

# Divergent series: from Thomas Bayes to resurgence via the rainbow 

## Michael Berry

H HWills Physics Laboratory, University of Bristol, UK
http://michaelberryphysics.wordpress.com
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the subject has been repeatedly reborn, more deeply each time


If the following observatione do not secen to you to be foe moniut, $g$ shou'd cotcom it as a faver if you wou'd plears to conomuwicale them to the rayal secith
that
It hat becn afourted by vome cmonent onaflemaficians, the aum of $4^{*}$ logarithmy of the numbers $1.2 .3 .4 .8 \%$ to $z$ is equal to $\frac{1}{2} \log , c+\overline{2+\frac{1}{2}} \times \log , 2$ lefoenced by the sorien

$$
z=\frac{1}{12 z}+\frac{1}{360 z^{3}}-\frac{1}{1260 z^{5}}+\frac{1}{1640 z^{7}}-\frac{1}{1188 z^{9}}+8 f \quad \text { if c denotc }
$$ the circumference of a circle whose radsut is unity. And it is true that thi exprefoion with very wearly approach to the value of that dum when $z$ is targe, $\$$ you take in onty a proper number of the first terms of the foregoing series: but the whole deried can never properly expiof any quantity at all; because aftor the $s^{\text {th }}$ term the cotffiensth begin to incruase, they afteowanis incriase at a greater rath than whal can be compinsalid by the increare of the powen of ty cowideaing the follewing mancer in which the cocfficicets of that

 If $f=2 d a+2 c b \quad 13 f=2 a a+2 d b+c^{2} \quad 15 g=2 f a \mathrm{~m} 2 \mathrm{f} b+2 d c$, \& , om, then take $A=a \quad B=2 b \quad C=2 \times 3 \times 4 C \quad D=2 \times 3 \times 4 \times 5.6 d$ $E=2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \in \&$ so on, \& $A, B, C, D, E, F$, fr with be the cosfficients of the forigoing swics: from whence it casily follorty that if any term in the itries aftor the 3 first be called $y$ * its distanic from the, st $t \mathrm{crm}$, the nest term immedialicy
 length the subrequent termy of this sewts arc greake than the precee= ding onct s increate in infinitut os thencforc the wertil can have wo ulfimate value whahococr.
Mueh tef can that scrict bhere any whimate value, which is deduced from it by taking $z=1$ is supprocd to bs equal to the legarithere of the Square root of the peripheiy of a circte whose noliue it urithy, 4. whet it said concoming the forcesing verift is hue tappeass to be Te, much in the same mancothioning the seorte for finding gut the sum of of the-togarithms of the odd nuriters $3,5,7.8 \%, z$ form these that ase given for finding our the swn of the infinite piogrefoens in which the sevesal kerms have the same numerator whitst their deno minatow are any cerhain pewer of numberd increaring sis arithmethcal

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 then take $A=a \quad B=2 b \quad C=2 \times 3 \times 4 \mathrm{C} \quad D=2 \times 3.4 \times 5 \times 6 \mathrm{~d}$ $E=2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \in \&$ an on $A, B, C, D, E, F$, fr wifl be the cosfficients of the forigoing swics: from whence it casily follorts that if any term in the itries after the 3 first be called $y$ * its distanice from the, st $t \mathrm{crm}$ n, the nest term immediakly
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## Thomas Bayes submitted I747, published I763

XLIII. A Letter from the late Reverend Mr. Thomas Bayes, F. R. S. to John Canton, M. A. and F. R.S.

## S I R,

Read Nov. 24, TF the following obfervations do not 1763. feem to you to be too minute, I fhould efteem it as a favour, if you would pleafe to communicate them to the Royal Society.

