Highly Effective Actions

John H. Schwarz

California Institute of Technology

Based on arXiv:1311.0305 and 1405.7444

ICTP50 Oct. 9, 2014

Introduction

The plan is to obtain the world-volume action for a D3-brane in $AdS_5 \times S^5$ and to interpret the result as an effective action for a SCFT on the Coulomb branch.

$$U(N+1) \to U(1) \times U(N)$$

I have conjectured that the D3-brane action is the *exact* effective action for the $U(1)$ factor. I call such an action a Highly Effective Action (HEA). This conjecture has not been proved, and many people are skeptical. I will discuss some of the issues that are involved.

The Main Examples

I have constructed the world-volume action for a probe p-brane in an $AdS_{p+2} \times M_q$ background with N units of flux, $\int_{M_q} F_q = N$, in the following cases:

- D3-brane in $AdS_5 \times S^5$
- M2-brane in $AdS_4 \times S^7/\mathbb{Z}_k$
- D2-brane in $AdS_4 \times CP^3$
- M5-brane in $AdS_7 \times S^4$

This talk will only describe the D3 example.

Brane actions involve well-known approximations:

- The probe approximation involves neglecting the back reaction of the brane on the geometry and the other background fields. Since the brane is a source for one unit of flux, this requires that N is large.
- D-brane actions include a Born–Infeld–like term that is a functional of a $U(1)$ field strength, $F_{\alpha\beta}$, on the brane. F is assumed to be *slowly varying*, so that its derivatives can be neglected.

Nevertheless, the brane actions have some beautiful exact properties.

- They precisely realize the isometry of the background as a world-volume superconformal symmetry given by $PSU(2,2|4)$, $OSp(6|4)$, etc.
- The brane actions exactly implement the duality symmetries of the background theories as dualities: $SL(2,\mathbb{Z})$ for the D3-brane example; and a duality relating the D2-brane and M2-brane examples.

The brane world-volume actions are naturally formulated with local symmetries: general coordinate invariance and kappa symmetry.

There is a natural gauge choice – called static gauge – that results in formulas with the expected field content: an abelian $\mathcal{N} = 4$ supermultiplet $(A_\mu,\ 4\psi,\ 6\phi)$ in the D3-brane case.

The Coulomb Branch

Let us consider $\mathcal{N} = 4$ SYM theory with the gauge group $U(N+1)$. It has an $SL(2,\mathbb{Z})$ duality group, as required by AdS/CFT. On the Coulomb branch,

$$U(N+1) \to U(1) \times U(N),$$

$2N$ supermultiplets acquire mass.

In principle, one can integrate out the massive fields exactly. The resulting action for the $U(1)$ factor ("the photon supermultiplet") defines the *highly effective action* (HEA). The D3-brane action is conjectured to be this HEA.

General Requirements for the HEA

- Field content is an abelian $\mathcal{N} = 4$ supermultiplet
- Global symmetries and dualities same as the original Coulomb branch theory
- Conformal symmetry spontaneously broken by vev of a massless scalar field
- The same BPS spectrum; the $SL(2, \mathbb{Z})$ multiplet containing the W particles and monopoles should arise as solitons of the HEA

The D3-brane in $\mathbf{AdS_5 \times S^5}$

The ten-dimensional metric $g_{MN}(x)dx^M dx^N$ is

$$ds^2 = R^2 \left(\phi^2 dx \cdot dx + \phi^{-2} d\phi^2 + d\Omega_5^2 \right)$$
$$= R^2 \left(\phi^2 dx \cdot dx + \phi^{-2} d\phi^I d\phi^I \right).$$

ϕ is the length of the six-vector ϕ^I, and we use standard AdS/CFT formulas

$$R^2 = \sqrt{4\pi g_{\rm s} N} l_{\rm s}^2 \quad \text{and} \quad \int_{S^5} F_5 \sim N.$$

The $SL(2,\mathbb{Z})$ modular parameter is

$$\tau = \chi + \frac{i}{g_s} = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}.$$

The D3-brane action has two terms: $S = S_1 + S_2$.

