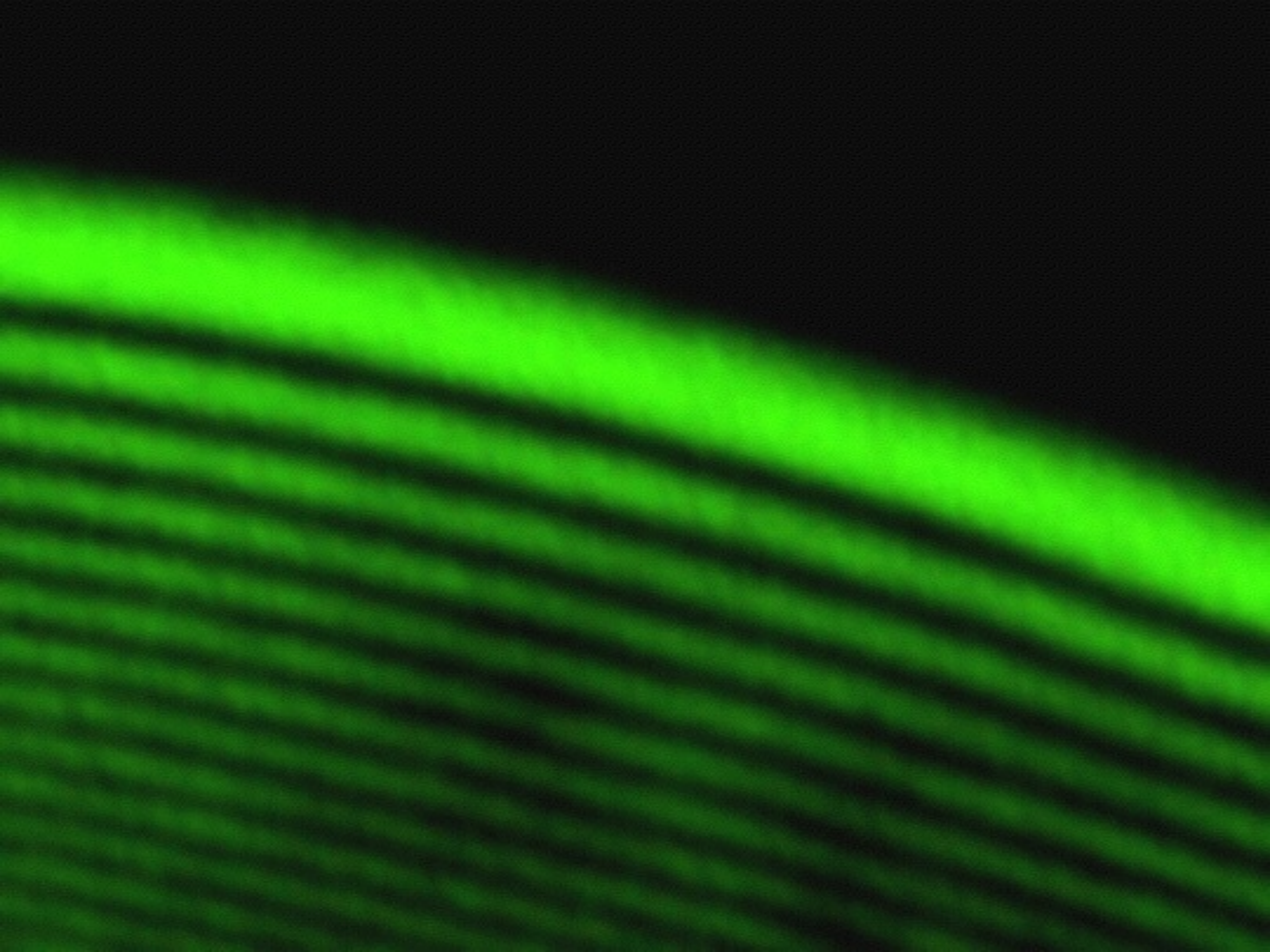

Divergent series: from Thomas Bayes to resurgence via the rainbow

Michael Berry

H H Wills Physics Laboratory, University of Bristol, UK

http://michaelberryphysics.wordpress.com

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the subject has been repeatedly reborn, more deeply each time

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submitted1747,
published 1763

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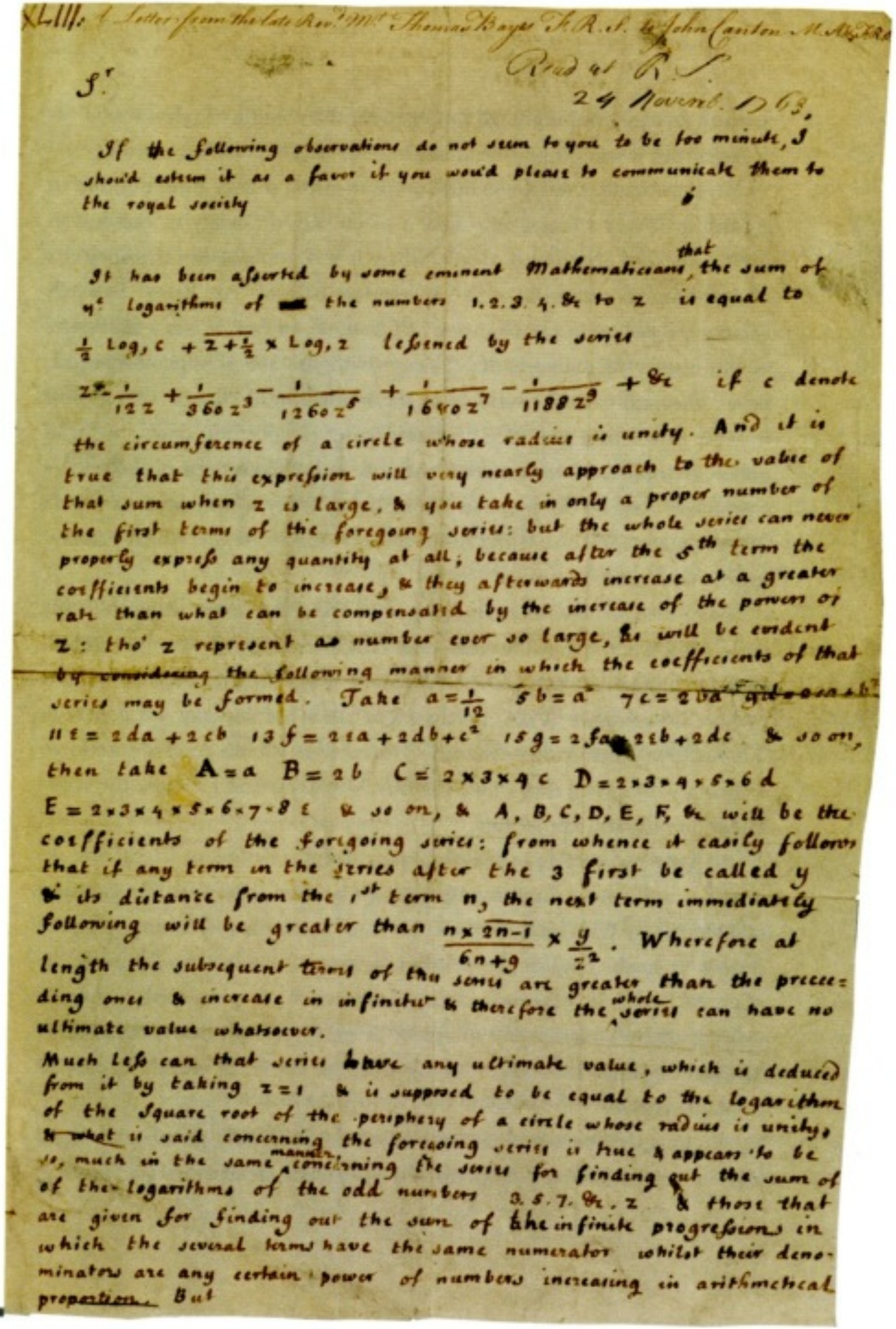
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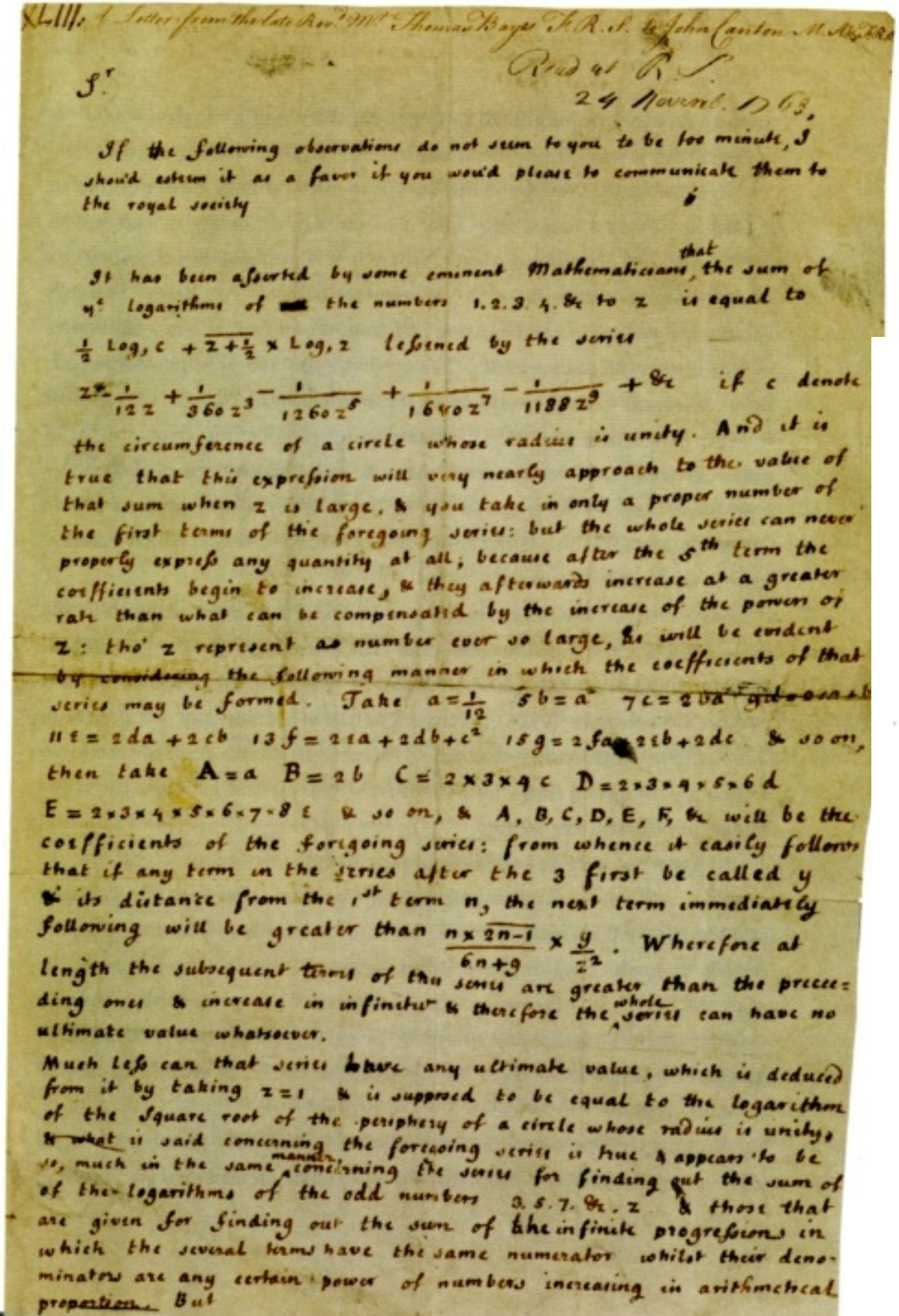

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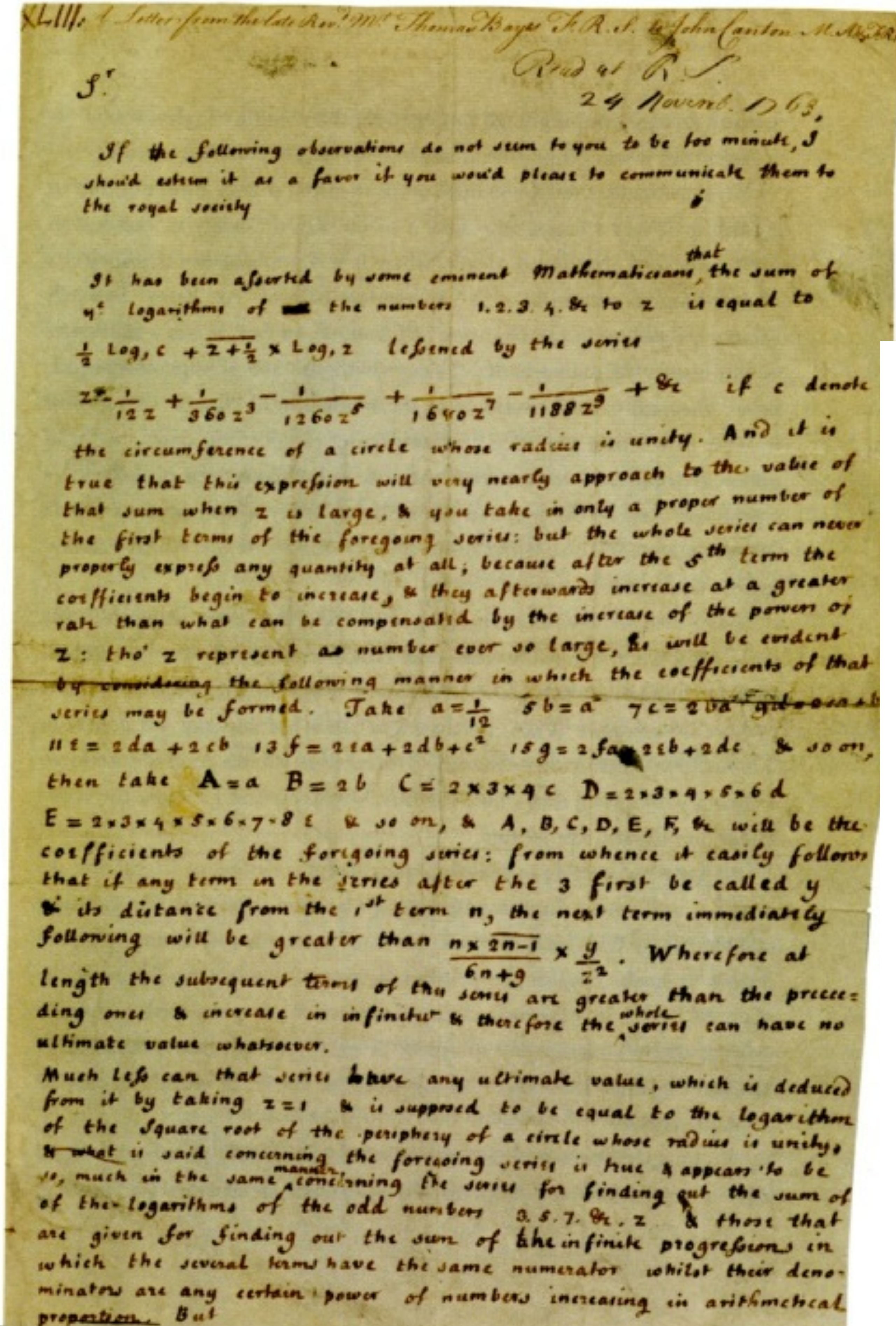
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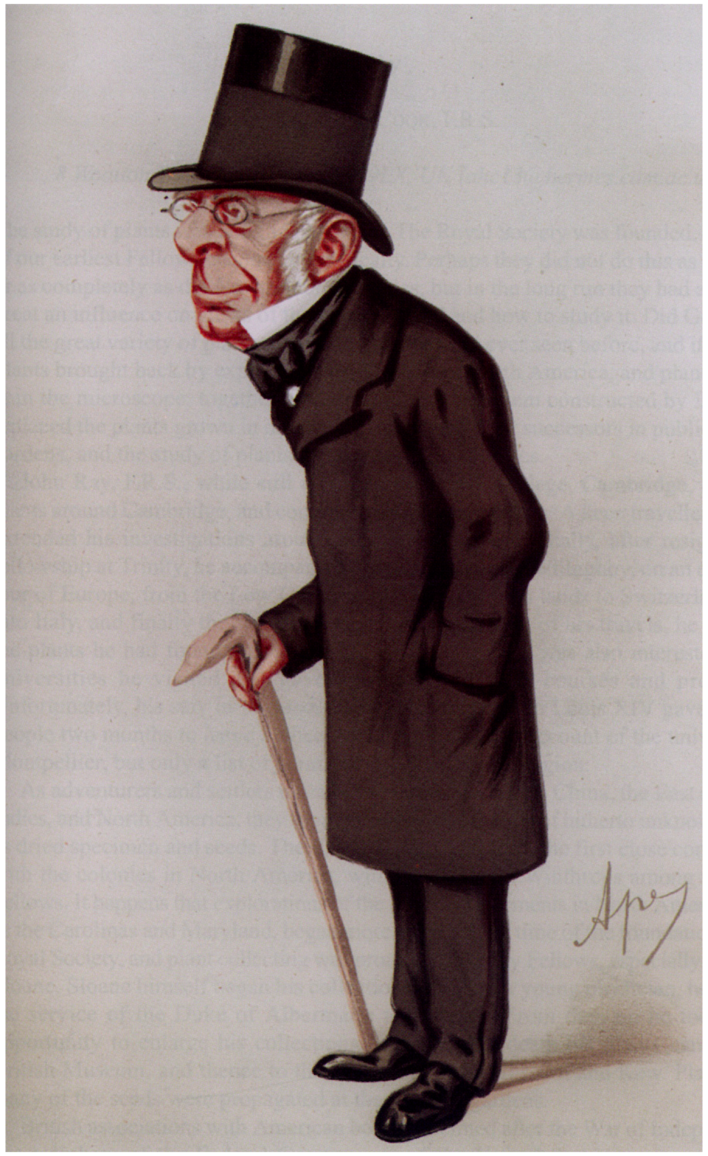
2+2=5, for sufficiently large values of 2

Airy and interference fringes in the rainbow



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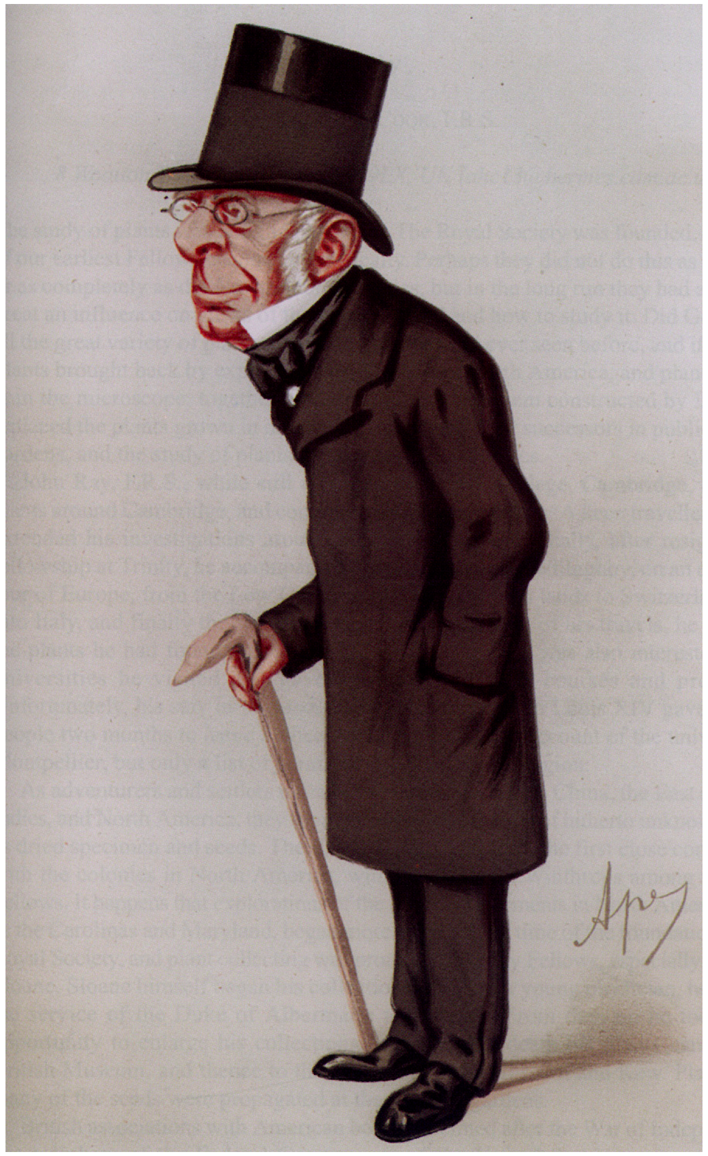

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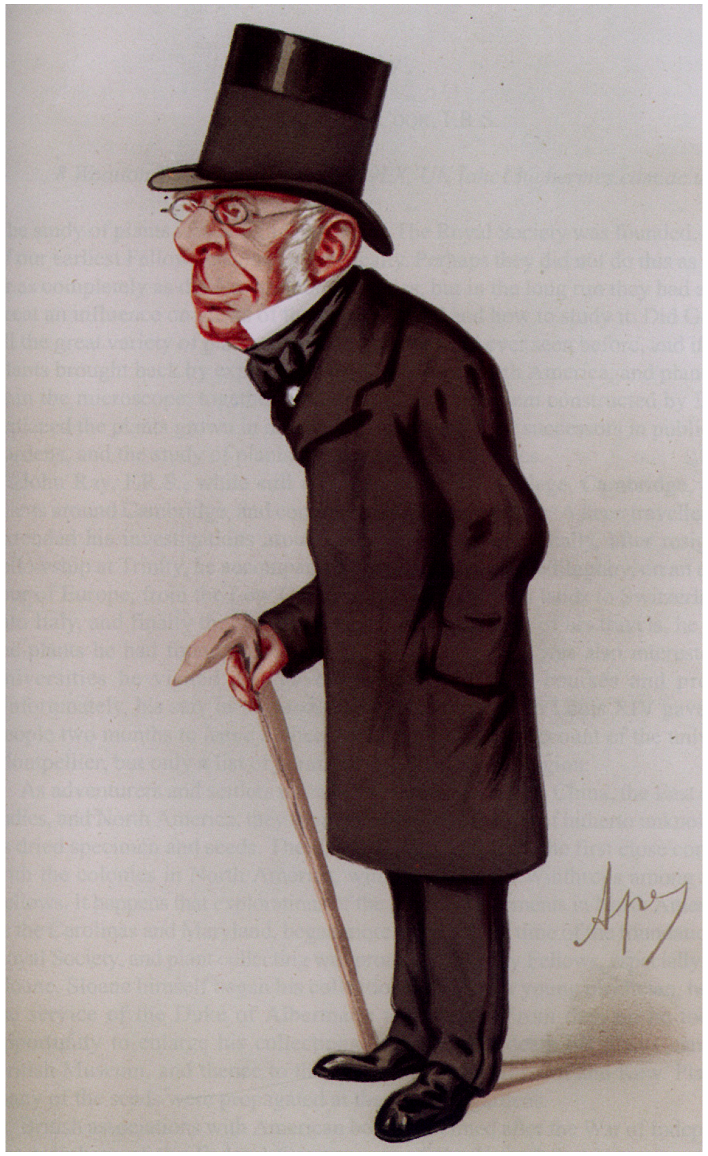

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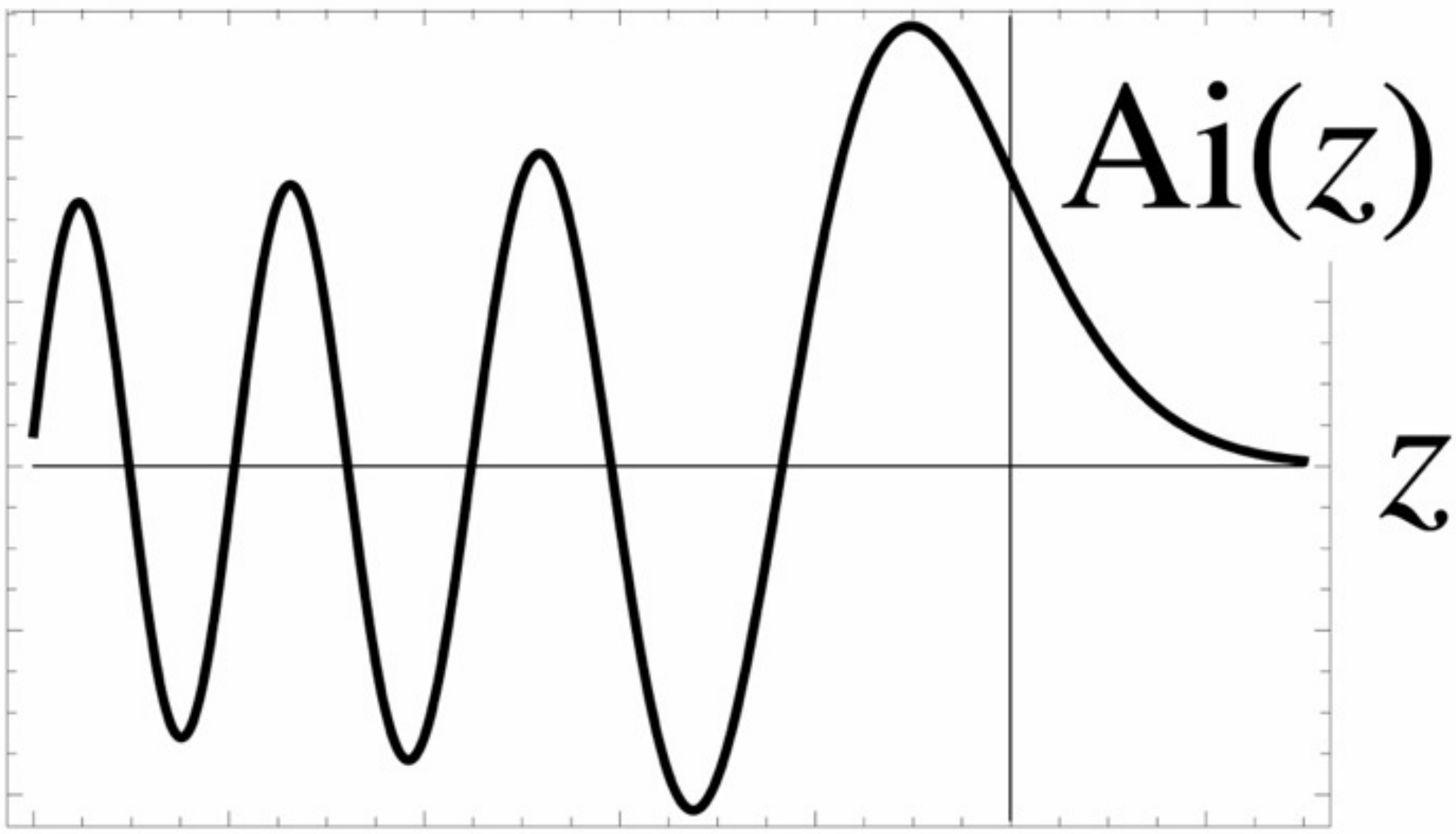

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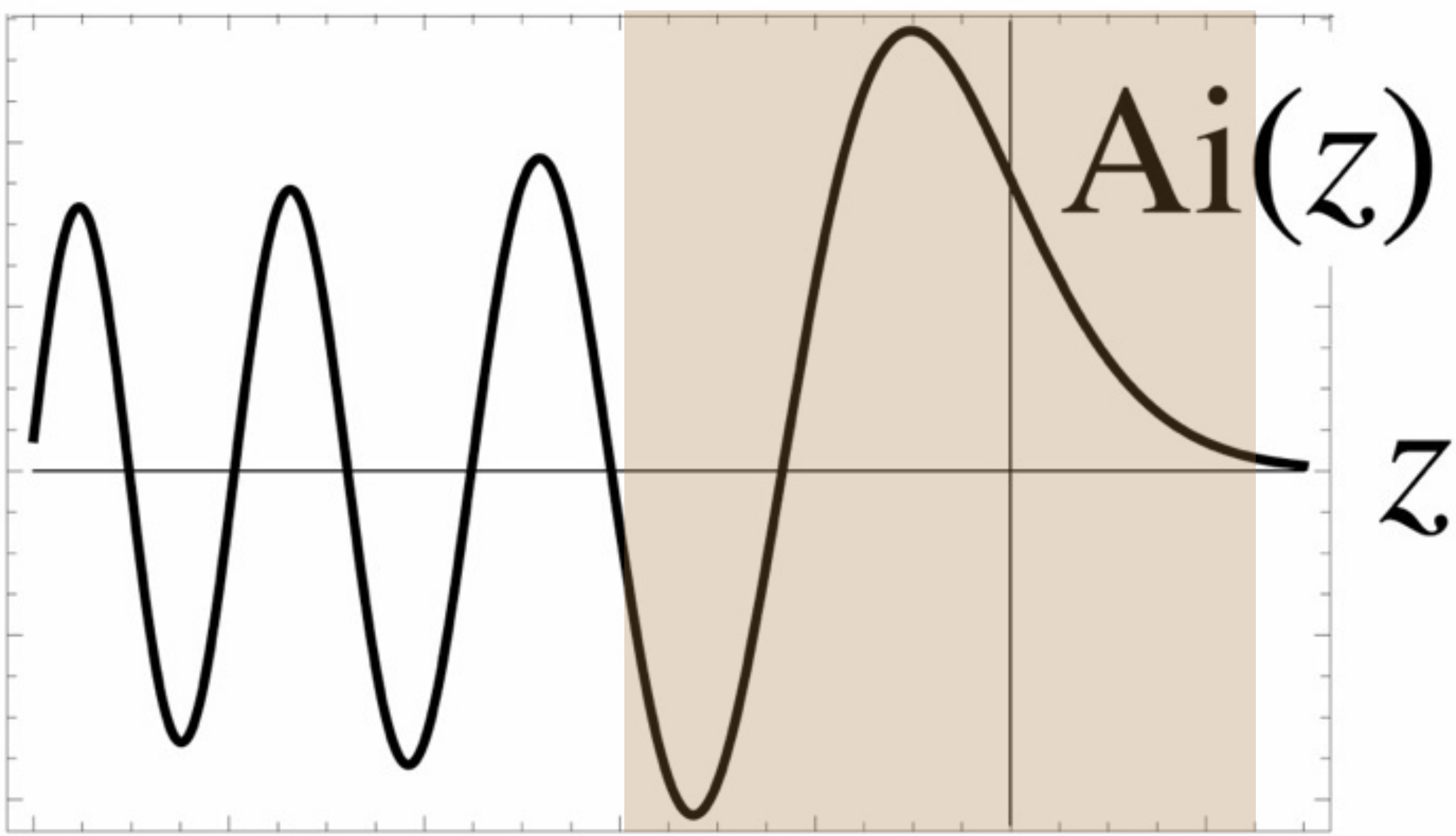

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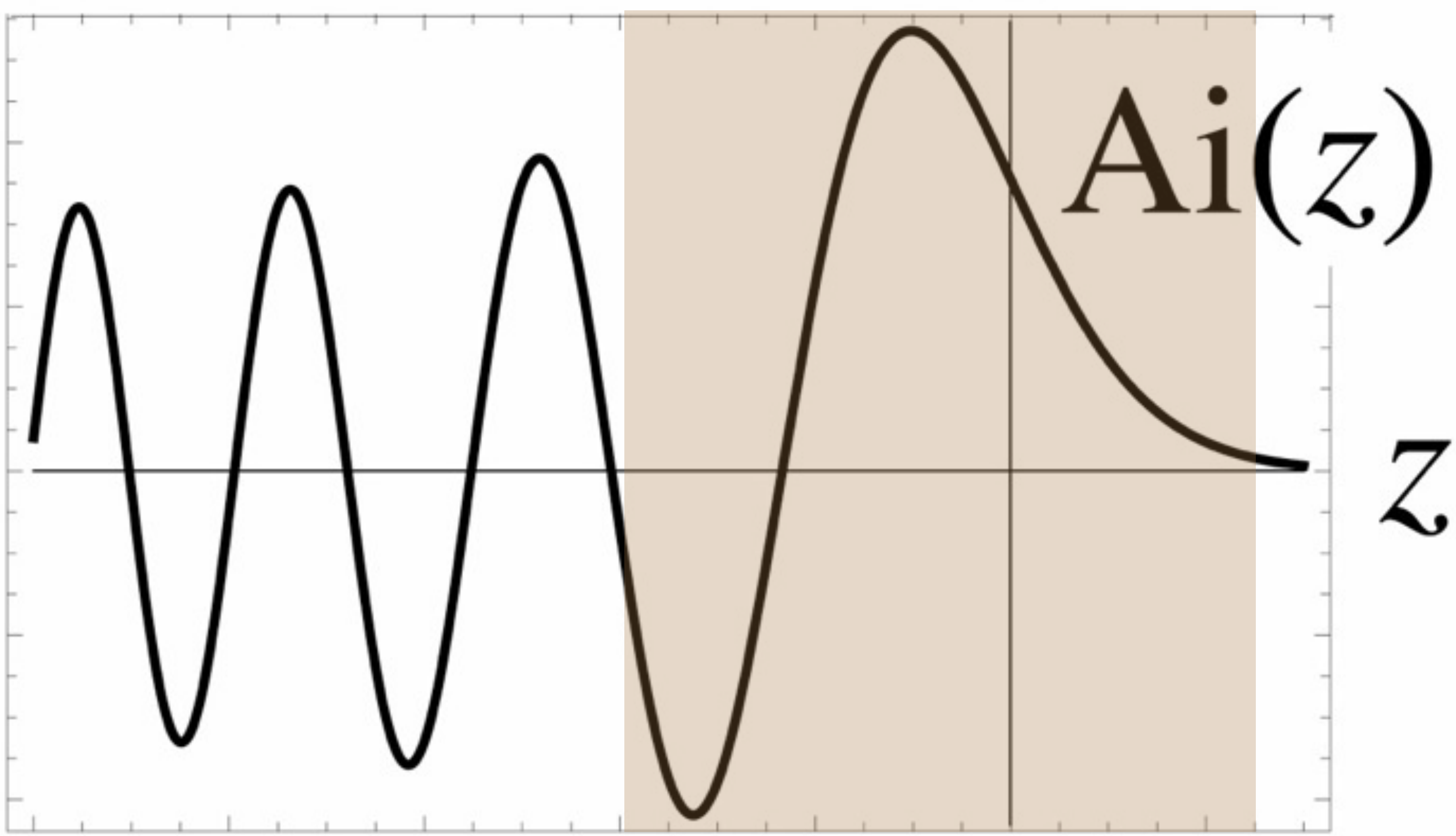

numerical experiment

Stokes's approximations for large $|z|$, e.g. on the dark side

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factorial divergence again

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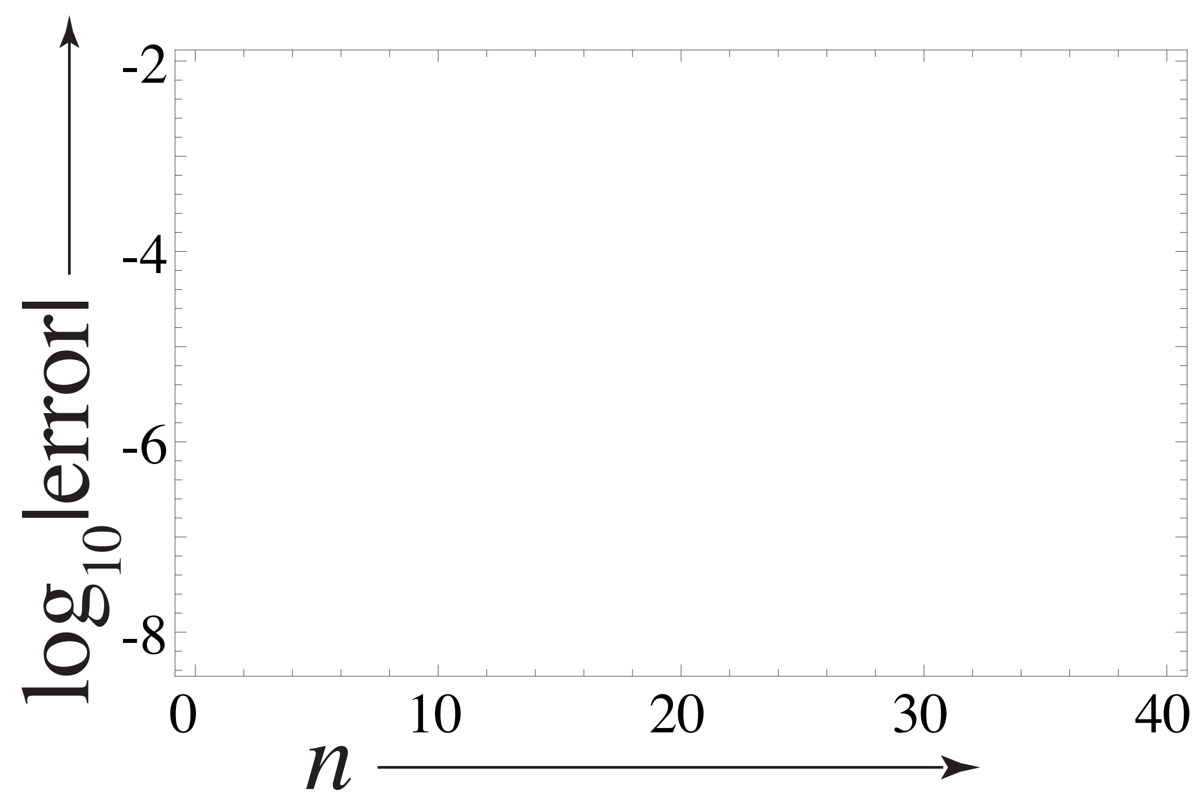

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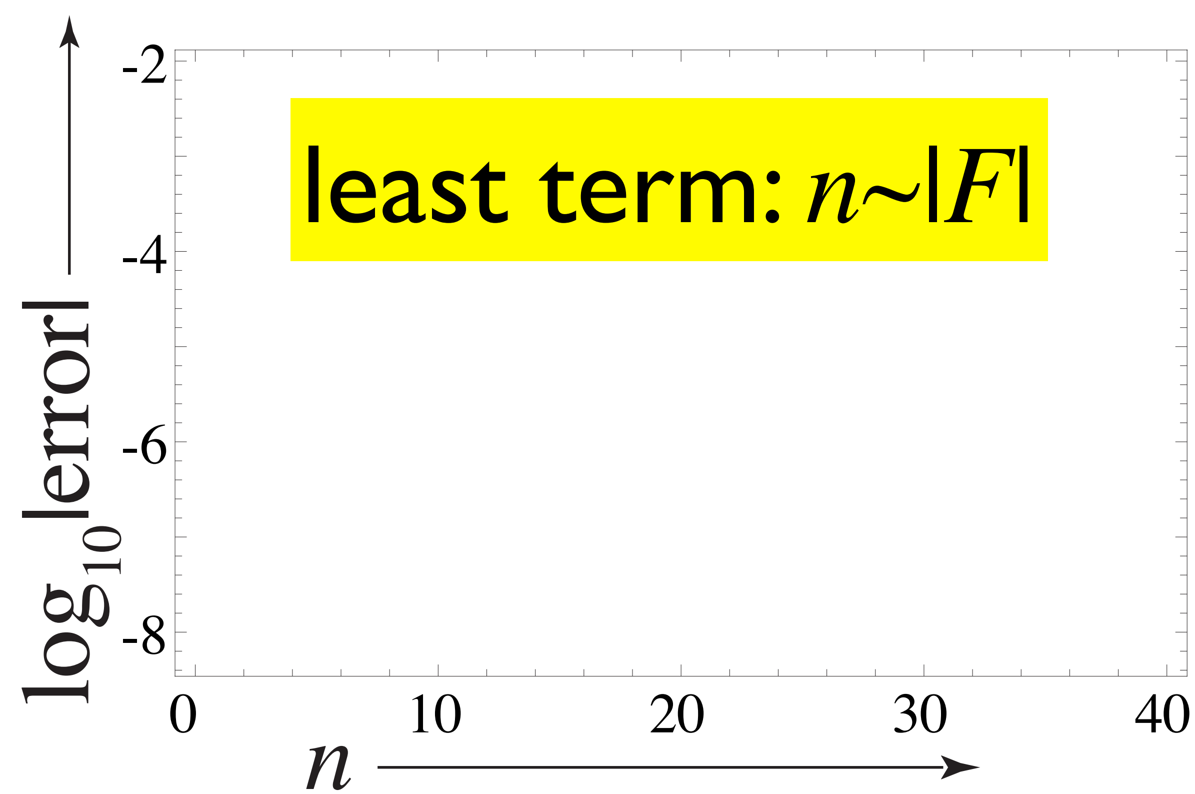

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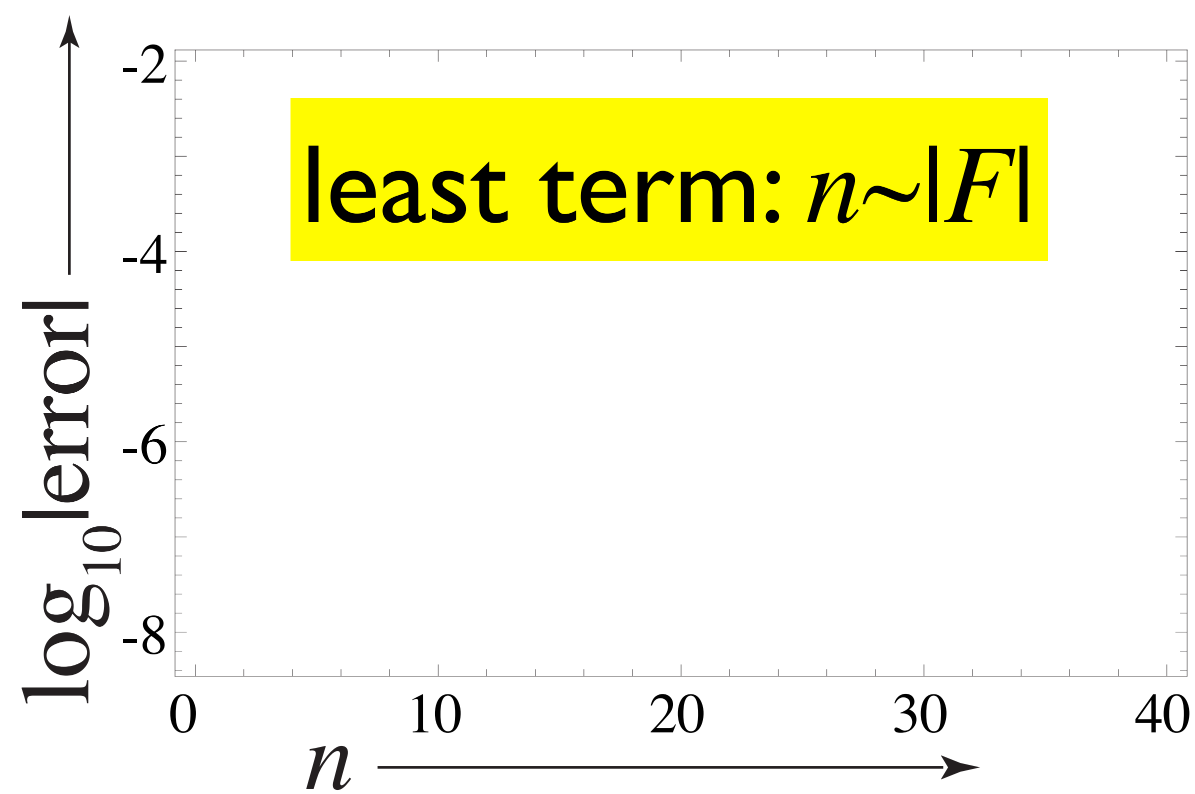