It has been afferted by fome eminent mathematicians, that the fum of the logarithms of the numbers I .2.3.4. \& c. to $z$, is equal to $\frac{1}{2} \log . c+\overline{z+\frac{1}{2}} \mathrm{X}$ $\log$. $z$ leffened by the feries $z-\frac{1}{12 z}+\frac{1}{360 z^{3}} \frac{1}{1260 z^{5}}+$ $\frac{1}{1680 z^{1}}-\frac{1}{1188 z^{9}}+\& c$. if $c$ denote the circumference of a circle whofe radius is unity. And it is true that this expreffion will very nearly approach to the value of that fum when $z$ is large, and you take in only a proper number of the firft terms of the foregoing feries: but the whole feries can never properly ex-

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 II $t$ may be formed. Jake $a=\frac{1}{12}$ then take $A=a \quad B=26 \quad C=2 \times 3 \times 4 C \quad D=2 \times 3,4 \times 5,6 d$ $E=2 \times 3 \times 4 \times 5 \times 6 \times 7 \cdot 8$ \& 10 on, \& $A, B, C, D, E, E$, w win be the corfficients of the forigoing suric: from whence it casily follown that if any torm in the zerries altur the 3 first be called $y$ * it distanze from the , ot torm n, the nest terme immediatily following will be greater than $\frac{n \times \frac{\pi}{2 n-1}}{6 n+9} \times \frac{g}{2^{2}}$. Wherefore at leng th the subscguent tions of thay sent are greake than the pricees
 $\omega$ thimate value whenosoce.
Much teso can that serice heve any uctimate value, which is deduces Fom it by taking $x=1$ a in ouppocd to be equal to the logaritthm

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\[

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\begin{gathered}
\log (z!)-\left(z+\frac{1}{2}\right) \log z+\log \sqrt{2 \pi}-z=\frac{1}{2 \pi^{2} z} \sum_{n=0}^{\infty}(-1)^{r} \frac{(2 r)!}{(2 \pi z)^{2 n}} \zeta(2 n+2) \\
\quad=\frac{1}{12 z}+\frac{1}{360 z^{3}}+\frac{1}{1260 z^{5}}+\frac{1}{1680 z^{7}}+\frac{1}{1188 z^{9}}+\frac{691}{360360 z^{11}}+\cdots
\end{gathered}
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\]

Bayes's discovery: Stirling's series for the factorial diverges factorially

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Euler 1755-I760: took divergent series seriously, as coded representations of functions

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Abel 1826 Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever.

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$2+2=5$, for sufficiently large values of 2

Airy and interference fringes in the rainbow


George Airy (1838)

Airy and interference fringes in the rainbow


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fundamental physics: waves near the singularities of ray optics (caustics)

Airy and interference fringes in the rainbow


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fundamental physics: waves near the singularities of ray optics (caustics)
the Airy function
the Airy function

$$
\mathrm{Ai}(\mathrm{z})=\frac{1}{2 \pi} \int^{0} \mathrm{~d} \operatorname{dexp}\left\{\left[\frac{1}{3} t^{3}+z t\right)\right\}
$$


the Airy function

$$
\mathrm{Ai}(z)=\frac{1}{2 \pi} \int^{0} d \operatorname{dexp}\left\{\left(\frac{1}{3} t^{3}+z t\right)\right\}
$$


the Airy function

$$
\operatorname{Ai}(z)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \mathrm{d} t \exp \left\{\mathrm{i}\left(\frac{1}{3} t^{3}+z t\right)\right\}
$$


numerical experiment

Stokes's approximations for large $|z|$, e.g. on the dark side


George Stokes (1847)

Stokes's approximations for large $|z|$, e.g. on the dark side


$$
\begin{aligned}
& \operatorname{Ai}(z)=\frac{\exp \left(+\frac{1}{2} F\right)}{2 \sqrt{\pi} z^{1 / 4}} \sum_{n=0}^{\infty} \frac{a_{n}}{F^{n}}, \quad F=-\frac{4}{3} z^{3 / 2} \\
& a_{0}=1, \quad a_{n}=\frac{\left(n-\frac{1}{6}\right)!\left(n-\frac{5}{6}\right)!}{2 \pi n!} \underset{r \rightarrow \infty}{\rightarrow} \frac{(n-1)!}{2 \pi}
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factorial divergence again

