GENERAL RELATIVITY AND GNSS

Jorge D. Taramona P.
AIR NAVIGATION DIVISION
MINISTRY OF TRANSPORT - PERU

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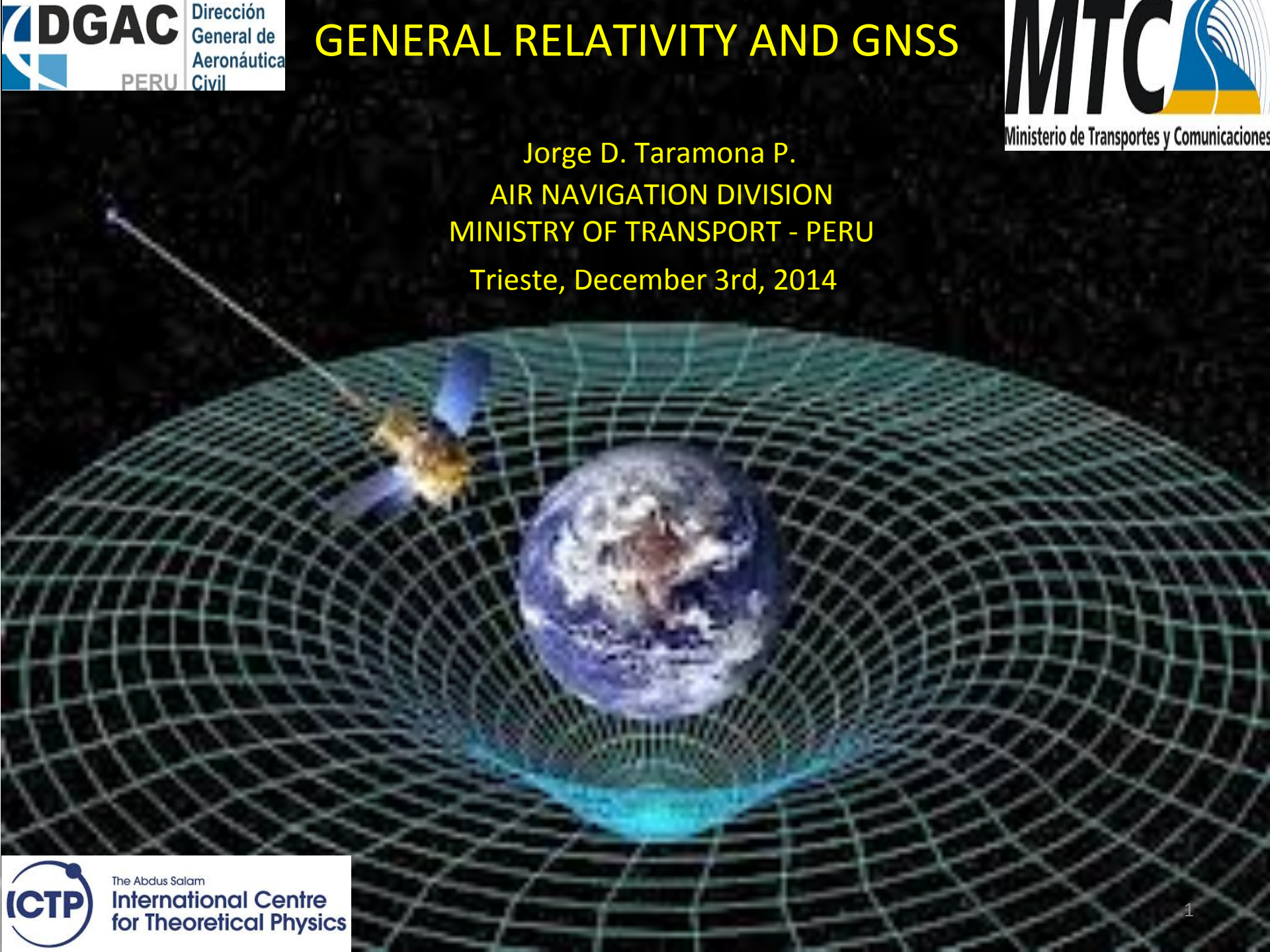