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factorial divergence again

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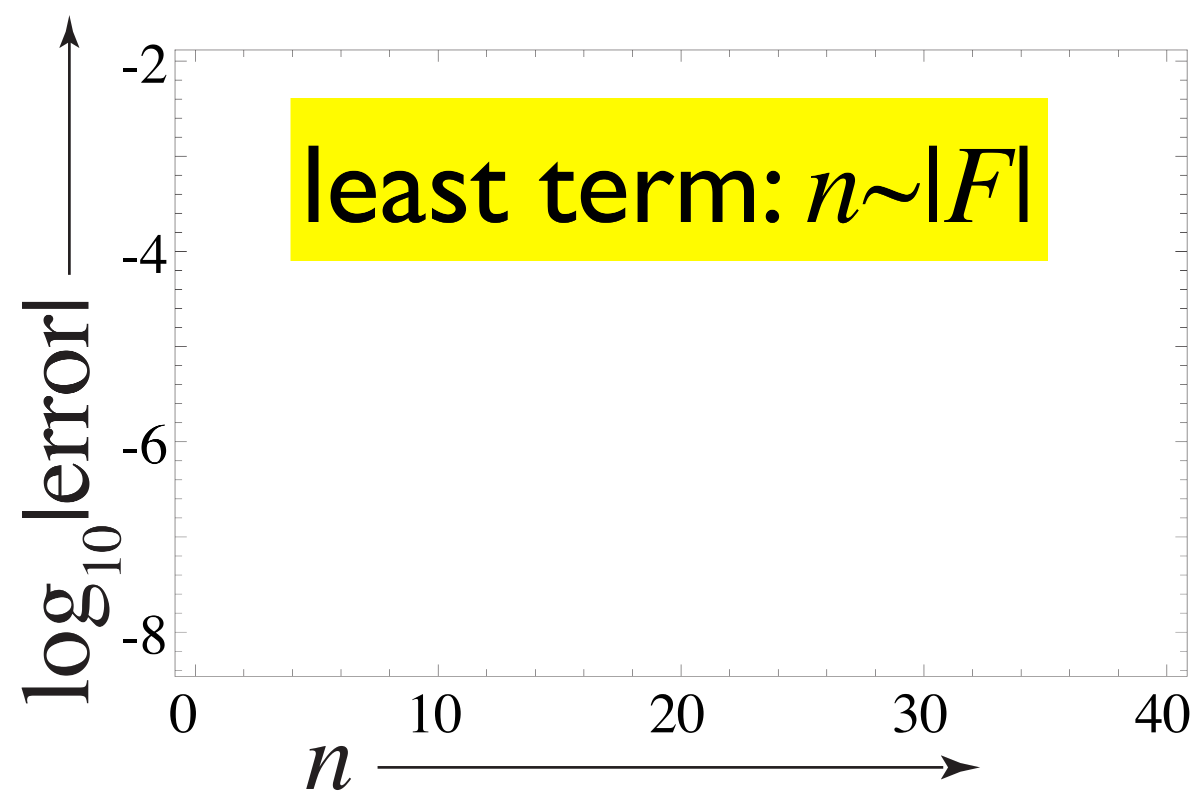
for z<<-1, pre-invented stationary phase



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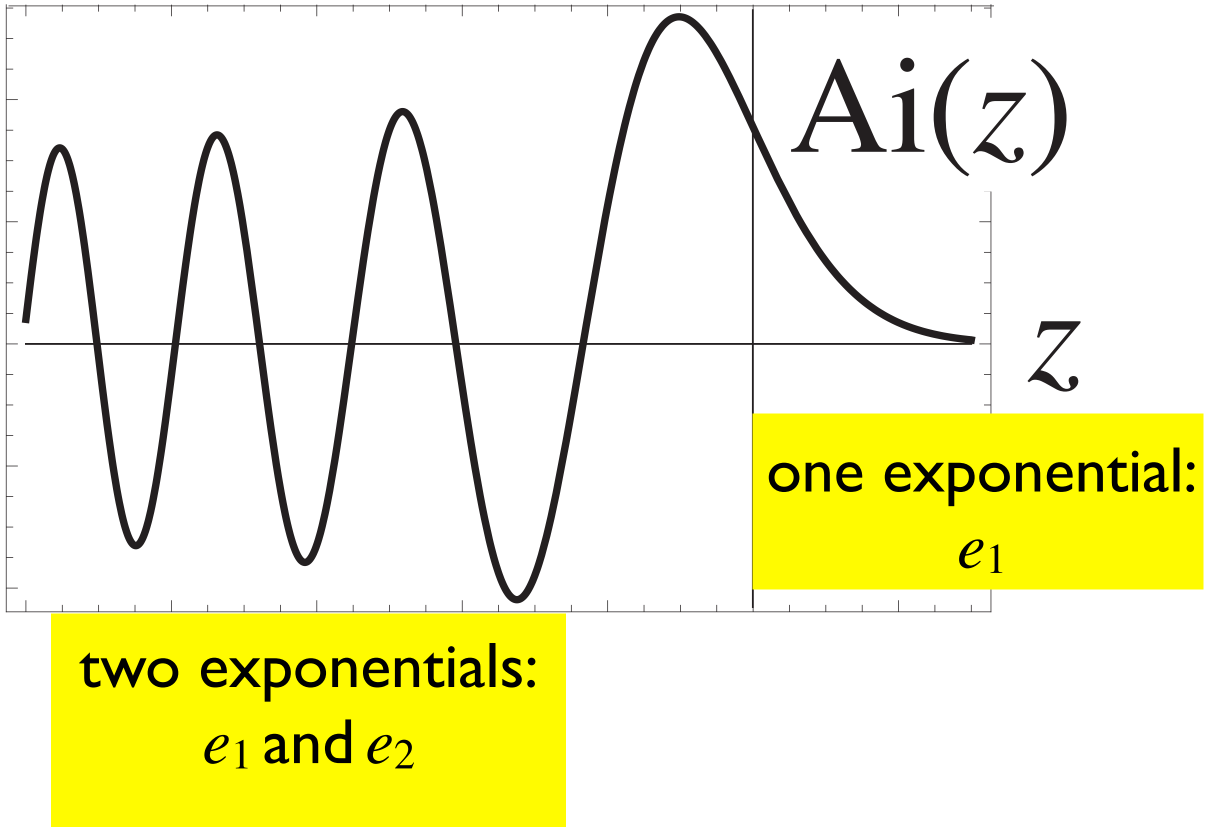

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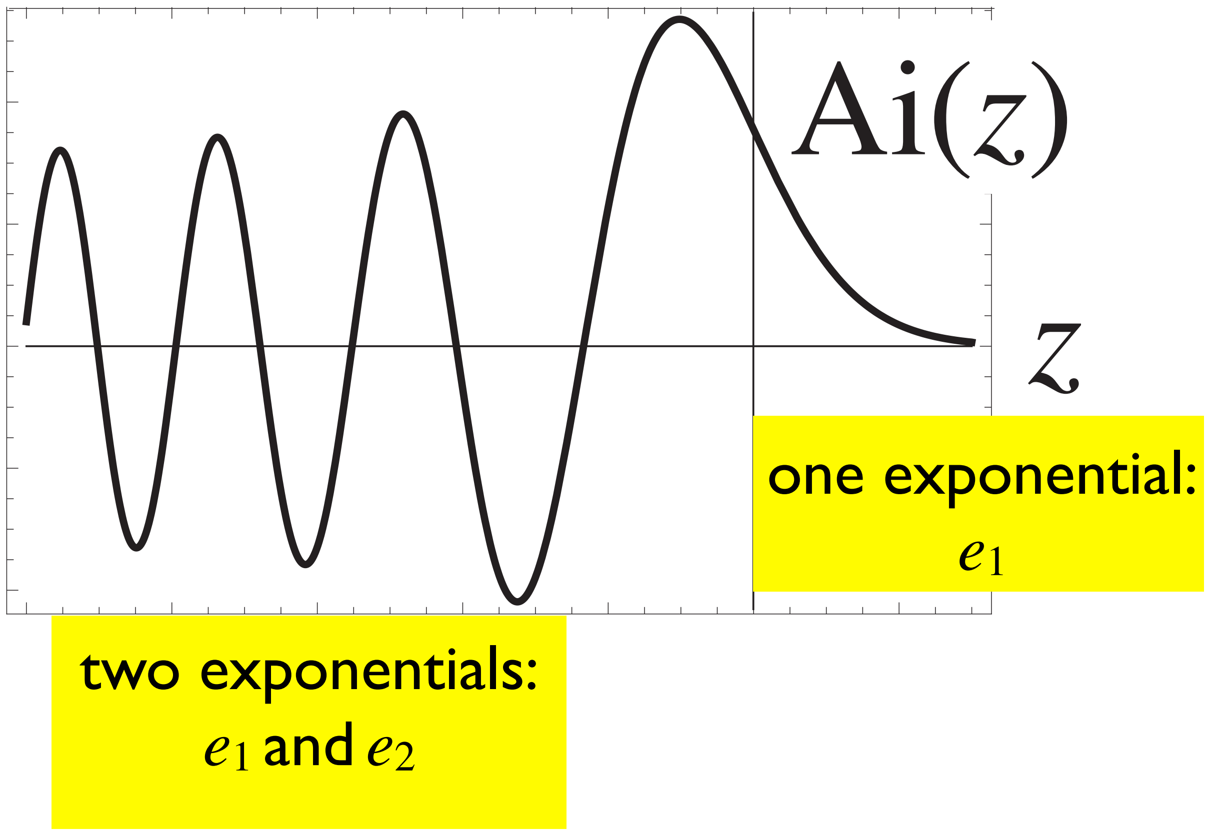

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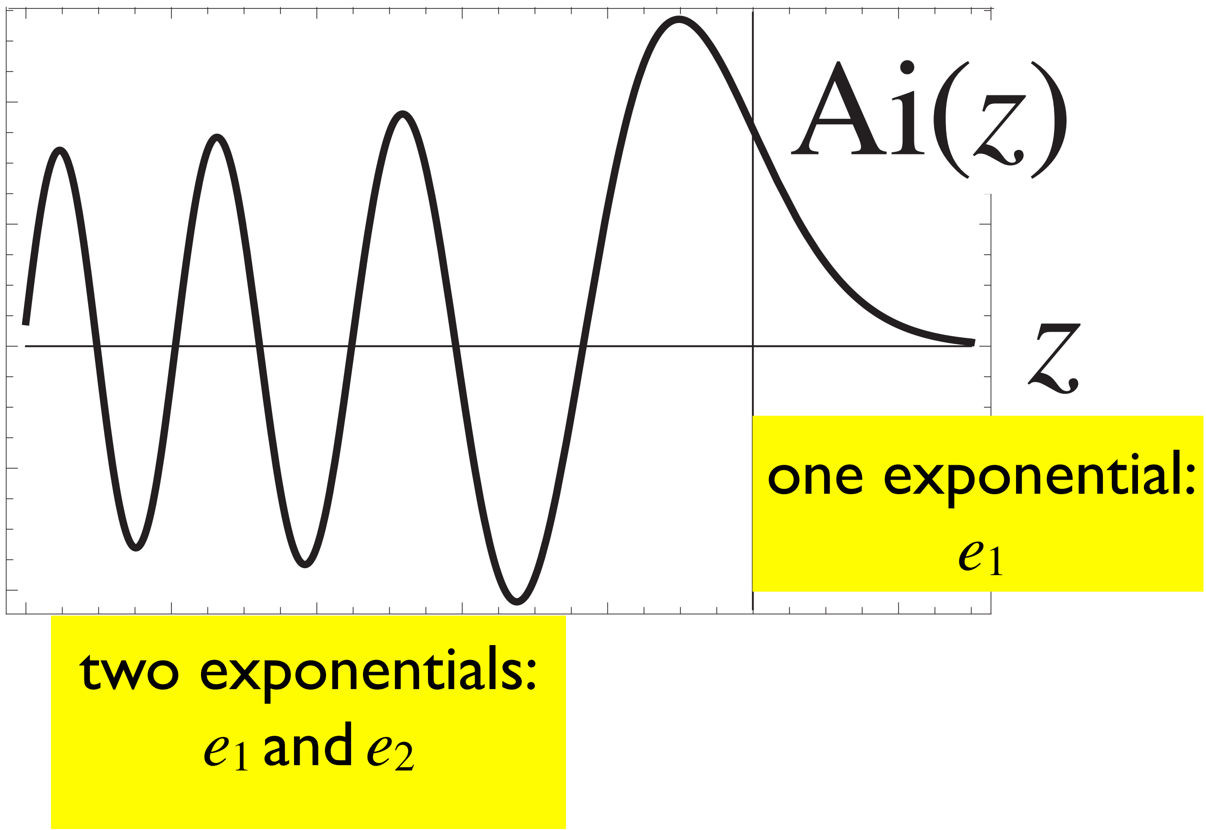

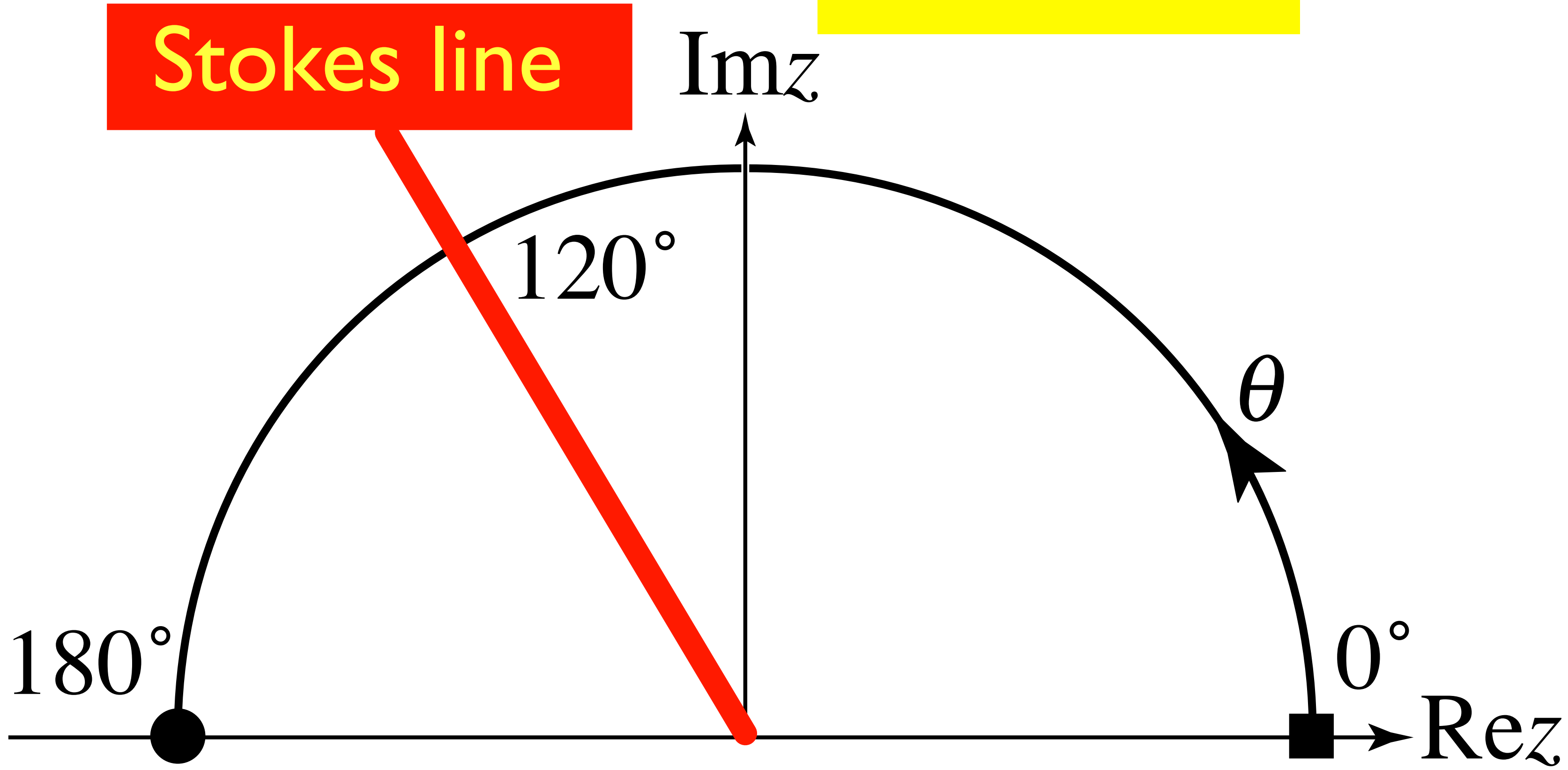

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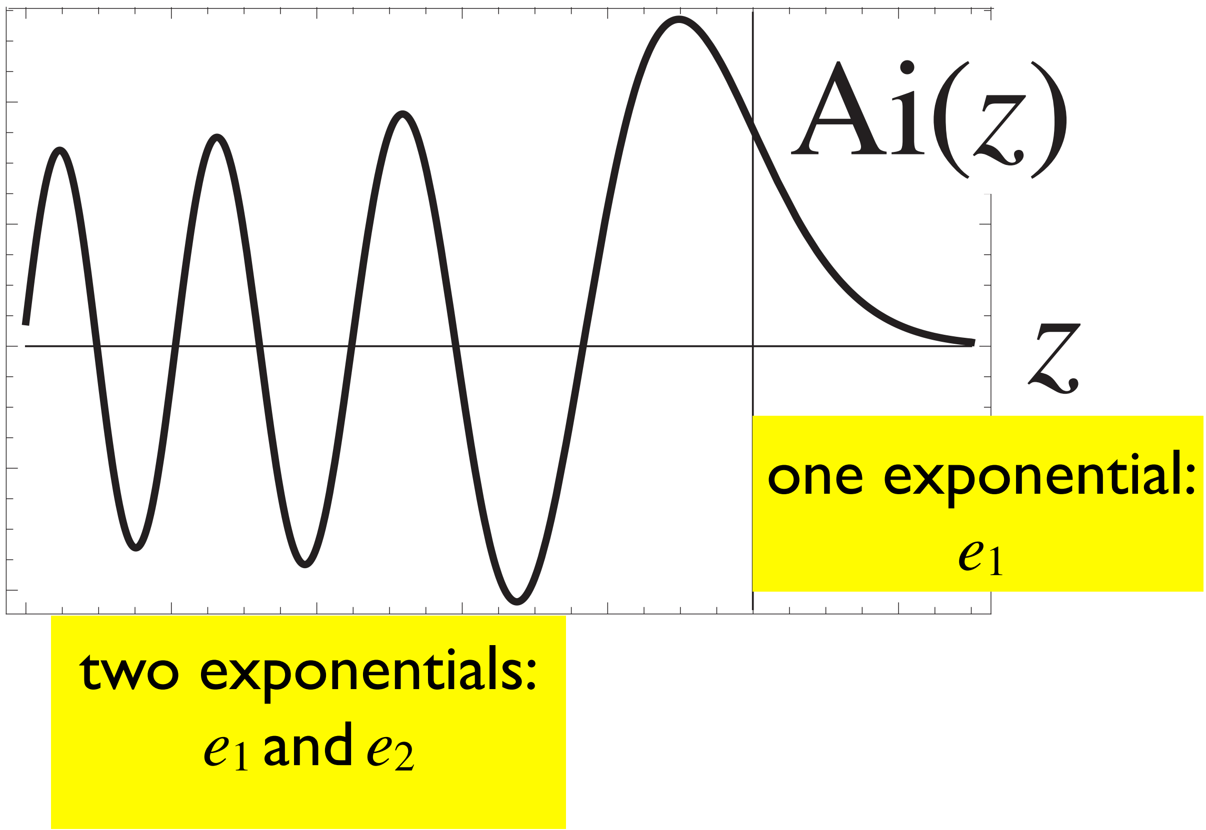

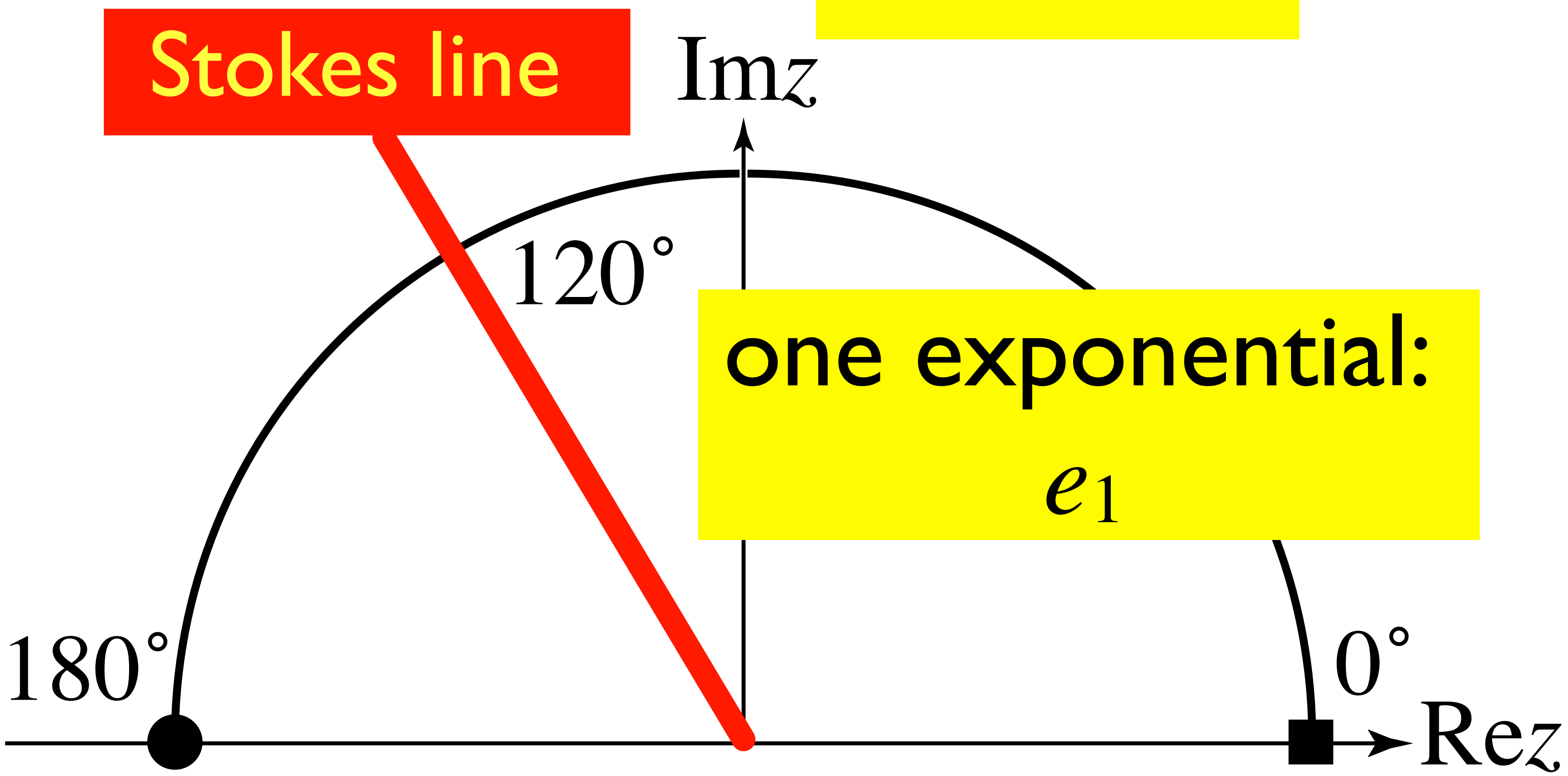

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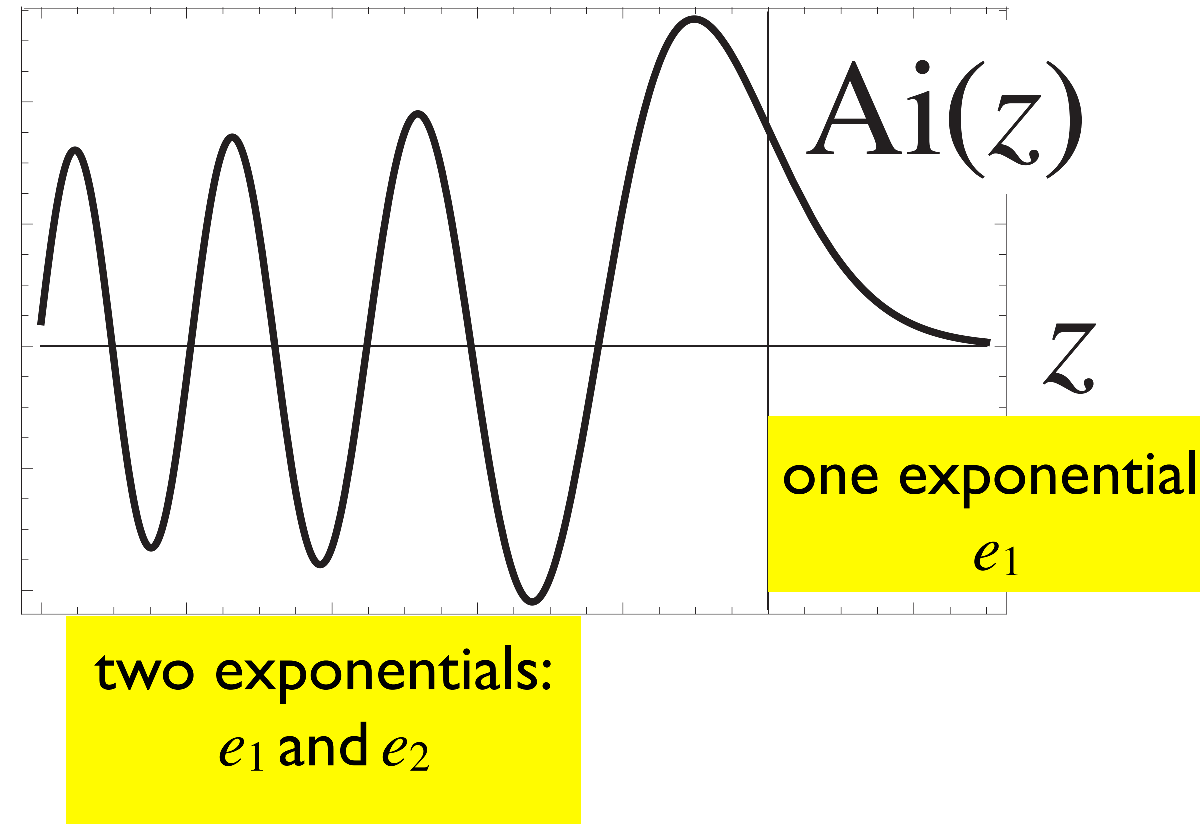

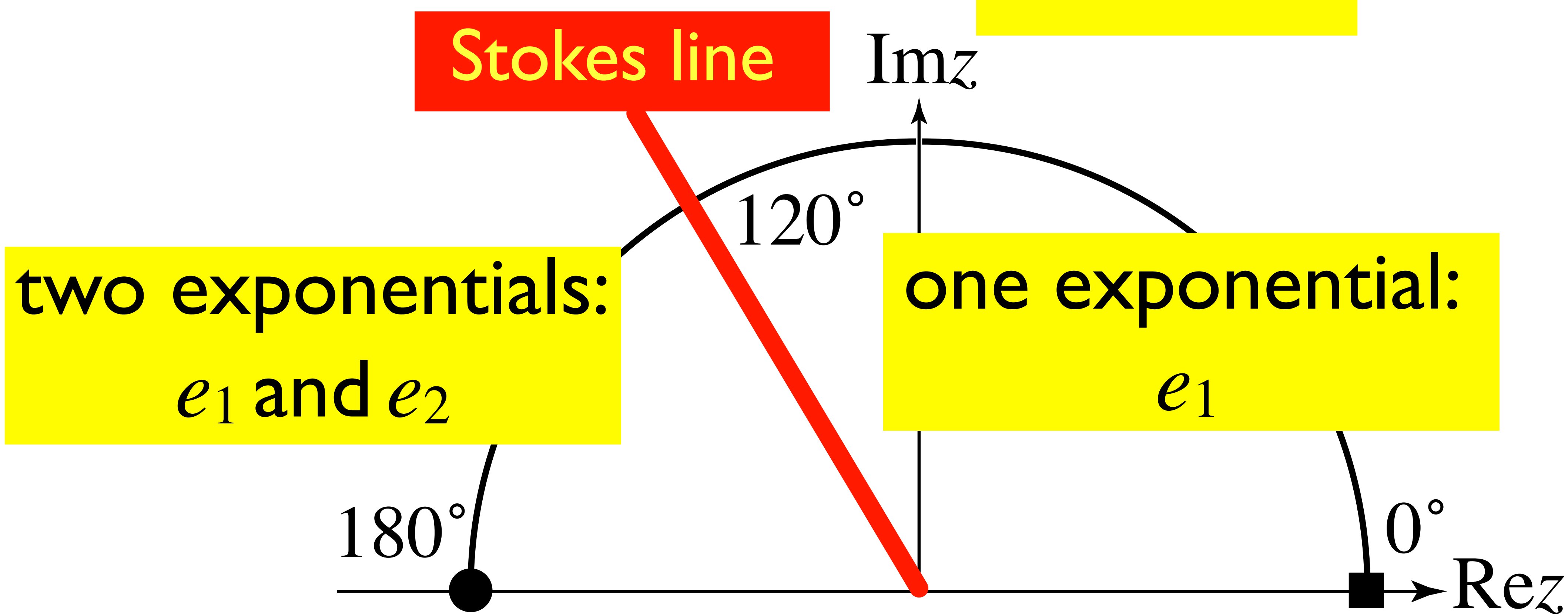

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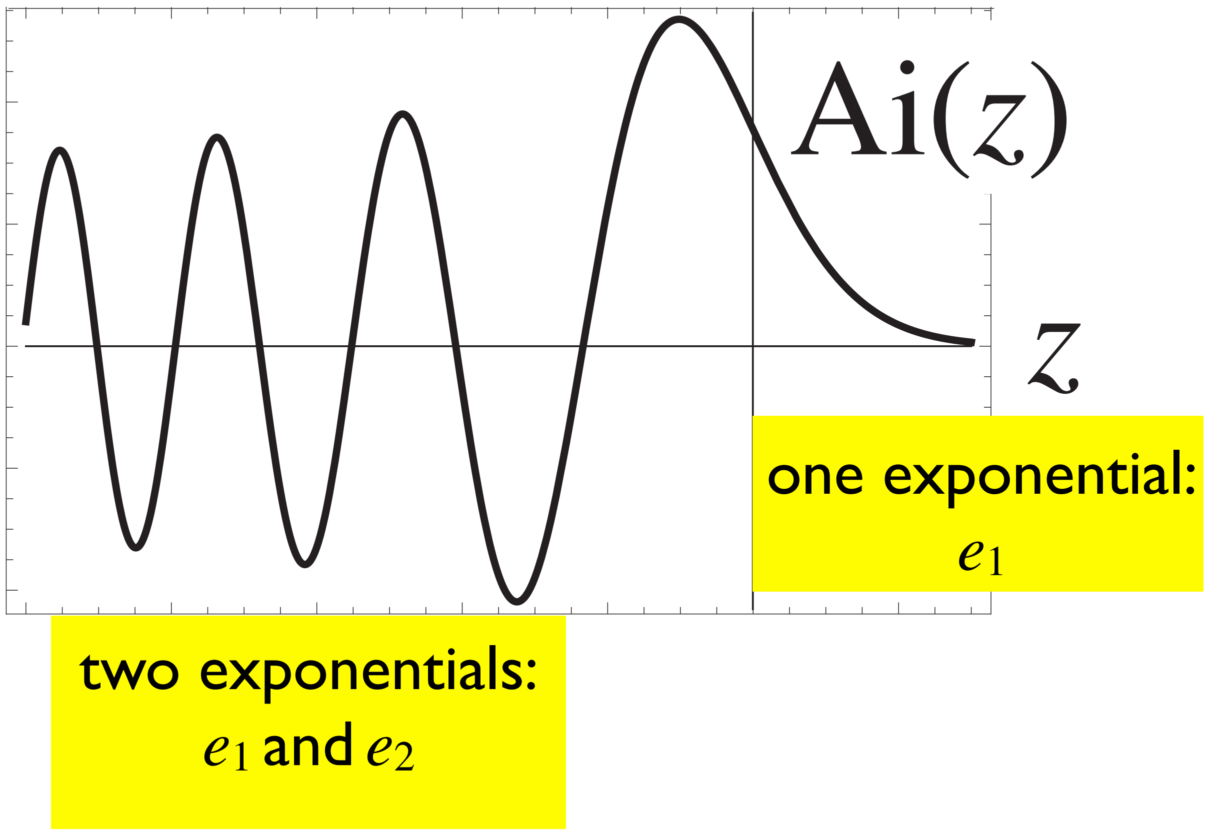

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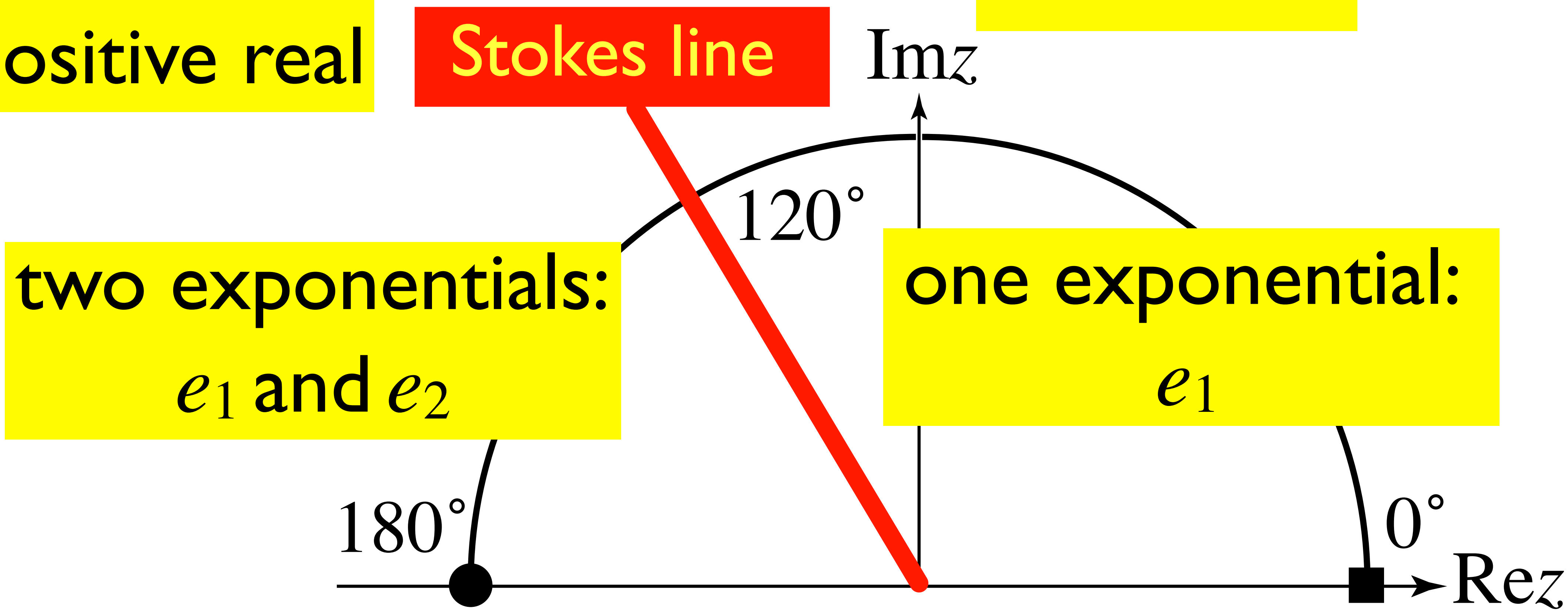

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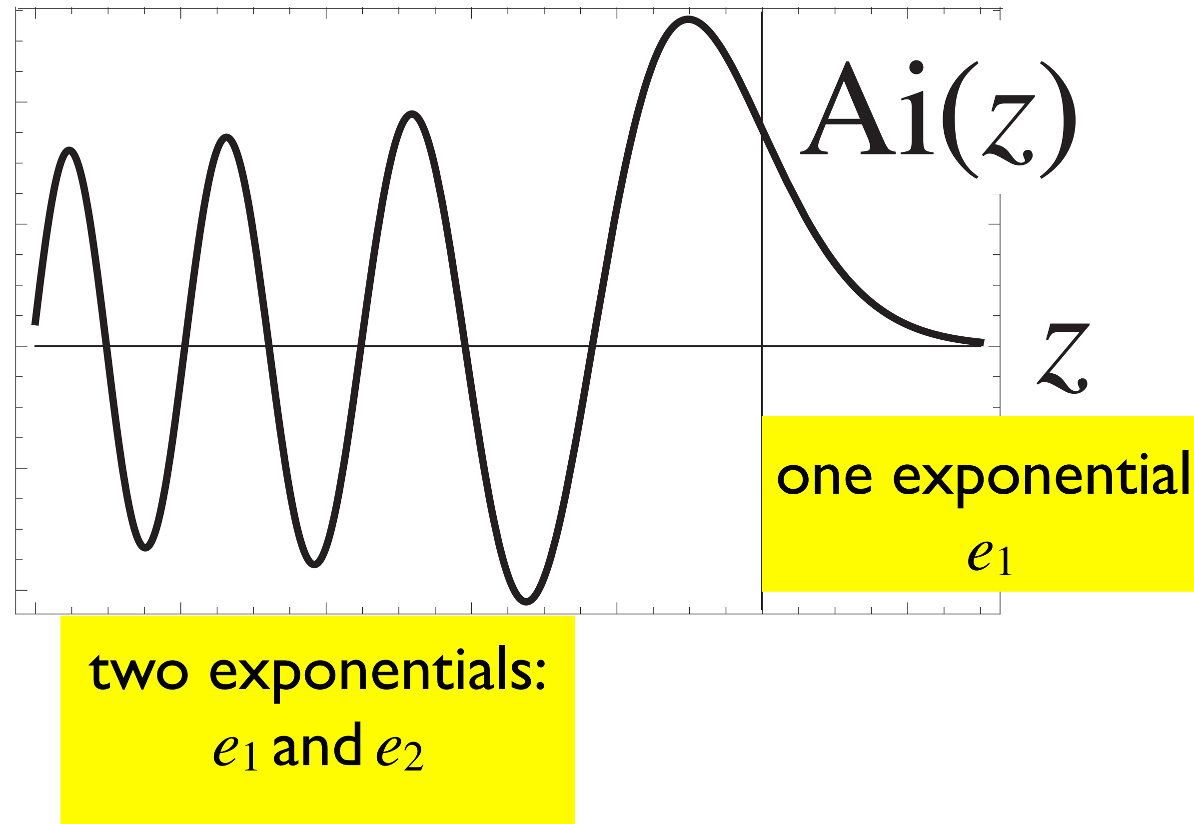

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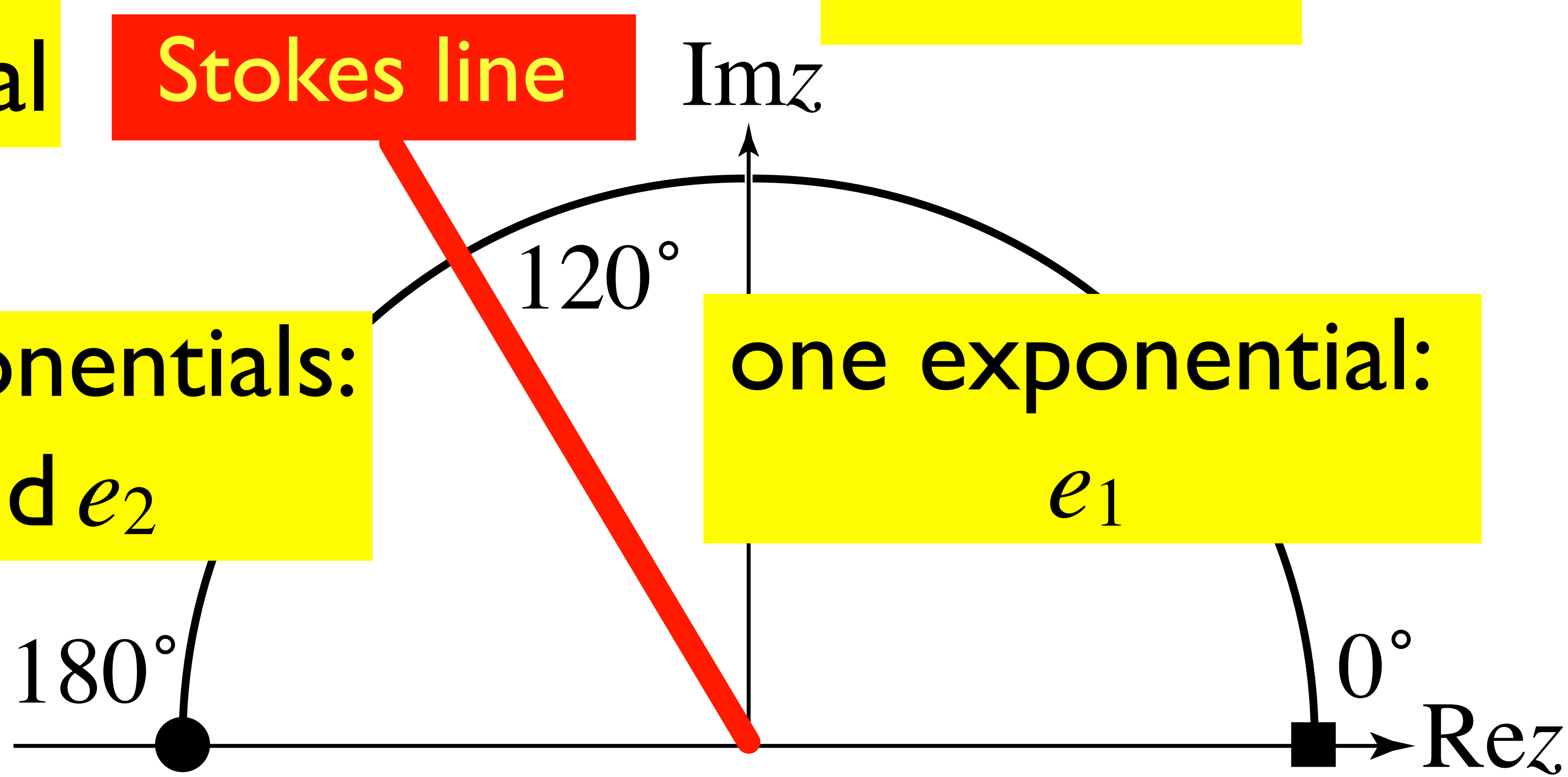

e_2 is born where it is maximally dominated by e_1

the quietly beating heart of asymptotics:
Stokes's phenomenon: the sudden appearance of a small exponential while hidden behind a large one, going from dark to bright 'around the rainbow' without passing $z=0$