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for $z \gg 1$, pre-invented WKB

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Stokes's approximations for large $|z|$, e.g. on the dark side


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for $z \gg 1$, pre-invented WKB for $z \ll-1$, pre-invented stationary phase

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factorial divergence again

puzzle of the two exponentials

## puzzle of the two exponentials


two exponentials:
$e_{1}$ and $e_{2}$

## puzzle of the two exponentials

$$
\begin{aligned}
& e_{1}=\exp \left(\frac{1}{2} F\right)=\exp \left\{-\frac{2}{3} \mathrm{i}(-z)^{3 / 2}\right\} \\
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$120^{\circ}$
one exponential:
$e_{1}$


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$F$ positive real Stoke
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\end{aligned}
$$


two exponentials: $e_{1}$ and $e_{2}$
$F$ positive real Stoke
two exponentials: $e_{1}$ and $e_{2}$

$e_{2}$ is born where it is maximally dominated by $e_{1}$

## the quietly beating heart of asymptotics:

 Stokes's phenomenon: the sudden appearance of a small exponential while hidden behind a large one, going from dark to bright 'around the rainbow' without passing $z=0$
## the quietly beating heart of asymptotics:

 Stokes's phenomenon: the sudden appearance of a small exponential while hidden behind a large one, going from dark to bright 'around the rainbow' without passing $z=0$Stokes phenomenon occurs throughout asymptotics in integrals, differential equations, integral equations, difference equations, series, near more general types of caustics...


## fold caustic

## cusp caustic

## fold caustic

wave pattern decorating a cusp caustic: Pearcey's integral

$$
\Psi_{\text {cusp }}(x, y)=\int_{-\infty}^{\infty} \mathrm{d} t \exp \left\{\mathrm{i}\left(\frac{1}{4} t^{4}+x t^{2}+y t\right)\right\}
$$

wave pattern decorating a cusp caustic: Pearcey's integral

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caustic
(cusp catastrophe)
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caustic
(cusp catastrophe)

maximum
$\uparrow$
for $\operatorname{Ai}(u)$, Stokes set occurs in the complex plane

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for higher diffraction catastrophes, the Stokes phenomenon can occur for real parameters

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cusp diffraction catastrophe (Wright 1980)


## two contrasting general phenomena, with exponents i $\Phi(t)$

## two contrasting general phenomena, with exponents $\mathrm{i} \Phi(t)$

bifurcation (caustic, catastrophe) set: real saddles collide

violent birth/death of real waves

$$
\begin{gathered}
\partial_{t} \Phi=0 \\
\partial_{t}{ }^{2} \Phi=0
\end{gathered}
$$

two contrasting general phenomena, with exponents $\mathrm{i} \Phi(t)$
bifurcation (caustic, catastrophe) set: real saddles collide

one evanescent wave (complex saddle)
violent birth/death of real waves

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\begin{aligned}
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stokes set

```
evanescent wave + dominant wave
```


gentle birth/death of evanescent waves
$\operatorname{Re}\left(\Phi_{1}-\Phi_{2}\right)=0$ nonlocal bifurcation
(strictly, change of coefficient of subdominant exponential)

Stokes's argument: the least term represents an irremovable vagueness in optimally-truncated asymptotic series, and the small exponential $e_{2}$ can enter only where it is smaller than this vagueness - which only happens very close to a Stokes line

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natural (large) variable $F$ : difference between exponents least term: order $n \sim|F|$
asymptotics: large $|F|$

$$
\sum_{n=0}^{\infty} \frac{a_{n}}{F^{n}}=\sum_{n=0}^{|F|-1} \frac{a_{n}}{F^{n}}+\sum_{n=|F|}^{\infty} \frac{a_{n}}{F^{n}}
$$