OUTLINE

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Introduction

- Today, GNSS, used as global positioning systems, are the GPS and the GLONASS. They are based on a Newtonian Model and hence, they are only operative when several and important effects are taken into account.
- **General Relativity Theory (GRT)** predicts that clocks in a stronger gravitational field will tick at a slower rate.
- **Special Relativity Theory (SRT)** predicts that moving clocks will appear to tick slower than no-moving ones.
- **An on-board clock keeps time to an accuracy** of about 4ns per day. Since, the speed of light is about 30cm/ns. This is the amazing capacity of the system to locate anything on Earth or near its surface.
- For GPS satellites (SVs), **GRT** predicts that the atomic clocks at SVs altitude will tick faster by about 45,700 ns/day (gravitational blueshift effect). **SRT** predicts that atomic clocks moving at SVs orbital speeds will tick slower by about 7110 ns/day than stationary ground clocks.

GPS works by triangulation using signals referenced to onboard atomic clocks

2

To triangulate, GPS measures distance using the travel time of a radio signal.

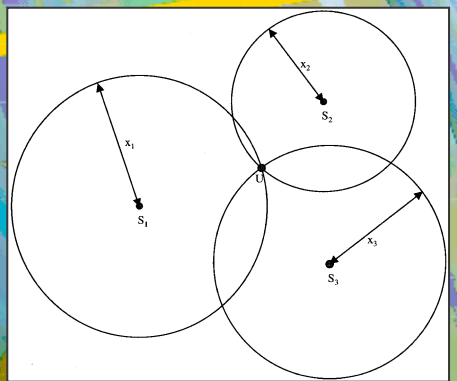
3

To measure travel time, GPS needs very accurate clocks.

4

In addition to knowing the distance to a satellite. a user needs to know the satellite's location.

$$|\vec{x}_o - \vec{x}_s|^2 - c^2(t_o - t_s)^2 = 0, \quad s = 1, 2, 3, 4.$$



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As the GPS signal travels through the ionosphere and troposphere, it gets delayed.

1

Triangulation from satellites is the basis of the system.

Prof. Robert A. Nelson, 2007

Relativity in GNSS

Special Relativity

General Relativity

The Principle of Relativity: *The laws of physics take the same form in every inertial frame*

The Principle of Constancy of the Speed of Light: *The speed of light c is the same in all inertial reference frame*

Gravitational Time Dilation

Gravitational Redshift

Time Dilation

Doppler Frequency Shift

The General Principle of Relativity: All systems of reference are equivalent with respect of the formulation of the fundamental laws of physics

Other Relativistic Effects

The Equivalence Principle: The outcome of any local non-gravitational experiment in a freely falling laboratory is independent of the velocity of the laboratory and its location in space-time.