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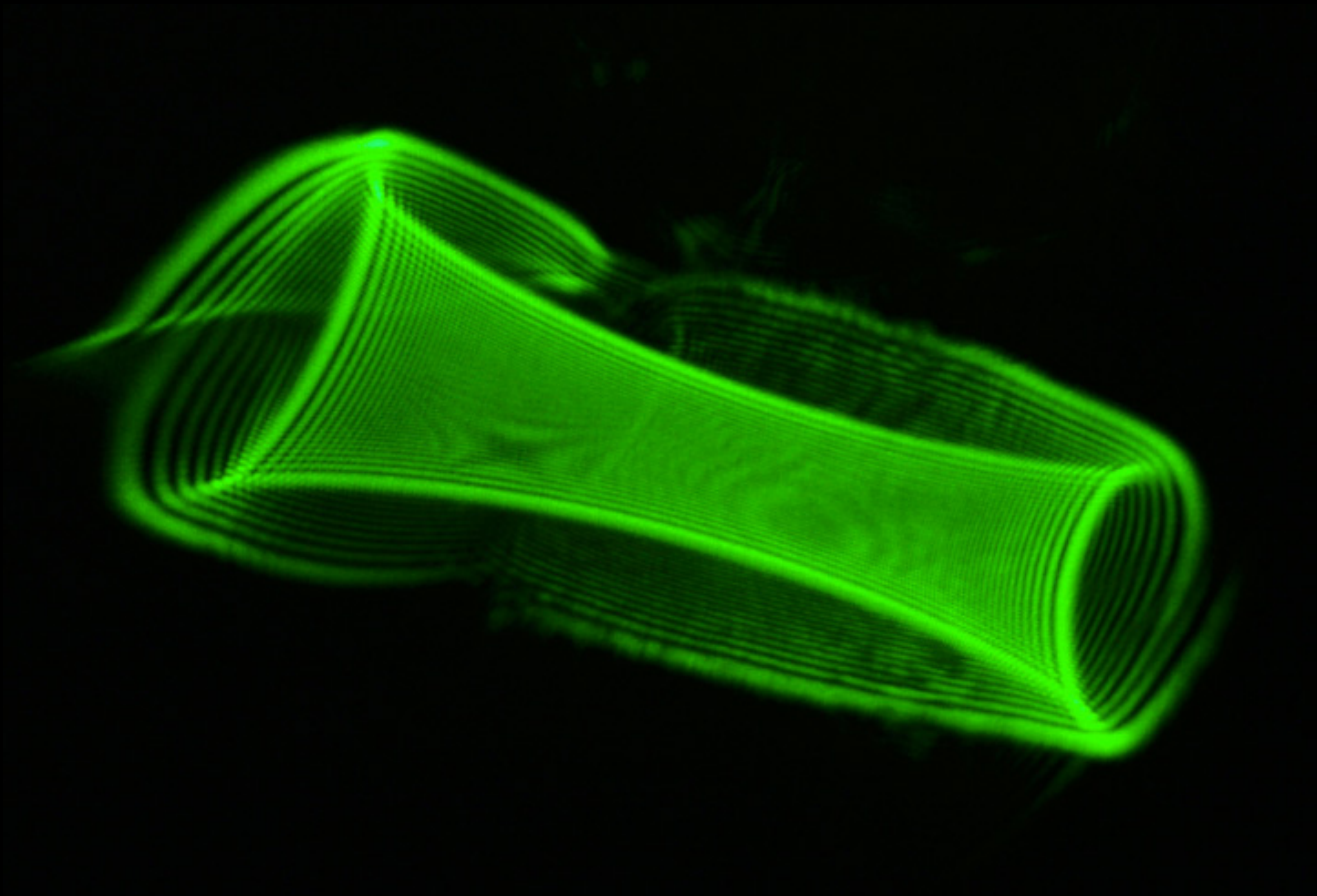
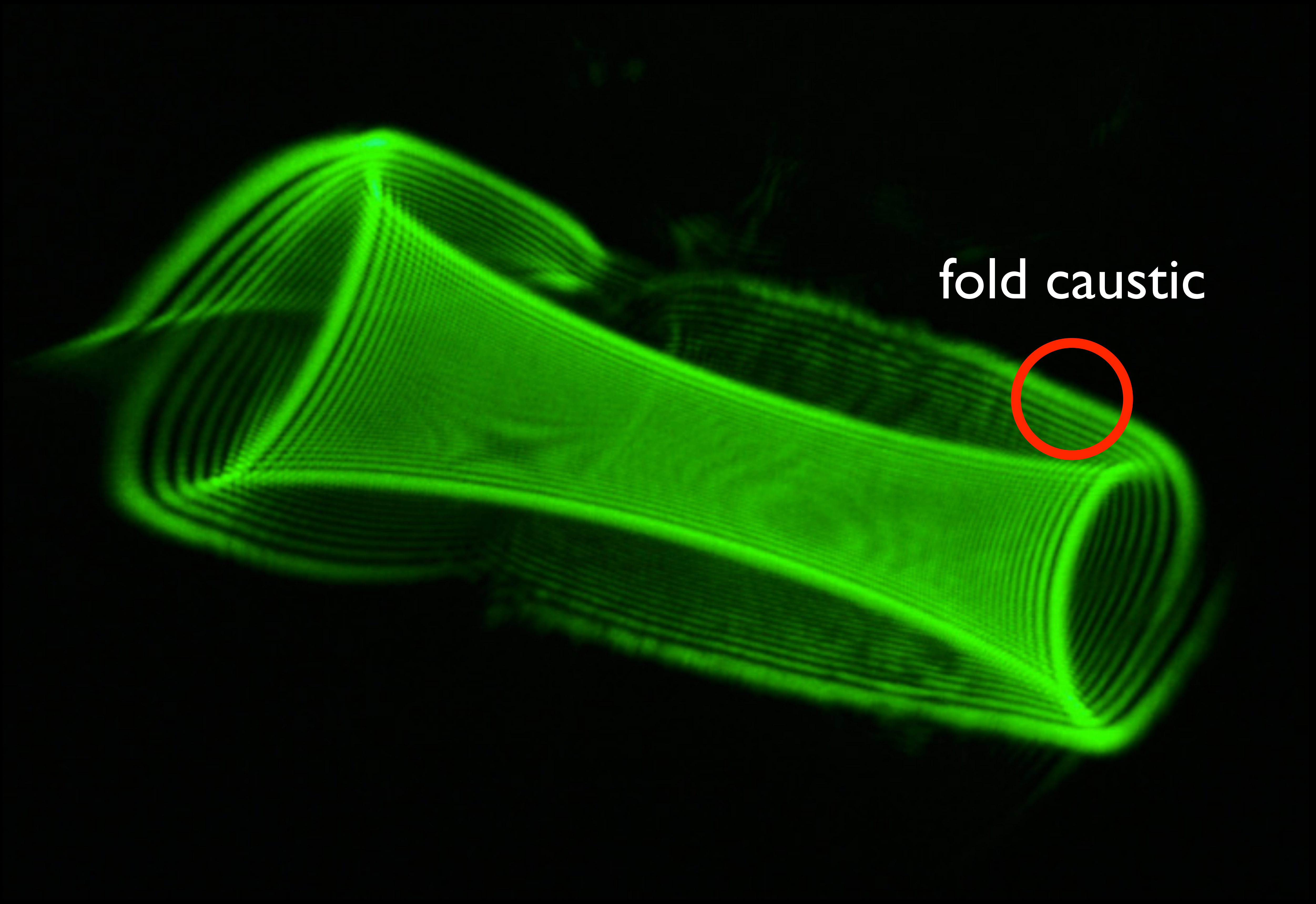
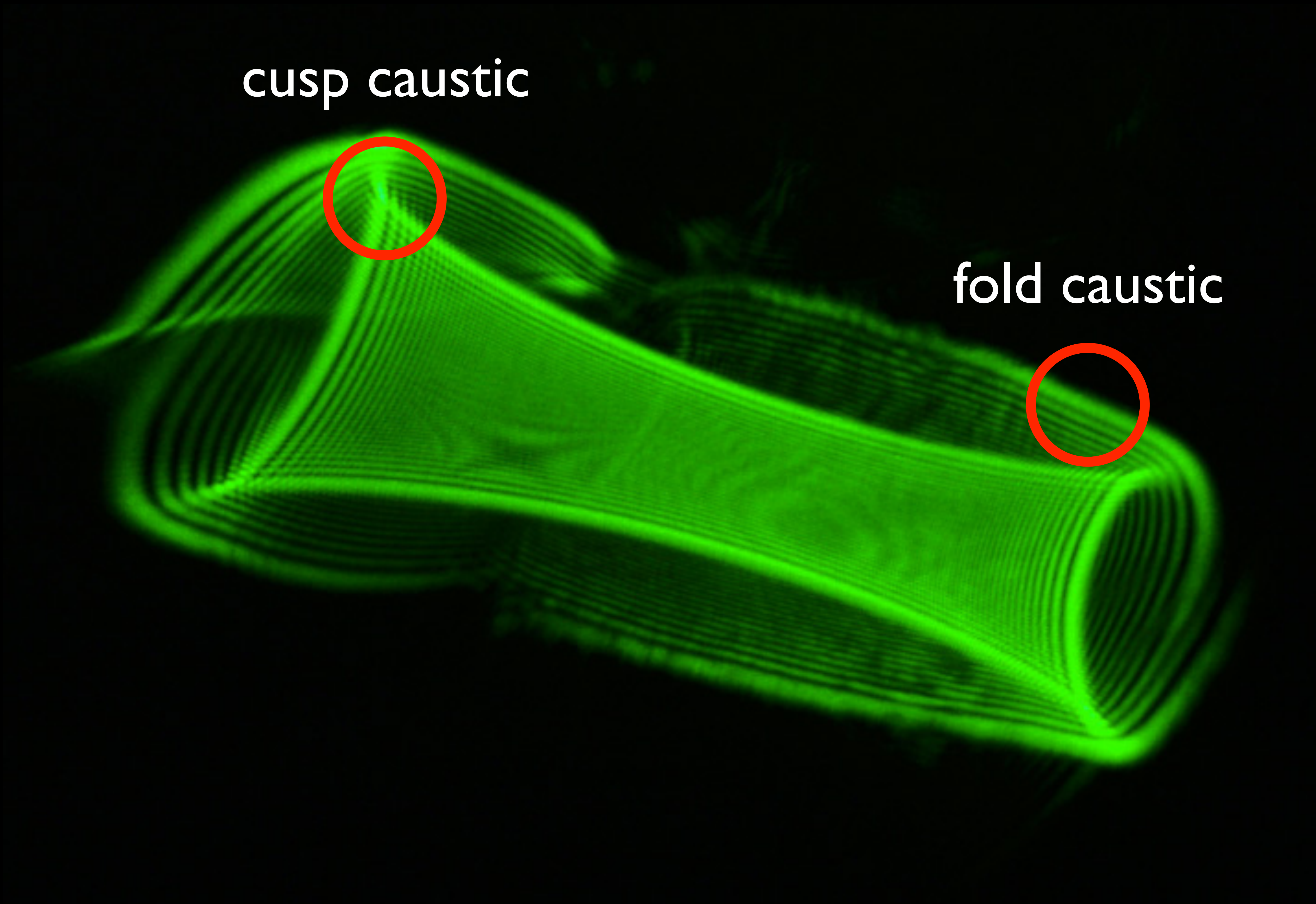
Stokes phenomenon occurs throughout asymptotics - in integrals, differential equations, integral equations, difference equations, series, near more general types of caustics...




fold caustic


cusp caustic
fold caustic

wave pattern decorating a cusp caustic: Pearcey's integral

$$\Psi_{\text{cusp}}(x,y)=\int_{-\infty}^{\infty}\mathrm{d}t\,\exp\left\{\mathrm{i}\left(\tfrac{1}{4}t^4+xt^2+yt\right)\right\}$$

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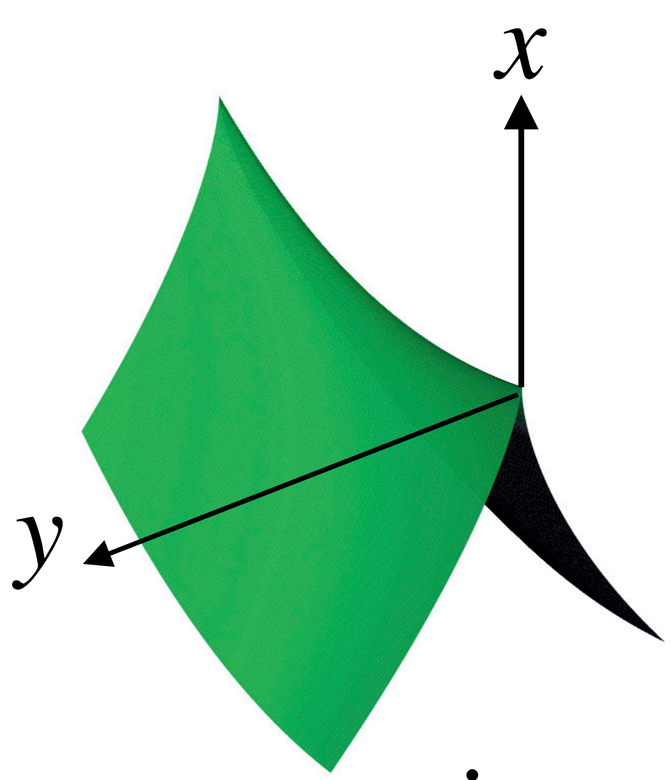

caustic
(cusp catastrophe)

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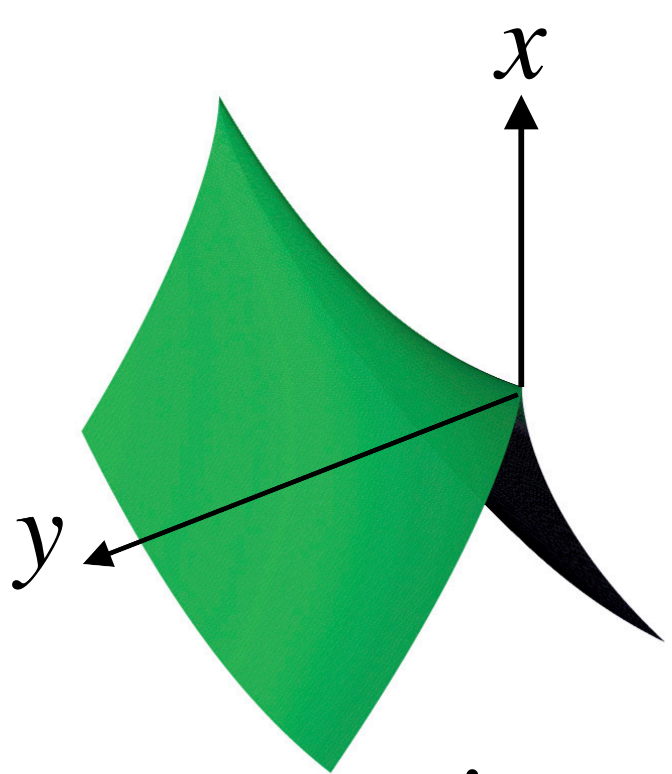

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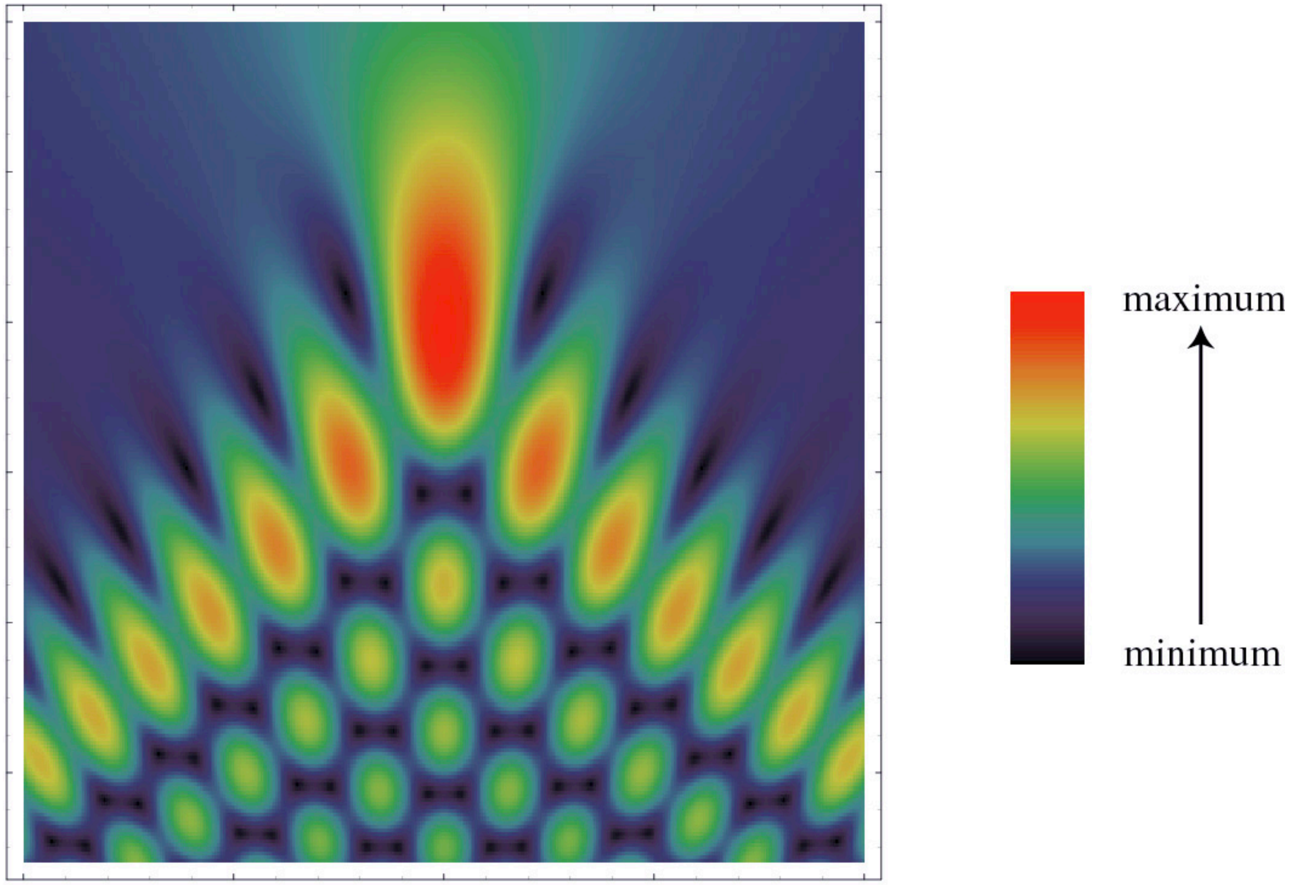
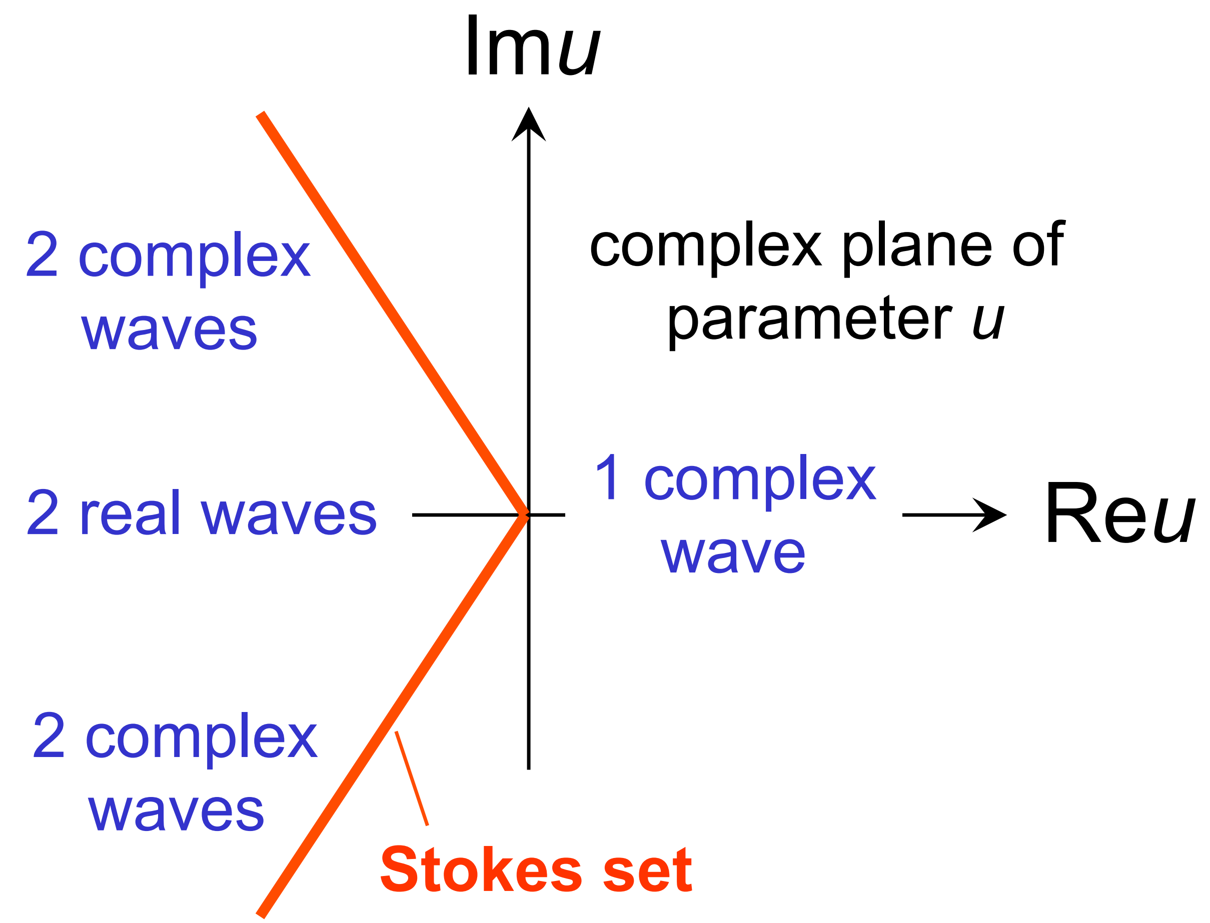


wave intensity

for $\mathrm{Ai}(u)$, Stokes set occurs in the complex plane



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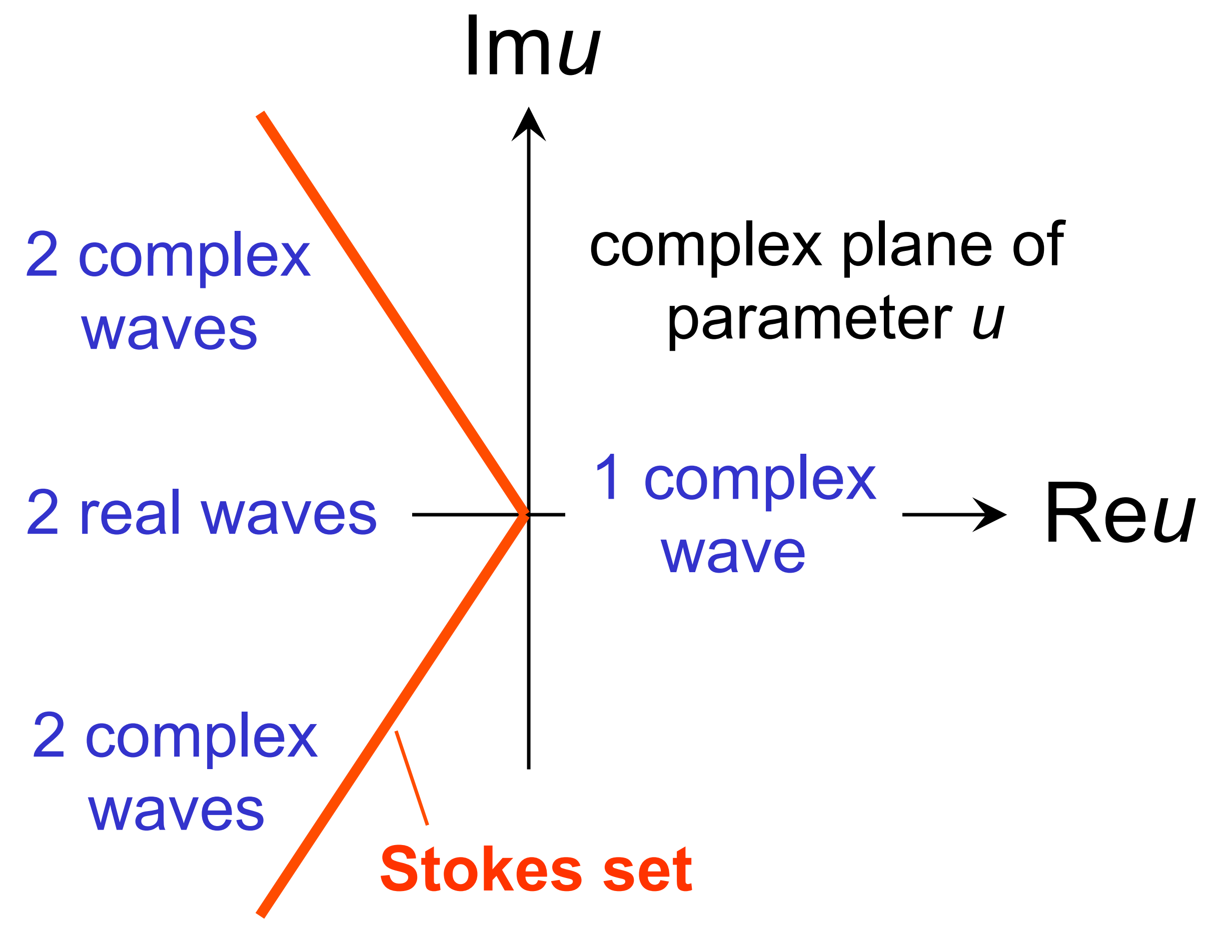

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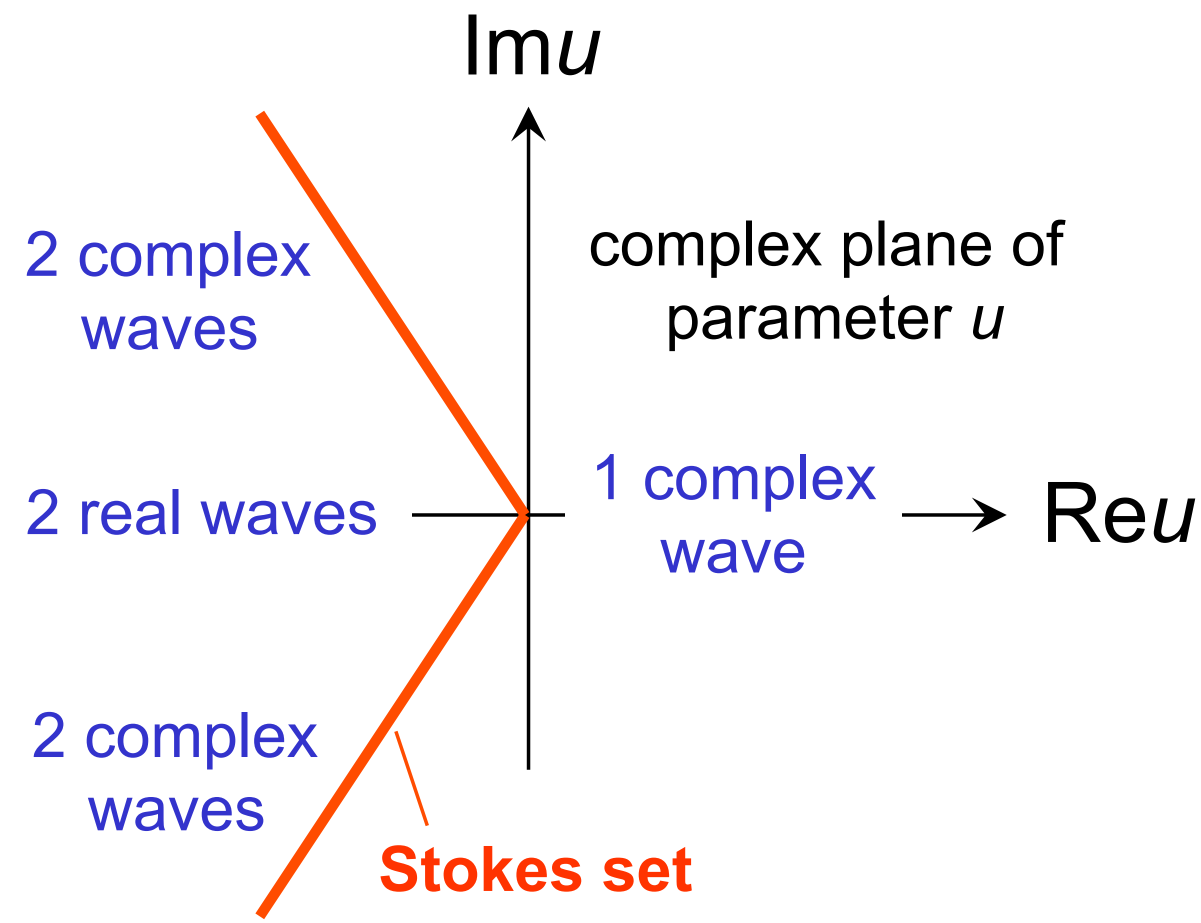
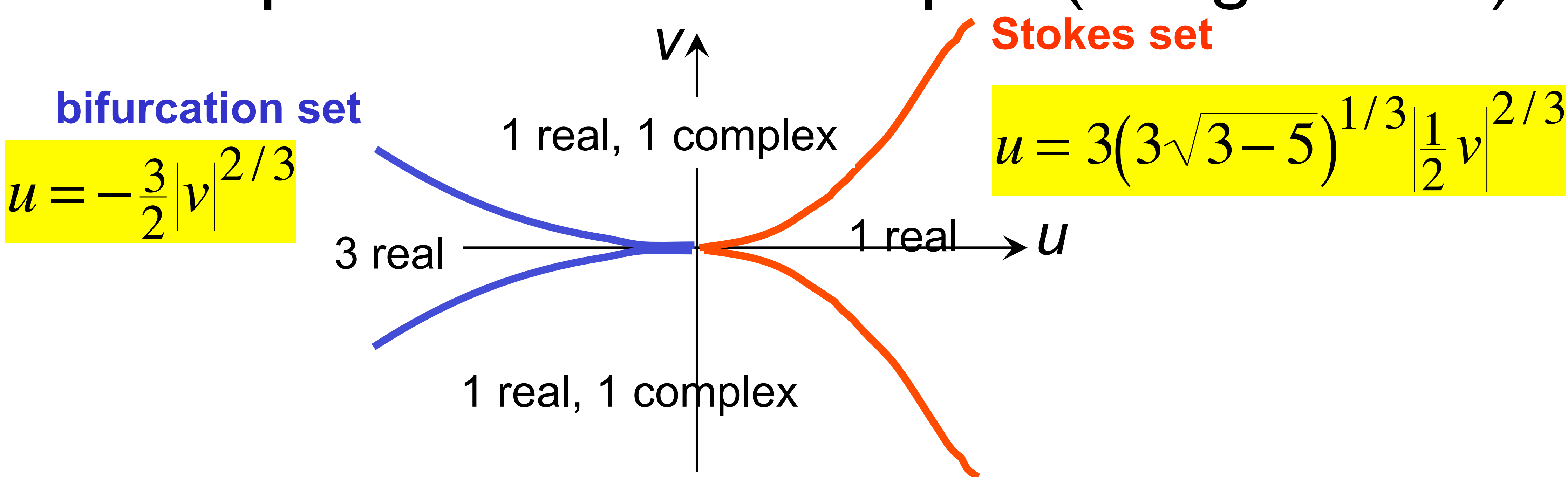


cusp diffraction catastrophe (Wright 1980)



two contrasting general phenomena, with exponents $\mathrm{i}\Phi(t)$

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(strictly, change of coefficient of subdominant exponential)

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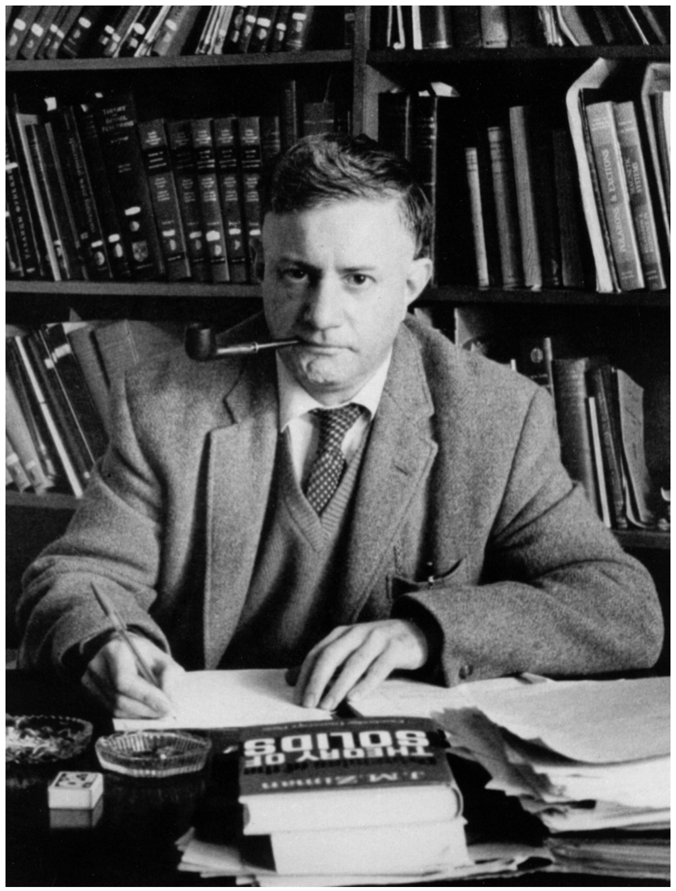

Robert Dingle

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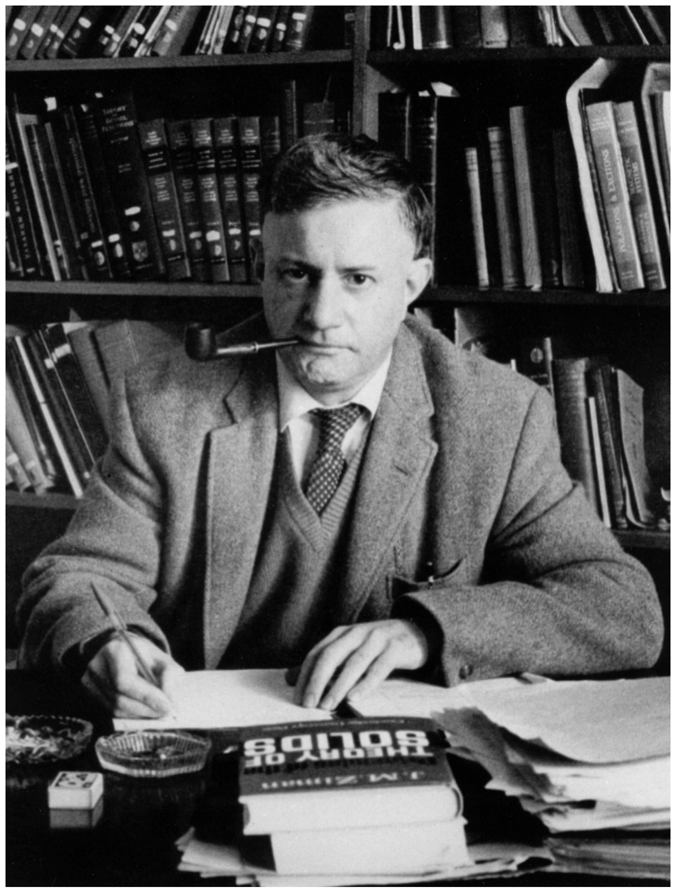

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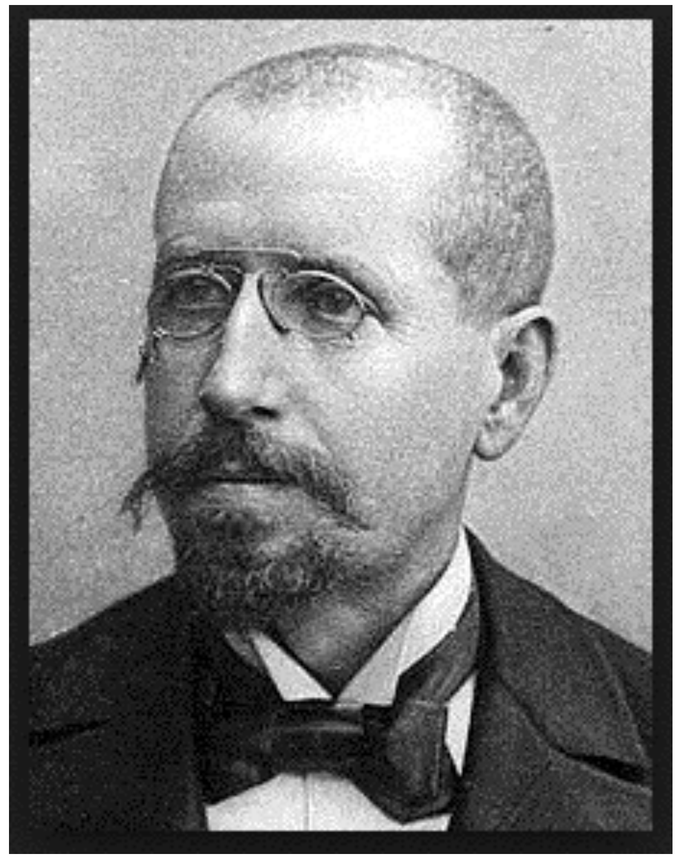

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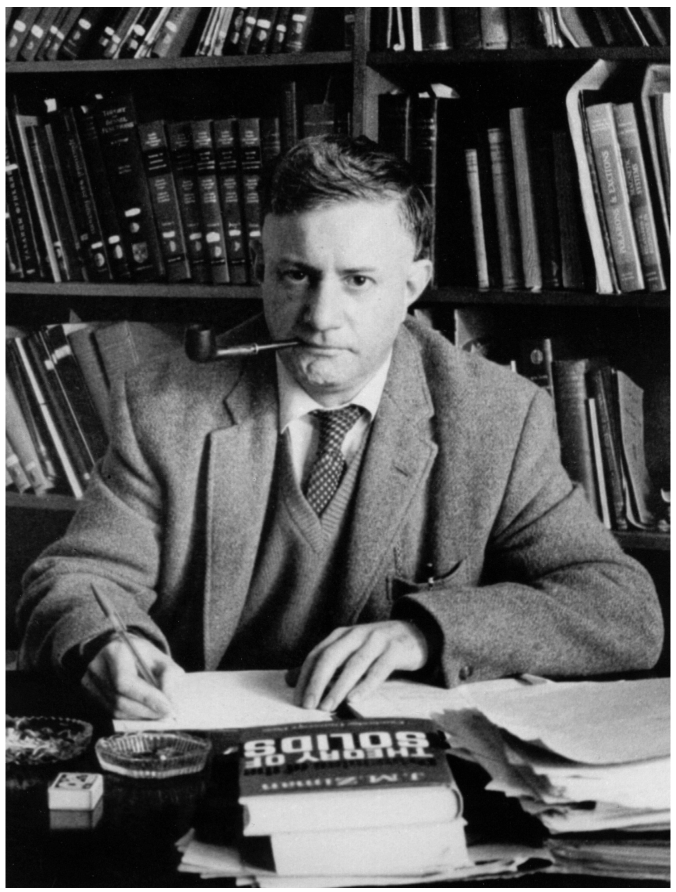

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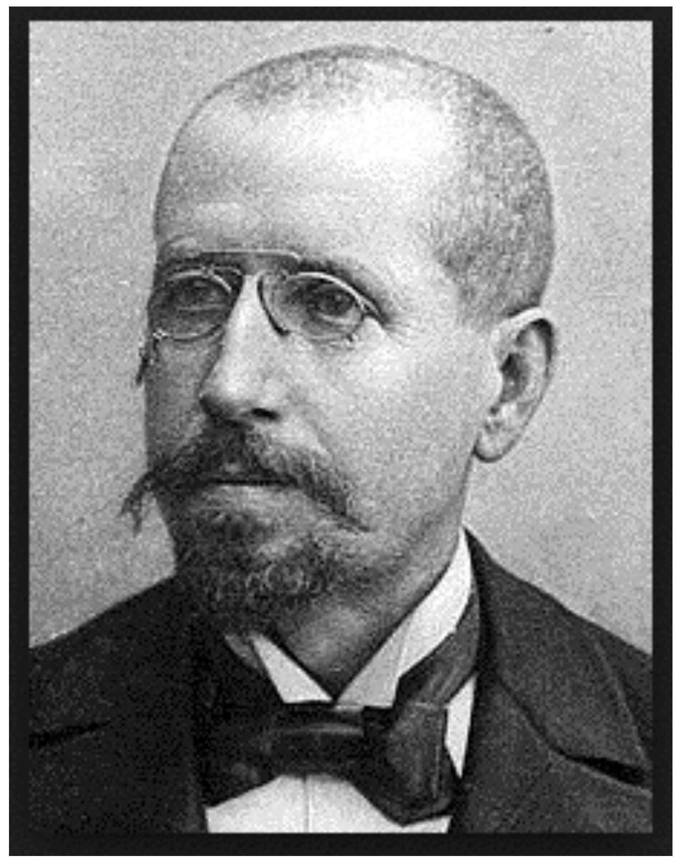

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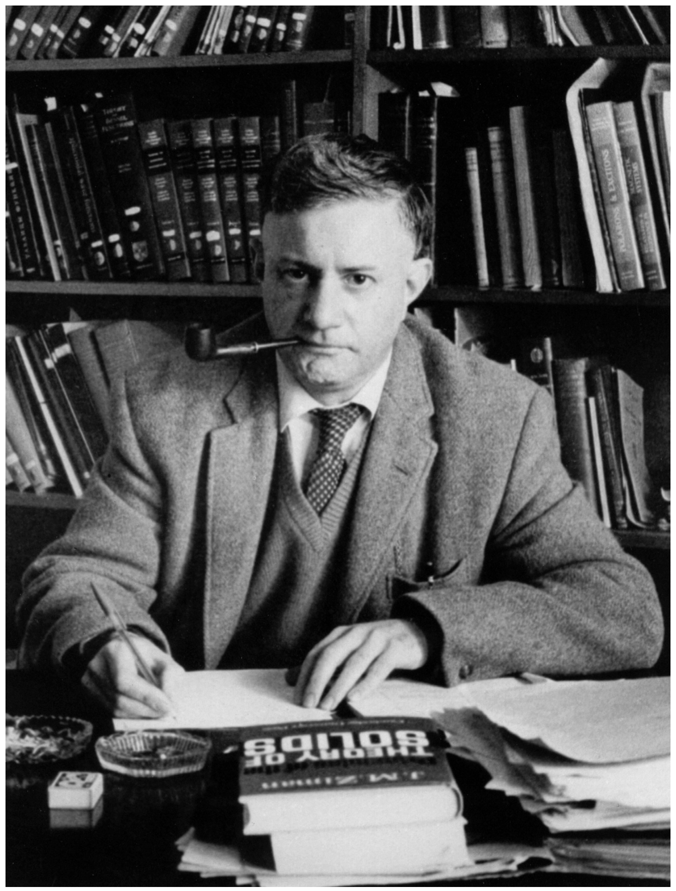