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\sum_{n=0}^{\infty} \frac{a_{n}}{F^{n}}=\begin{gathered}
\text { head } \\
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$$

# asymptotics of the <br> asymptotics: large $n$ 

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universality of factorial divergence of high orders (Dingle, based on Darboux)
asymptotics of the asymptotics: large $n$


Robert Dingle
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Gaston Darboux
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$$
\begin{aligned}
& \text { tail }=\sum_{n=|F|}^{\infty} \frac{a_{n}}{F^{n}} \\
& \approx C \sum_{n=|F|}^{\infty} \frac{(n-\alpha)!}{F^{n}}
\end{aligned}
$$

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huge simplification
because exact terms
rapidly get
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universality of factorial divergence of high orders (Dingle, based on Darboux)


Robert Dingle

$C_{20}(z)=\frac{332727711 C_{0}(z)}{274877906944 \pi^{10}}+\frac{117753804989 C_{0}^{(4)}(z)}{3298534883328 \pi^{12}}+\frac{13899745416281 C_{0}^{(8)}(z)}{69269232549880 \pi^{14}}+\frac{311274631265011 C_{0}^{(12)}(z)}{164583696538533888 \pi^{16}}+$ $\frac{2431103703048530417 C_{0}^{(16)}(z)}{44931349155019751424000 \pi^{18}}+\frac{232544268738862214941 C_{0}^{(20)}(z)}{373186948553264049684480000 \pi^{20}}+$
huge simplification because exact terms rapidly get complicated

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$$

$\frac{361888761444289010497 C_{0}^{(24)}(z)}{106489993378346112059965440000 \pi^{22}}+\frac{66540631045322715923177 C_{0}^{(28)}(z)}{6843046974492521160973379174400000 \pi^{24}}+$
$\frac{391261681973226653 C_{0}^{(32)}(z)}{25057539453517190072893440000000 \pi^{26}}+\frac{1259995823308801 C_{0}^{(36)}(z)}{85717193786692288213155840000000 \pi^{28}}+$
$\frac{713214794639 C_{0}^{(40)}(z)}{85717193786692288213155840000000 \pi^{30}}+\frac{50407933481 C_{0}^{(44)}(z)}{17650884544555675988853050572800000 \pi^{32}}+$
$\frac{1039499 C_{0}^{(48)}(z)}{1768363201124316332834999500800000 \pi^{34}}+\frac{22411 C_{0}^{(52)}(z)}{321391973789793928418521000181760000 \pi^{36}}+$
$1768363201124316332834999500800000 \pi^{34}$
$\frac{59 C_{0}^{(56)}(z)}{13636202316509828105757248150568960000 \pi^{38}}+\frac{C_{0}^{(60)}(z)}{9327162384492722424337957734989168640000 \pi^{40}}$

## asymptotics of the asymptotics of the asymptotics

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resumming the tail by Borel summation, giving an integral


Émile Borel

## asymptotics of the asymptotics of the asymptotics

resumming the tail by Borel summation, giving an integral uniform approximation of the integral across a Stokes line


Émile Borel

## asymptotics of the asymptotics of the asymptotics

resumming the tail by Borel summation, giving an integral uniform approximation of the integral across a Stokes line

the small exponential is born not suddenly but smoothly, according to a universal scaling in terms of


Émile Borel an error function

## subtract the large exponential series

$$
\text { tail }=-\operatorname{iexp}\left(\frac{1}{2} F\right)\left(2 \sqrt{\pi} z^{1 / 4} \mathrm{Ai}(z)-\exp \left(\frac{1}{2} F\right) \sum_{n=0}^{\text {int }|F|} \frac{a_{n}}{F^{n}}\right)
$$

$$
\left(F=-\frac{4}{3} z^{3 / 2}\right)
$$

subtract the large exponential series

$$
\left(F=-\frac{4}{3} z^{3 / 2}\right)
$$

subtract the large exponential series
tail $=-\operatorname{iexp}\left(\frac{1}{2} F\right)\left(\begin{array}{c}\text { big } \\ \sqrt{2 \sqrt{\pi} z^{1 / 4}} \mathrm{Ai}(z)- \\ \exp \left(\frac{1}{2} F\right) \sum_{n=0}^{\text {in } \mid F /} \frac{a_{n}}{F^{n}}\end{array}\right)$