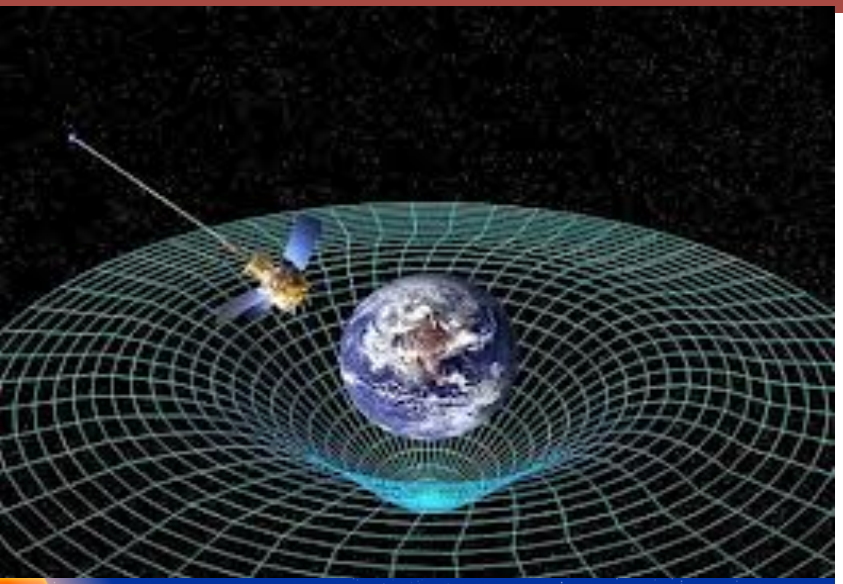
Relativistic path range effect

Relativistic Earth Rotation Effects

Relativistic effects due to orbit eccentricity

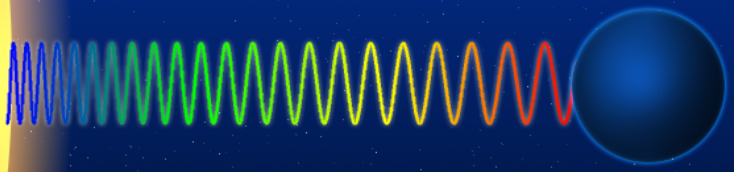
Acceleration of the satellite in the Space-Time