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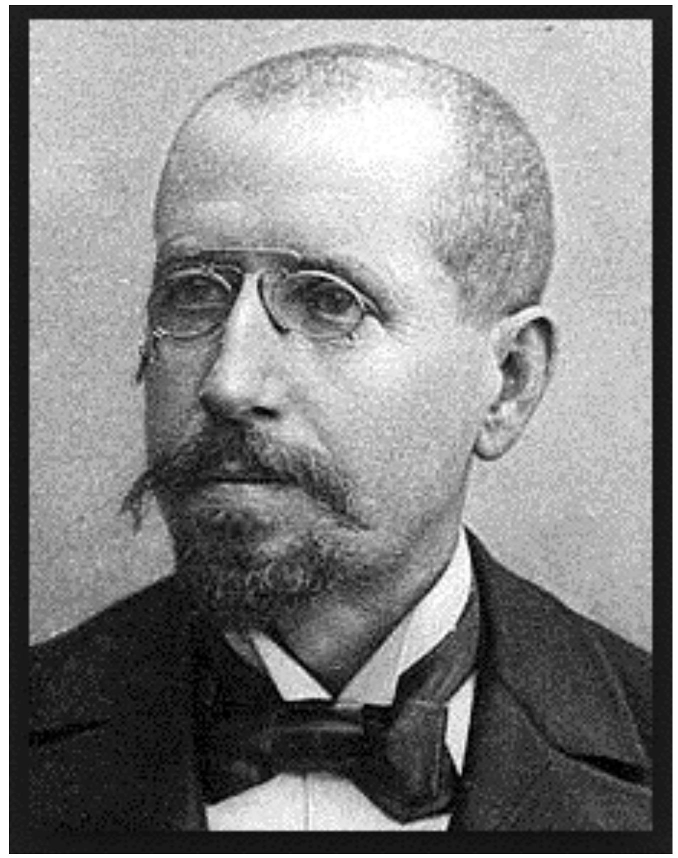

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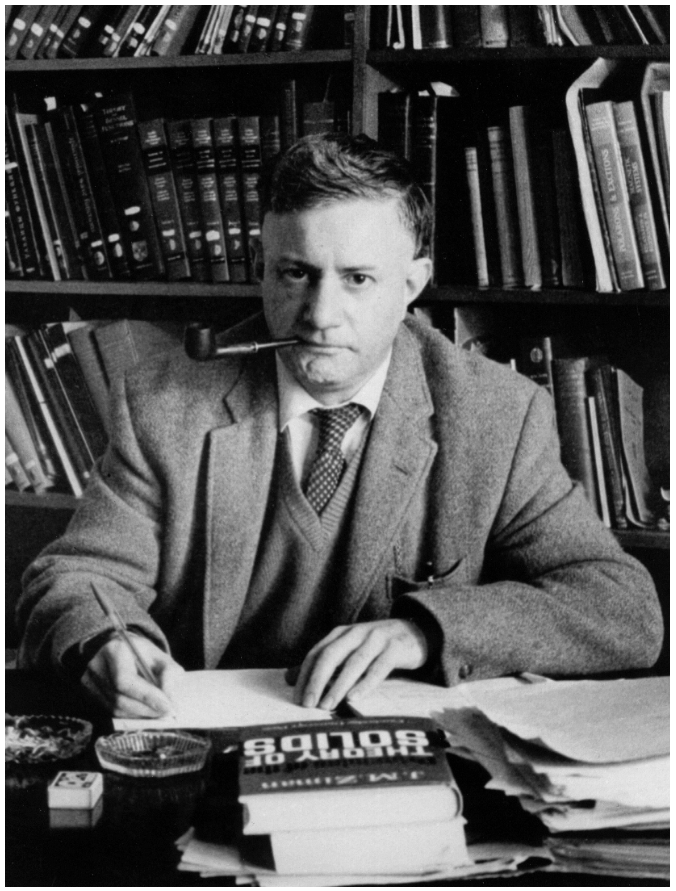

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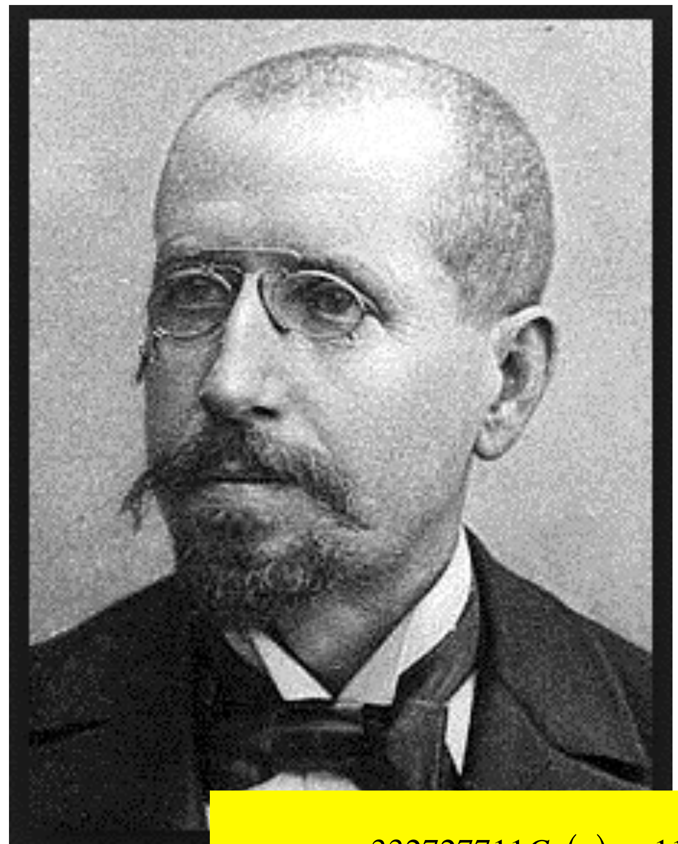

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$$\frac{243110370304853041 7C_0^{(16)}(z)}{44931349155019751424000\pi^{18}} + \frac{2325442687388622149 41C_0^{(20)}(z)}{3731869485532640496844800 00\pi^{20}} +$$
$$\frac{361888761444289010497 C_0^{(24)}(z)}{10648999337834611205996544 0000\pi^{22}} + \frac{665406310453227159231 77C_0^{(28)}(z)}{684304697449252116097337917 4400000\pi^{24}} +$$
$$\frac{391261681973226653 C_0^{(32)}(z)}{25057539453517190072893440000000\pi^{26}} + \frac{1259995823308801 C_0^{(36)}(z)}{857171937866922882131558400000 00\pi^{28}} +$$
$$\frac{713214794639 C_0^{(40)}(z)}{85717193786692288213155840000000\pi^{30}} + \frac{50407933481 C_0^{(44)}(z)}{176508845445556759888530505728000 00\pi^{32}} +$$
$$\frac{1039499 C_0^{(48)}(z)}{176836320112431633283499950080 0000\pi^{34}} + \frac{22411 C_0^{(52)}(z)}{32139197378979392841852100018176 0000\pi^{36}} +$$
$$\frac{59 C_0^{(56)}(z)}{136362023165098281057572481505689 60000\pi^{38}} + \frac{C_0^{(60)}(z)}{932716238449272242433795773498916864 0000\pi^{40}}$$

asymptotics of the asymptotics of the asymptotics

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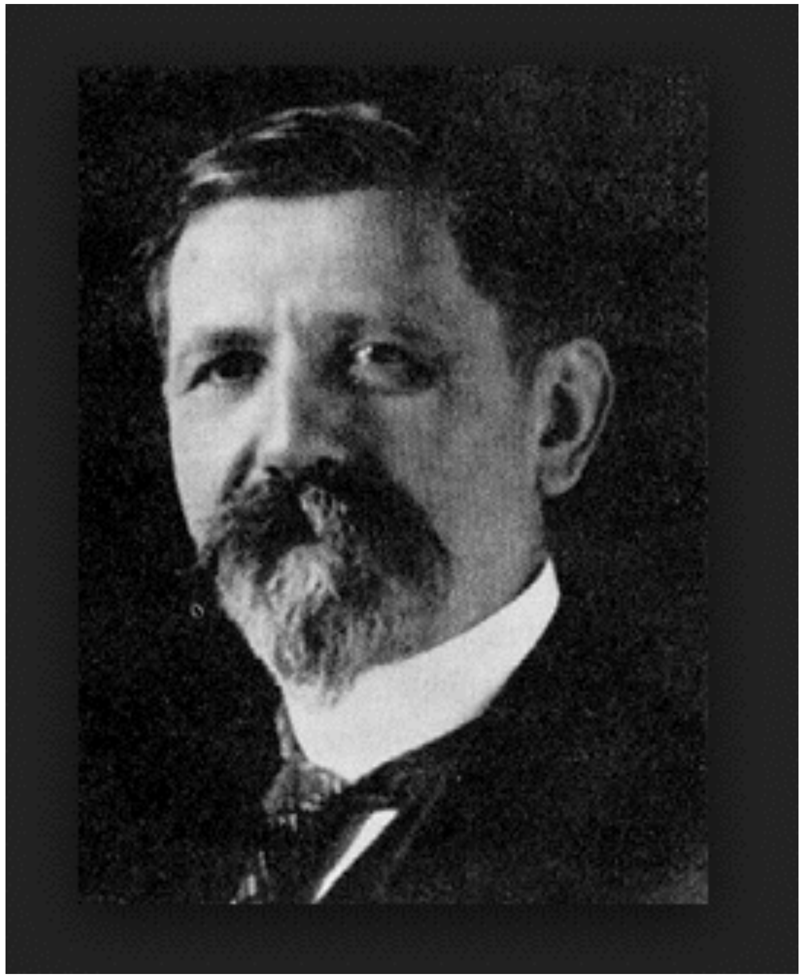
resumming the tail by Borel summation, giving an integral



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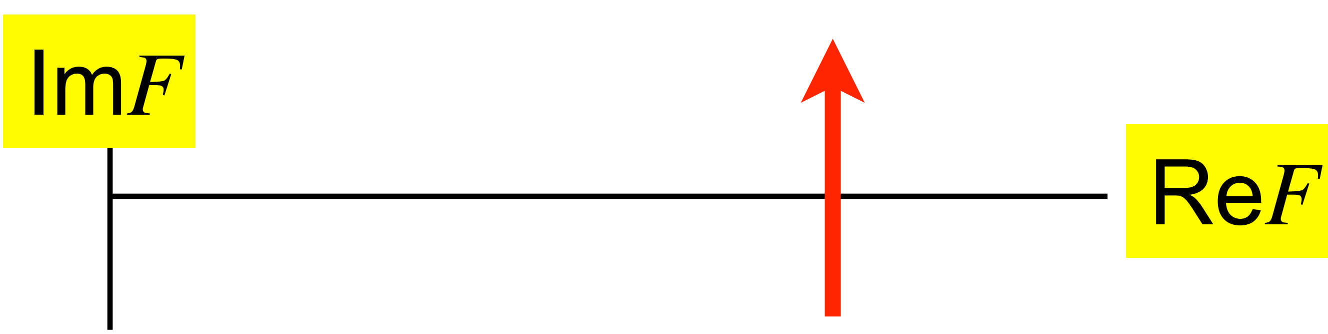
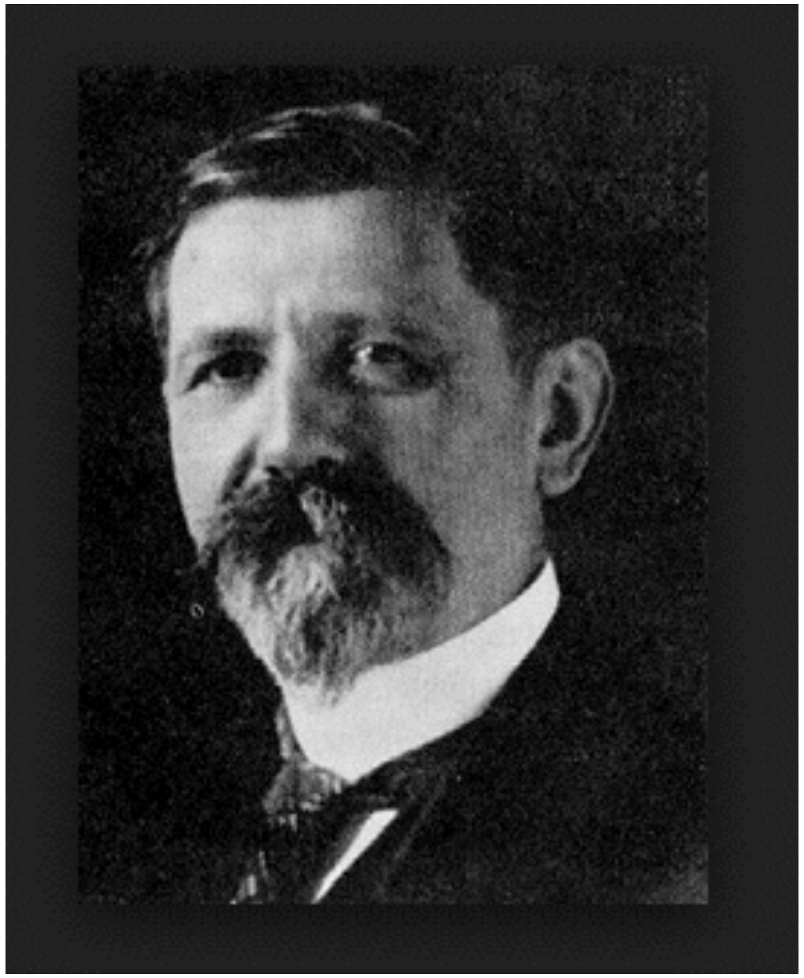
uniform approximation of the integral across a Stokes line





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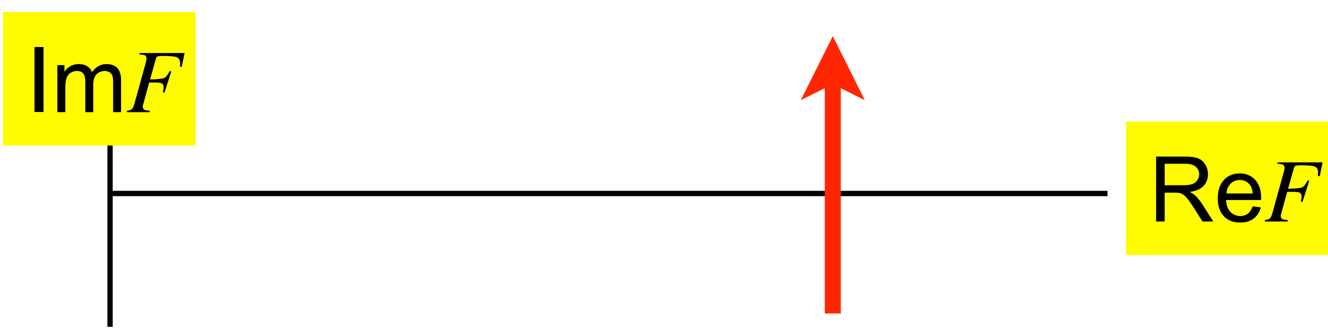
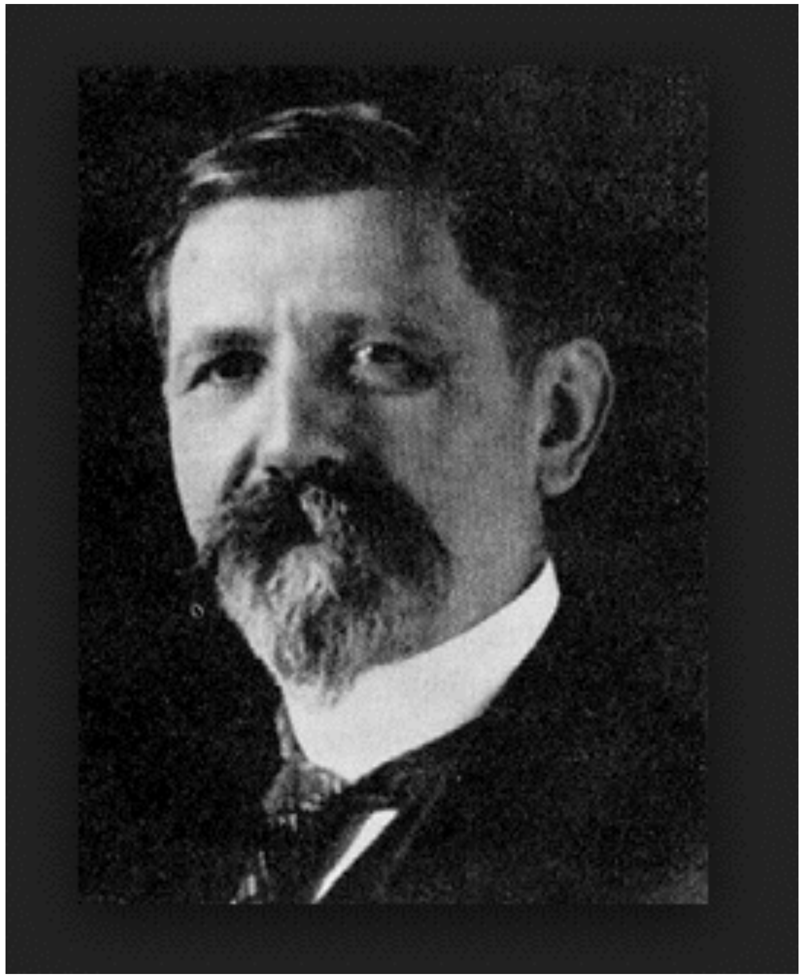


the small exponential is born not suddenly but smoothly, according to a ***universal scaling*** in terms of an ***error function***



Émile Borel

subtract the large exponential series

$$\text{tail} = -\mathrm{i}\exp\left(\tfrac{1}{2}F\right)\left(2\sqrt{\pi}z^{1/4}\mathrm{Ai}(z) - \exp\left(\tfrac{1}{2}F\right)\sum_{n=0}^{\mathrm{int}|F|}\frac{a_n}{F^n}\right)$$

$$\left(F = -\tfrac{4}{3}z^{3/2}\right)$$

subtract the large exponential series

big

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difference small

$$\left(F = -\tfrac{4}{3}z^{3/2}\right)$$

subtract the large exponential series

big big big

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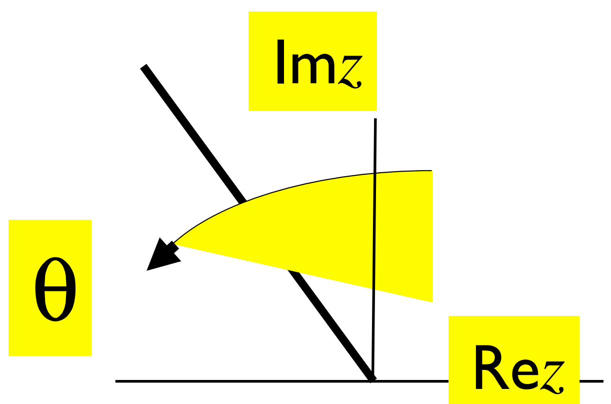

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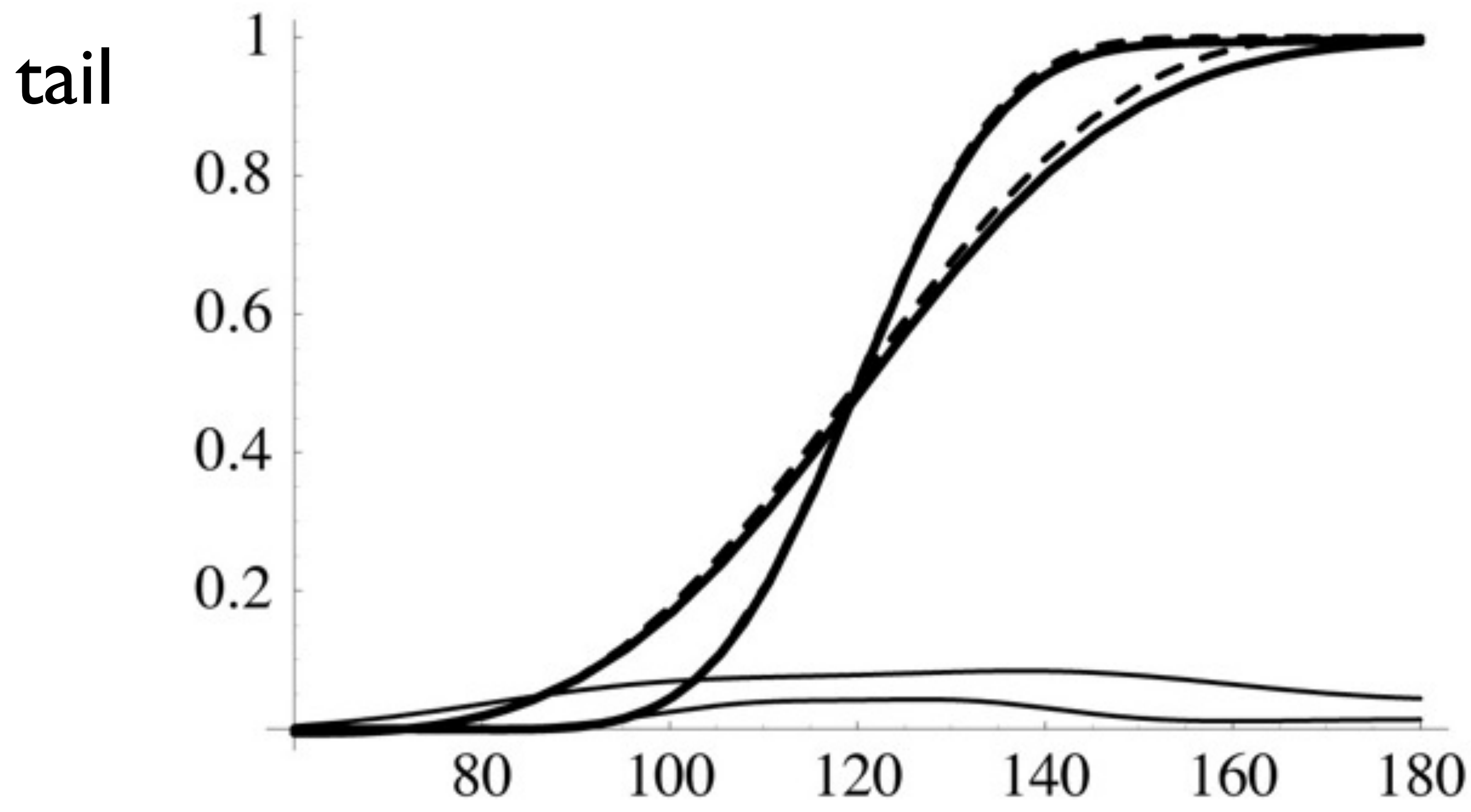

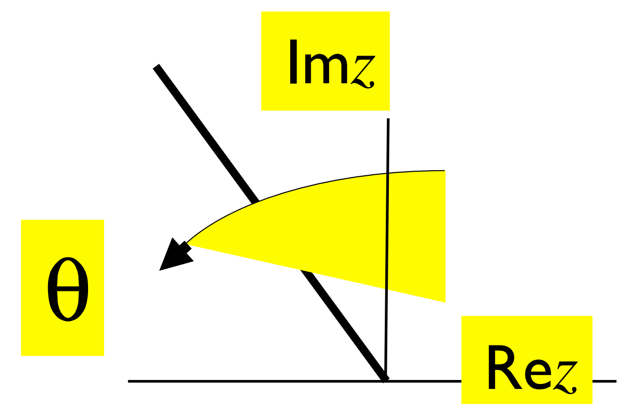

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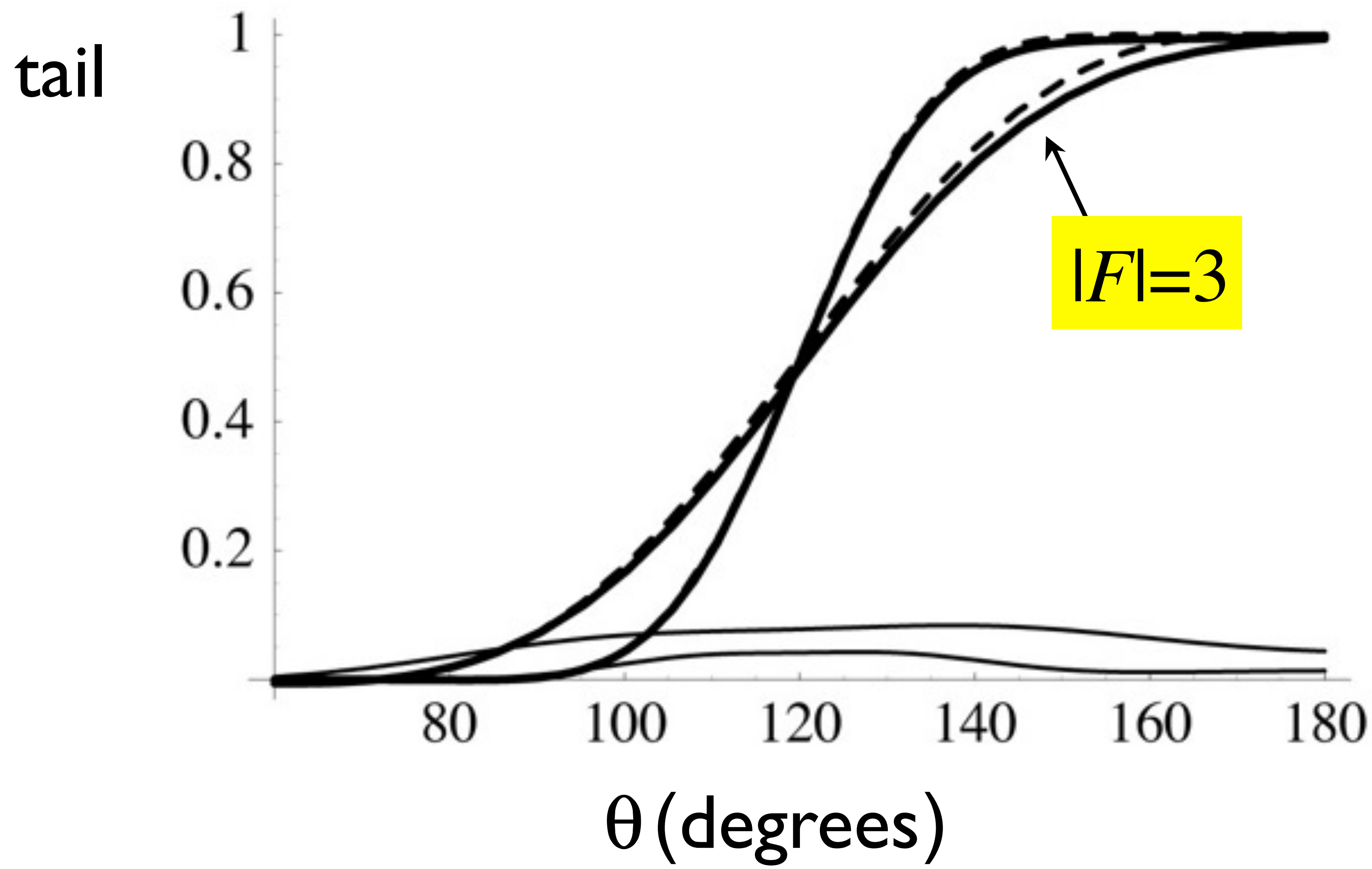

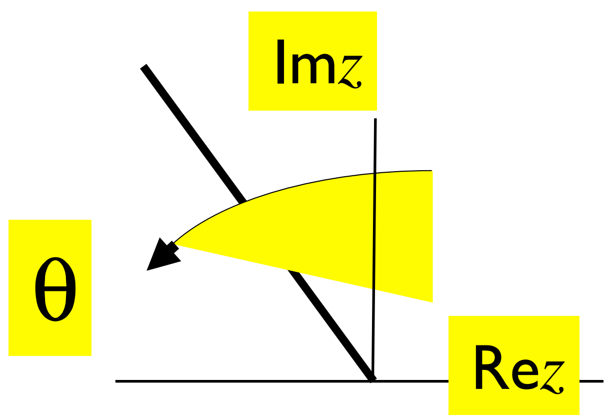

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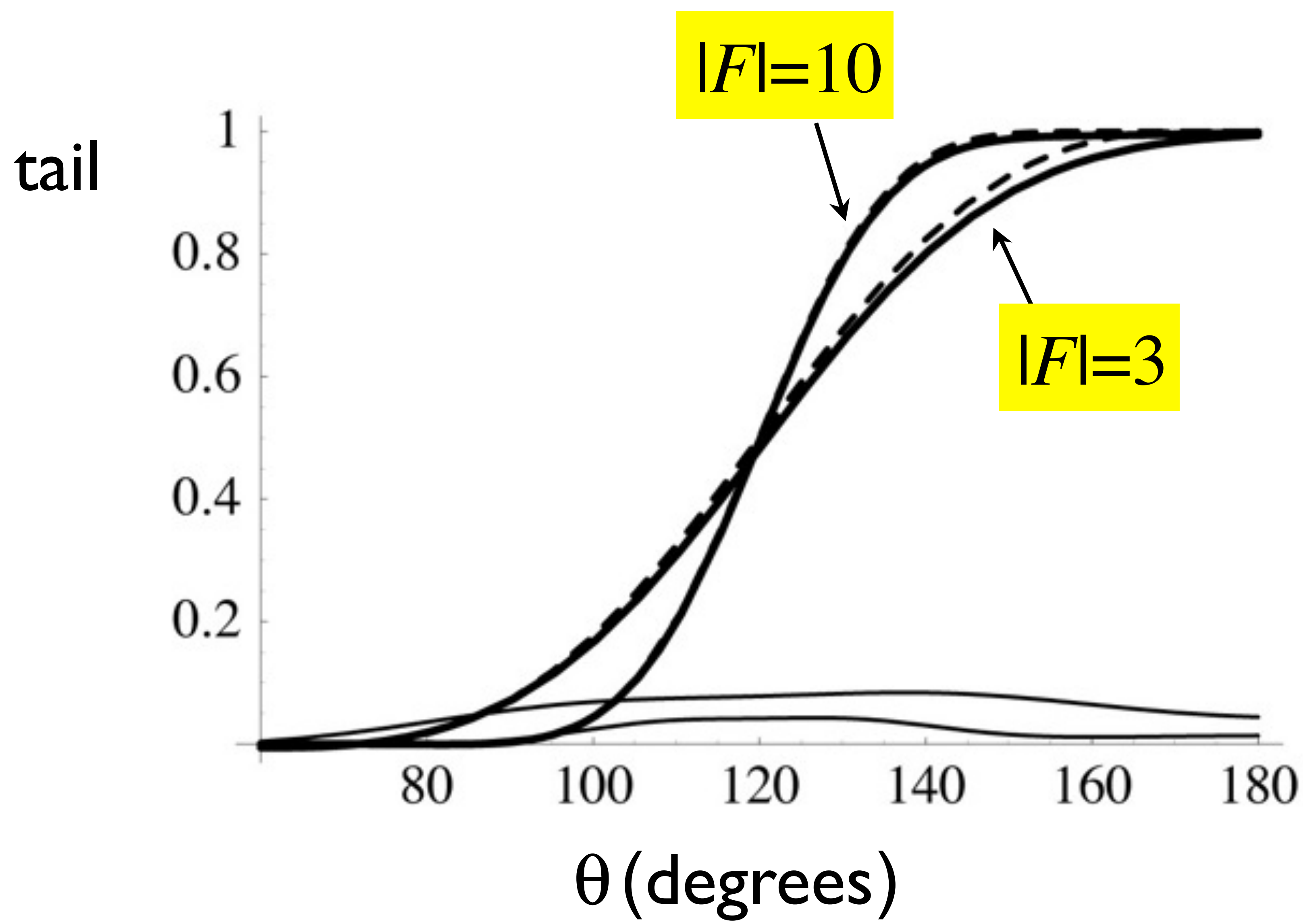

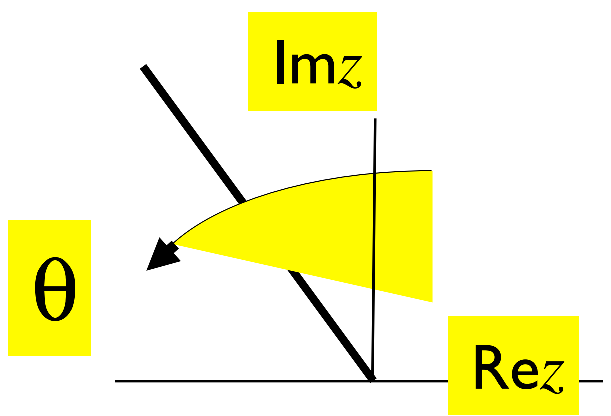

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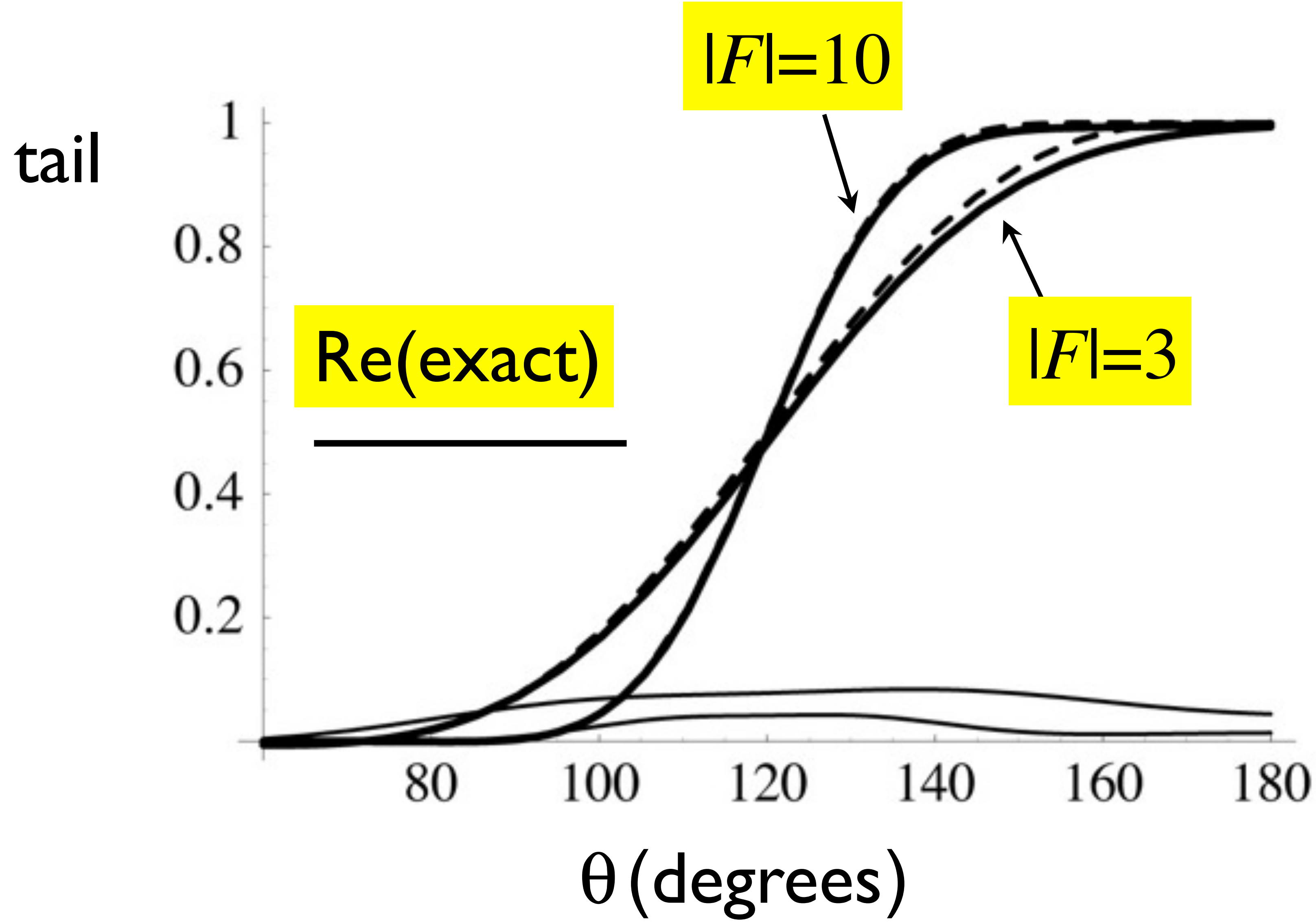

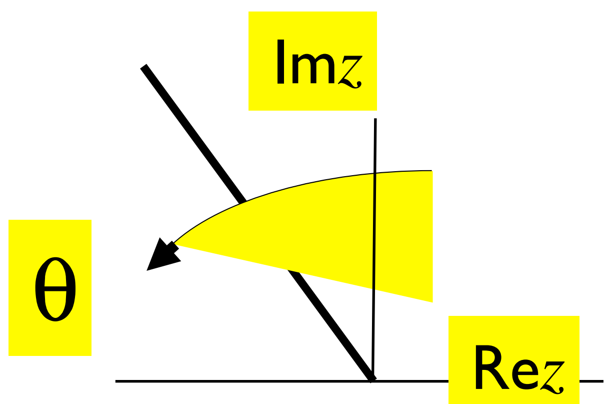

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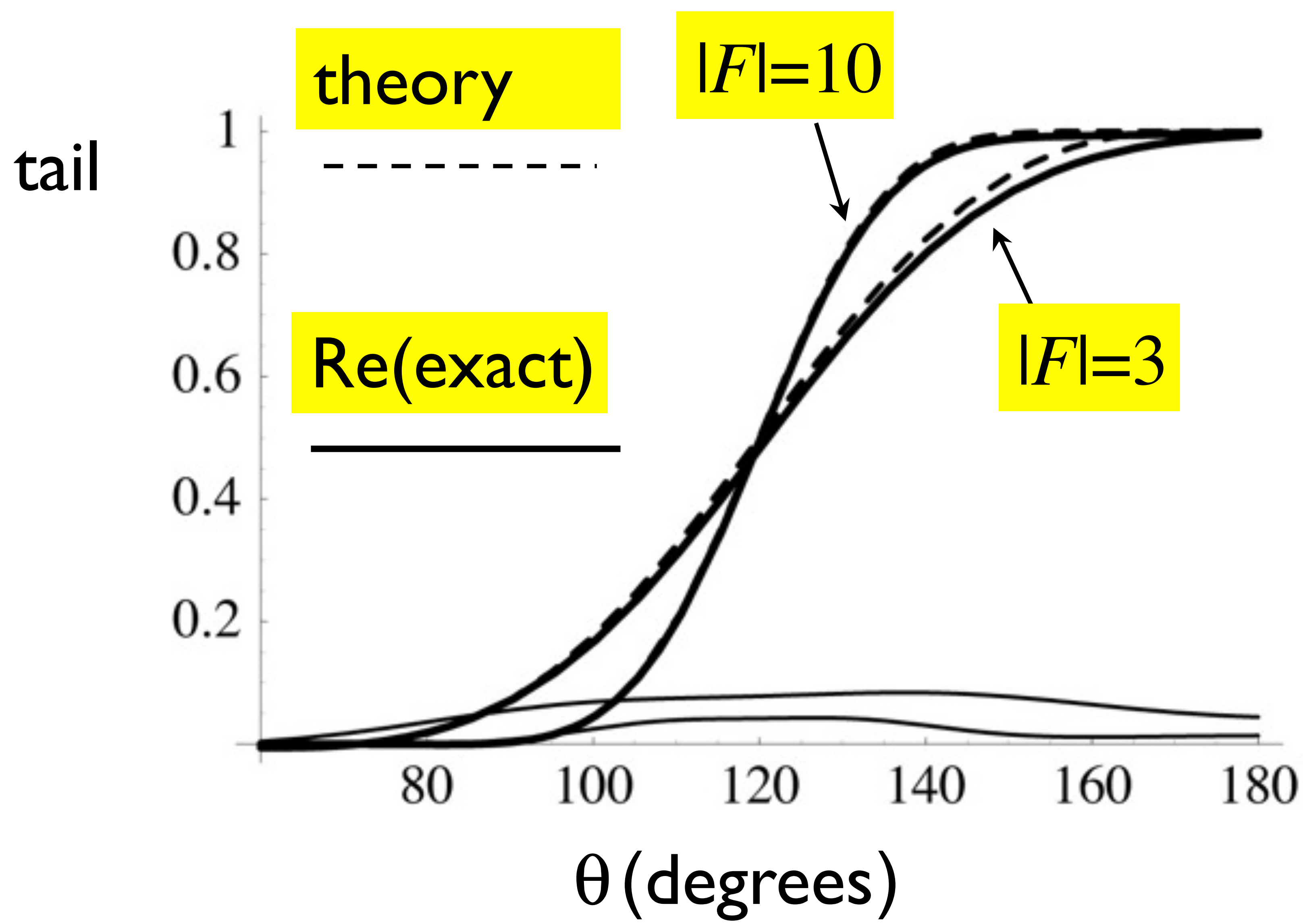

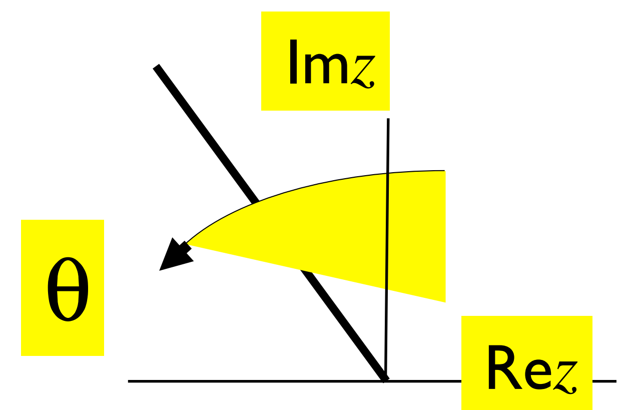

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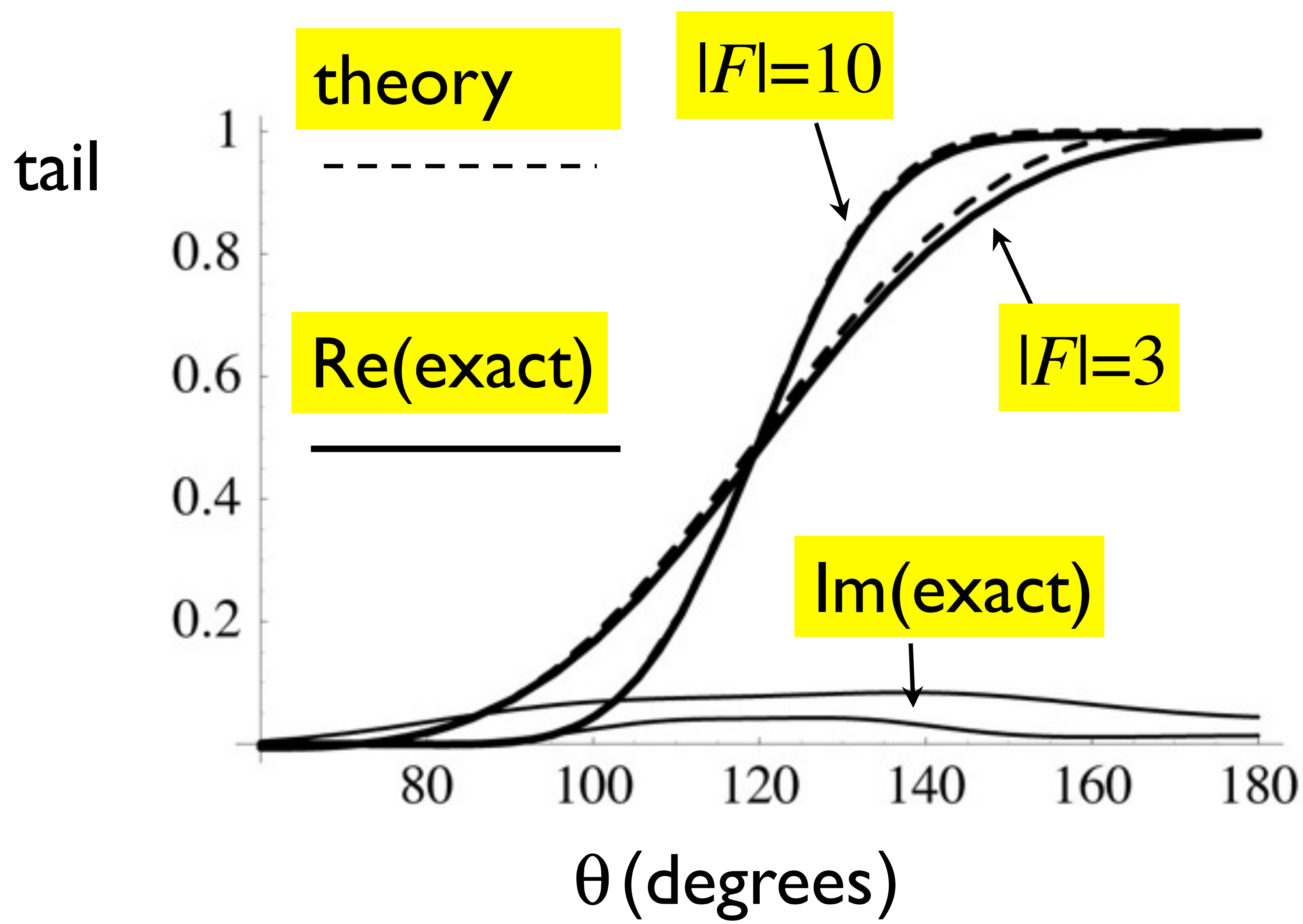