$$
\left(F=-\frac{4}{3} z^{3 / 2}\right)
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subtract the large exponential series

tail $=$| big |
| :---: |
| $-\mathrm{iexp}\left(\frac{1}{2} F\right)$ |
| $2 \sqrt{\pi} z^{1 / 4} \mathrm{Ai}(z)-$ |
| $\left.-\exp \left(\frac{1}{2} F\right) \sum_{n=0}^{\text {in } \mid F /} \frac{a_{n}}{F^{n}}\right)$ |

$$
\left(F=-\frac{4}{3} z^{3 / 2}\right)
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subtract the large exponential series


$$
\left(F=-\frac{4}{3} z^{3 / 2}\right)
$$

difference small
subtract the large exponential series
tail $=\frac{\text { big }}{-\operatorname{iexp}\left(\frac{1}{2} F\right)}\left(\sqrt{\left.2 \sqrt{\pi} z^{1 / 4} \mathrm{Ai}(z)-\exp \left(\frac{1}{2} F\right) \sum_{n=0}^{\text {int }|F|} \frac{a_{n}}{F^{n}}\right)=\frac{1+\operatorname{erf} \sigma}{2}, ~}\right.$

$$
\left(F=-\frac{4}{3} z^{3 / 2}\right)
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difference small
subtract the large exponential series
big
tail $=\frac{\text { big }}{-\operatorname{iexp}\left(\frac{1}{2} F\right)( }\left(\begin{array}{l}\text { big } \\ \left.2 \sqrt{\pi} z^{1 / 4} \mathrm{Ai}(z)-\exp \left(\frac{1}{2} F\right) \sum_{n=0}^{\text {int }|F|} \frac{a_{n}}{F^{n}}\right)=\frac{1+\operatorname{erf} \sigma}{2}, ~\end{array}\right.$

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$$
\sigma=\frac{\operatorname{Im} F}{\sqrt{2 \operatorname{Re} F}}
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subtract the large exponential series

$$
\text { tail }=-\operatorname{big}\left(\frac{\text { big }}{-\operatorname{iexp}\left(\frac{1}{2} F\right)}\left(\sqrt{\left.2 \sqrt{\pi} z^{1 / 4} \mathrm{Ai}(z)-\exp \left(\frac{1}{2} F\right) \sum_{n=0}^{\text {int }|F|} \frac{a_{n}}{F^{n}}\right)}\right)=\frac{1+\operatorname{erf\sigma }}{2}\right.
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subtract the large exponential series tail $=-\operatorname{liexp}\left(\frac{1}{2} F\right)\left(\sqrt{\text { big }}\left(\sqrt{2 \sqrt{\pi} z^{1 / 4} \mathrm{Ai}(z)}-\exp \left(\frac{1}{2} F\right) \sum_{n=0}^{\text {int }|F|} \frac{a_{n}}{F^{n}}\right)\right)=\frac{1+\operatorname{erf\sigma }}{2}$

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subtract the large exponential series tail $\left.=\frac{\text { big }}{-\operatorname{iexp}\left(\frac{1}{2} F\right)}\left(\sqrt{2 \sqrt{\pi} z^{1 / 4} \operatorname{Ai}(z)}-\exp \left(\frac{1}{2} F\right) \sum_{n=0}^{\text {int }|F|} \frac{a_{n}}{F^{n}}\right)\right)=\frac{1+\operatorname{erf\sigma }}{2}$

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subtract the large exponential series
 $\left(F=-\frac{4}{3} z^{3 / 2}\right)$
difference small