S_1 is a DBI functional of the embedding functions $x^M(\sigma^\alpha)$ and a world-volume $U(1)$ gauge field $A_\beta(\sigma^\alpha)$ with field strength $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$:

$$S_1 = -T_{D3} \int \sqrt{-\det\left(G_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}\right)}\, d^4\sigma,$$

where $G_{\alpha\beta}$ is the induced 4d world-volume metric

$$G_{\alpha\beta} = g_{MN}(x)\partial_\alpha x^M \partial_\beta x^N.$$

As usual, $\alpha' = l_{\rm s}^2$ and the D3-brane tension is

$$T_{D3} = \frac{2\pi}{g_{\rm s}(2\pi l_{\rm s})^4}.$$

Only dimensionless combinations occur in the brane action:

$$R^4 T_{D3} = \frac{N}{2\pi^2} \quad \text{and} \quad 2\pi\alpha'/R^2 = \sqrt{\pi/g_{\rm s}N}.$$

General coordinate invariance allows one to choose the static gauge

$$x^\mu(\sigma) = \delta^\mu_\alpha \sigma^\alpha.$$

In this gauge $\phi^I(\sigma)$ and $A_\mu(\sigma)$ become functions of x^μ.

In static gauge we obtain

$$S_1 = -\frac{1}{2\pi g_s k^2} \int \sqrt{-\det\left(G_{\mu\nu} + kF_{\mu\nu}\right)}\, d^4x$$

where $k = \sqrt{g_s N/\pi}$ and $G_{\mu\nu} = \phi^2 \eta_{\mu\nu} + k^2 \partial_\mu \phi^I \partial_\nu \phi^I / \phi^2$.

Rescaling all fields by $\sqrt{2\pi g_s}$, their normalization becomes canonical leaving

$$S_1 = -\frac{1}{\gamma^2} \int \sqrt{-\det\left(h_{\mu\nu} + \gamma^2 \partial_\mu \phi^I \partial_\nu \phi^I / \phi^2 + \gamma F_{\mu\nu}\right)}\, d^4x$$

where $\gamma = \pi^{-1}\sqrt{N/2}$ and $h_{\mu\nu} = \phi^2 \eta_{\mu\nu}$. Note that the dependence on $g_s = g^2/(4\pi)$ has disappeared!

The Chern–Simons Term

$$S_2 \sim \int C_4 + \chi \int F \wedge F$$

The RR four-form potential C_4 has a self-dual field strength $F_5 = dC_4$.

$$F_5 \sim \mathrm{vol}(S^5) + \mathrm{vol}(AdS_5)$$

The constant χ is the value of the RR 0-form C_0. It is proportional to a theta angle, $\chi = \theta/(2\pi)$.

S_1 contains a "potential" term $\int \phi^4 d^4x$, which would give a net force acting on the brane. It is canceled by the $\int C_4$ term in S_2. The coefficients work perfectly.

The complete action for canonically normalized fields in static gauge (aside from fermions) is

$$S = \frac{1}{\gamma^2} \int \phi^4 \left(1 - \sqrt{-\det M_{\mu\nu}}\right) d^4x + \frac{1}{4} g_s \chi \int F \wedge F,$$

where $\gamma = \sqrt{\frac{N}{2\pi^2}}$ and

$$M_{\mu\nu} = \eta_{\mu\nu} + \gamma^2 \frac{\partial_\mu \phi^I \partial_\nu \phi^I}{\phi^4} + \gamma \frac{F_{\mu\nu}}{\phi^2}.$$

S duality

The $\tau \to -1/\tau$ duality of the $U(N+1)$ theory on the Coulomb branch has not been proved in the formulation with W fields, but I have proved it for the D3-brane action.

As one would expect, $\tau \to -1/\tau$ is accompanied by a nonlocal field redefinition that exchanges electric and magnetic fields. Then S_1 and S_2 are separately invariant. In the S_2 case $g_s\chi$ and $\int F \wedge F$ change signs simultaneously.

Solitons

For spherically symmetrical static solutions, centered at $r = 0$, we require that $E_i = F_{0i}$ and $B_i = \frac{1}{2}\varepsilon_{ijk}F_{jk}$ only have radial components, denoted E and B, and that E, B, and ϕ are functions of r only. It then follows that

$$-\det(M_{\mu\nu}) = -\det\left(\eta_{\mu\nu} + \gamma^2 \frac{\partial_\mu \phi \partial_\nu \phi}{\phi^4} + \gamma \frac{F_{\mu\nu}}{\phi^2}\right)$$

$$= \left(1 + \gamma^2 \frac{(\phi')^2 - E^2}{\phi^4}\right)\left(1 + \gamma^2 \frac{B^2}{\phi^4}\right).$$

This results in the Lagrangian density

$$\mathcal{L} = \frac{1}{\gamma^2}\phi^4\left(1 - \sqrt{\left(1 + \gamma^2\frac{(\phi')^2 - E^2}{\phi^4}\right)\left(1 + \gamma^2\frac{B^2}{\phi^4}\right)}\right).$$