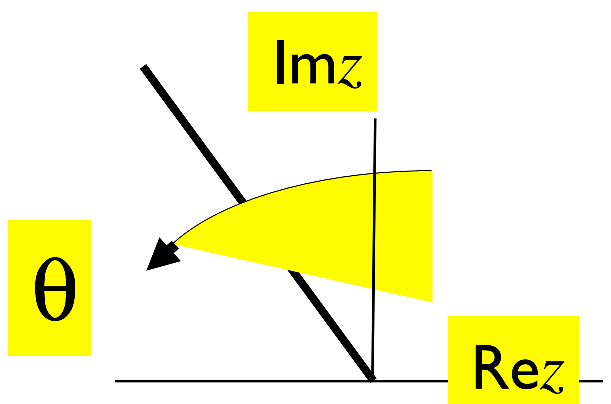

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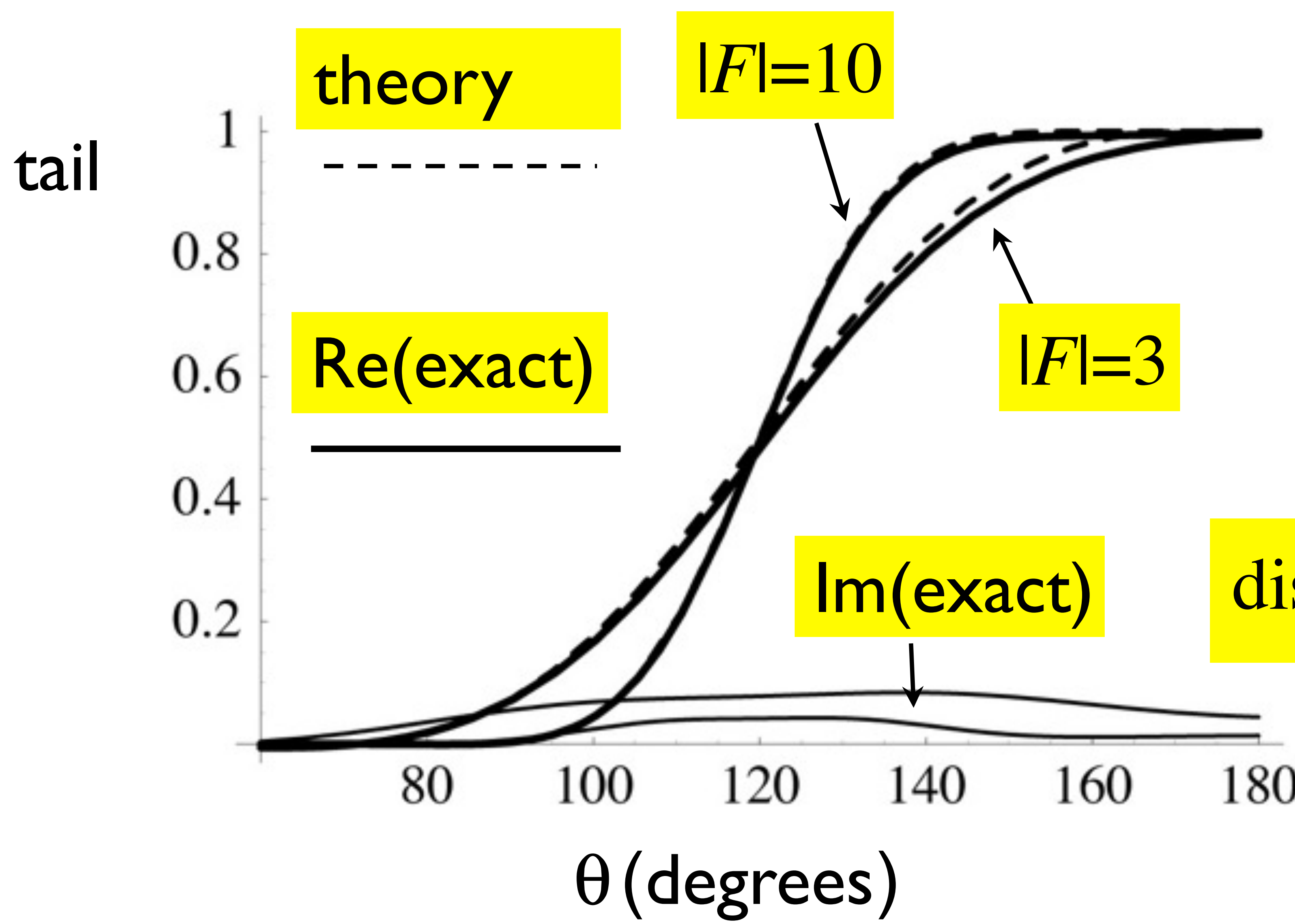

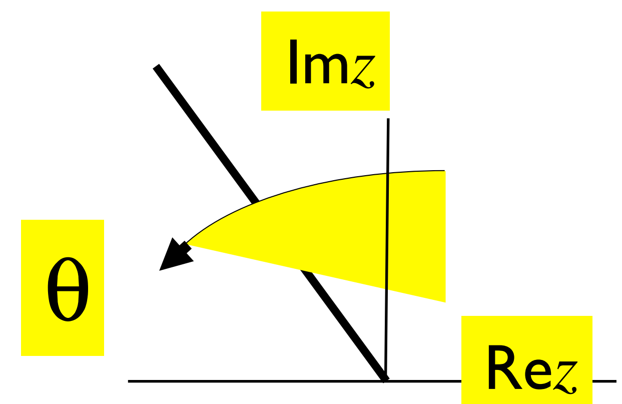

$$\text{disparity}\ \exp\left|F=10\right| \approx 22000$$

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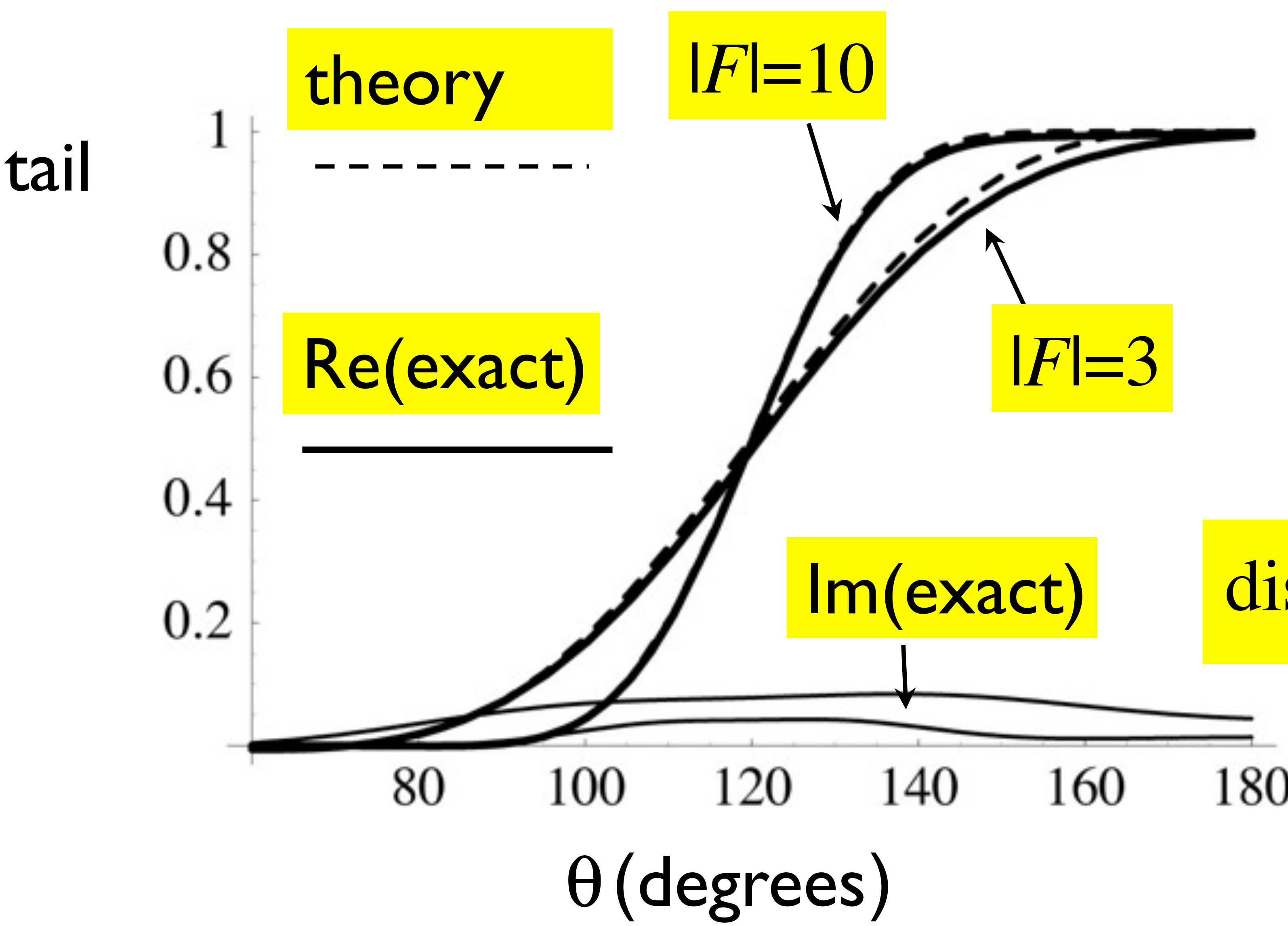

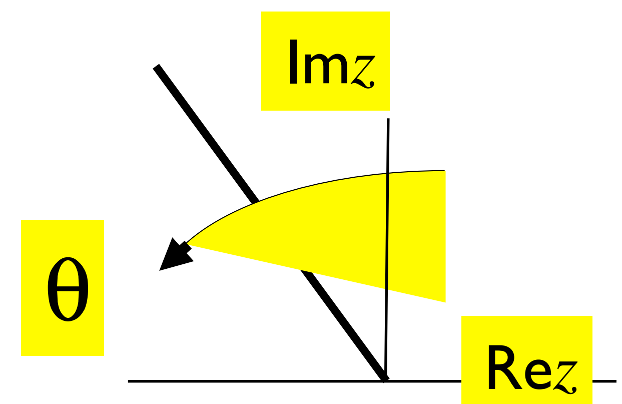

disparity $\exp|F=10| \approx 22000$

the error function
in the Airy function

many applications in mathematics, to the approximation of a variety of functions: the error function in

- Bessel
- hypergeometric
- gamma
- even the error function itself
- integrals with coalescing saddles
- Riemann zeta function
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in physics, applications to

- reflection of waves by refractive-index gradients
- histories of quantum jumps induced by slowly-changing external forces, and particle pair creation
- breakdown of slow manifold in slow-fast systems

histories of quantum transitions driven by slowly-changing hamiltonians

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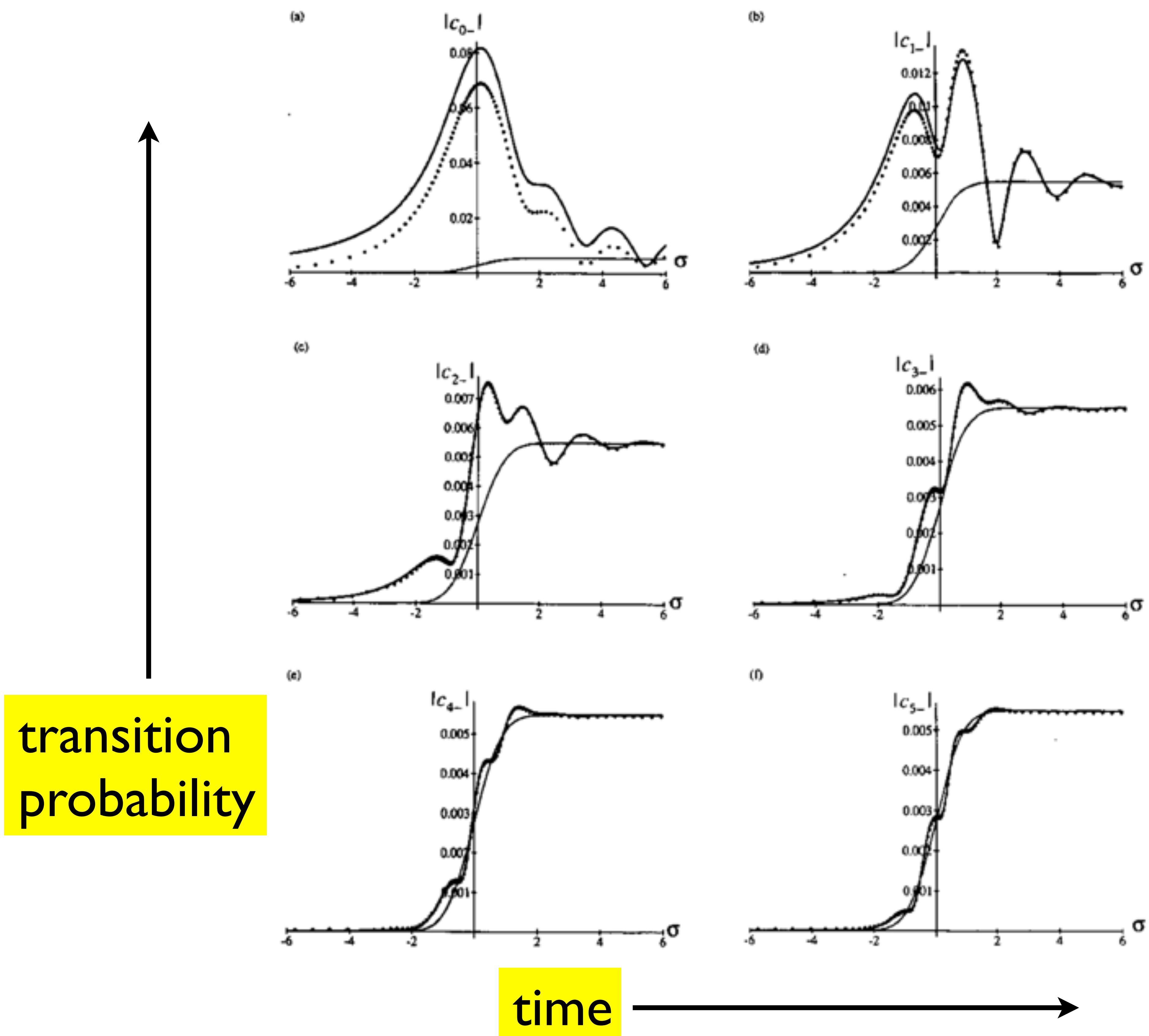

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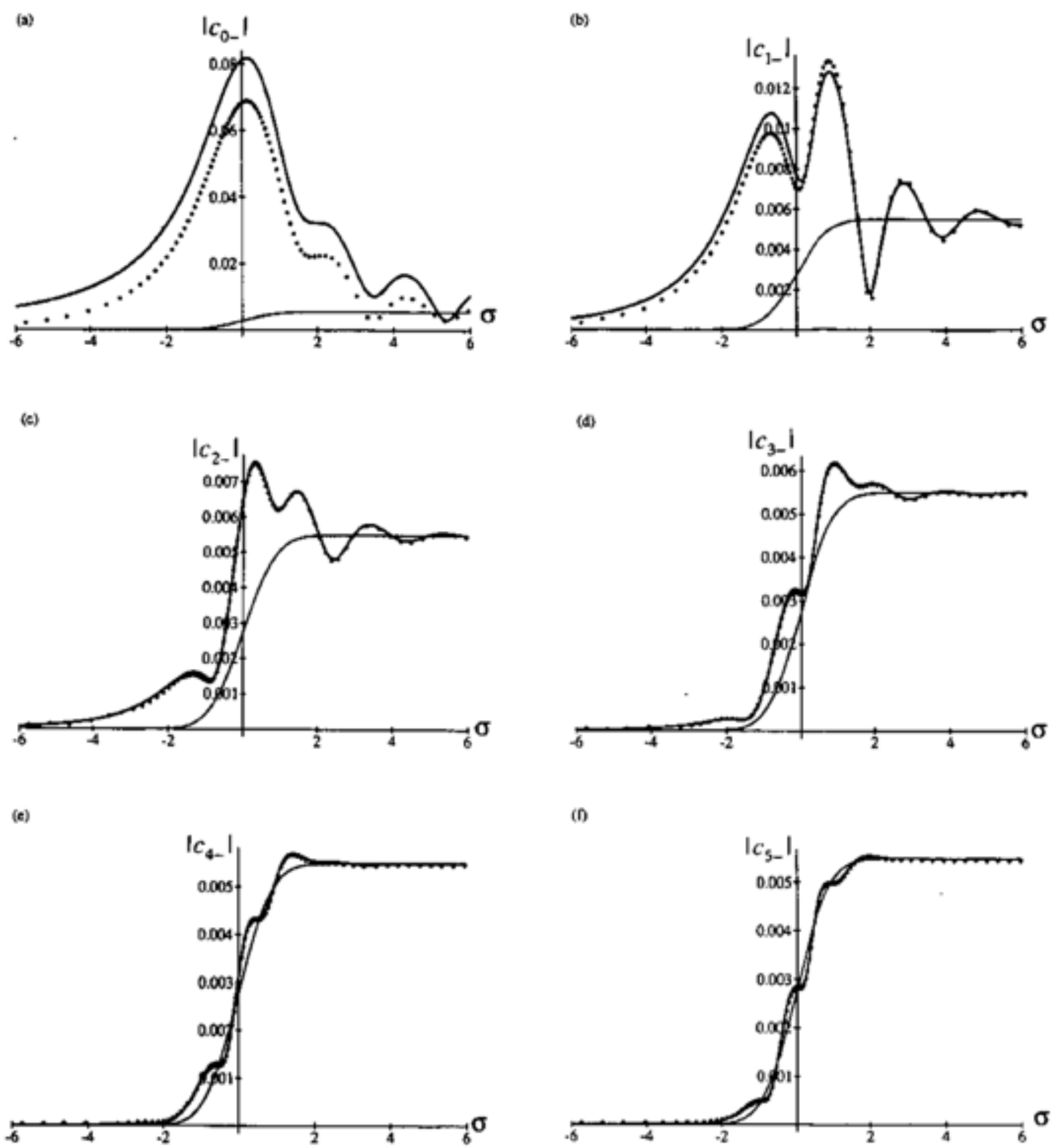
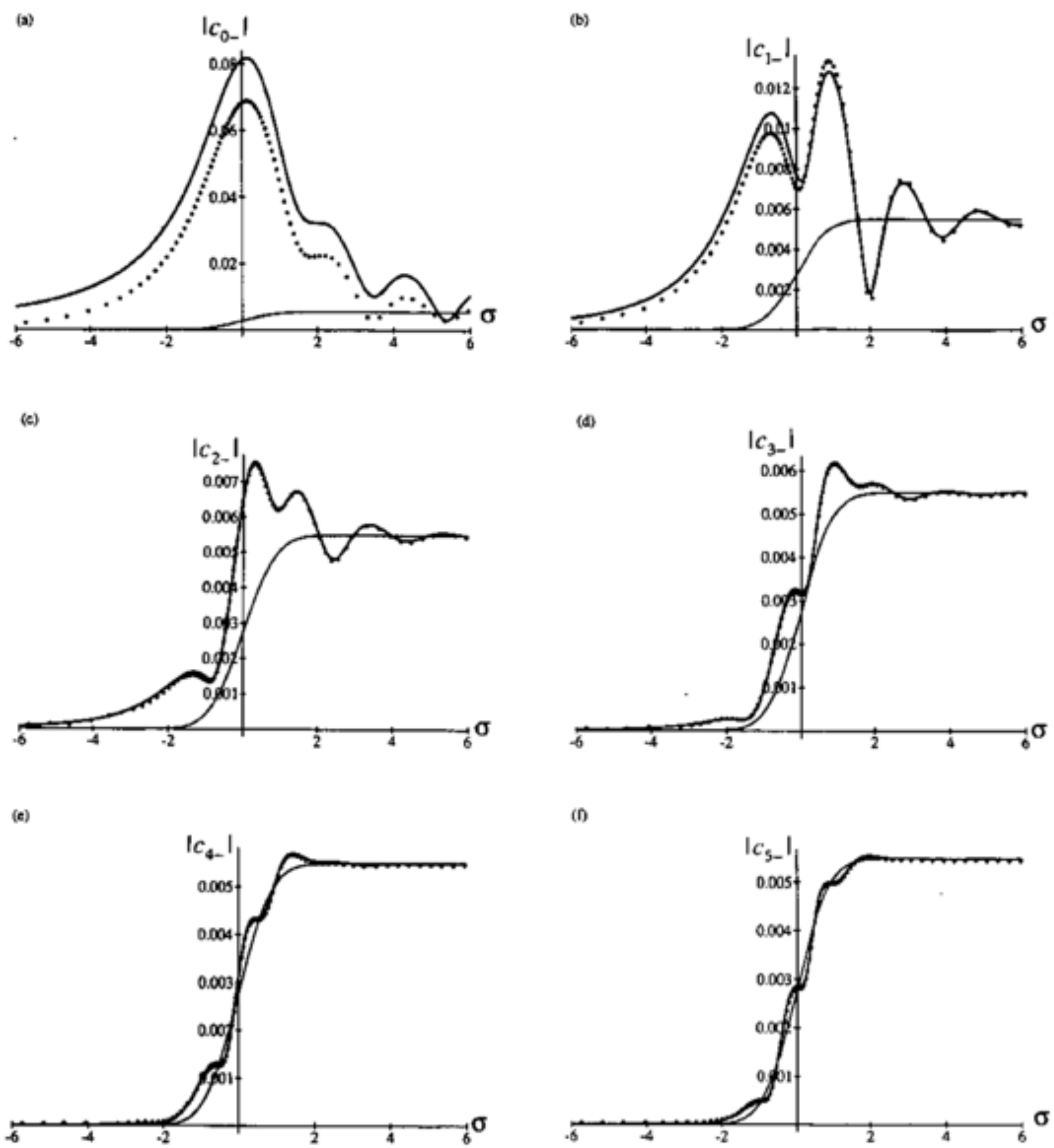


*n*th order superadiabatic bases

transition probability

time

histories of quantum transitions driven by slowly-changing hamiltonians



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final probability is exponentially small: exp(-const/ε)

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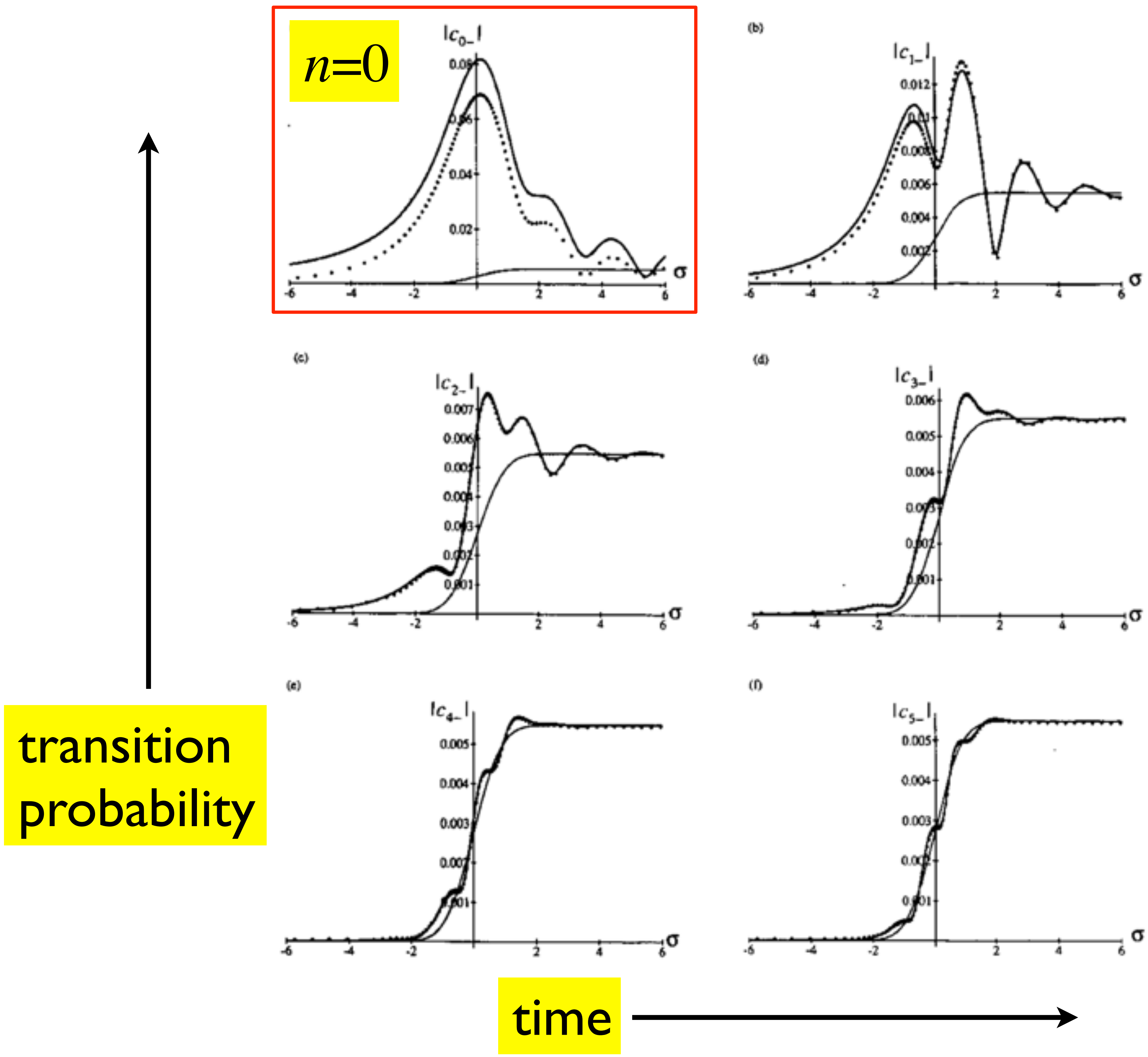

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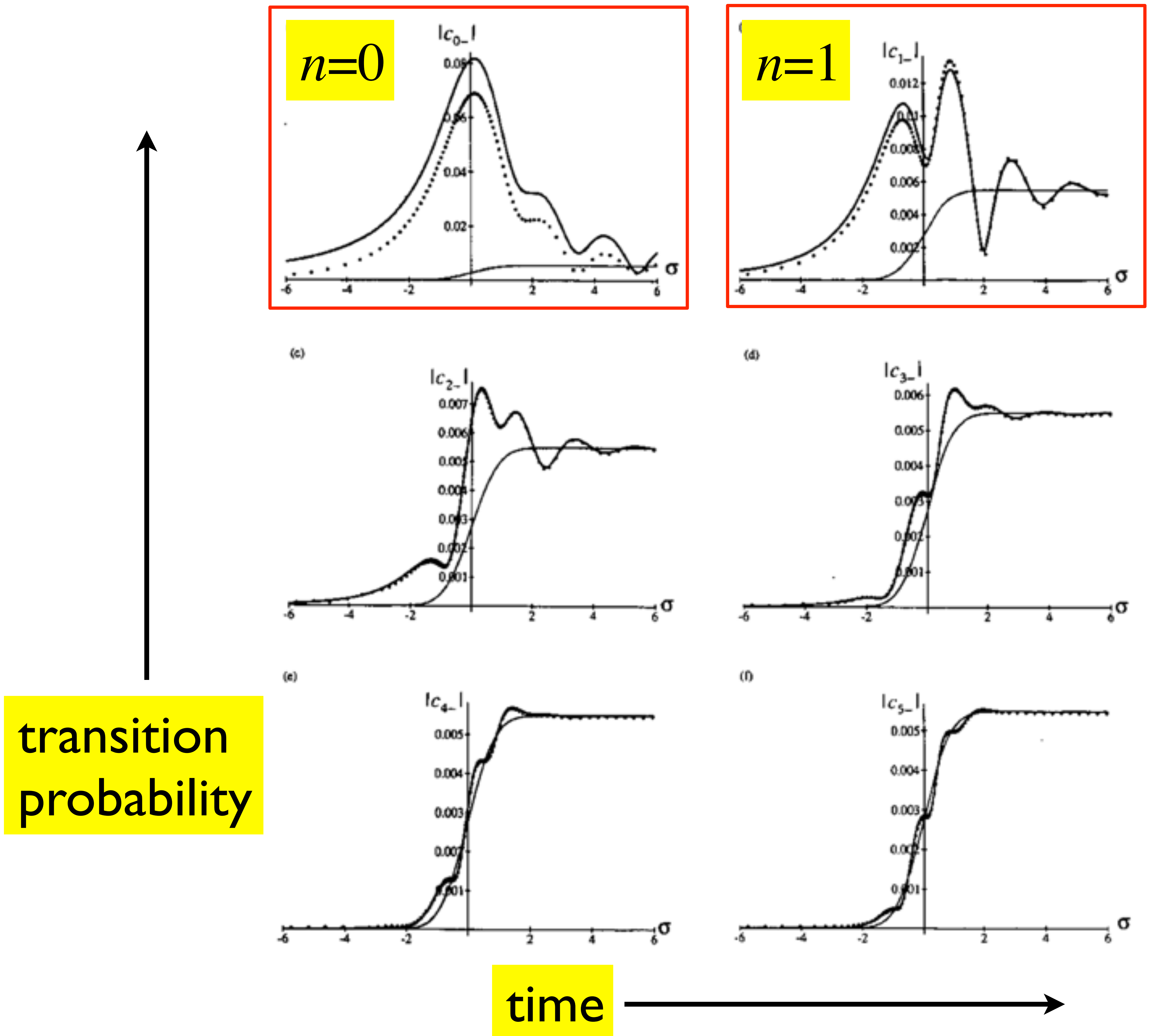

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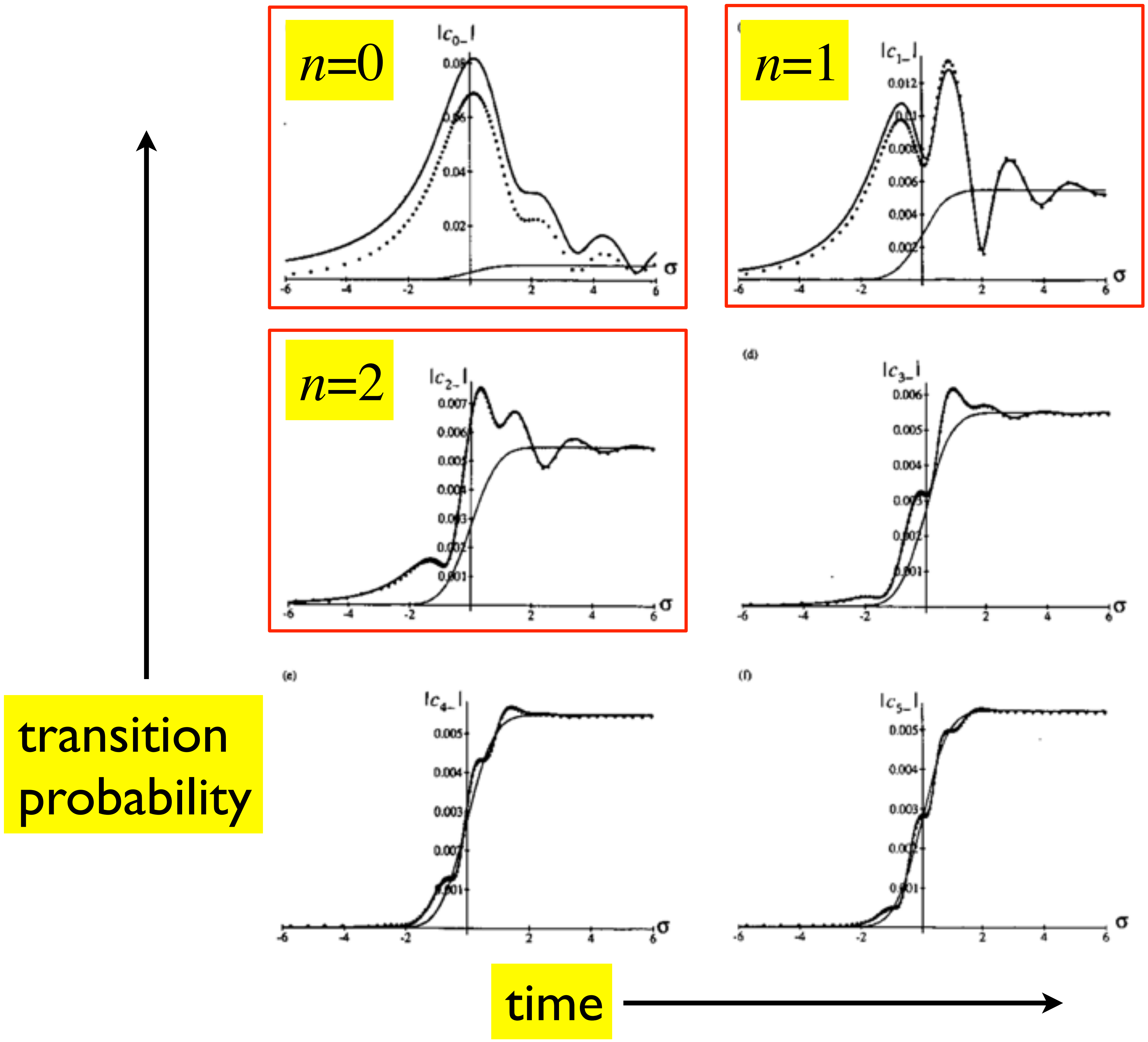

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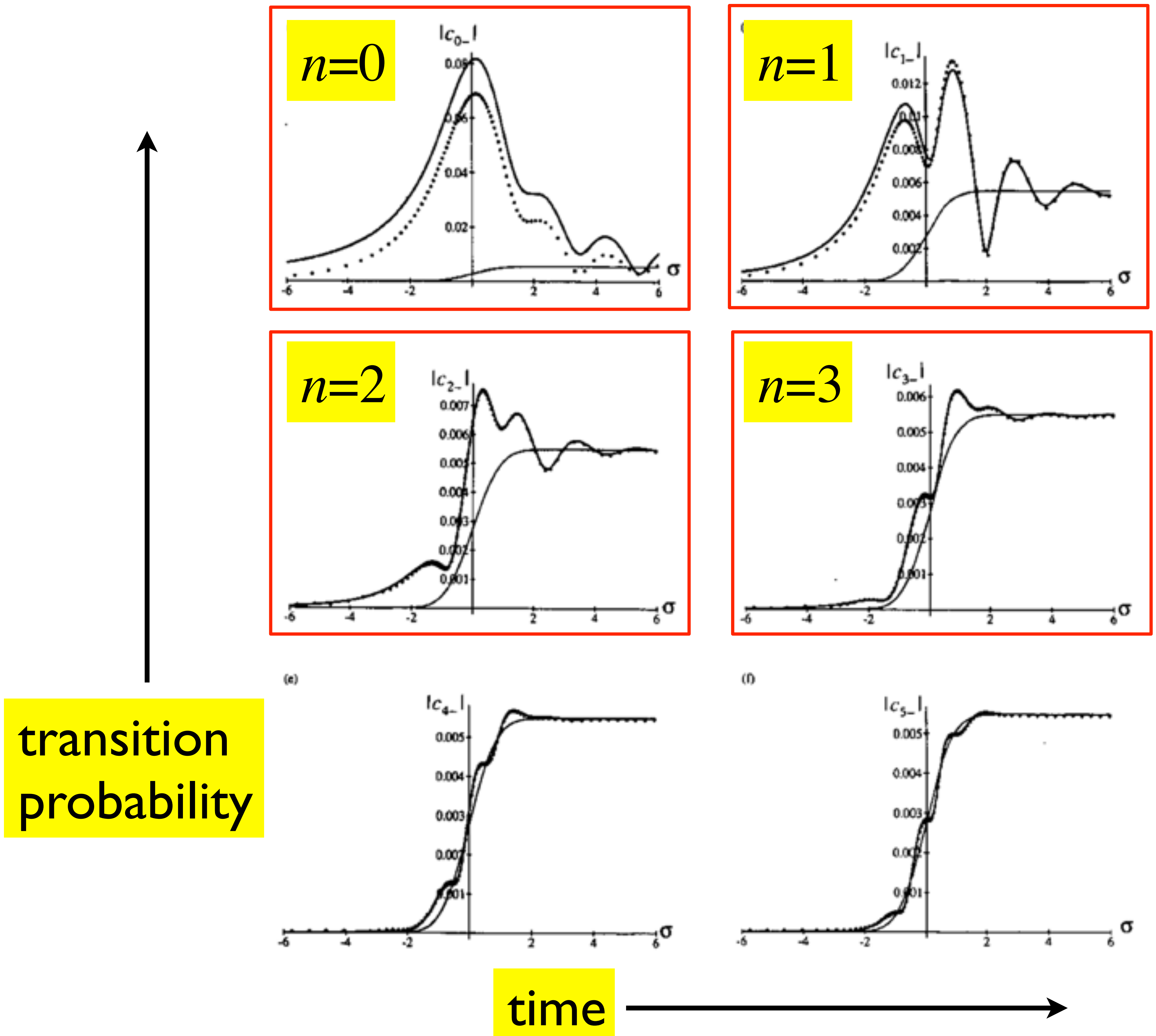

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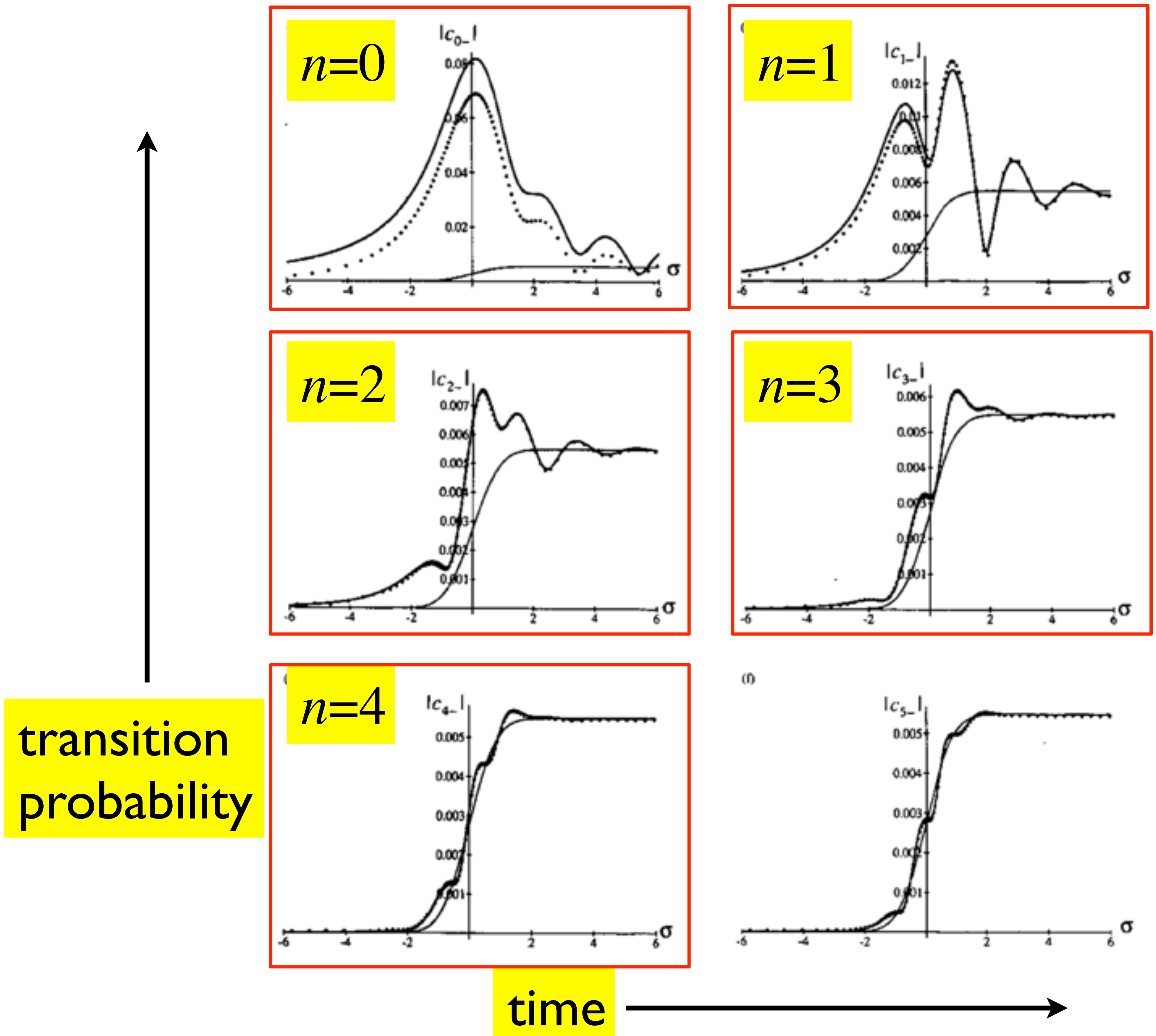

