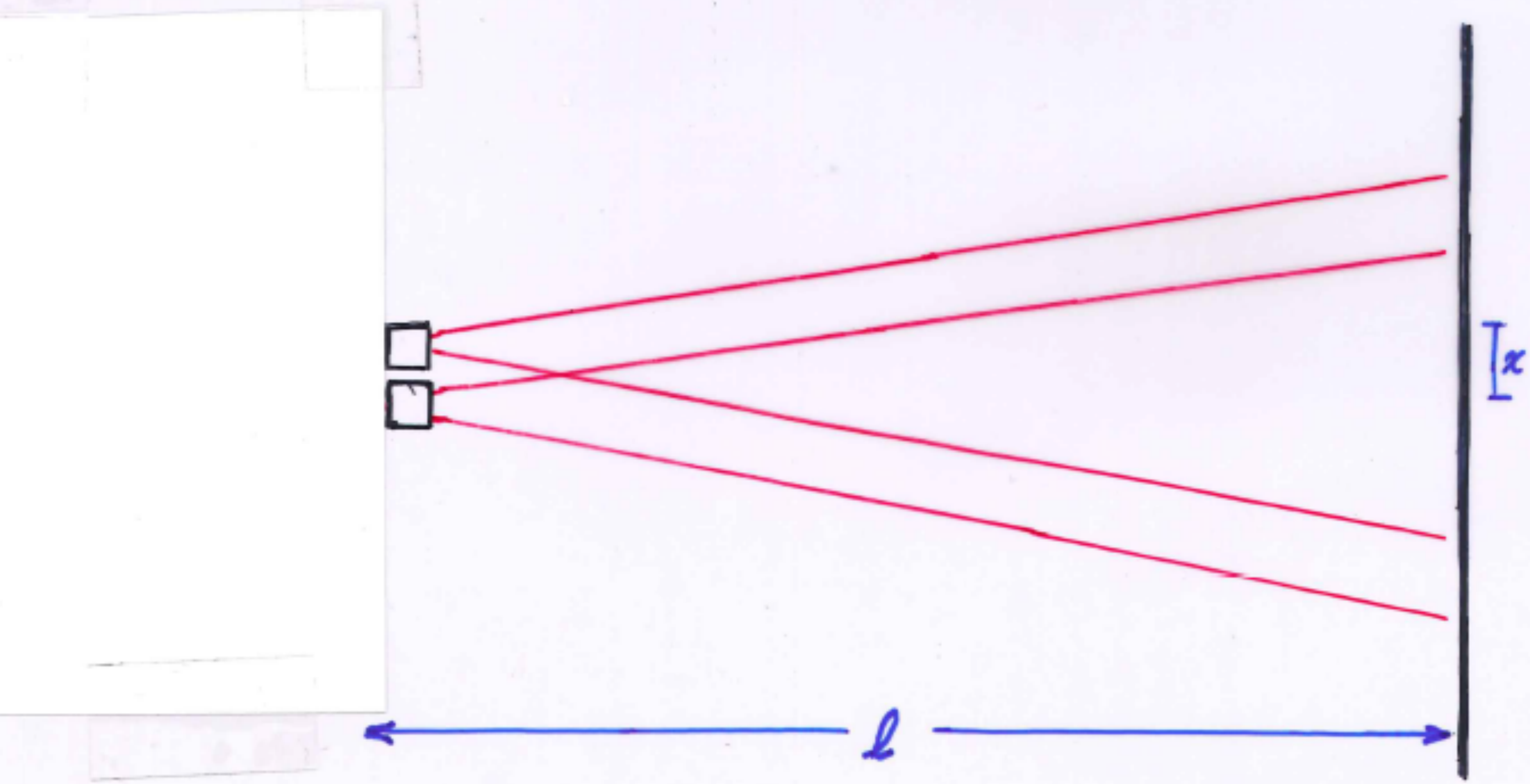
phase diff. at x: $\frac{2\pi x d}{\lambda \ell}$

Intensity $\sim \cos^2\frac{\pi x d}{\lambda \ell}$



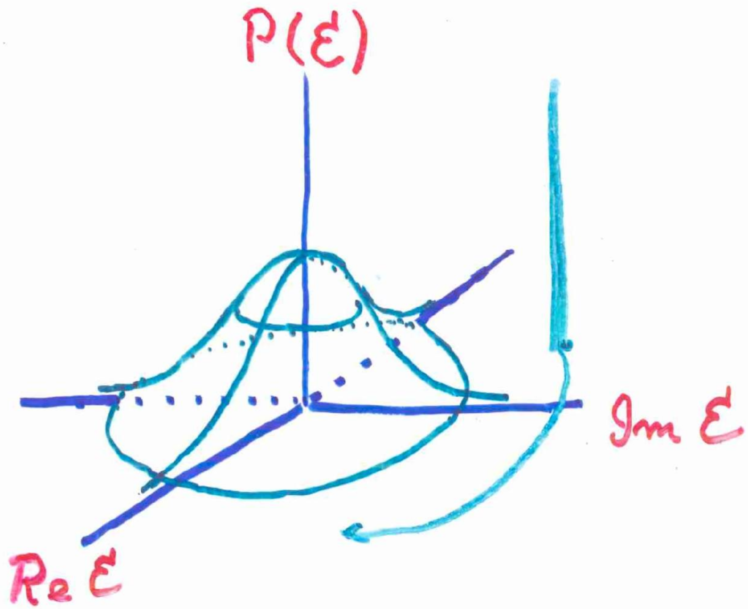
Quantum Young Exp't

– with Two Amplifiers



Quantum Young Exp't

– with Two Amplifiers



P(ℰ)
Im ℰ
Re ℰ