Time dilation, Doppler and Sagnac effect

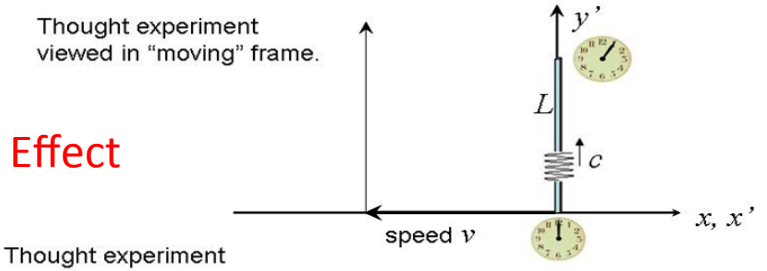


Gravitational Time Dilation

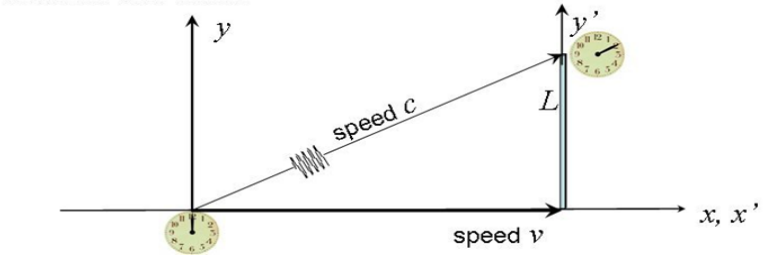
GRT

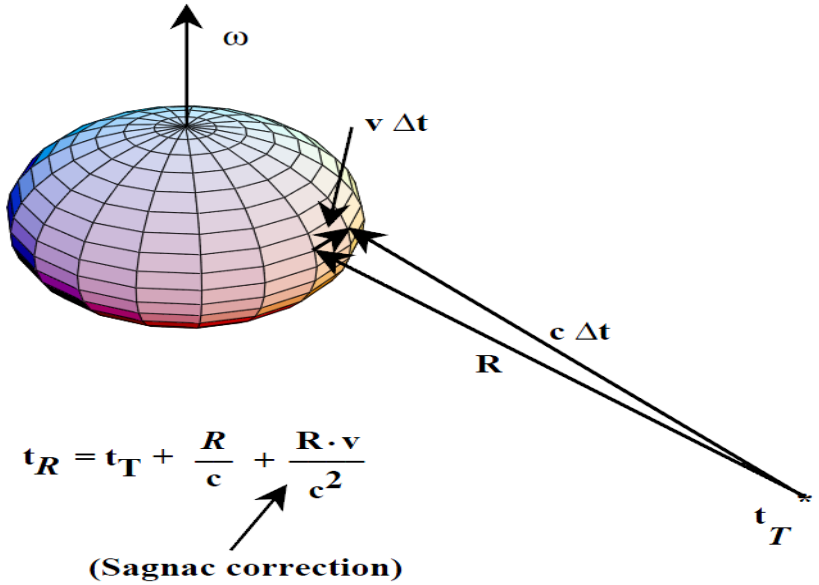


Doppler Effect



SRT





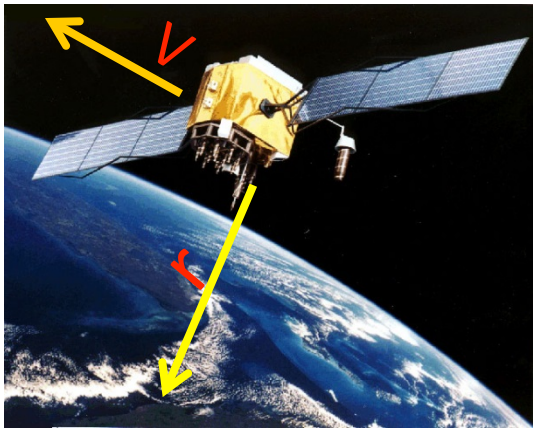
FORMULAS FOR RELATIVISTIC EFFECTS

Schwarzschild Metric in Spherical Coordinates

$$ds^2 = (1 - 2GM/rc^2)\, c^2dt^2 - [1 - 2GM/rc^2]^{-1}\, dr^2 - r^2\, d\theta - r^2\, \mathrm{sen}^2\, \theta\, d\phi^2$$

$v = r d\varphi / dt$ $\quad d\tau = ds/c$ $\quad dr = d\theta = 0$

Circular equatorial orbit



$$d\tau_E = dt\left[\sqrt{(1 - 2GM/r_E c^2)} + v_E^2/c^2\right]$$

MacLaurin Series

$$\sqrt{1 - x} = 1 - \frac{1}{2}x + \cdots$$

$$d\tau_S = dt\left[\sqrt{(1 - 2GM/r_S c^2)} + v_S^2/c^2\right]$$

Total Sagnac Effect

$$\Delta t_{\text{Sagnac}} \approx \frac{1}{c^2}\int_A^B (\boldsymbol{\omega} \times \mathbf{r}) \cdot d\mathbf{r} = \frac{2\,\omega\, A}{c^2}$$

Sagnac Correction term

$$d\tau_E < d\tau_S$$

Formula for Doppler and gravitational time dilation

$$\frac{d\tau_E}{d\tau_S} = 1 - \frac{GM}{r_E\, c^2} - \frac{v_E^2}{2\, c^2} + \frac{GM}{r_S\, c^2} + \frac{v_S^2}{2\, c^2}$$

$$\phi = -GM/r$$

Amplitude of Orbit Eccentricity

$$\Delta t_{rel} = \frac{2}{c^2}\sqrt{GM\, a}\; e \sin E = \frac{2\,\mathbf{r}\cdot\mathbf{v}}{c^2}$$

WORKED EXAMPLE BEIDOU I

EARTH AND PHYSICAL PARAMETERS

$GM = 3{,}986004418 \times 10^{14}\ m^3\ s^{-1}$

$v_E = 465\ ms^{-1}$

$c = 299792458\ ms^{-1}$

$r_E = 6371000\ m$

$\omega = 7{,}2921151467 \times 10^{-5}\ \text{rad}\, s^{-1}$

BEIDOU PARAMETERS

$r_s = 27521000\ m$

$v_s = 3806\ \text{m}\, s^{-1}$

GRAVITATIONAL TIME DILATION (GRT) AND DOPPLER TIME DILATION (SRT)

$$\frac{d\tau_E}{d\tau_S} = 1 - \frac{GM}{r_E\, c^2} - \frac{v_E^2}{2\, c^2} + \frac{GM}{r_S\, c^2} + \frac{v_S^2}{2\, c^2}$$

$d\tau_E / d\tau_S = 1 - 4{,}5559 \times 10^{-10}$

$d\tau_E / d\tau_S = 1 - 6{,}96127 \times 10^{-10} - 1{,}20291 \times 10^{-12} + 1{,}61151 \times 10^{-10} + 8{,}05872 \times$

$\delta_{rel} = 4{,}5559 \times 10^{-10} \times 24 \times 3600 \times 10^{9} = 39363{,}124\ ns/day$

$\varepsilon_{rel} = 39363{,}124 \times 10^{-9} \times 299792458 \times 10^{-3} = 11{,}801\ Km/day$

$d\tau_E / d\tau_S = 0{,}999999999544078$

$f_{rel} = d\tau_E / d\tau_S \times 10{,}23\ MHZ = 10{,}229999953393\ MH$

WORKED EXAMPLE BEIDOU II

$$ds^2 = (1 - 2GM/rc^2)c^2\,dt^2 - (1 + 2GM/rc^2)dr^2 - r^2\,d\theta^2 \qquad a_E = 6378137\ m$$