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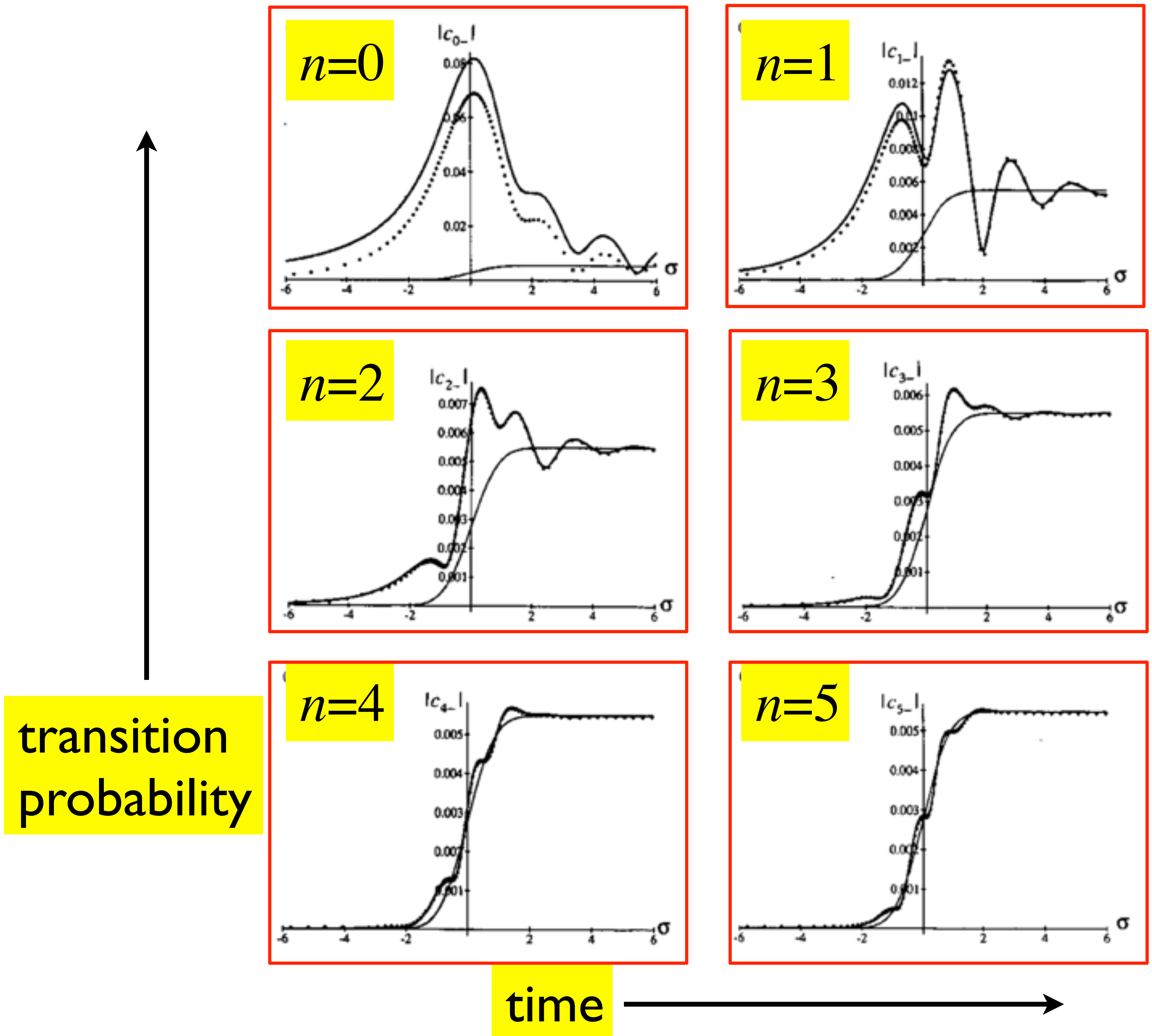

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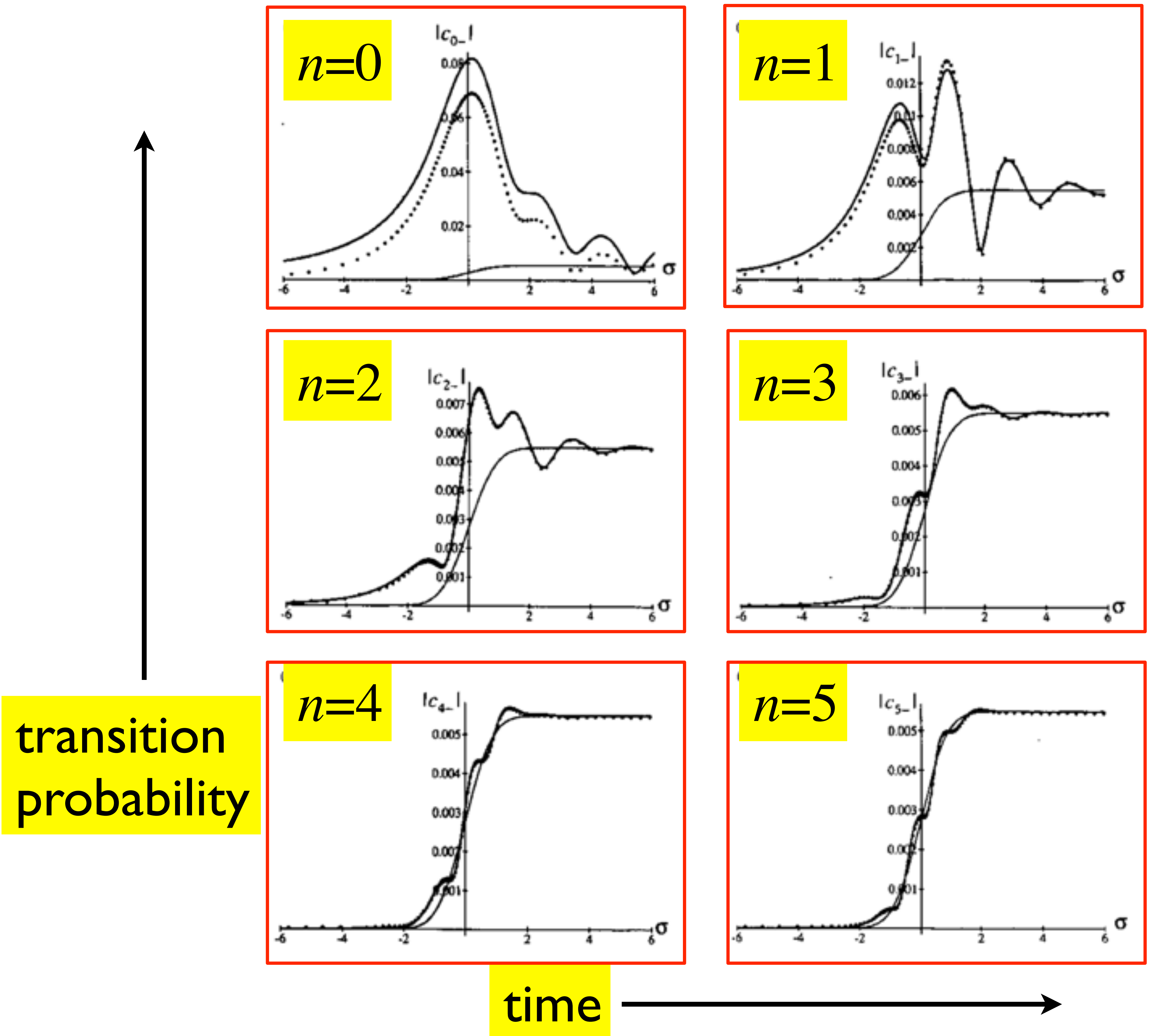

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large oscillations en route (O($\sqrt{\varepsilon}$)), getting smaller as optimal order (*n*=5) is approached

Poincaré asymptotics: summing to a fixed order

Henri Poincaré

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capturing small exponentials: Kruskal, 'asymptotics beyond all orders

hyperasymptotics:
repeated resummation, based on the principle of ***resurgence*** (Dingle 1960s, Écalle 1980s)

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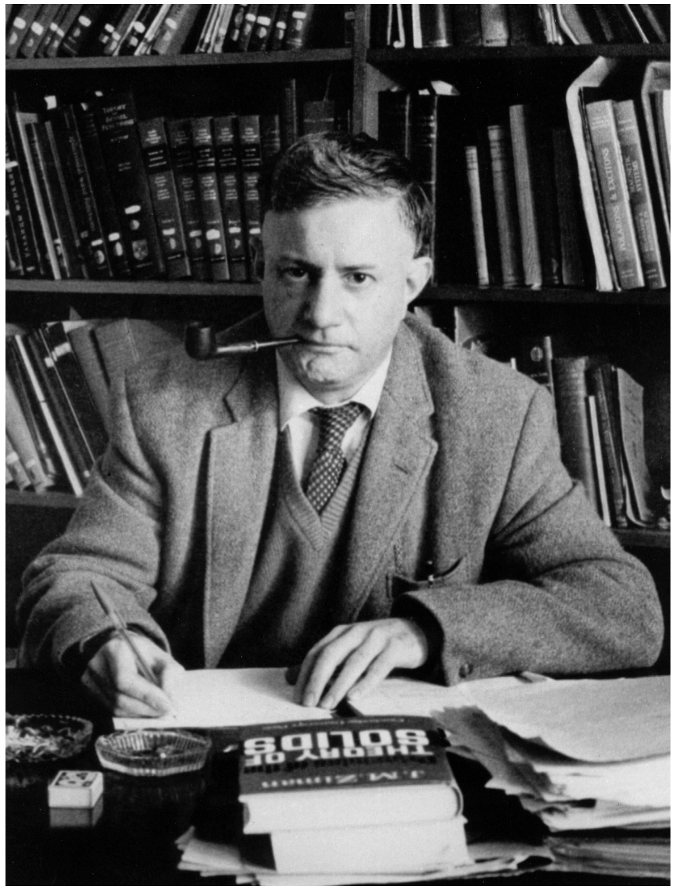

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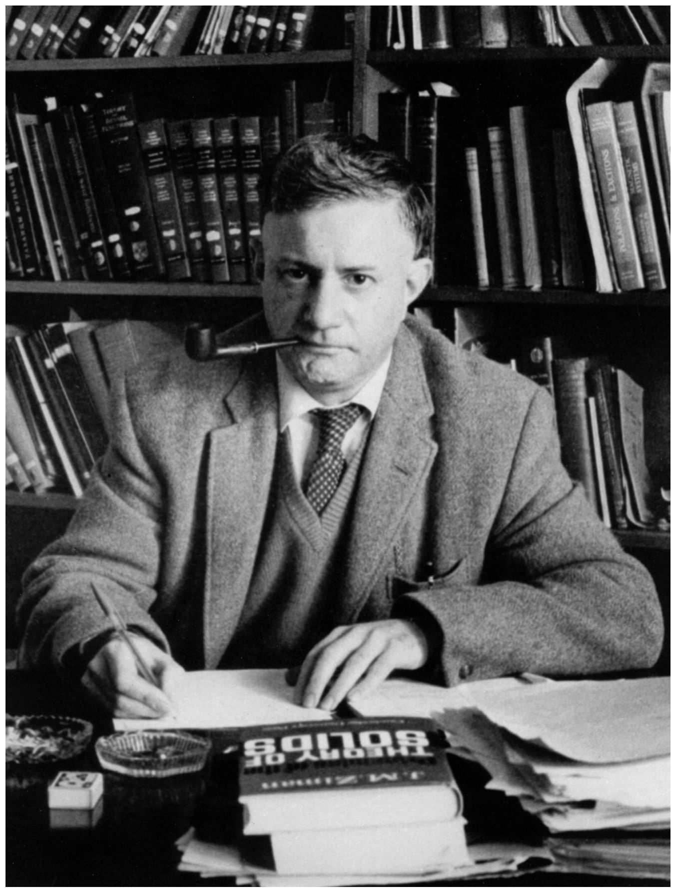

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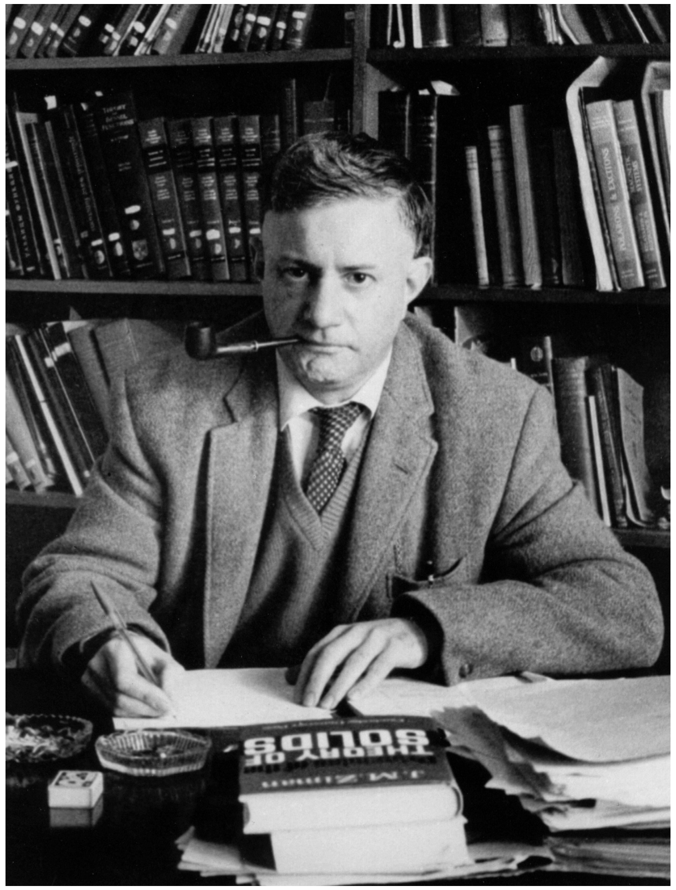

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the divergence of a series must reflect its cause

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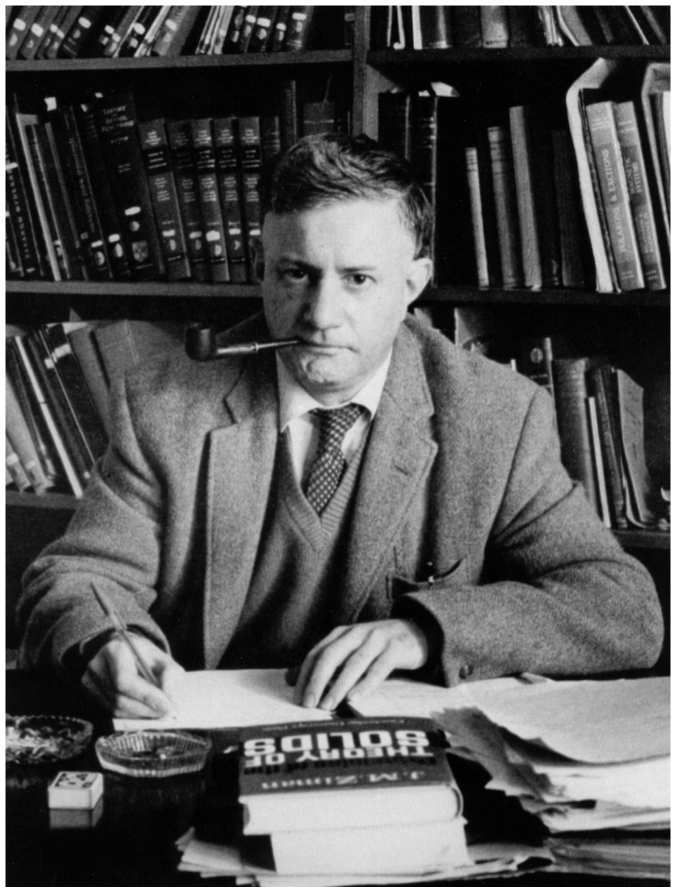

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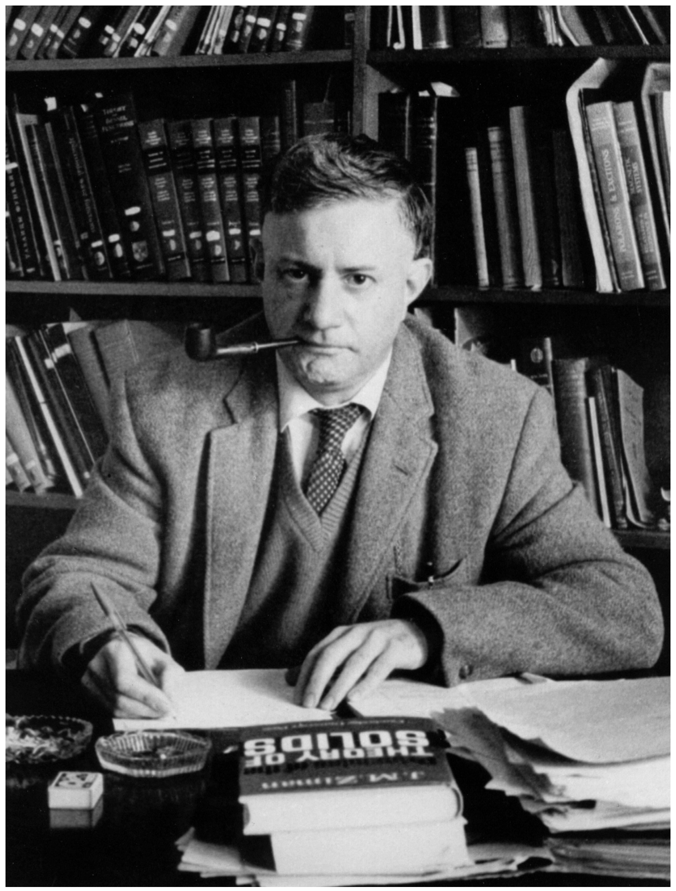

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the divergence of a series must reflect its cause

moreover, each component series must contain, coded into its high orders, information about all the other exponentials, and all terms of the series multiplying them simplest case: only two exponentials $\exp\left(\pm\tfrac{1}{2}F\right)$

in $S=\sum_{n=0}^{\infty}\frac{a_n}{F^n}$

$$a_n \underset{n\to\infty}{\longrightarrow} \tfrac{1}{2\pi}(n-1)!\left[a_0-\frac{a_1}{(n-1)}+\right.$$
$$\left.+\frac{a_2}{(n-1)(n-2)}-\frac{a_3}{(n-1)(n-2)(n-3)}\cdots\right]$$

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hyperasymptotic scheme for sum S as a series of series:

- primitive asymptotics - only a_0
- sum series to least term - S_0 (superasymptotics)
- integral representation for remainder
- asymptotic series for remainder, summed to least term (S_1)
- asymptotic series for new remainder, truncated (S_2) ...

hyperasymptotics for Ai for F=-16, i.e. z=5.2414827884177932413

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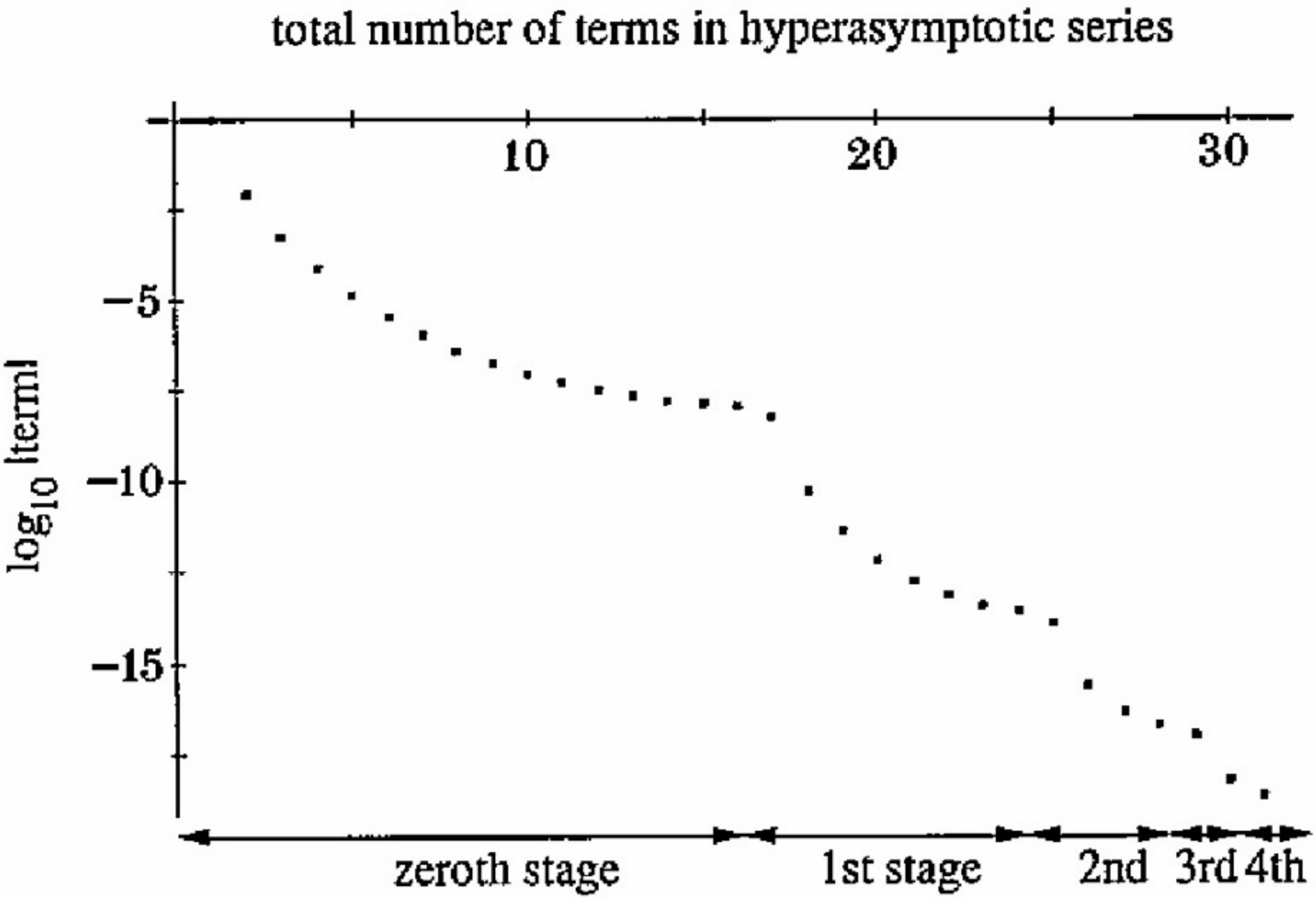

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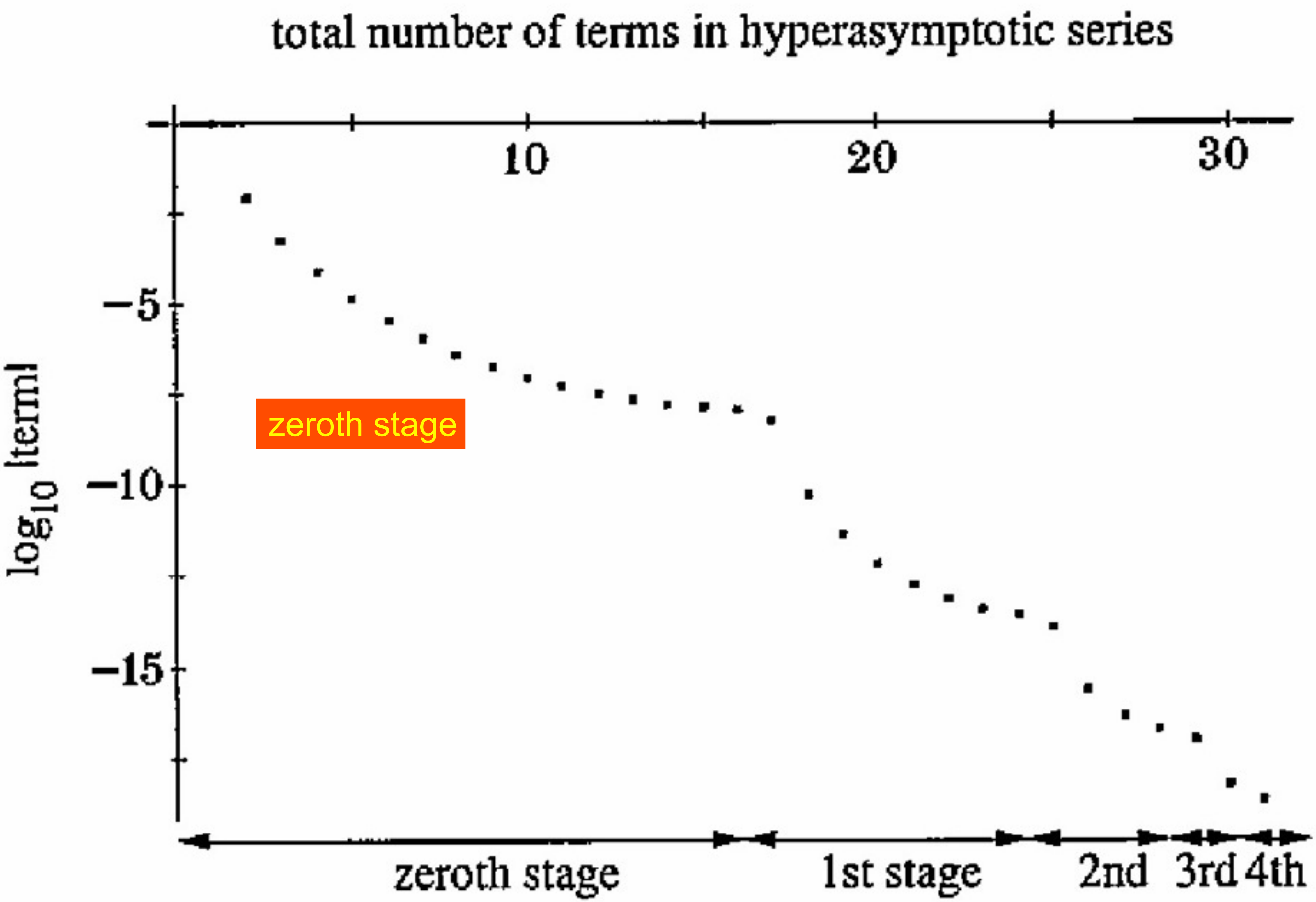

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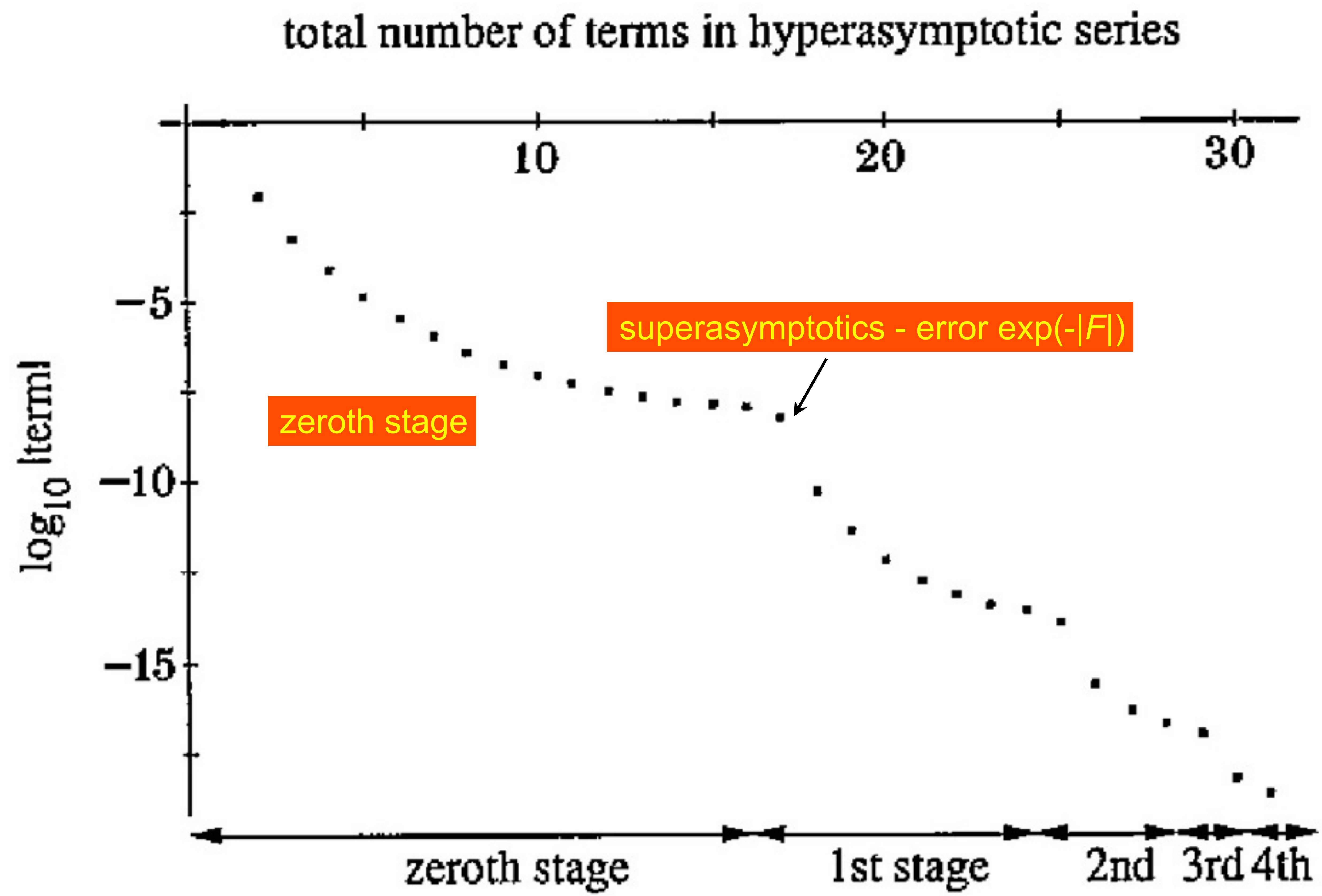

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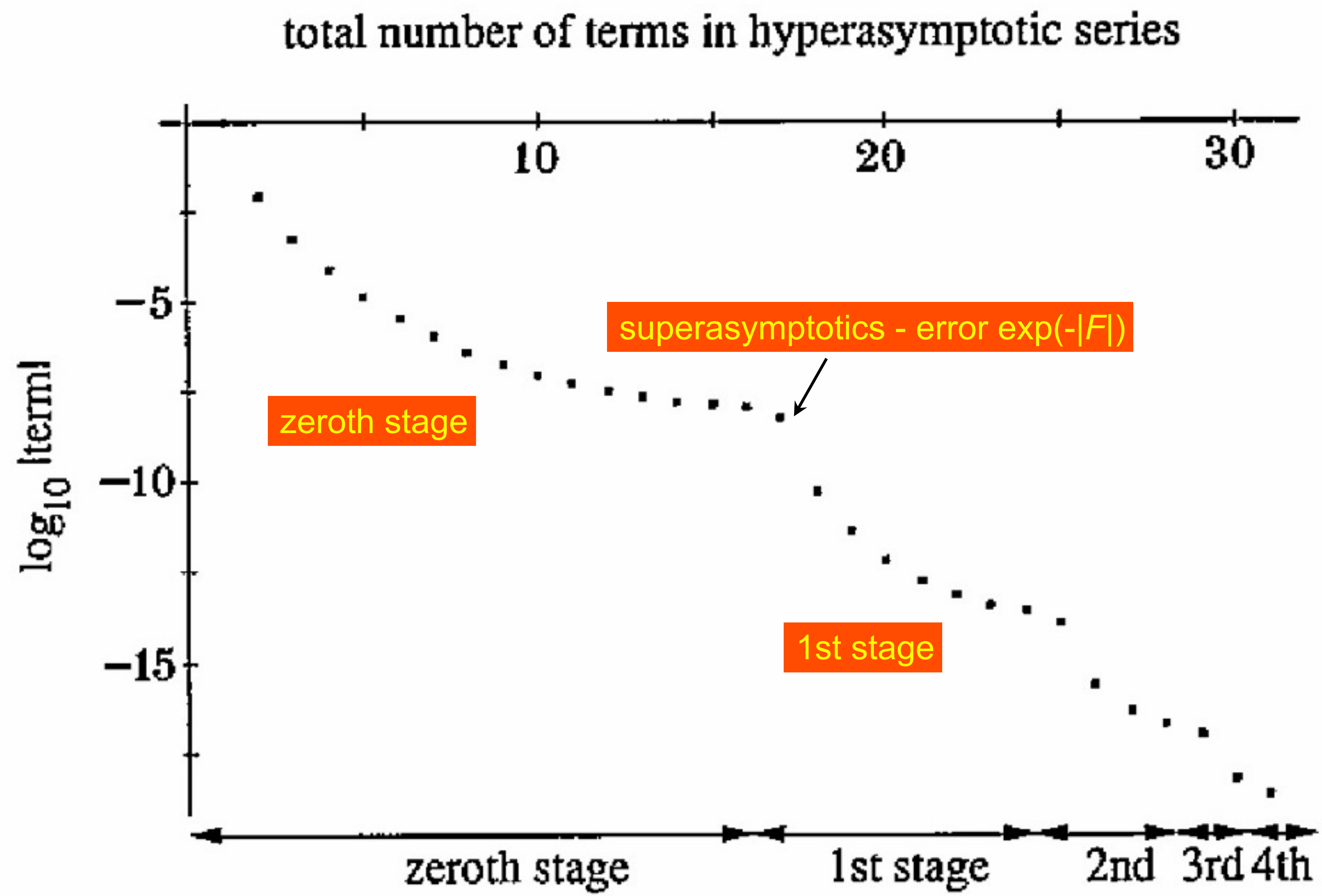

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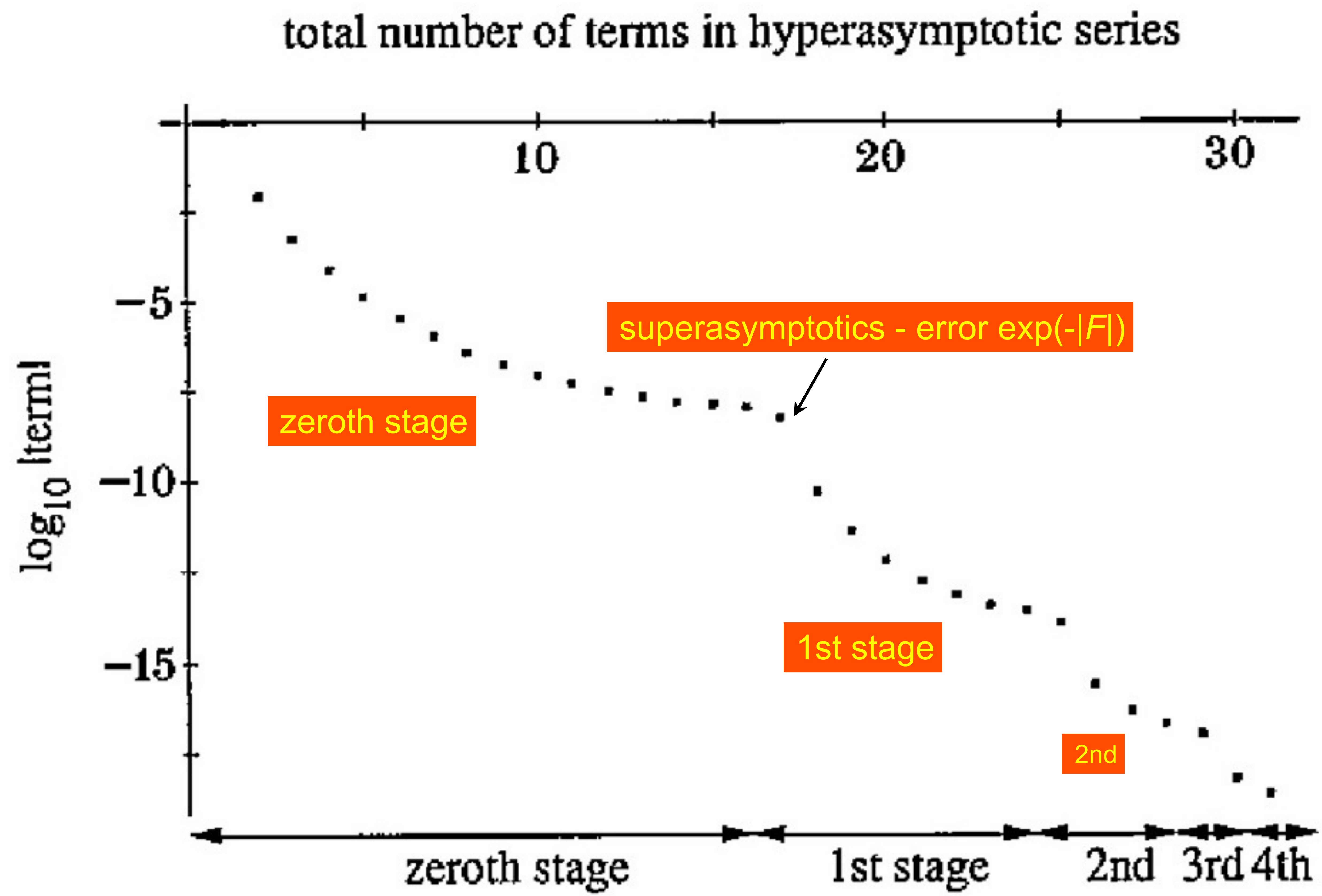

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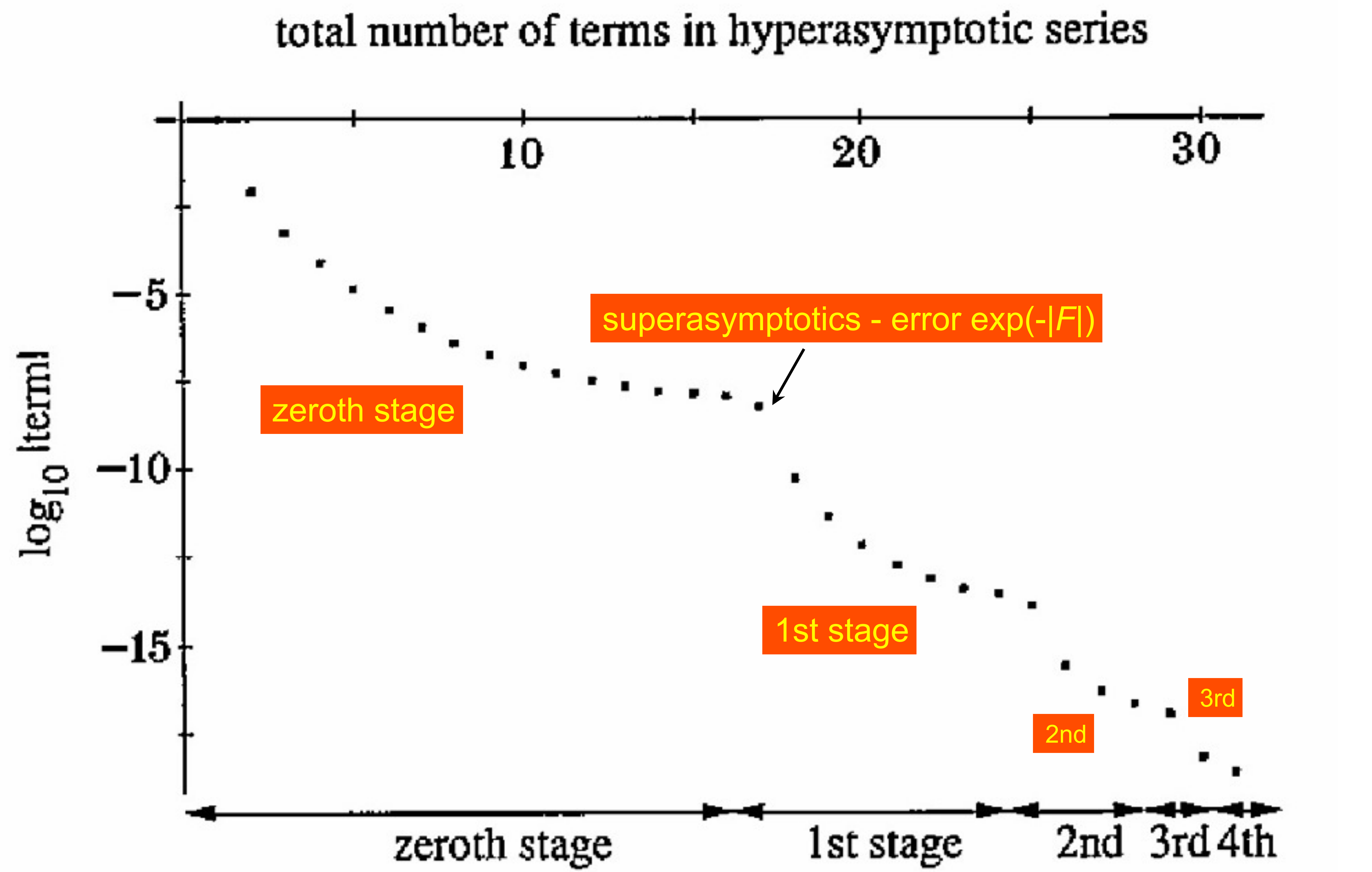

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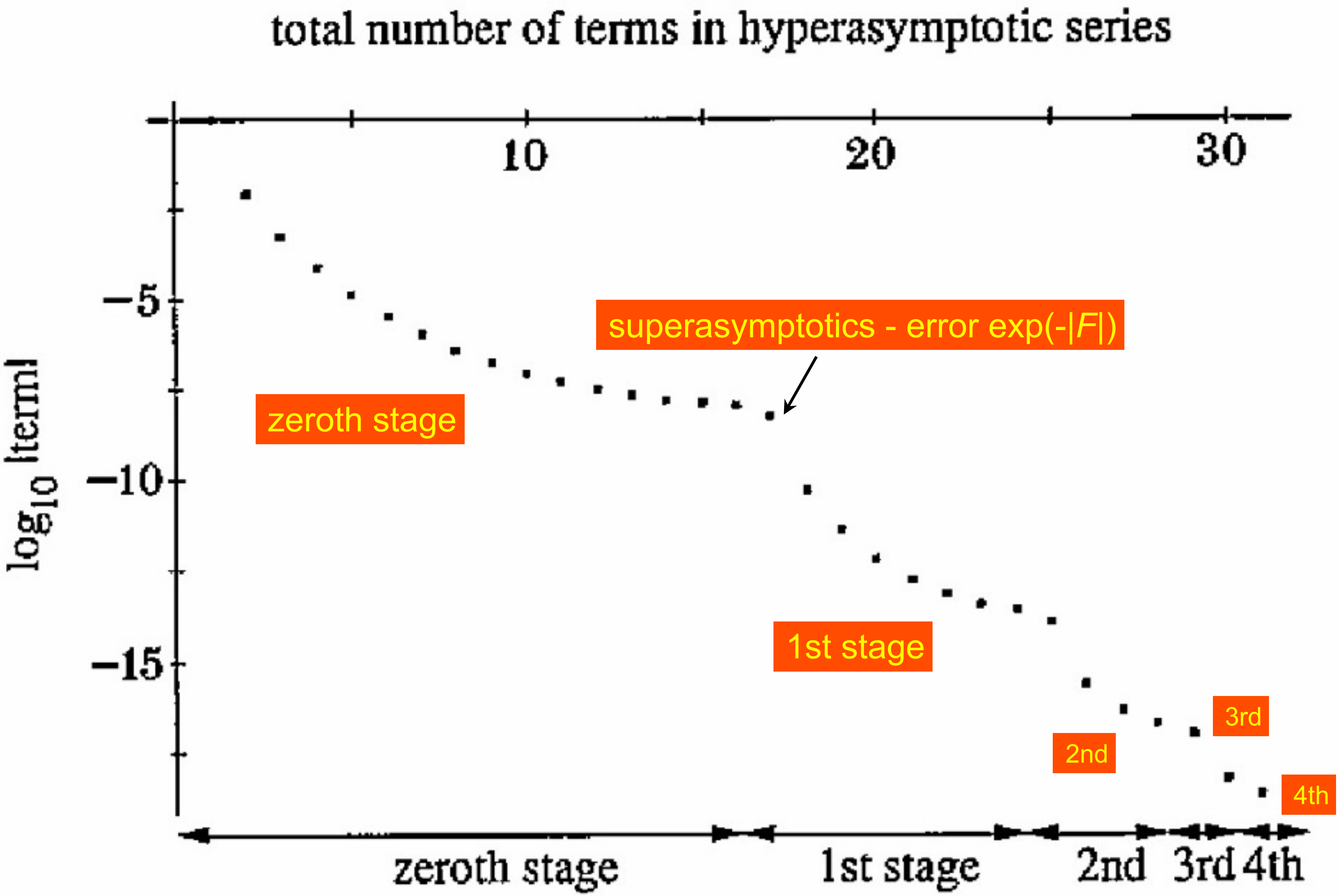