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subtract the large exponential series tail $=-\frac{\text { big }}{-\mathrm{iexp}\left(\frac{1}{2} F\right)}\left(\frac{\text { big }}{\left.\left(2 \sqrt{\pi} z^{1 / 4} \mathrm{Ai}(z)-\exp \left(\frac{1}{2} F\right) \sum_{n=0}^{\operatorname{inn}|F|} \frac{a_{n}}{F^{n}}\right)\right)=\frac{1+\operatorname{erf\sigma }}{2}, ~}\right.$

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subtract the large exponential series


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$$
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subtract the large exponential series tail $=-\mathrm{iexp}\left(\frac{1}{2} F\right)\left(\sqrt{\text { big }}\left(\sqrt{2 \sqrt{\pi} z^{1 / 4} \mathrm{Ai}(z)}-\exp \left(\frac{1}{2} F\right) \sum_{n=0}^{\mathrm{inn}|F|} \frac{a_{n}}{F^{n}}\right)\right)=\frac{1+\operatorname{erf\sigma }}{2}$

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\left(F=-\frac{4}{3} z^{3 / 2}\right)
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$$
\sigma=\frac{\operatorname{Im} F}{\sqrt{2 \operatorname{Re} F}}
$$


disparity $\exp |F=10| \approx 22000$
subtract the large exponential series

many applications in mathematics, to the approximation of a variety of functions: the error function in

- Bessel
- hypergeometric
- gamma
- even the error function itself
- integrals with coalescing saddles
- Riemann zeta function
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- Bessel
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in physics, applications to
- reflection of waves by refractive-index gradients
- histories of quantum jumps induced by slowlychanging external forces, and particle pair creation
- breakdown of slow manifold in slow-fast systems
histories of quantum transitions driven by slowly-changing hamiltonians

transition probability
histories of quantum transitions driven by slowly-changing hamiltonians


(e)



(1)

time
transition probability
histories of quantum transitions driven by slowly-changing hamiltonians

(b)

$n$th order superadiabatic bases
final probability is
 exponentially small: $\exp (-c o n s t / \varepsilon)$
transition probability

(0)

time
histories of quantum transitions driven by slowly-changing hamiltonians


transition probability



final probability is exponentially small: $\exp (-c o n s t / \varepsilon)$
$n$th order superadiabatic bases
time
histories of quantum transitions driven by slowly-changing hamiltonians


(a)


( 1

$n$th order superadiabatic bases
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transition probability
time
histories of quantum transitions driven by slowly-changing hamiltonians




## $n$th order

 superadiabatic bases

(0)

final probability is exponentially small: $\exp (-c o n s t / \varepsilon)$

transition probability
time
histories of quantum transitions driven by slowly-changing hamiltonians


## $n$th order

 superadiabatic bases



(0)
final probability is exponentially small: $\exp (-$ const $/ \varepsilon)$
transition probability
time
histories of quantum transitions driven by slowly-changing hamiltonians

histories of quantum transitions driven by slowly-changing hamiltonians

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## Poincaré asymptotics: summing to a fixed order



Henri Poincaré

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cannot capture exponentially
small terms

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cannot capture exponentially
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does not distinguish divergent from from convergent series

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superasymptotics: summing to the least term $r \sim|F|$ :
Stokes and the smoothing of the Stokes discontinuity

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cannot capture exponentially
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Henri Poincaré
superasymptotics: summing to the least term $r \sim|F|$ :
Stokes and the smoothing of the Stokes discontinuity
capturing small exponentials: Kruskal,'asymptotics beyond all orders

## hyperasymptotics:

repeated resummation, based on the principle of resurgence (Dingle 1960s, Écalle 1980s)

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Robert Dingle

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Jean Écalle
Robert Dingle

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Jean Écalle
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the divergence of a series must reflect its cause

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Jean Écalle
the series multiplying each exponential must diverge, in order to accommodate the other exponentials

Robert Dingle
the divergence of a series must reflect its cause

## hyperasymptotics:

repeated resummation, based on the principle of resurgence (Dingle I960s, Écalle 1980s)