Earth Equatorial radius

$$t = t' \; ; \qquad r = r' \; ; \qquad \theta = \theta' \; ; \qquad \varphi = \varphi' + \omega t'$$

$$ds^2 = (1 - 2GM/rc^2)c^2\,dt^2 - 2\omega r^2 \sin^2\theta\, d\varphi dt - (1 + 2GM/rc^2)dr^2 - r^2\,d\theta^2 -$$

$$\Delta t_{\text{Sagnac}} \approx \frac{1}{c^2}\int_A^B (\boldsymbol{\omega}\times\mathbf{r})\cdot d\mathbf{r} = \frac{2\omega A}{c^2}$$

$$\Delta t_{Sagnac} \approx 2\omega A/c^2 = 207{,}38\ ns \qquad A = \pi * a_E^2$$

$$\Delta t_{Sagnac} \approx (\Delta r_s\, v_E/c^2) \times 10^9 = 142{,}3888$$

Sagnac Effect Correction

Amplitude of Orbit Eccentricity

$$E - e\sin E = \sqrt{GM/a^3}\ (t - t_p) \qquad dE/dt = \sqrt{GM/a^3}\ /1 - e\cos E$$

$$\Delta t_{rel} = +2\sqrt{GMa}/c^2\ e\sin E + constant$$

$$\Delta t_{rel} = \frac{2}{c^2}\sqrt{GM\,a}\ e\sin E = \frac{2\,\mathbf{r}\cdot\mathbf{v}}{c^2}$$

$$\Delta t_{rel} = +4{,}4428\times10^{-10} \times 0{,}02 \times \sqrt{27521000} \times 10^9 = 46{,}6143\ ns$$