The equation of motion for A_0 is $\frac{\partial}{\partial r}(r^2 D) = 0$, where

$$D = \frac{\partial \mathcal{L}}{\partial E} = E\sqrt{\frac{1 + \gamma^2 B^2/\phi^4}{1 + \gamma^2[(\phi')^2 - E^2]/\phi^4}}.$$

For a soliton centered at $r = 0$, with p units of electric charge g and q units of magnetic charge g_m, where $g_m = 4\pi/g$, we have

$$D = \frac{pg}{4\pi r^2} \quad \text{and} \quad B = \frac{qg_m}{4\pi r^2}.$$

Thus, $D^2 + B^2 = Q^2/r^4$, where (adding a θ angle)

$$Q = \frac{g}{4\pi}|p + q\tau|,$$

and

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}.$$

Eliminating E in favor of D gives the Hamiltonian $H = 4\pi \int \mathcal{H} r^2 dr$, where $\mathcal{H} = DE - \mathcal{L}$ is

$$\mathcal{H} = \frac{1}{\gamma^2}\left(\sqrt{(\phi^4 + \gamma^2(\phi')^2)(\phi^4 + \gamma^2 X^2)} - \phi^4\right),$$

and

$$X = \sqrt{D^2 + B^2} = Q/r^2.$$

We want to find functions $\phi(r)$ that give BPS extrema of H with $\phi \to v$ as $r \to \infty$. The BPS condition turns out to require that the two factors inside the square root are equal, which implies that $\mathcal{H} = (\phi')^2$.

The BPS equation $(\phi')^2 = Q^2/r^4$, together with the B. C. $\phi \to v$ as $r \to \infty$, has two solutions

$$\phi_\pm = v \pm Q/r,$$

where (as before) $Q = \frac{g}{4\pi}|p + q\tau|$ and $\mathcal{H} = Q^2/r^4$.

The ϕ_+ solution is similar to the flat space ($\mathbb{R}^{9,1}$) case studied by Callan and Maldacena in 1997. It describes a funnel-shaped protrusion of the D3-brane extending to the boundary of AdS at $\phi = +\infty$. It gives infinite mass (proportional to $\int dr/r^2$) and is not the solution I am after.

The ϕ_- solution is different. $\phi = 0$ corresponds to the horizon of the Poincaré patch of AdS_5. This means that the ϕ_- solution must be cut off at

$$r_0 = Q/v.$$

Thus, the masses of BPS solitons are given by

$$M = 4\pi \int_{r_0}^{\infty} \mathcal{H} r^2 dr = \frac{4\pi Q^2}{r_0} = vg|p + q\tau|.$$

As expected, the $(p, q) = (\pm 1, 0)$ solitons are $W^\pm$ with mass vg and the $(p, q) = (0, \pm 1)$ solitons are magnetic monopoles with mass $4\pi v/g$ (for $\theta = 0$).

Interpretation

The charge of the ϕ_- solution is uniformly spread on the sphere $r = r_0$, which we call a soliton bubble. The interior of the bubble does not contribute to the mass of the soliton. So, how should we interpret this?

The only sensible interpretation is that the gauge theory is in the ground state of the *conformal phase* of $U(N+1)$ inside the sphere. This implies that the bubble is a phase boundary.

Multi-soliton Solutions

It is easy to derive the generalization to the case of n solitons of equal charge. Since the forces between them should cancel, their centers can be at arbitrary positions. The key formula is

$$\phi(\vec{x}) = v - Q \sum_{k=1}^{n} \frac{1}{|\vec{x} - \vec{x}_k|}.$$

The surfaces of the bubbles are given by $\phi(\vec{x}) = 0$. The fields $\vec{D}$ and $\vec{B}$ are then proportional to $\vec{\nabla}\phi$, with coefficients determined by the charges. This is much simpler than the usual nonabelian multi-monopole analysis!

Magnetic Bags

By considering multi-monopole solutions of *large* magnetic charge in the nonabelian gauge theory on the Coulomb branch, Bolognesi (hep-th/0512133) deduced the existence of "magnetic bags" with properties that are very close to those of the soliton bubbles that were found here. He also pointed out the analogy to black holes.

The analysis in terms of nonabelian gauge fields is much more complicated, subtle, and approximate than the analysis of the abelian effective action.

Conclusion

The action of a probe D3-brane in $AdS_5 \times S^5$ is a candidate for the HEA of a $U(1)$ factor on the Coulomb branch. It incorporates all the required symmetries and dualities, and it gives the expected BPS soliton solutions.

Even so, we have not proved that it is the HEA. If the conjecture that the theory defined by the probe D3-brane action is the HEA is not correct, then we have found an interesting new theory.