Jean Écalle
the series multiplying each exponential must diverge, in order to accommodate the other exponentials

Robert Dingle
the divergence of a series must reflect its cause moreover, each component series must contain, coded into its high orders, information about all the other exponentials, and all terms of the series multiplying them

## simplest case: only two exponentials $\exp \left( \pm \frac{1}{2} F\right)$

in $S=\sum_{n=0}^{\infty} \frac{a_{n}}{F^{n}}$

$$
a_{n} \rightarrow \frac{1}{n \rightarrow \infty}(n-1)!\left[a_{0}-\frac{a_{1}}{(n-1)}+\right.
$$

$$
\left.+\frac{a_{2}}{(n-1)(n-2)}-\frac{a_{3}}{(n-1)(n-2)(n-3)} \cdots\right]
$$

simplest case: only two exponentials $\exp \left( \pm \frac{1}{2} F\right)$
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$$

$$
\left.+\frac{a_{2}}{(n-1)(n-2)}-\frac{a_{3}}{(n-1)(n-2)(n-3)} \cdots\right]
$$

hyperasymptotic scheme for sum $S$ as a series of series:

- primitive asymptotics - only $a_{0}$
- sum series to least term $-S_{0}$ (superasymptotics)
- integral representation for remainder
- asymptotic series for remainder, summed to least term $\left(S_{1}\right)$
- asymptotic series for new remainder, truncated $\left(S_{2}\right)$...
hyperasymptotics for Ai for $F=-16$, i.e. $z=5.2414827884177932413$ total number of terms in hyperasymptotic series

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hyperasymptotics for Ai for $F=-16$, i.e. $z=5.2414827884177932413$ total number of terms in hyperasymptotic series
 optimally truncated hyperseries get shorter
with more than two exponentials, graph structure of higher approximants, e.g. multisaddle integrals:
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with more than two exponentials, graph structure of higher approximants, e.g. multisaddle integrals:

- basic saddle
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〇adjacent saddles
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reached on descent paths from as $\arg F$ varies
with more than two exponentials, graph structure of higher approximants, e.g. multisaddle integrals:


- basic saddle

〇adjacent saddles
reached on descent paths from as $\arg F$ varies

- non-adjacent saddles
with more than two exponentials, graph structure of higher approximants, e.g. multisaddle integrals:

- basic saddle
$\bigcirc$ adjacent saddles reached on descent paths from as $\arg F$ varies
- non-adjacent saddles
more adjacent saddles introduced at successive stages of hyperasymptotics
example: Pearcey integral

$$
P(x, y)=\int_{C} \mathrm{~d} t \exp \left\{\mathrm{i}\left(\frac{1}{4} t^{4}+\frac{1}{2} x t^{2}+y t\right)\right\} \quad x=7, \quad y=1+\mathrm{i}
$$

## example: Pearcey integral

$$
P(x, y)=\int_{C} \mathrm{~d} t \exp \left\{\mathrm{i}\left(\frac{1}{4} t^{4}+\frac{1}{2} x t^{2}+y t\right)\right\} \quad x=7, \quad y=1+\mathrm{i}
$$


three saddles
example: Pearcey integral

$$
P(x, y)=\int_{C} \mathrm{~d} t \exp \left\{\mathrm{i}\left(\frac{1}{4} t^{4}+\frac{1}{2} x t^{2}+y t\right)\right\} \quad x=7, \quad y=1+\mathrm{i}
$$


hyperasymptotics generates a sequence of series, from 'scatterings' between saddles






level
approximation to $\mathrm{P}(7,1+\mathrm{i}) \quad$ |approx./exact - 1|
lowest
super.
$0.779703507027512+\mathrm{i} 0.765551648542315$

```
1.496\times10-2
\(2.916 \times 10^{-6}\)
```