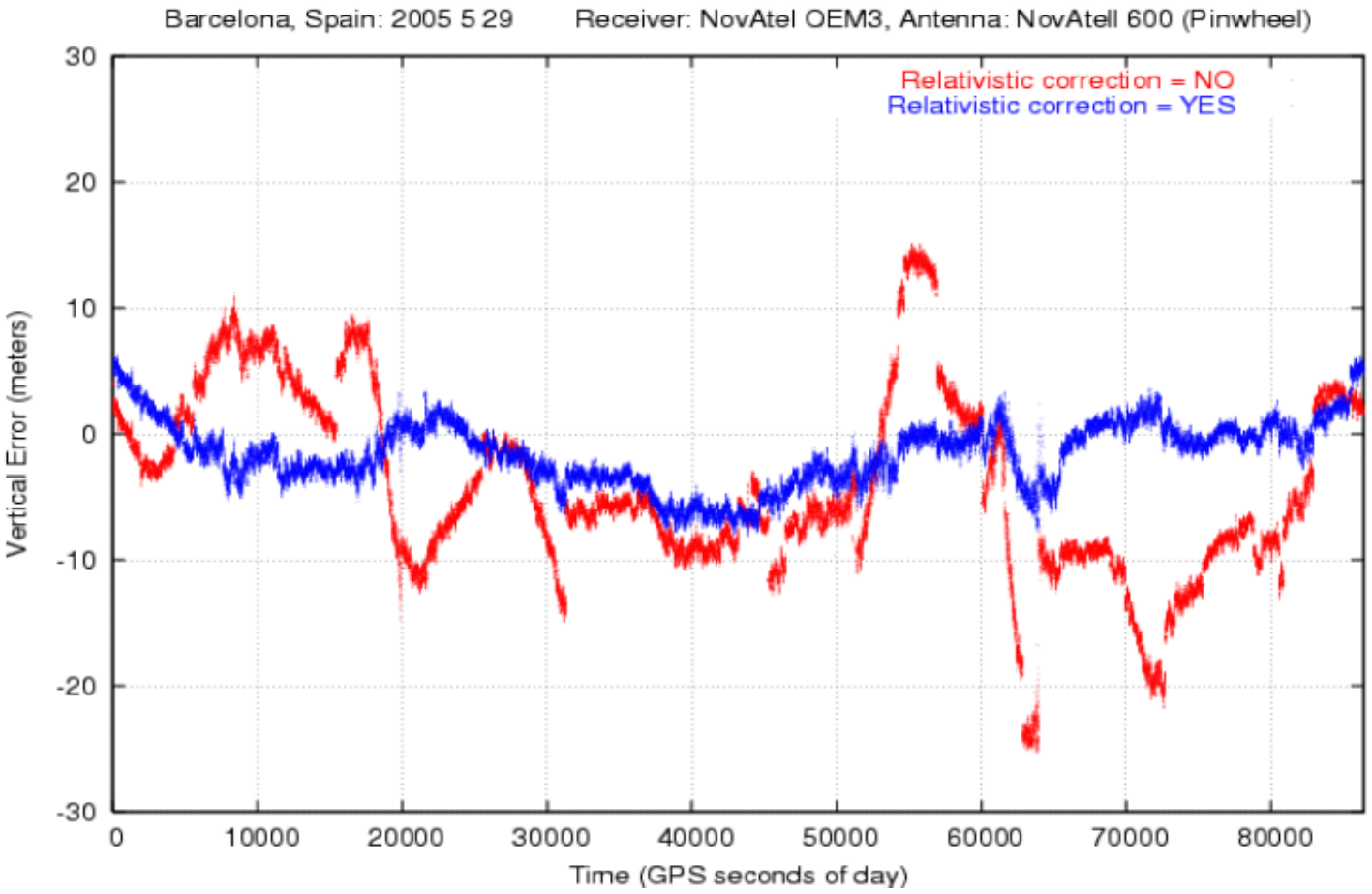
GNSS PARAMETERS

SATELLITE		ISS	GLONASS	GPS	BEIDOU	GALILEO	GEO
SEMIMAJOR AXIS	Km	6775	25471	26562	27521	29994	42164
ECCENTRICITY	Km	0.01	0.02	0.02	0.02	0.02	0.01
INCLINATION	deg	51.65	63.15	54.28	53.51	55.17	43.5
ARGUMENT OF PERIGEE	deg	270	171.97	356.91	250.63	270.56	270
APOGEE ALTITUDE	Km	425	19100	20715	21150	24216	35954
PERIGEE ALTITUDE	Km	416	18900	19652	20900	23016	35632
ASCENDING/ DESCENDING NODE ALTITUDE	Km	420.5	19000	20173	21025	23604	35793
PERIOD OF REVOLUTION	s	5570.4	40140	43082	46380	51697	86163
MEAN VELOCITY	Km/ s	7.660	3.953	3.874	3.806	3.645	3.075