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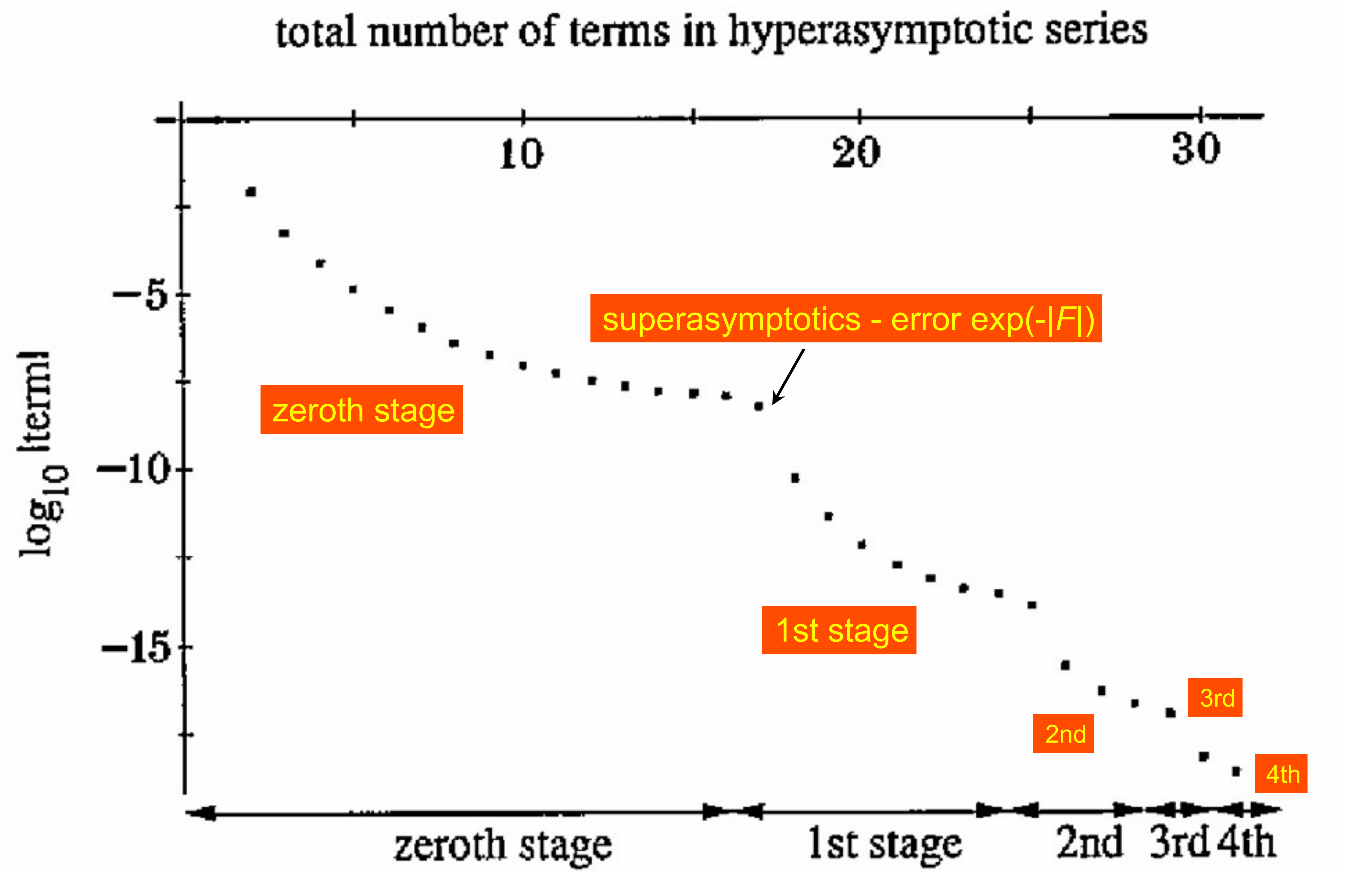

optimally truncated hyperseries get shorter

with more than two exponentials, graph structure of higher approximants, e.g. multisaddle integrals:

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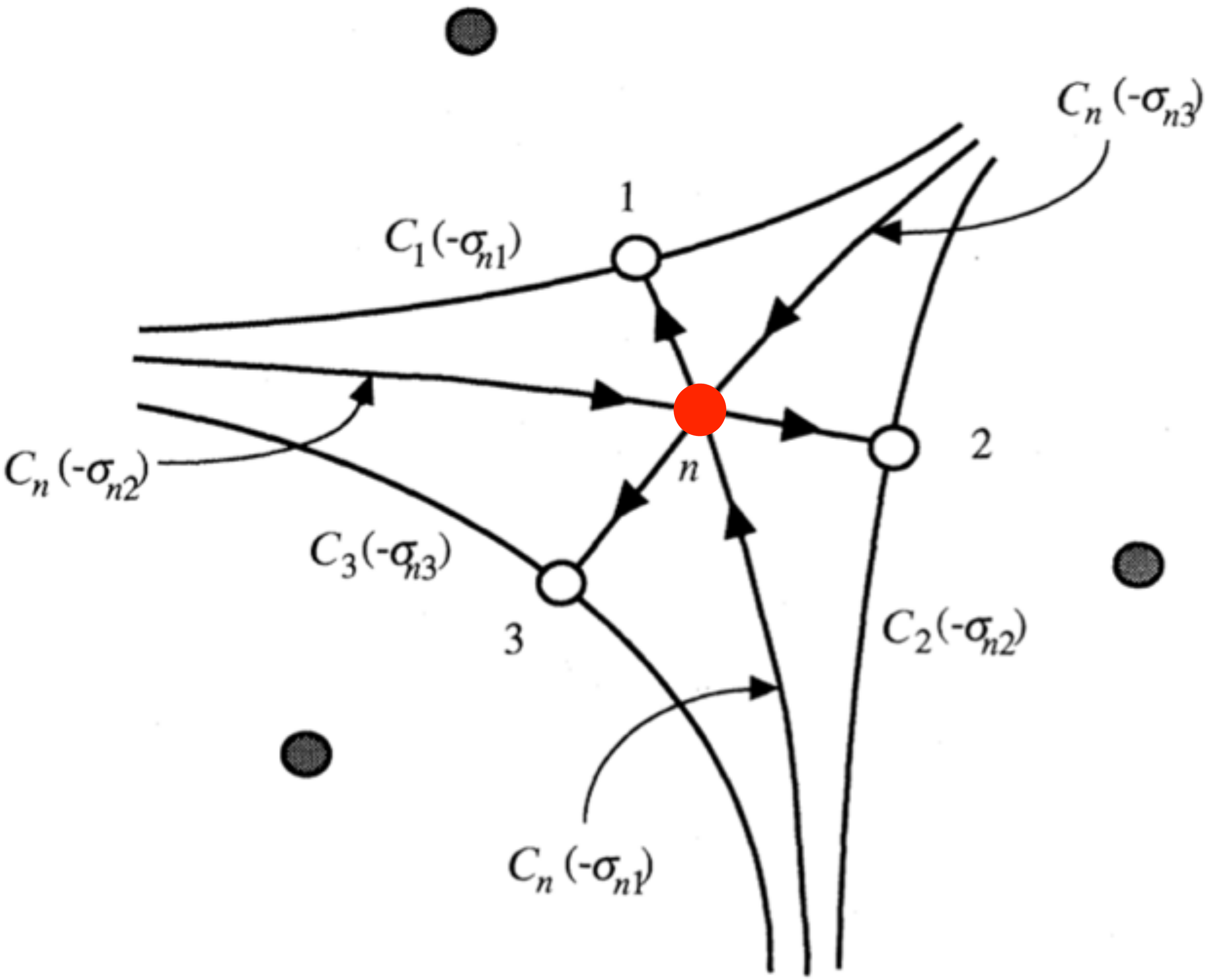
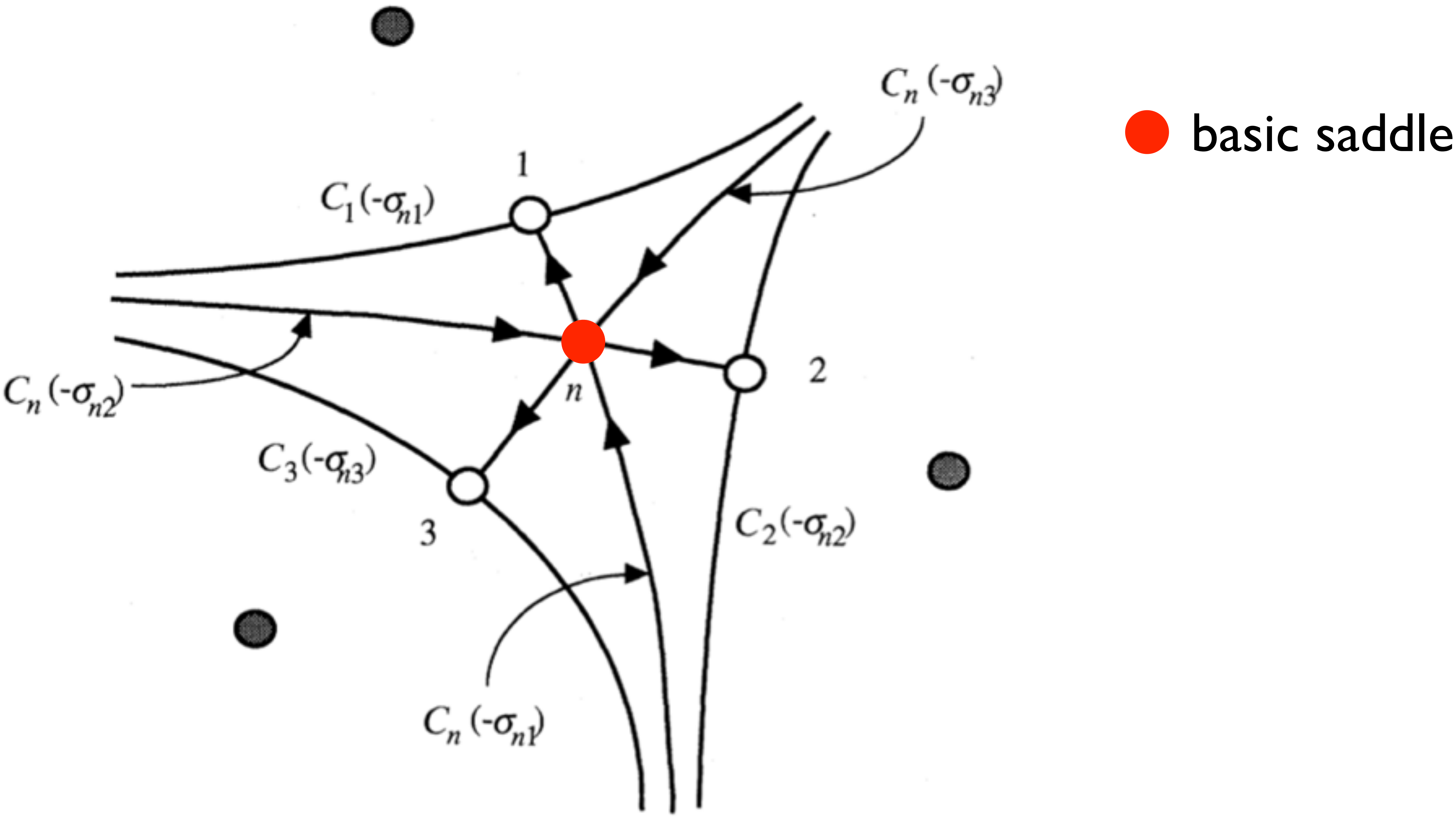


with more than two exponentials, graph structure of higher approximants, e.g. multisaddle integrals:



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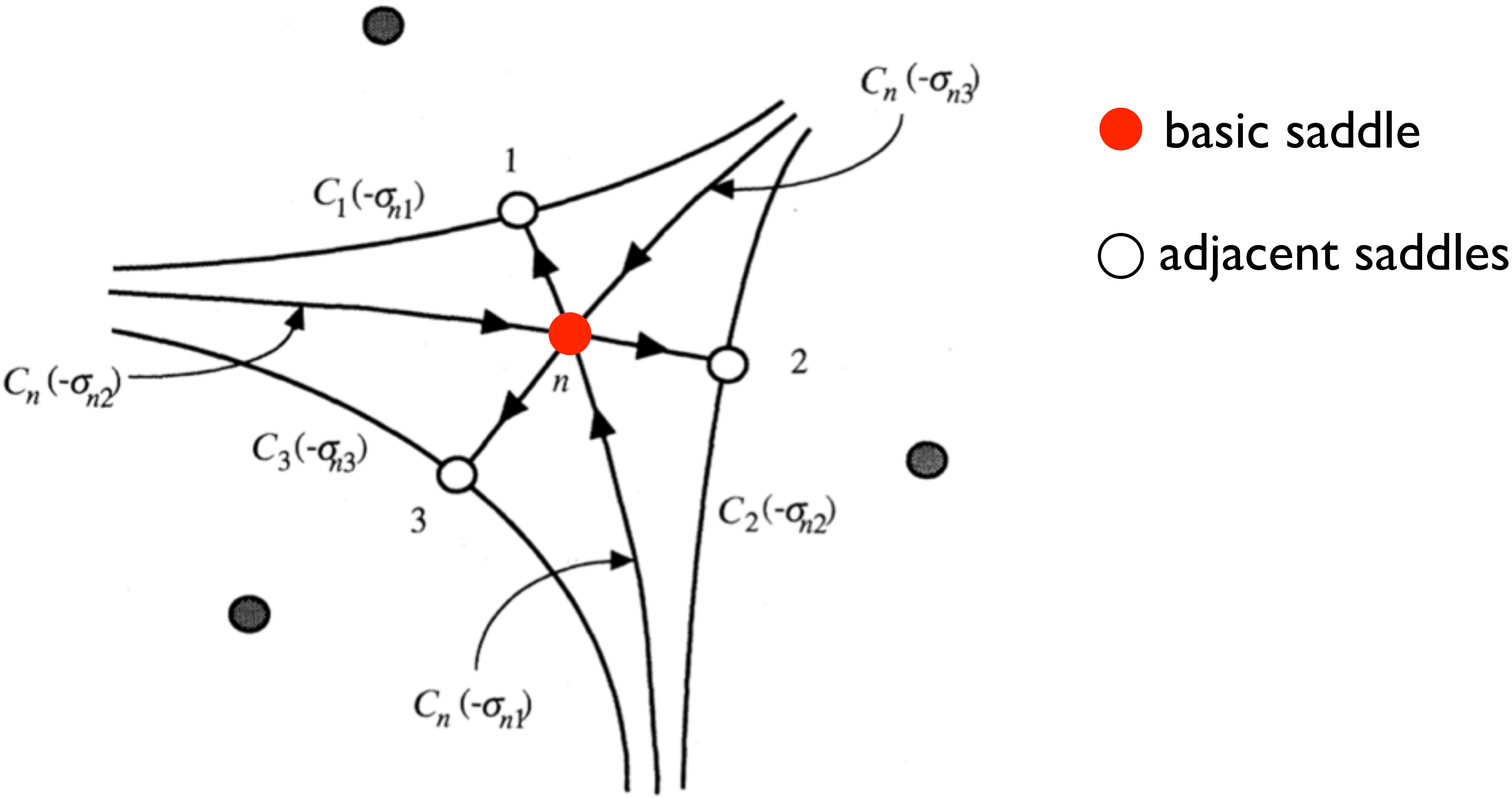

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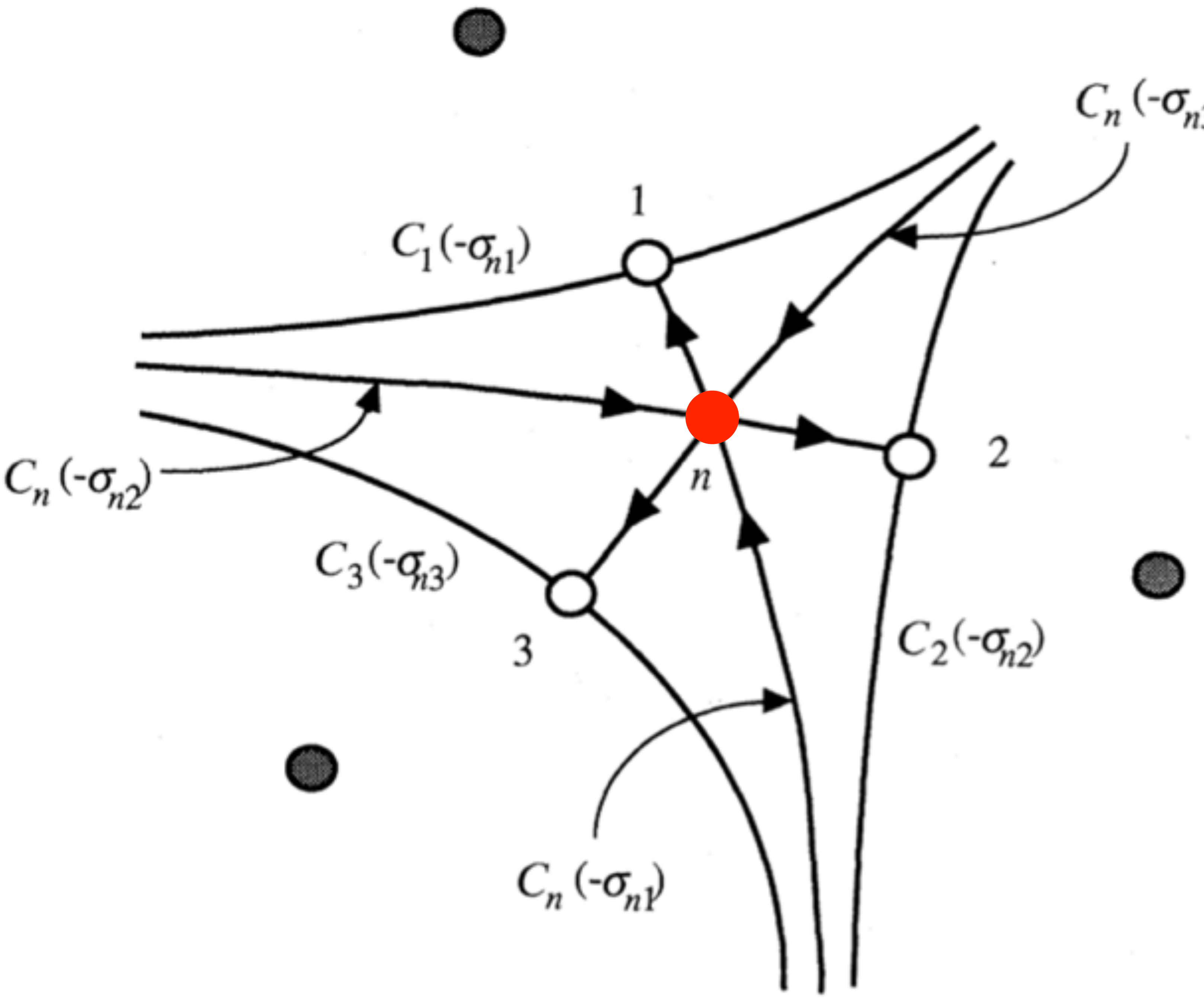

● basic saddle

○ adjacent saddles reached on descent paths from ● as $\arg F$ varies

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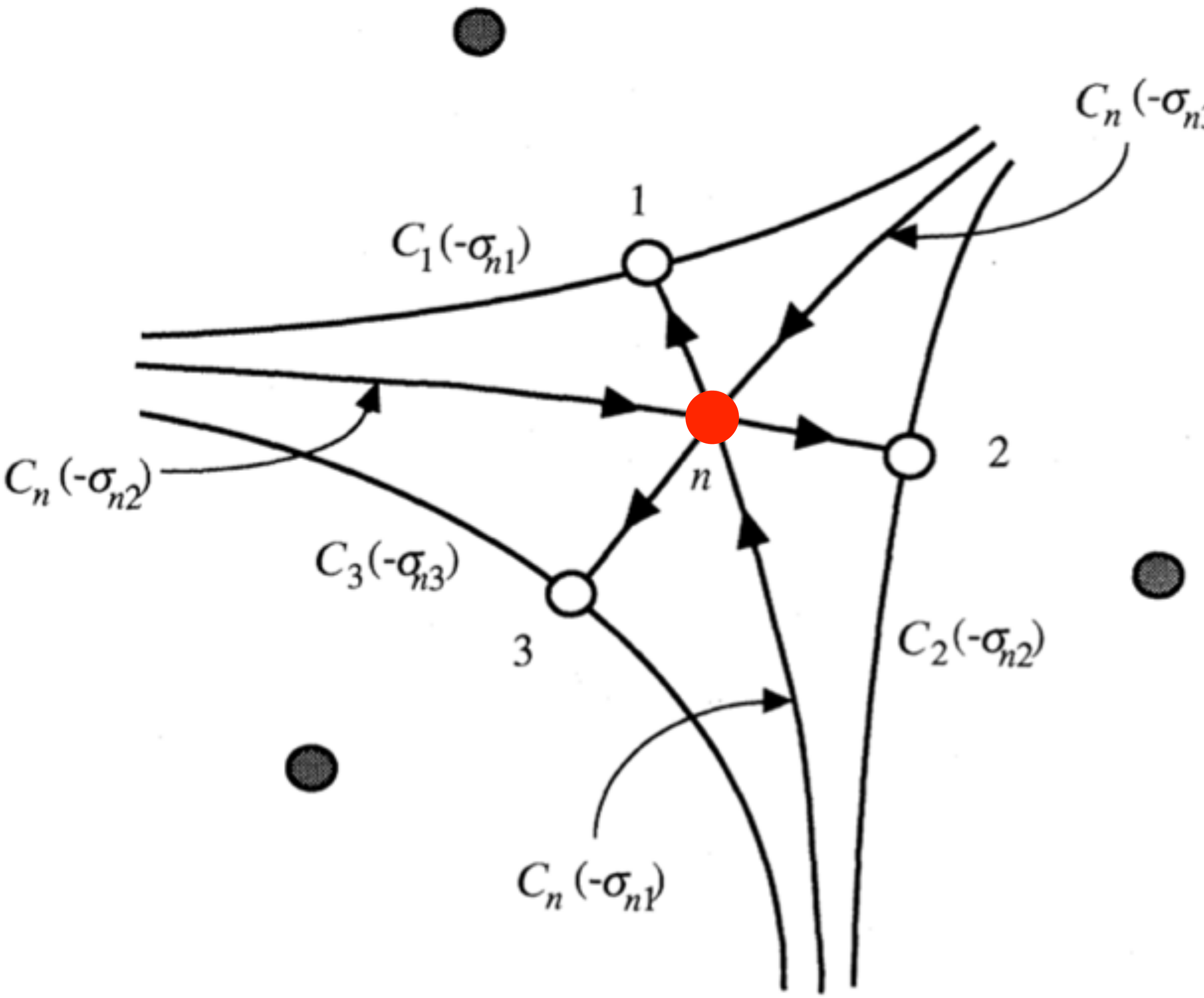
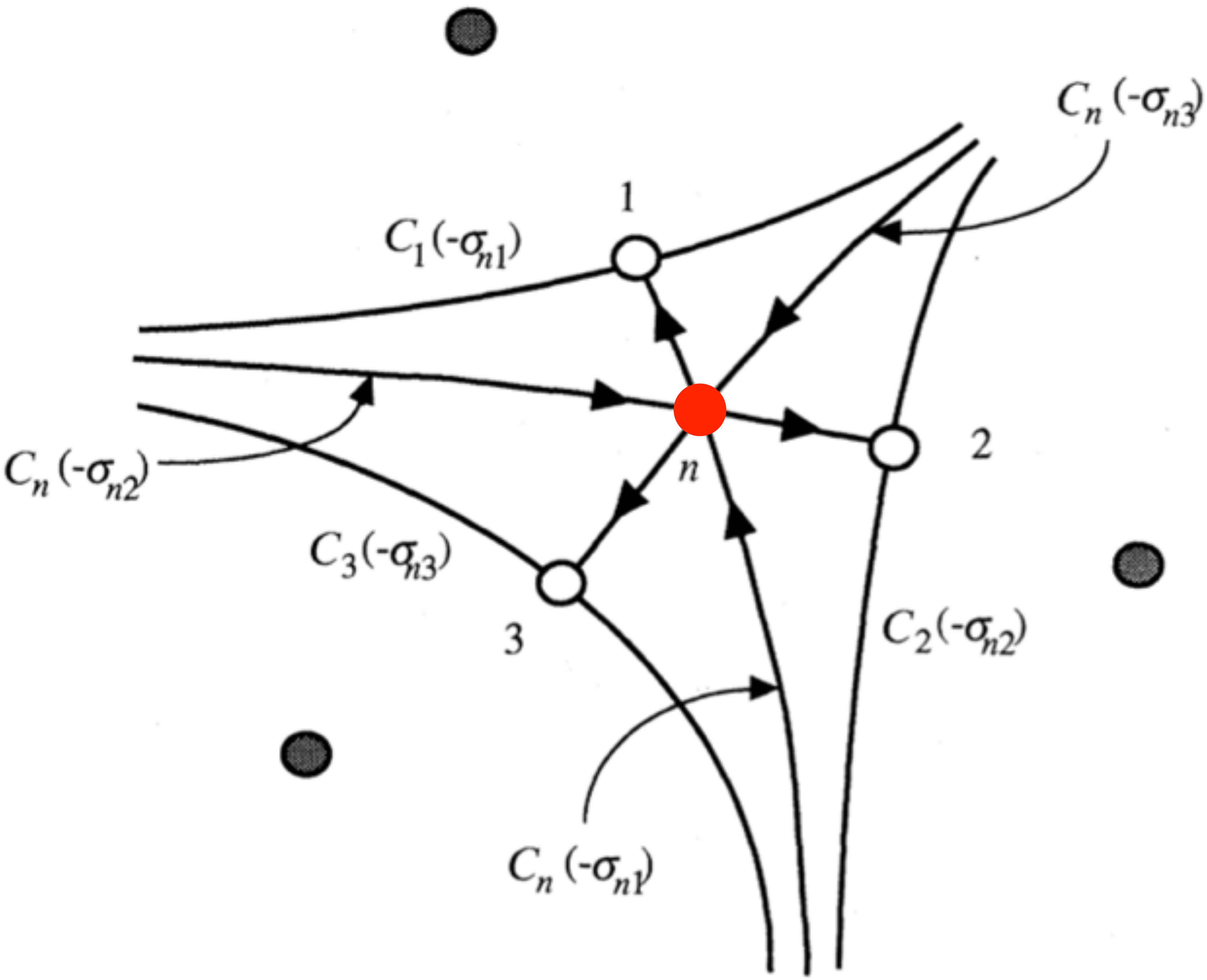


basic saddle

adjacent saddles reached on descent paths from [red dot] as $\arg F$ varies

non-adjacent saddles

with more than two exponentials, graph structure of higher approximants, e.g. multisaddle integrals:



basic saddle

adjacent saddles reached on descent paths from basic saddle as $\arg F$ varies

non-adjacent saddles

more adjacent saddles introduced at successive stages of hyperasymptotics

example: Pearcey integral

$$P(x,y) = \int_C \mathrm{d}t \exp\left\{\mathrm{i}\left(\tfrac{1}{4}t^4 + \tfrac{1}{2}xt^2 + yt\right)\right\}$$

$$x = 7, \quad y = 1 + \mathrm{i}$$

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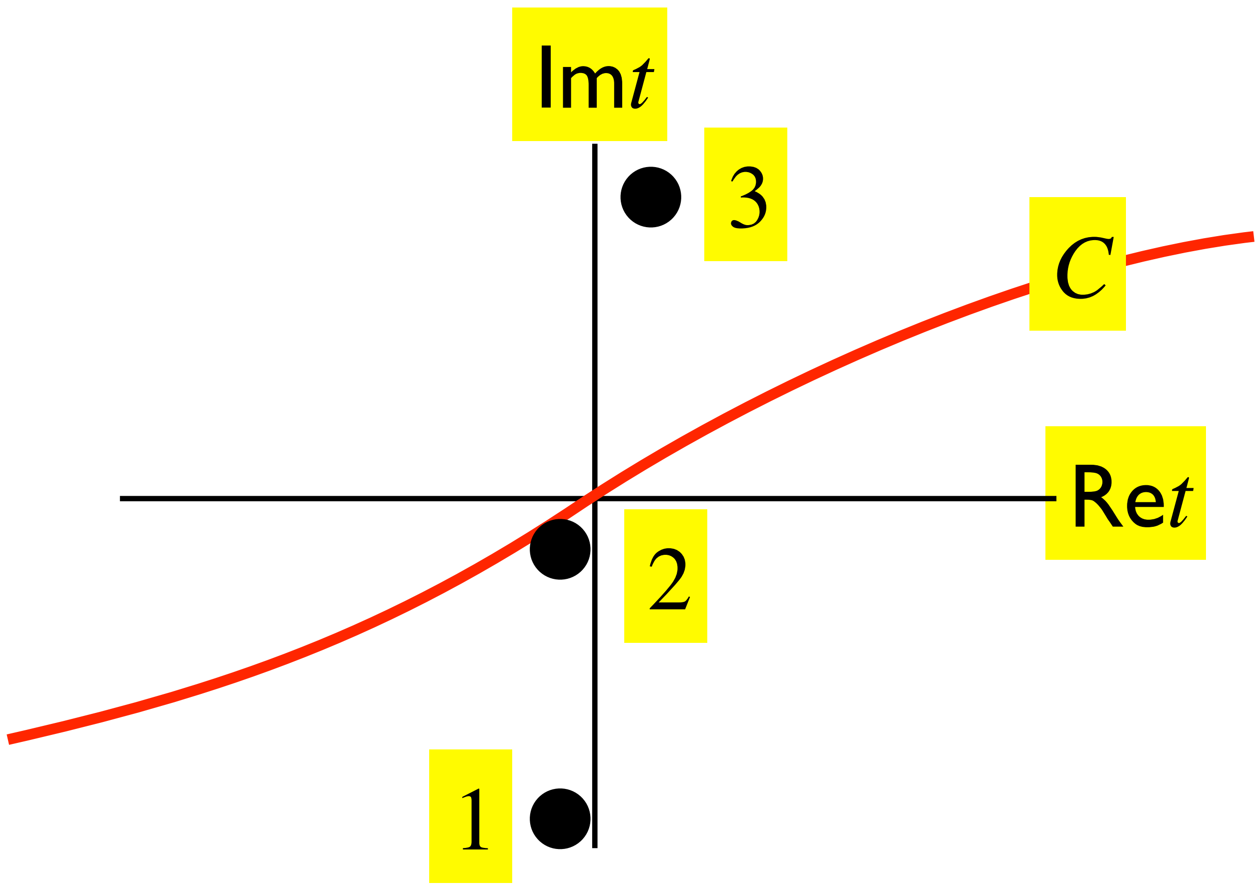

three saddles ●

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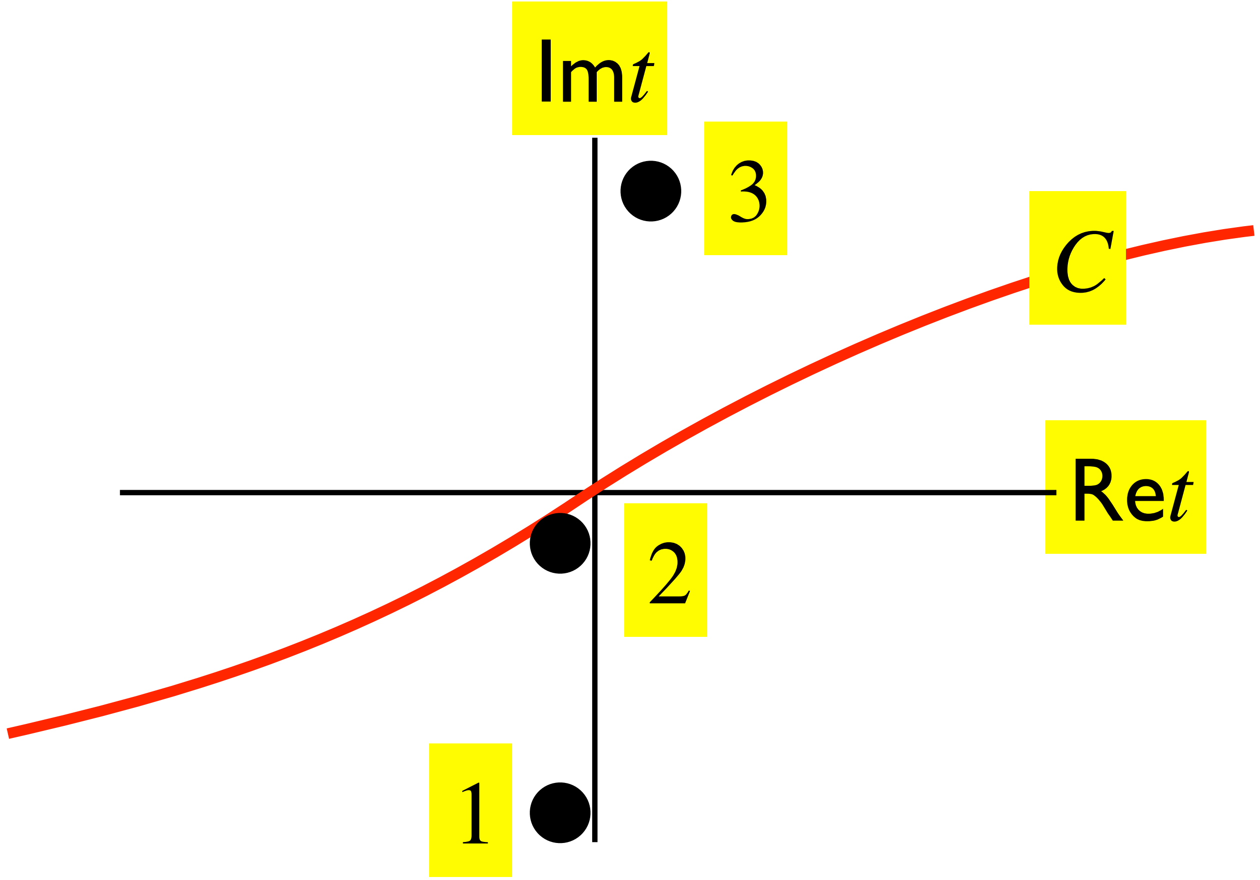
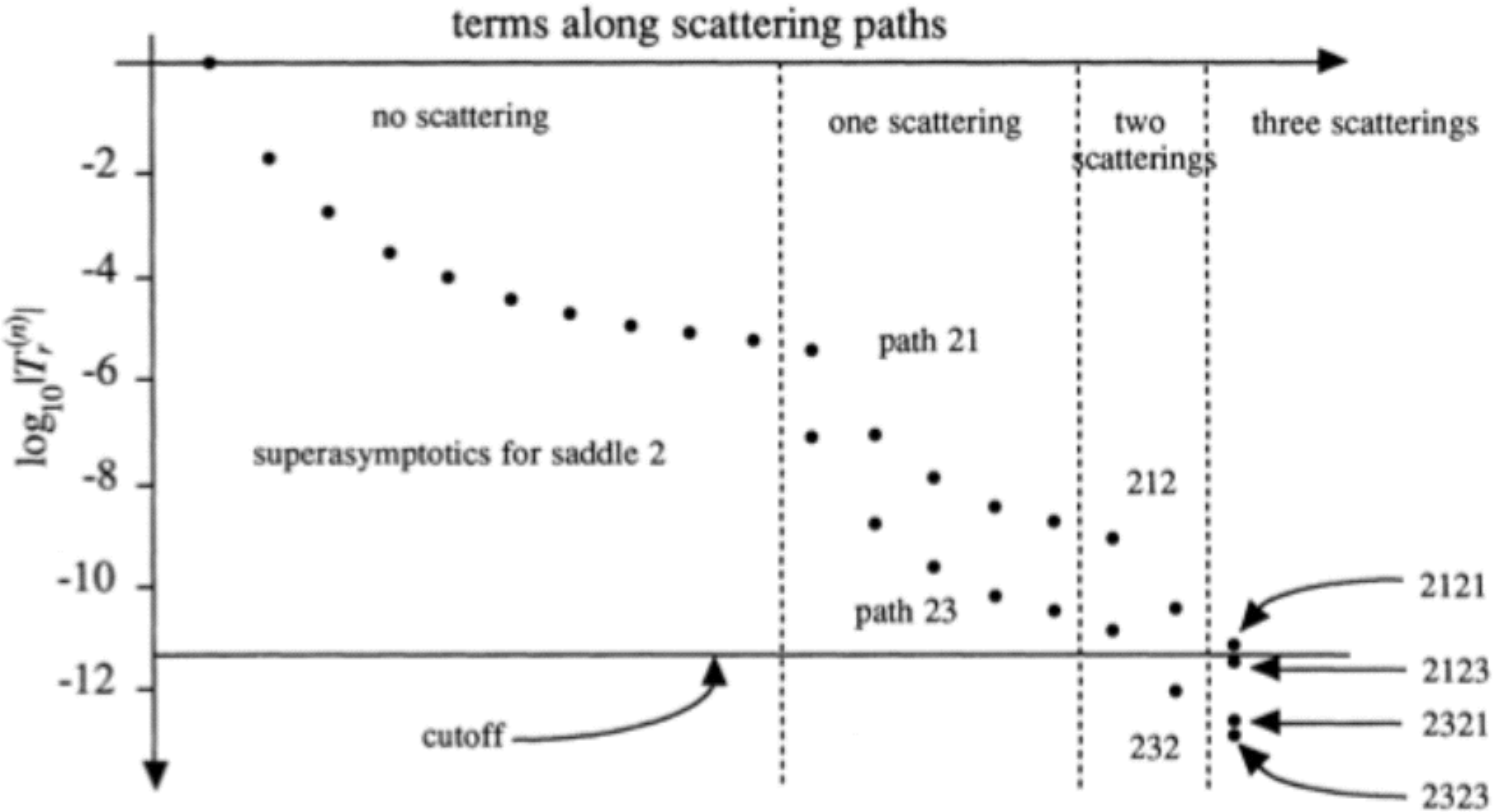
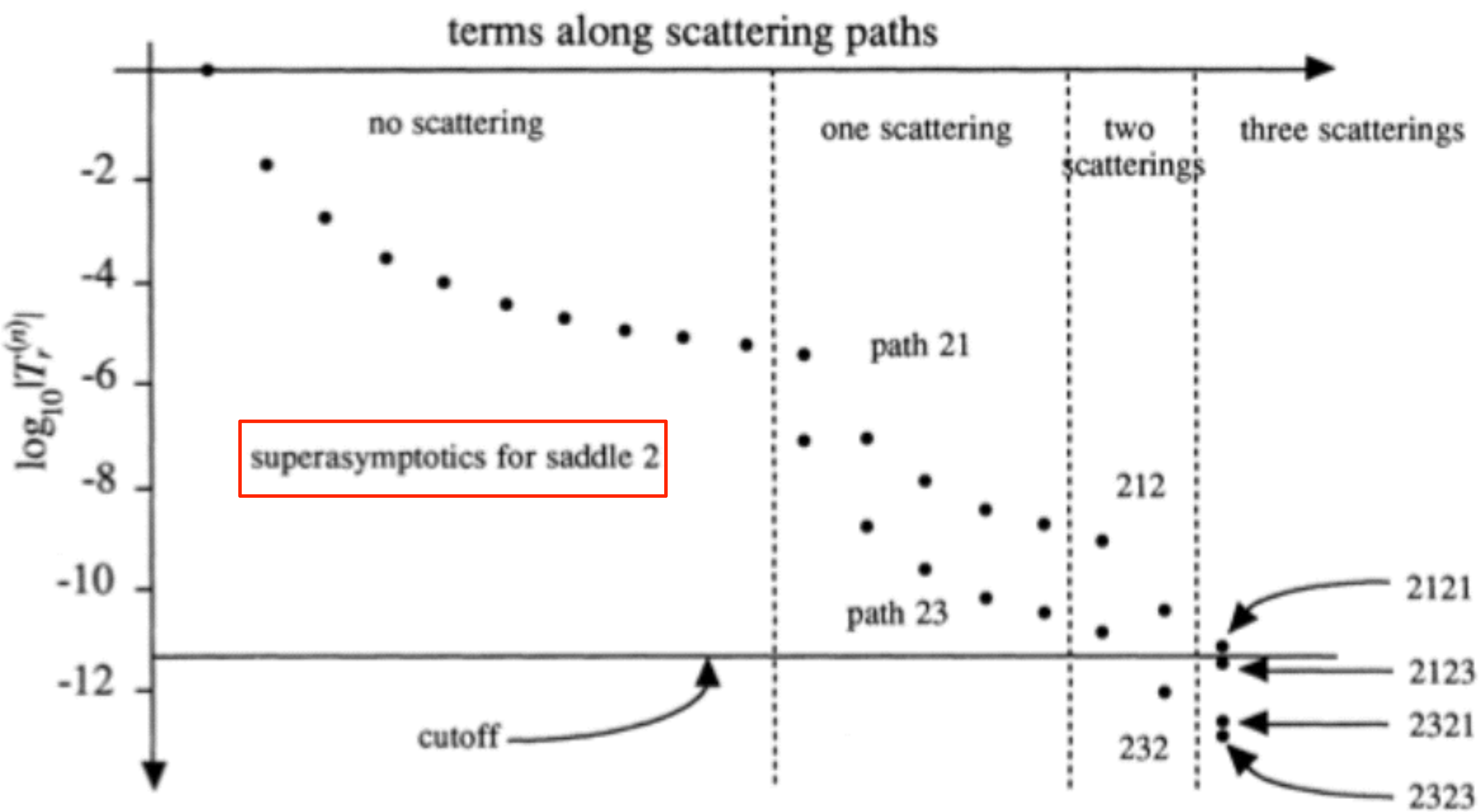


three saddles ●

hyperasymptotics generates a sequence of series, from 'scatterings' between saddles