$1.535 \times 10^{-12}$
$0.788920520763900+\mathrm{i} 0.752101783262683$
ultimate hyper.
$0.788922837595360+\mathrm{i} 0.752103959759701$
exact
$0.788922837596969+\mathrm{i} 0.752103959759243$

level approximation to $\mathrm{P}(7,1+\mathrm{i}) \quad$ |approx./exact -1 |

| lowest | $0.779703507027512+\mathrm{i} 0.765551648542315$ | $1.496 \times 10^{-2}$ |
| :--- | :--- | :---: |
| super. | $0.788920520763900+\mathrm{i} 0.752101783262683$ | $2.916 \times 10^{-6}$ |
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| exact | $0.788922837596969+\mathrm{i} 0.752103959759243$ | 0 |


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| lowest | $0.779703507027512+\mathrm{i} 0.765551648542315$ | $1.496 \times 10^{-2}$ |
| :--- | :--- | :---: |
| super. | $0.788920520763900+\mathrm{i} 0.752101783262683$ | $2.916 \times 10^{-6}$ |
| ultimate hyper. | $0.788922837595360+\mathrm{i} 0.752103959759701$ | $1.535 \times 10^{-12}$ |
| exact | $0.788922837596969+\mathrm{i} 0.752103959759243$ | 0 |


level
approximation to $\mathrm{P}(7,1+\mathrm{i})$
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Legacy from Euler, Dingle, Écalle... from Stokes's insistence on understanding how the rainbow's dark side is connected to the interference fringes on its bright side:

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a divergent series is not meaningless, or a nuisance, but an essential and informative coded representation of the function

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in the 1990s, 2000s, much new mathematics originating from resurgence, etc: Boyd, Chapman, Delabaere, Dunster, Ecalle, Howls, Kruskal, Olde Daalhuis, Lutz, McLeod, Paris, Olver, Ramis, Pham, Segur, Temme, Voros, Wong, Wood...
now, in 2010 s , resurgence of interest in divergence, resurgence, resummation... applications to field and string theory

# now, in 2010 s, resurgence of interest in divergence, resurgence, resummation... applications to field and string theory 

Analytic Continuation Of Chern-Simons Theory

Edward Witten

School of Natural Sciences, Institute for Advanced Study
Einstein Drive, Princeton, NJ 08540 USA

The semi-classical expansion and resurgence in gauge theories: new perturbative, instanton, bion, and renormalon effects

Philip C. Argyres ${ }^{1}$ and Mithat Ünsal ${ }^{2}$

Resurgence and Trans-series in Quantum Field Theory: The $\mathbb{C P}^{N-1}$ Model

Gerald V. Dunne ${ }^{1}$ and Mithat Ünsal ${ }^{2}$

Lectures on non-perturbative effects in large $N$ theory, matrix models and topological strings

Marcos Mariño
Département de Physique Théorique et Section de Mathématiques,
Université de Genève, Genève, CH-1211 Suitzerland

Introduction to 1 -summability and the resurgence theory

## phenomena, new saddle points and Lefschetz thimbles

## Resurgence and Transseries in Quantum, Gauge and String Theories

## from 30 June 2014 to 4 July 2014 (Europe/Zurich)

CERN
Europe/Zurich timezone

## Overview

Scientific Programme
Timetable
Registration
$\perp$ Registration Form
Participant List

> The goal of this CERN TH-Institute/Conference is to bring together researchers in Mathematics (with backgrounds in resurgence, transseries, asymptotic analysis), with researchers in Theoretical Physics (with backgrounds in quantum mechanics, gauge theory, string theory) working on topics where resurgent methods are promising.
> This meeting will have an interdisciplinary nature, where interactions between different communities are likely to foster new advances and results in this exciting subject.

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[^0]:    To, much in the same mancontioning the suit for finding gut the to be of the- logarithms of the odd nuriters a.s.7. Or, $z$ gut the sum of are given for finding out the sum of the infinite otogrebores in which the several hems have the Jame numerator whiter their in pinatas are any certain power of numbers increasing sit aritfinctical