RELATIVISTIC EFFECTS ON CLOCKS AND SIGNALS FOR GNSS

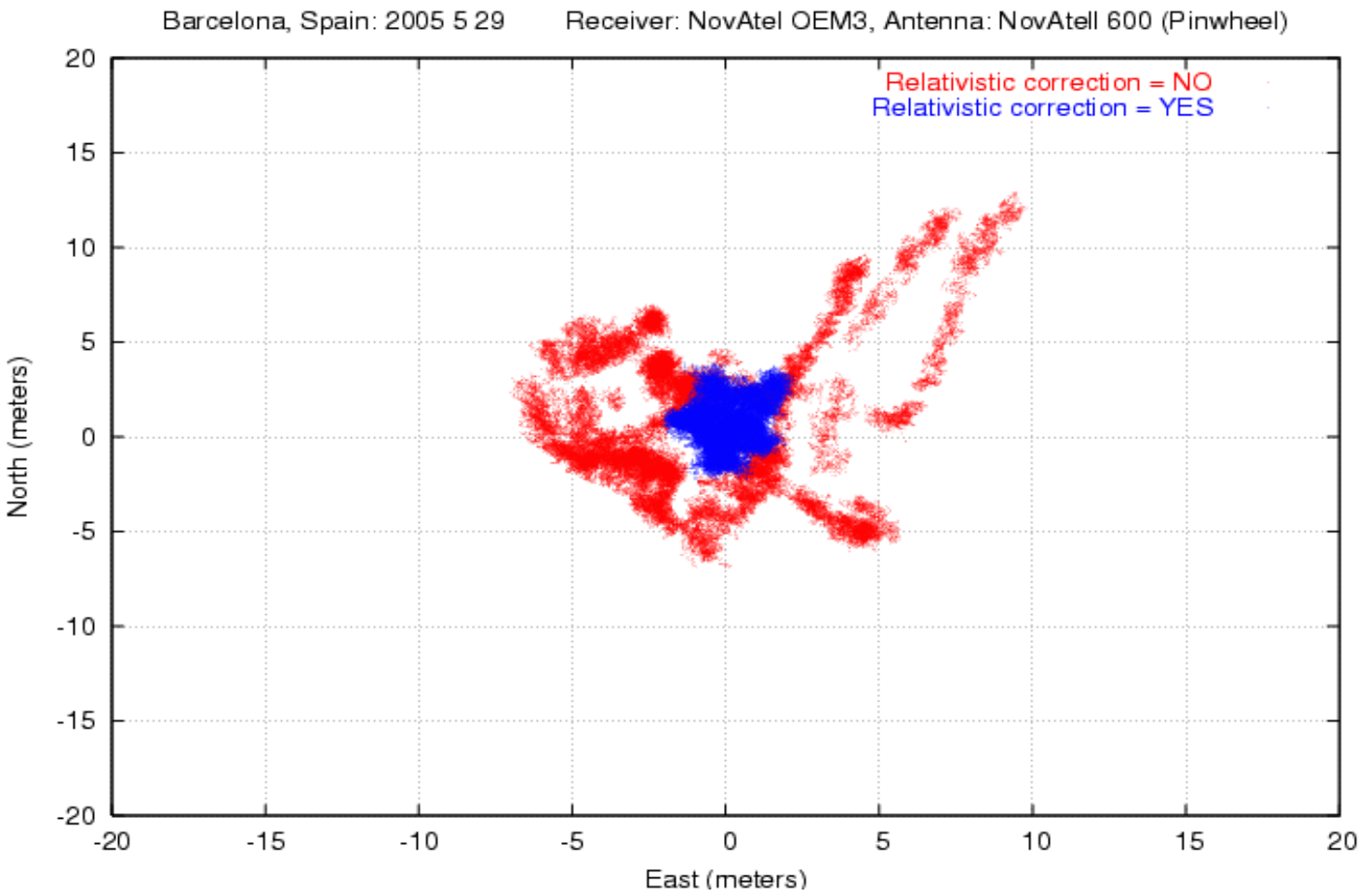
CLOCK EFFECTS		ISS	GLONASS	GPS	BEIDOU	GALILEO	GEO
DOPPLER TIME DILATION	μs/d	-28.099	-7.407	-7.110	-6.859	-6.282	-4.441
GRAVITATIONAL REDSHIFT	μs/d	3.587	45.101	45.154	46.222	47.370	51.057
NET SECULAR EFFECT	μs/d	-24.512	37.694	38.044	39.363	41.088	46.616
AMPLITUDE OF ECCENTRICITY EFFECT	ns	11.56	44.84	45.79	46.61	48.66	28.85
SIGNAL PROPAGATION							
MAXIMUM SAGNAC EFFECT	ns	35.05	131.78	137.43	142.39	155.18	218.15

VERTICAL POSITIONING ERROR (SRT and GRT)



Courtesy by:
http://www.gage.es/sites/default/files/TEACHING_MATERIAL/THEORY_SLIDES/Lecture_UPC_3.v0.0.pdf

HORIZONTAL POSITIONING ERROR (SRT and GRT)



Courtesy by:
http://www.gage.es/sites/default/files/TEACHING_MATERIAL/THEORY_SLIDES/Lecture_UPC_3.v0.0.pdf