terms along scattering paths
no scattering
one scattering
two scatterings
three scatterings
-2
-4
-6
-8
-10
-12
$\log_{10}|T_r^{(n)}|$
path 21
superasymptotics for saddle 2
212
2121
path 23
2123
2321
cutoff
232
2323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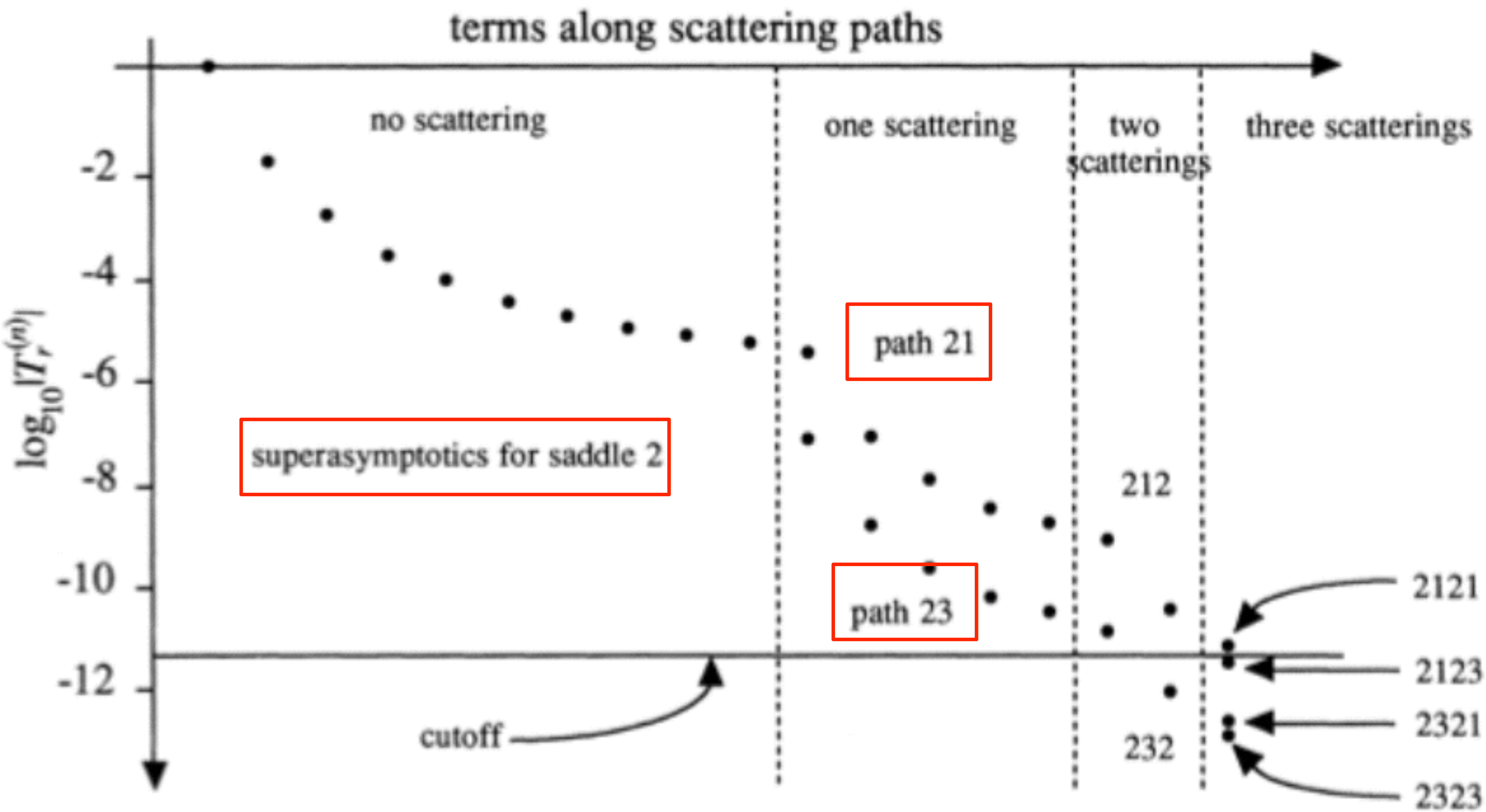
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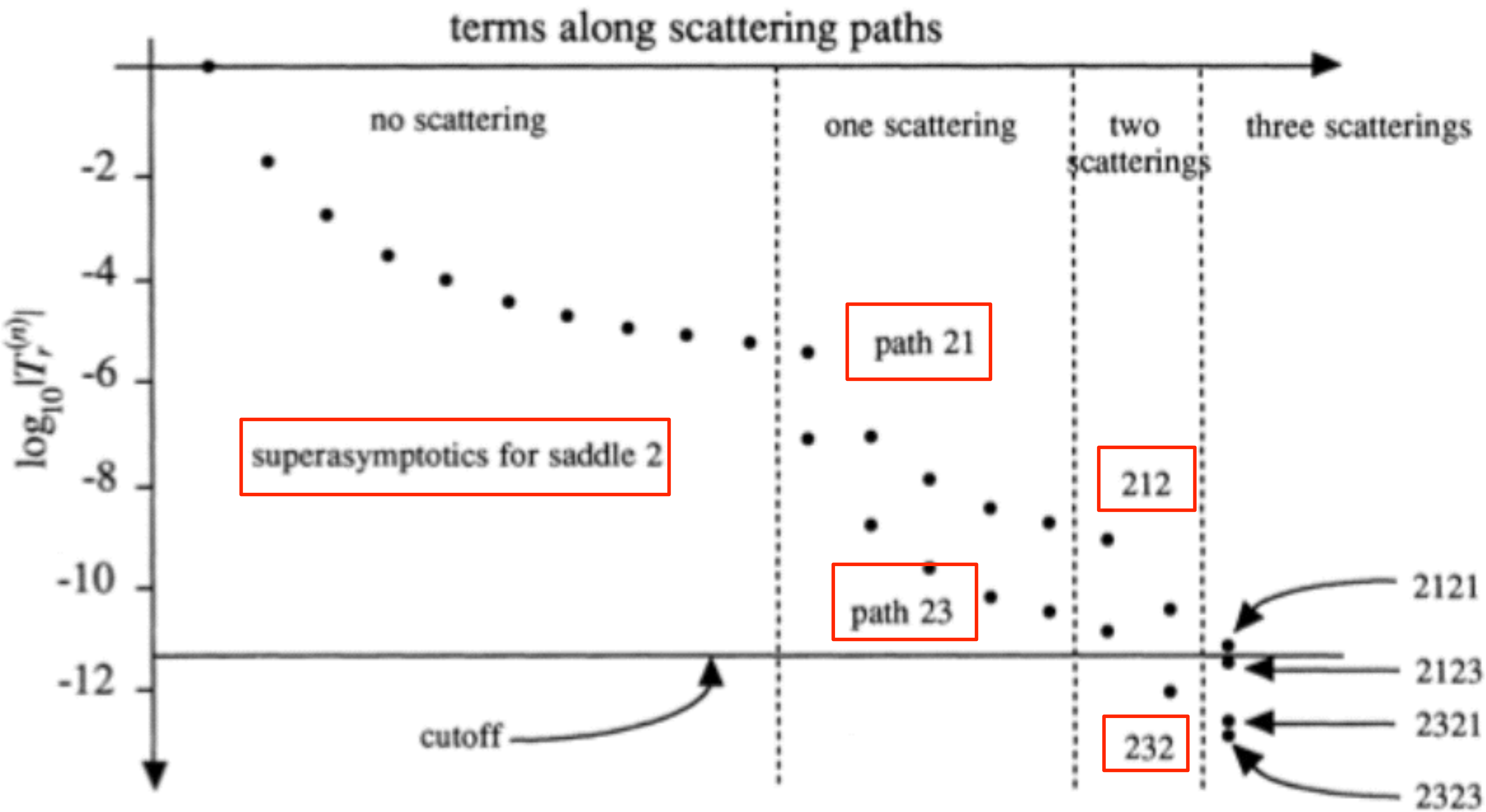

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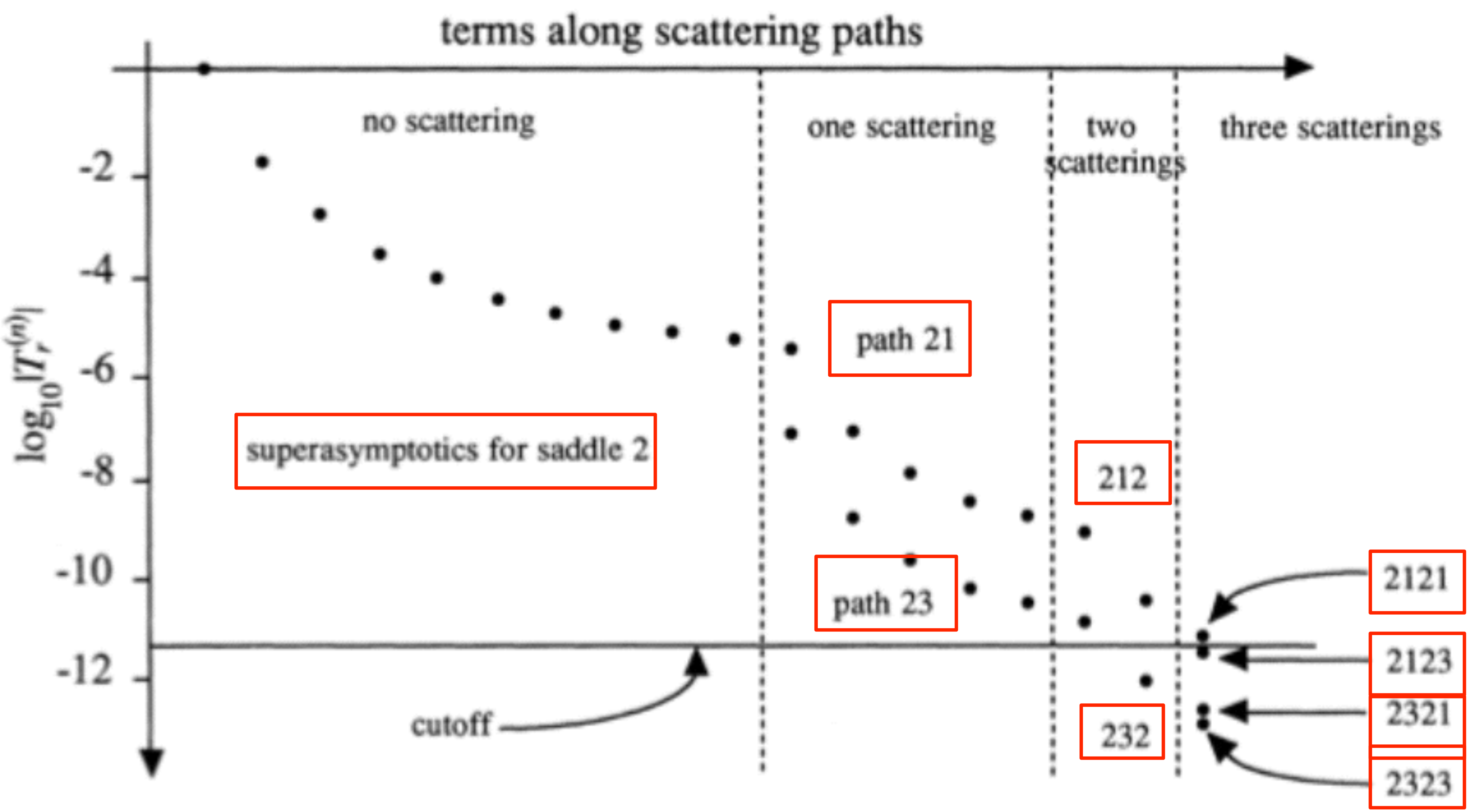

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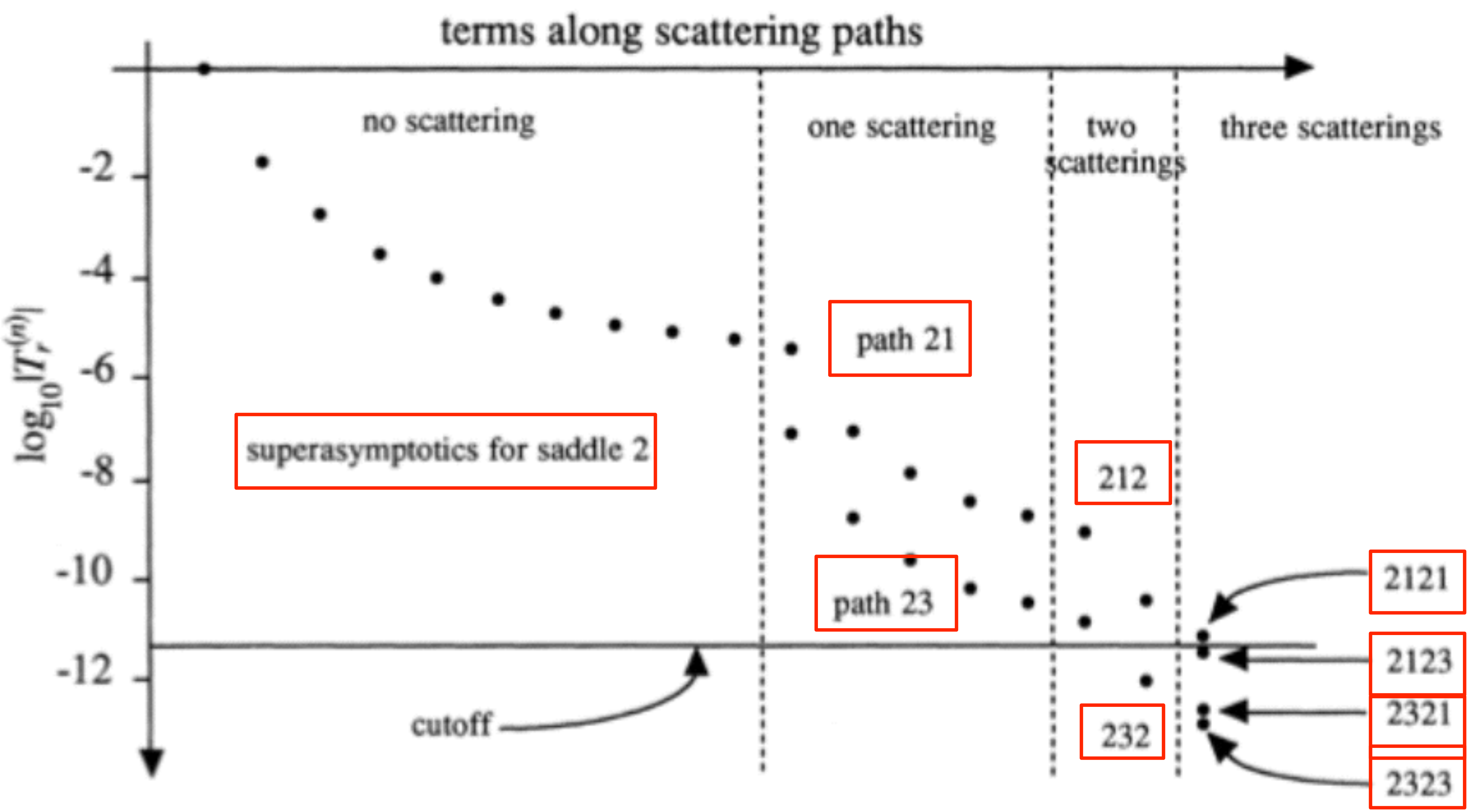
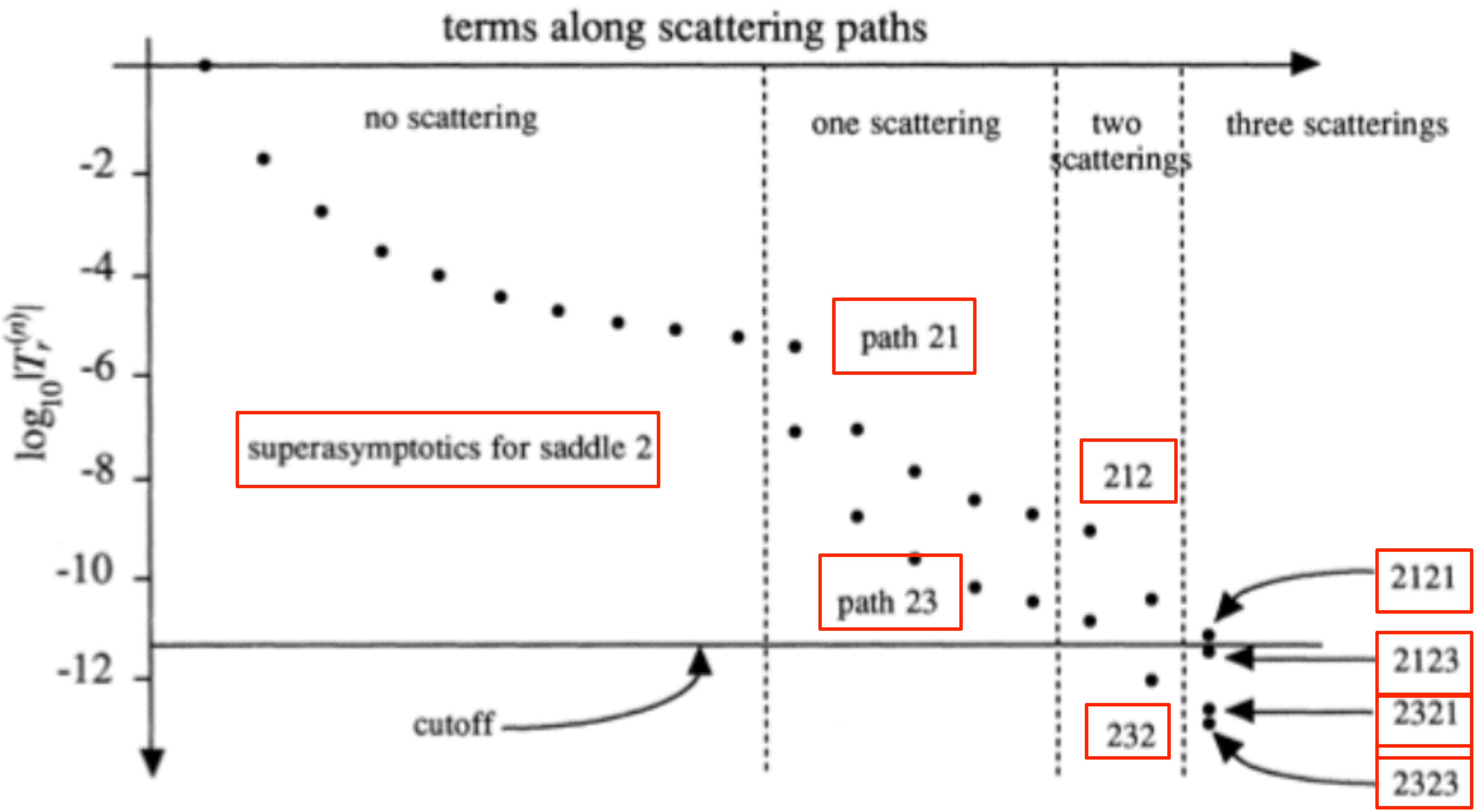


level	approximation to P (7, 1+i)	|approx./exact − 1|
lowest	0.779703507027512 + i 0.765551648542315	1.496×10^{-2}
super.	0.788920520763900 + i 0.752101783262683	2.916×10^{-6}
ultimate hyper.	0.788922837595360 + i 0.752103959759701	1.535×10^{-12}
exact	0.788922837596969 + i 0.752103959759243	0



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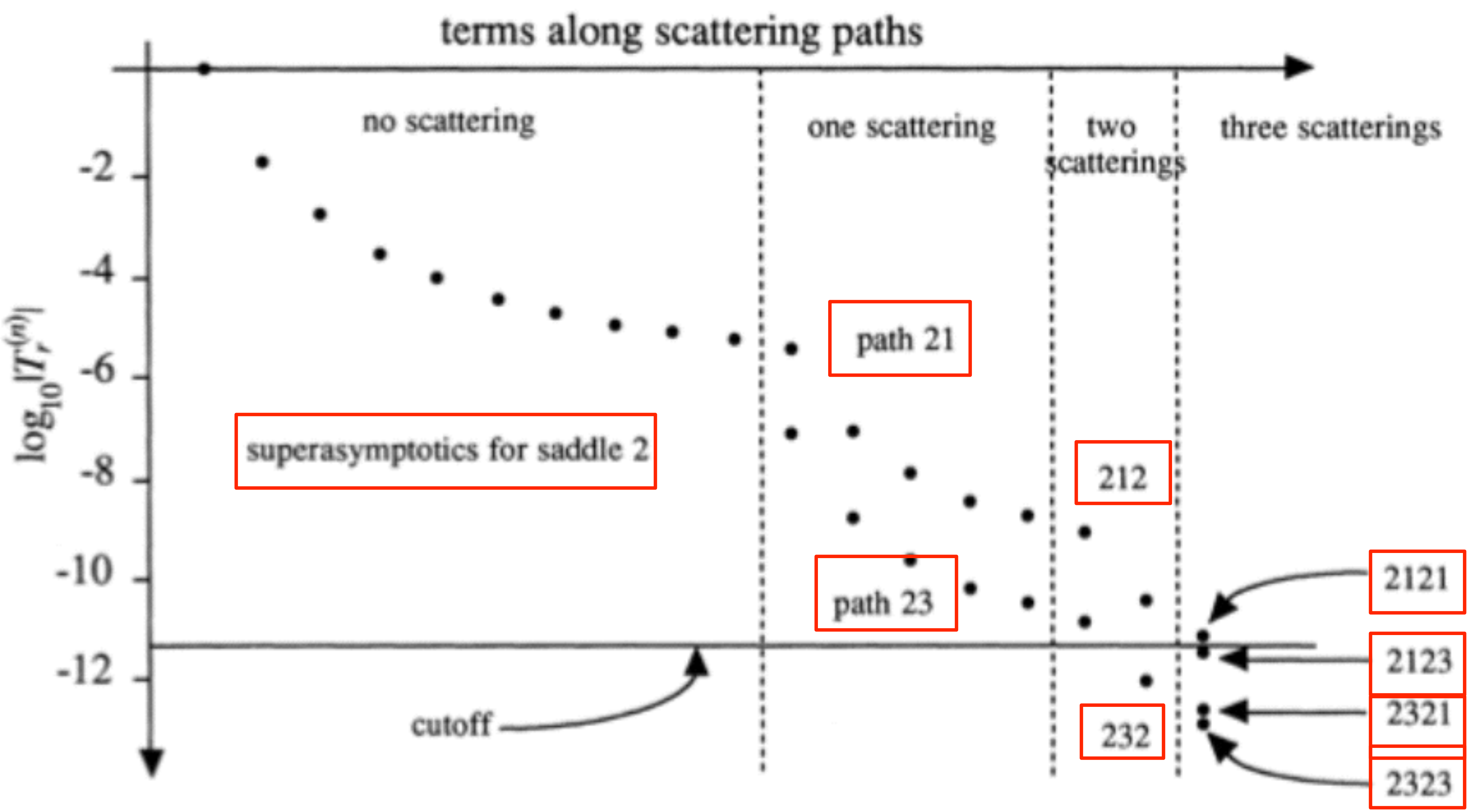

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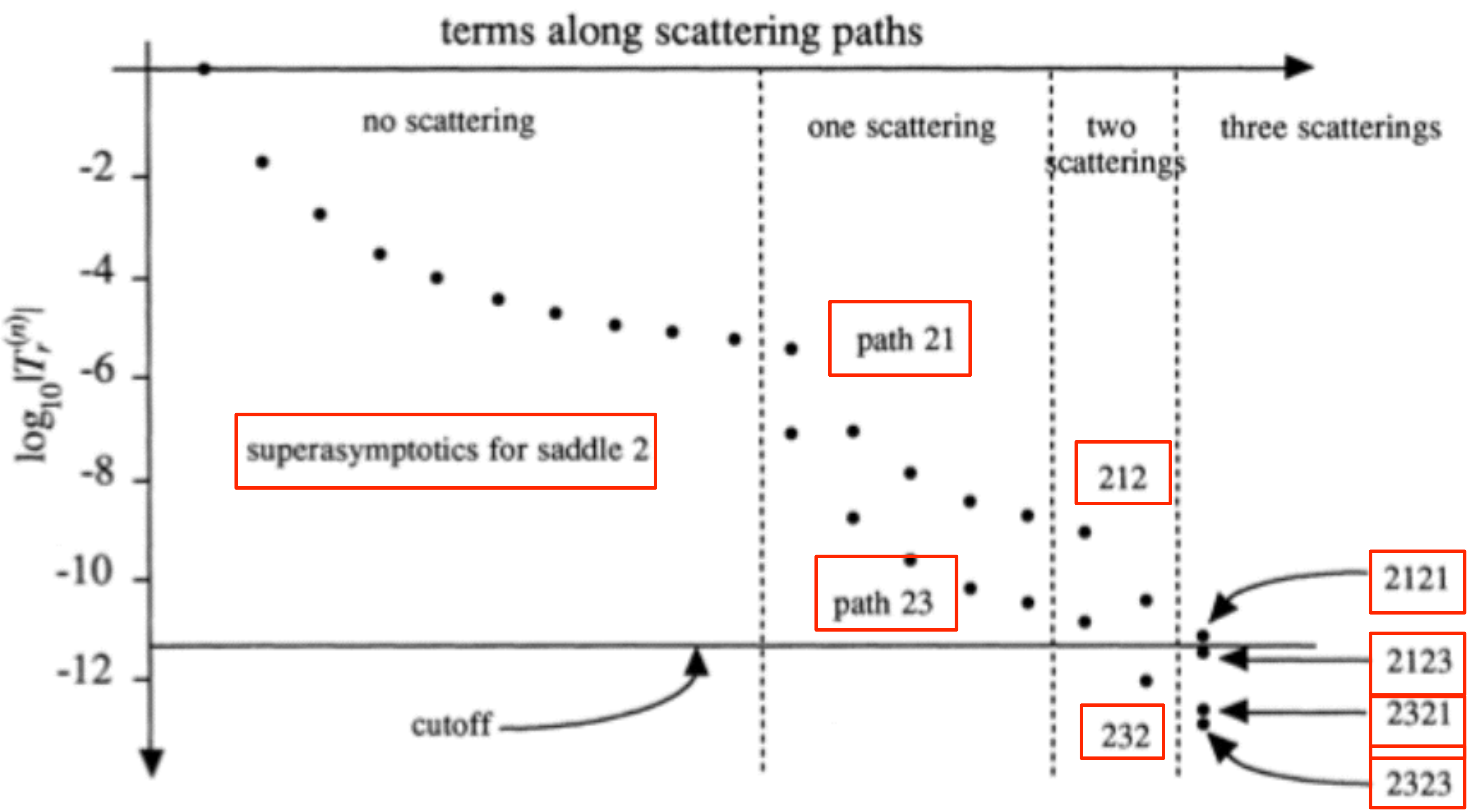

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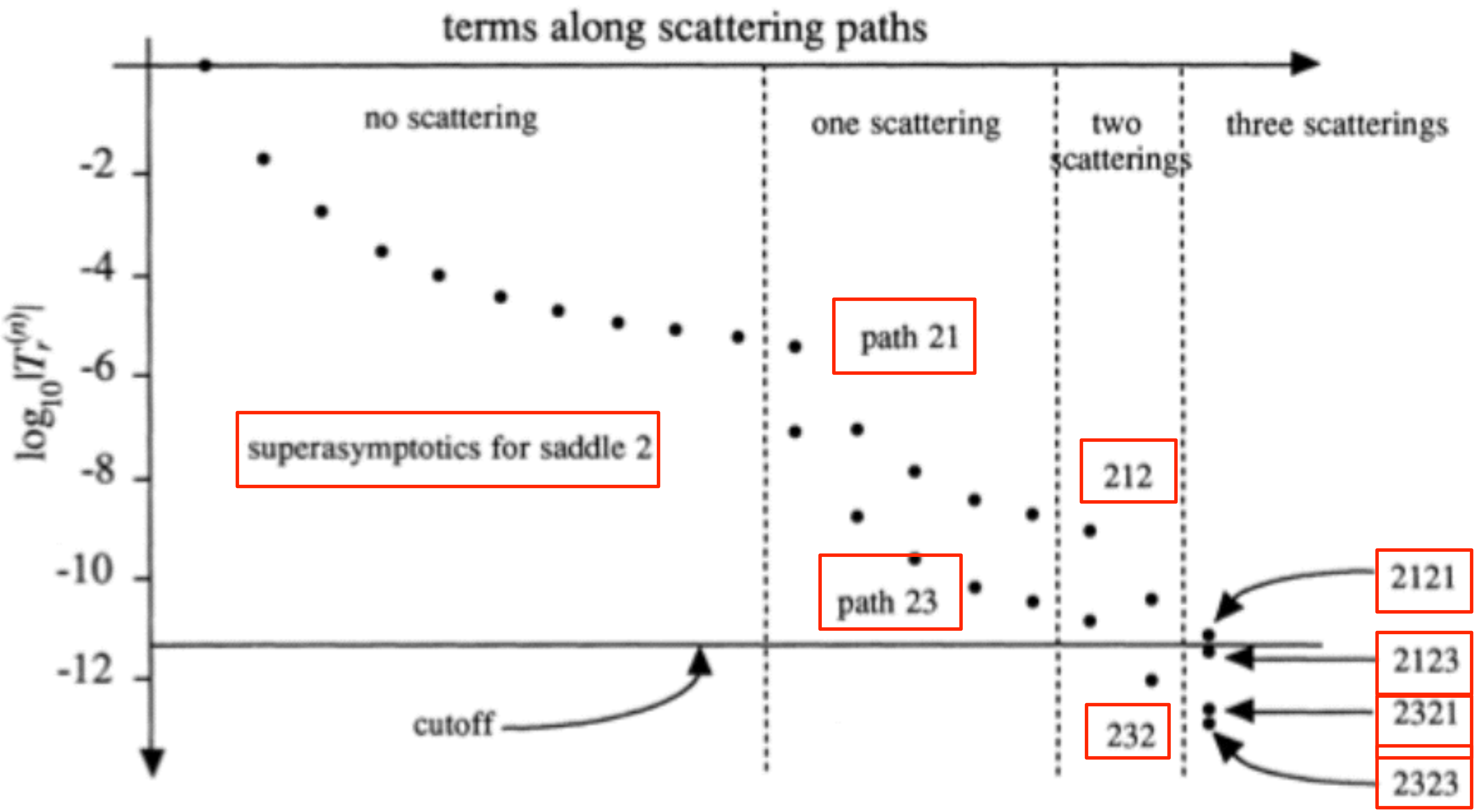

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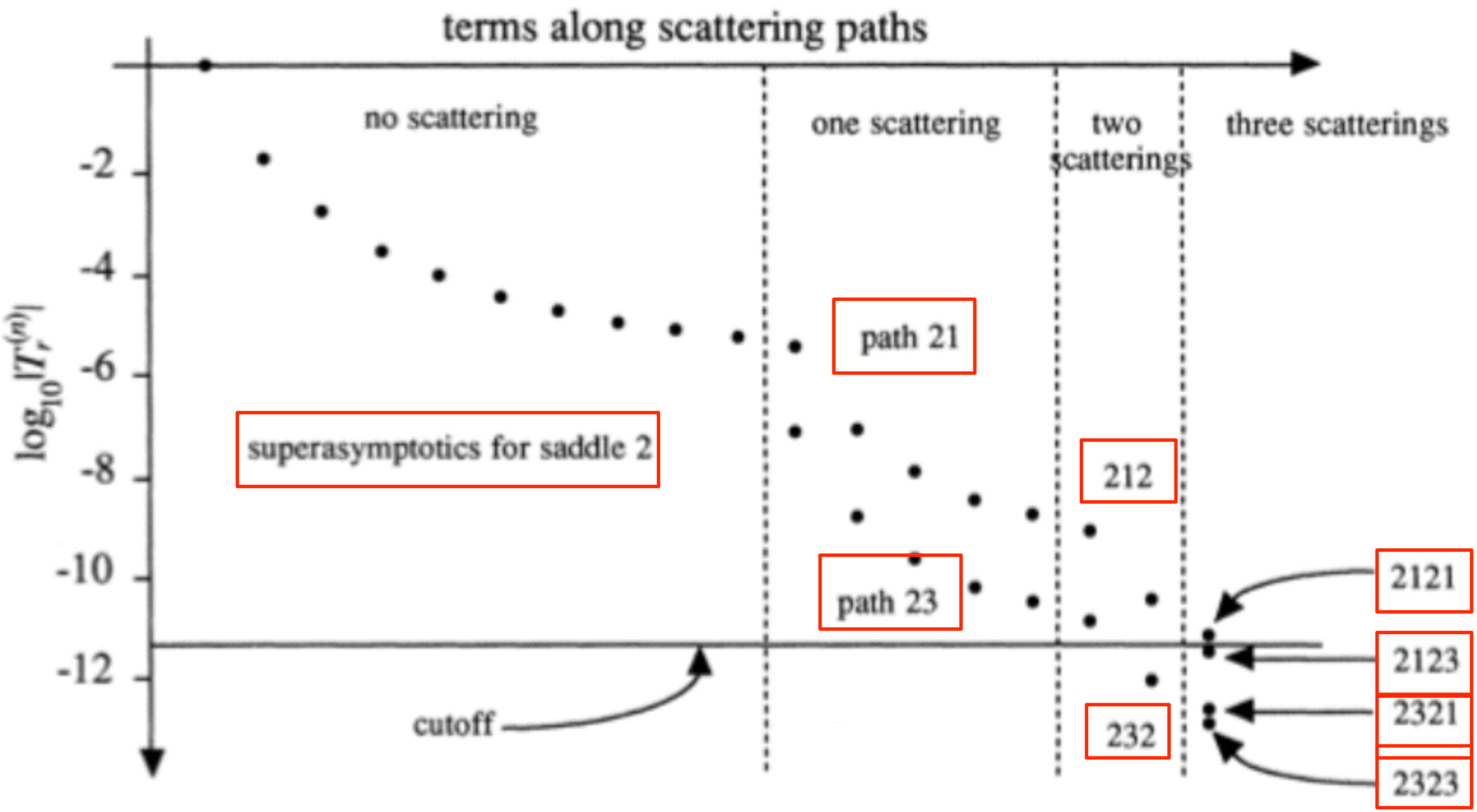

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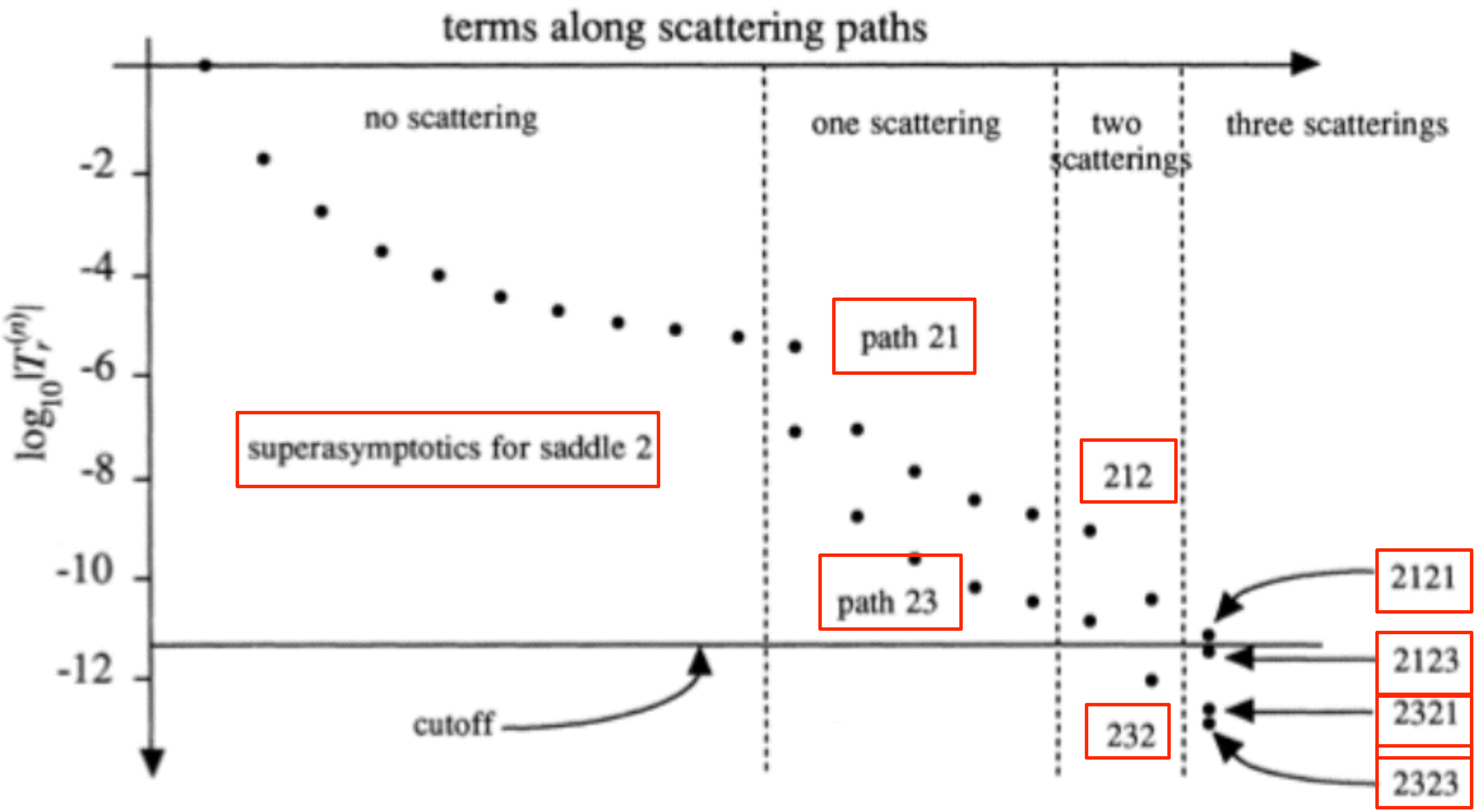

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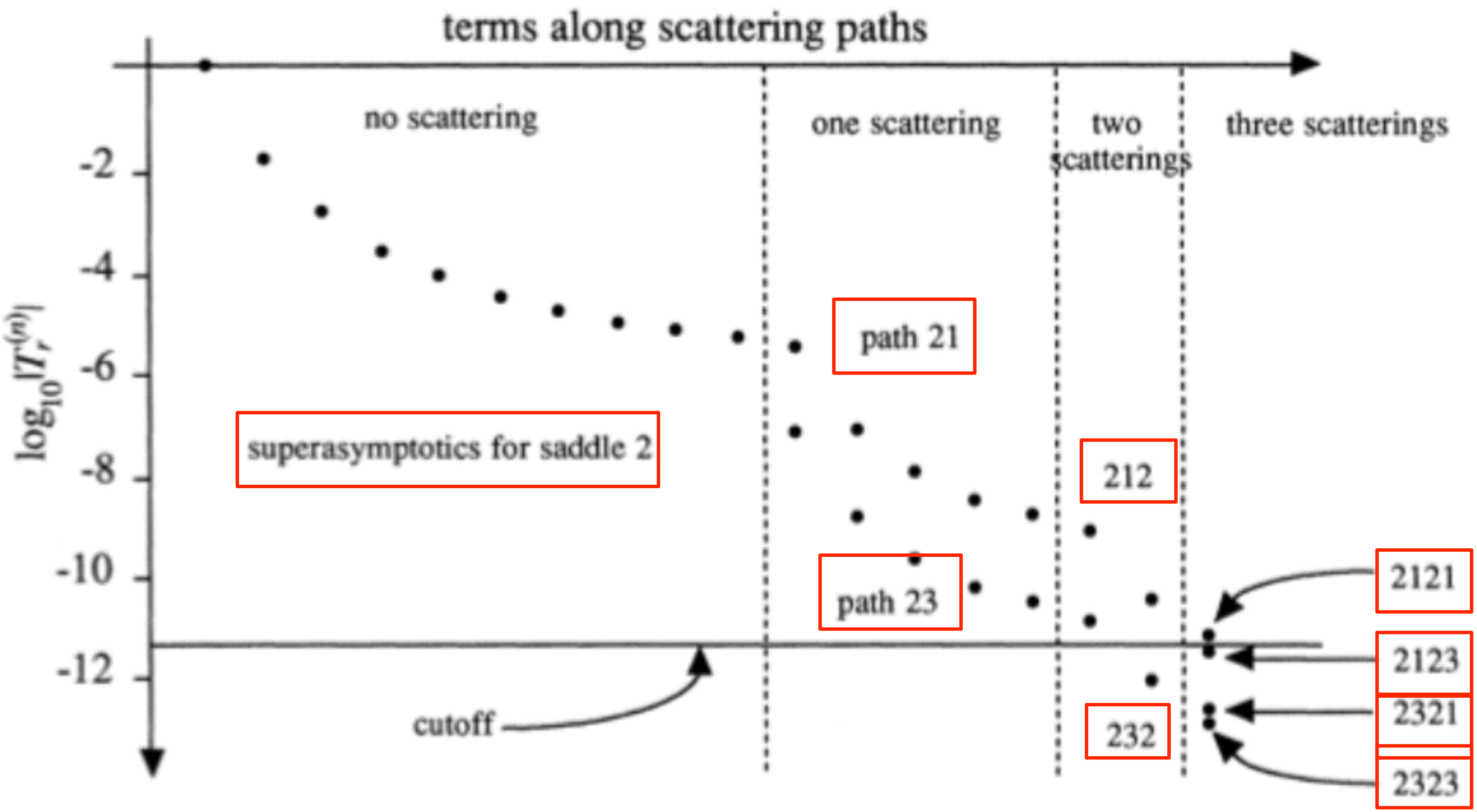

level	approximation to P (7, 1+i)	|approx./exact − 1|
lowest	0.779 703 507 027 512 + i 0.765 551 648 542 315	1.496×10^{-2}
super.	0.788 920 520 763 900 + i 0.752 101 783 262 683	2.916×10^{-6}
ultimate hyper.	0.788 922 837 595 360 + i 0.752 103 959 759 701	1.535×10^{-12}
exact	0.788 922 837 596 969 + i 0.752 103 959 759 243	0

Legacy from Euler, Dingle, Écalle... from Stokes's insistence on understanding how the rainbow's dark side is connected to the interference fringes on its bright side:

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in the 1990s, 2000s, much new mathematics originating from resurgence, etc:
Boyd, Chapman, Delabaere, Dunster, Ecalle, Howls, Kruskal, Olde Daalhuis, Lutz, McLeod, Paris, Olver, Ramis, Pham, Segur, Temme, Voros, Wong, Wood...

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Analytic Continuation Of Chern-Simons Theory

Edward Witten

School of Natural Sciences, Institute for Advanced Study
Einstein Drive, Princeton, NJ 08540 USA

The semi-classical expansion and resurgence in gauge theories: new perturbative, instanton, bion, and renormalon effects

Philip C. Argyres[1] and Mithat Ünsal[2]

Lectures on non-perturbative effects in large N theory, matrix models and topological strings

Marcos Mariño

Département de Physique Théorique et Section de Mathématiques,
Université de Genève, Genève, CH-1211 Switzerland

Resurgence and Trans-series in Quantum Field Theory: The $\mathbb{CP}^{N-1}$ Model

Gerald V. Dunne[1] and Mithat Ünsal[2]

Introduction to 1-summability and the resurgence theory

David Sauzin

Decoding perturbation theory using resurgence: Stokes phenomena, new saddle points and Lefschetz thimbles

Aleksey Cherman,[1] Daniele Dorigoni[2] and Mithat Ünsal[3]

Resurgence and Transseries in Quantum, Gauge and String Theories

from 30 June 2014 to 4 July 2014 (Europe/Zurich)
CERN
Europe/Zurich timezone

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The goal of this CERN TH-Institute/Conference is to bring together researchers in Mathematics (with backgrounds in resurgence, transseries, asymptotic analysis), with researchers in Theoretical Physics (with backgrounds in quantum mechanics, gauge theory, string theory) working on topics where resurgent methods are promising.

This meeting will have an interdisciplinary nature, where interactions between different communities are likely to foster new advances and results in this exciting subject.

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- Overview
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- Timetable
- Registration
  - Registration Form
- Participant List

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steps in humanity's long struggle to understand infinity