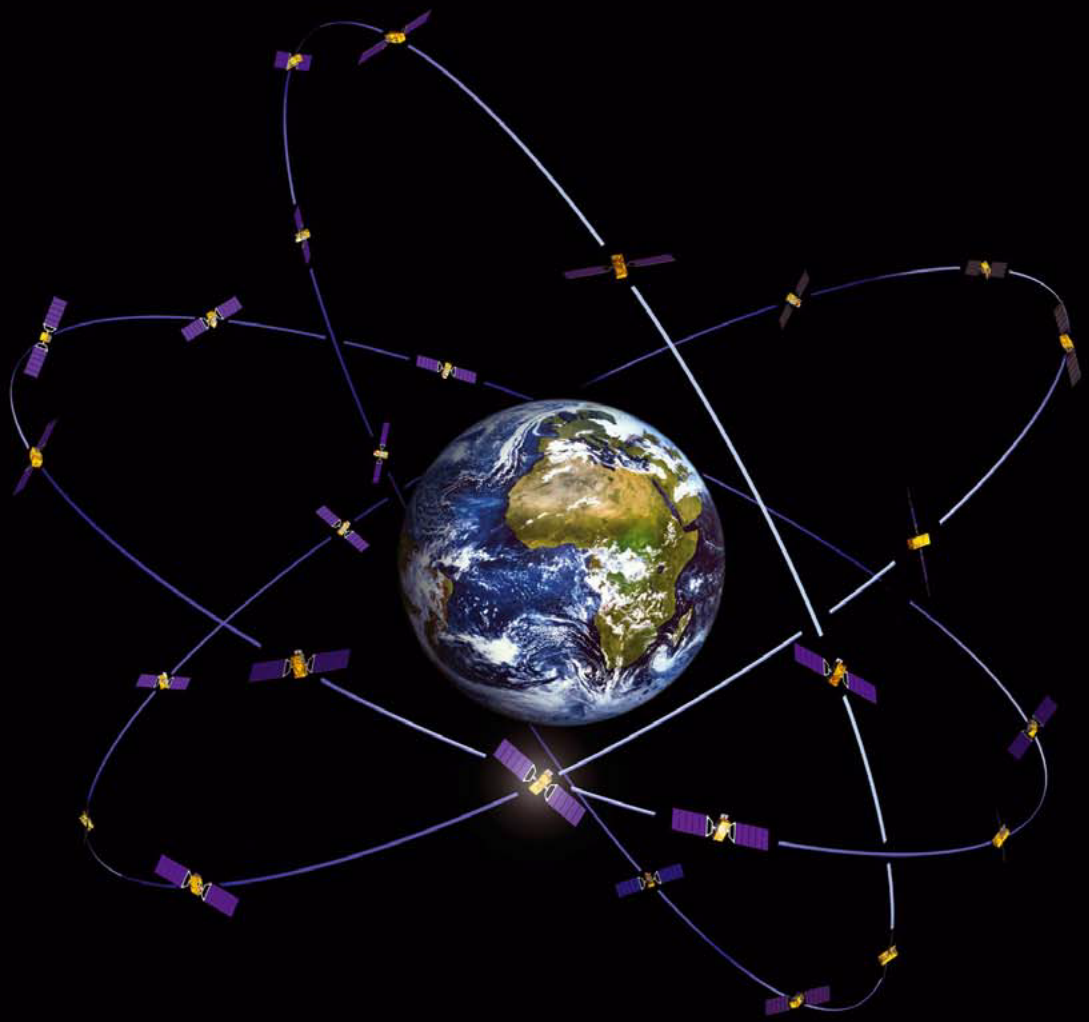
FINAL COMMENTS

- Coll and Tarantola (2009) have proposed a fully relativistic model, but require a theoretical development. Still, there are four teams actively working on these ideas, they are in Ljubljana, Syrte (Observatory of Paris), University of Valencia and ESA (Advanced Concepts Team). They are determined to turn this idea into a deployment for the Galileo System.

- This is an approach where we can see frequency variation of the signal emitted by a satellite relative to the other in the surface of the earth, due to the high speed of the satellite relative to the Earth observer and also because the difference of gravitational potential between them. Both combined effects yield 39,363 ms/day of advance in time of the satellite's clock with respect to the ground one. It represents a 11,801 km / day in positioning error for the case of Beidou (MEO).

- Finally, I am not taking into account DeSitter precession of the order of 2x10 ^ -11 m/s^2 and the Lense-Thirring Frame Dragging of the order of 1x10^-12 m/s^2 (Steigenberger, 2009) due to limitations in its detection, except in the experiment Gravity Probe B, (see the link below http://einstein.stanford.edu/). This is only as motivation for future researchers.

•"The most beautiful thing we can experience is the mysterious. It is the source of all true art and all science. He to whom this emotion is a stranger, who can no longer pause to wonder and stand rapt in awe, is as good as dead: his eyes are closed." Albert Einstein



GRACIAS